A Constructive Proof of the Existence of a Utility in Revealed Preference Theory

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Abstract

Within the context of the standard model of rationality within economic modelling we show the existence of a utility function that rationalises a demand correspondence, hence completely characterizes the associated preference structure, by taking a dense demand sample. This resolves the problem of revealed preferences under some very mild assumptions on the demand correspondence which are closely related to a number of established axioms in preference theory. The proof establishes the existence of a limit of a sequences of indirect utilities that rationalise finite data sets, where the sample size increases to infinity. This limiting utility provides a rationalisation of the demand relation. Up to a rescaling this limiting indirect utility is unique and continuous on a set of full measure.

1 Introduction

1.1 Posing the question

A demand relation $X(p)$ describes what can be, in principle, observed about the preferences of an agent such as an individual consumer: There are n commodities and a positive budget, that we normalise to 1, that constitutes the consumer's wealth. Then given a vector of n nonnegative prices p (with $p \neq 0$) $X(p)$ returns a commodity bundle x that the consumer is *observed* to buy with their budget. In other words, out of all commodity bundles that are within budget, none is more preferred by the consumer than $x \in X(p)$. When $x \in X(p)$ and $\langle p, x - y\rangle \geq 0$ then y is within budget but not chosen. We then say that x is a (directly) revealed preference to y and denote this by $x \succeq_R y$.

As not all observations are directly revealed, a preference relation may be deduced via transitivity of the directly revealed preferences, i.e., $x \succeq_R y$ and $y \succeq_R z$ would imply that x is preferred to z. This is only possible if observed behavior is at least consistent under this transitive closure and this has become axiomatic in the theory, known as the *Generalized Axiom of Revealed Preferences* (GARP). As in other studies of revealed preference theory we will posit this as a standing assumption.

We do not know *a priori* whether there exists a utility function representing such a preference relation, an existence problem known as the *problem of revealed preferences.* Typically in revealed preference theory [[Houthakker (1950)], [Samuelson (1947)], [Varian (2006)], [Varian (1982)]], additional structure of the demand relation is postulated to derive a preference relation. The debate about which axioms are essential or most justifiable is ongoing but these are critical to the building of utility functions. In the main such theories have been relational in nature, often giving rise to geometric constructions. The role of the axioms in revealed preference theory in relation to (topological) convergence issues has not before been fully appreciated and the study of these issues will constitute the main contribution of this paper.

Our specific goal is to understand when demand relations, and therefore preferences, can be described by a utility function and, in particular, when such a utility can be reconstructed from any appropriate sample of price-commodity pairs $\{(p_i, x_i)\}_{i=1}^{\infty}$ where $x_i \in X(p_i)$ for each i. Not having such a utility to study in the first place, we construct it by approximation: for a finite sample

$\{(p_i, x_i)\}_{i=1}^m$ of m price-commodity bundles satisfying the demand relation, we may construct a *finite rationalisation*, namely, a utility u_m that is consistent with this data[1]. Such rationalisations are well known thanks to Afriat's classical construction [Afriat (1967)]; alternative rationalisations include the recent work of [Crouzeix et al. (2011)]. What may be surprising is that although Afriat's finite rationalisations are concave, there may not exist a concave rationalisation of the entire demand relation (see example 7 in section 2.1). Thus an understanding of the asymptotics of finite rationalisations, i.e., as the sample size m grows to ∞, is a challenge that needs techniques beyond convex analysis.

A final introductory point is that our approach to the asymptotics of sampling preference relations is via indirect utilities, which are dual to direct utilities in that they describe the maximum utility that could be gained within a unit budget. The notation for this follows.

1.1.1 Our assumptions and main result

We seek a utility that rationalises $X(\cdot)$. That is a quasiconcave function u on $\mathbb{R}^n_+$ that prescribes to each $x \in \mathbb{R}^n_+$ a number which represents the degree of preference, namely $x \in \mathbb{R}^n_+$ is preferred to $y \in \mathbb{R}^n_+$ if $u(x) \geq u(y)$. Given a (normalized) unit budget, we define the *indirect utility* $v(p)$ by

$$v(p) := \max_{x \in \mathbb{R}^n_+} \{u(x) \mid \langle p, x \rangle \leq 1\}. \tag{1}$$

This assigns the greatest utility that can be attained, within a budget of 1, at the price p. Under mild conditions we can recover u from v via the duality identity

$$u(x) = \inf_{p \in \mathbb{R}^n_+} \{v(p) \mid \langle p, x \rangle \leq 1\}, \tag{2}$$

see section 2.3. Similarly when such a utility exists we can recover the demand relation via

$$X(p) \;=\; \underset{x \in \mathbb{R}^n_+}{\operatorname{argmax}} \{u(x) \,:\, \langle p, x \rangle \leq 1\} \;:=\; \{x \in C \mid u(x) \geq v(p)\}. \tag{3}$$

Indeed we can recover $X(\cdot)$ from the indirect utility v by the connecting identity (see Proposition 18 in section 2.3)

$$X(p) \;=\; -N_v(p) \cap \{x \mid \langle x, p \rangle = 1\}, \tag{4}$$

where $N_v(p)$ is the normal cone to the (lower) level set of v given by

$$S_v(p) \;:=\; \{p' \mid v(p') \leq v(p)\}.$$

Convexity of these level sets also holds — that is, v is *quasiconvex* — which allows application of convex analysis (e.g., the usual normal cone construction above). Given a set $S \subset \mathbb{R}^n$, $\operatorname{co} S$ denotes its convex hull, and $\operatorname{cone} S := \cup_{\lambda \geq 0} \lambda S$ is its cone.

We will construct an indirect utility v that satisfies (4) as a kind of limit of a sequence of indirect utilities v_m each of which rationalises the data set $\{(p_i, x_i)\}_{i=1}^m$, that is $-x_i \in N_{v_m}(p_i)$ with $\langle x_i, p_i \rangle = 1$ for $i = 1, \ldots, m$. More generally, we say v_m is an *indirect finite rationalisation* or simply a *finite rationalisation* when it is clear that it is an indirect utility, if it is quasiconcave and the previous inclusion holds for $i = 1, \ldots, m$.

To date neither the existence of topological limit points of finite rationalisations $\{v_m\}$, as the sample size m goes to infinity, nor uniqueness of limit points has been studied. (Likewise, the asymptotics of the associated sequence of direct utilities has not been analysed.) These issues form the core questions around which this paper is directed. Critically we must make certain assumption on the nature of X from which we are sampling. We state our assumptions here and, after giving some notation, summarize our main results. Then we will briefly review these assumptions by placing them in the literature on revealed preference theory. Recall that the graph of $X(\cdot)$ is the set

$$\operatorname{graph}(X) \;:=\; \{(p, x) \mid x \in X(p)\}$$

so that a countable sample from this demand relation is given by the inclusion $\{(p_i, x_i)\}_{i=1}^\infty \subset \operatorname{graph}(X)$.

Demand relation assumptions

[1] Each u_m rationalises $\{(p_i, x_i)\}_{i=1}^m$ in that $x_i \in \operatorname{argmax}_{x \in \mathbb{R}^n_+} \{u_m(x) \mid \langle p_i, x \rangle \leq 1\}$ for $i = 1, \ldots, m$.

A1 The demand relation $X(\cdot)$ satisfies the generalised axiom of revealed preference, GARP (see section 1.2.1).

A2 The images $X(p)$ are nonempty and convex.

A3 The graph of $X(\cdot)$ is closed (i.e., if $x_i \in X(p_i)$ and $\lim_{i\to\infty}(p_i, x_i) = (x, p)$, then $x \in X(p)$).

Sampling assumption

A4 1. D, which denotes closure of $\{p_i\}_{i=1}^{\infty}$, is a convex set in $\mathbb{R}_+^n$ that has nonempty interior, $\operatorname{int} D$, and does not contain zero.

2. C, which denotes the closed, convex hull $\overline{\text{co}}\{x_i\}_1^{\infty}$, is a set in $\mathbb{R}_+^n$ that is bounded and does not contain zero.

Our first main result says that a dense sample of a demand relation $X(\cdot)$ is enough to reconstruct the entire relation; more than that, we can give a direct utility function u and indirect utility function v representing $X(\cdot)$. We make a few introductory remarks to link the assumptions and conclusions of Theorem 1. Axiom A1 implies that the level sets $S_{-u}(x)$ and $S_v(p)$ are convex: quasiconvexity of v and quasi-concavity of u. In concert with A2 and A3, we also anticipate a lower semicontinuity property of v, namely, the level sets $S_v(p)$ and strict level sets $\tilde{S}_v(p)$, defined by

$$\tilde{S}_v(p) \;:=\; \{p' \mid v(p') < v(p)\},$$

have the same closures when $v(p) > \inf_{p'} v(p')$. This closure property is essentially a consequence of being able to recover a preference relation by taking limits of dense sample data. Quasi-convex functions with this closure property are called *g-pseudo-convex* [Crouzeix et al. (2008)]. Interiority in Axiom A4 implies v is *solid*, i.e., $S_v(p)$ has interior when $v(p) > \inf_{p'} v(p')$. It turns out that $-u$ is also solid, in short u is *solid g-pseudo-concave*. Finally we note that the class of utilities u, or indirect utilities v, that represent a demand relation has at least one degree of freedom: for any strictly increasing function $k : \mathbb{R} \to \mathbb{R}$ we have that $k \circ u$ or $k \circ v$ represents, indirectly or directly, the same preferences (we will expand on this in section 1.1.2). We therefore focus on *normalised* utilities: v is said to be normalised in some direction $e \in \mathbb{R}^n$, or *e-normalised* for short, if $v(te) = -t$.

Theorem 1 *Under Assumptions A1-A4:*

1. *There exists an indirect utility function $v : D \to \mathbb{R}_+$ that rationalises $X(\cdot)$, i.e.,* (4) *holds, and that is solid and g-pseudo-convex and continuous almost everywhere in the interior of D.*

2. *For $e \in \operatorname{int} D$, being a fixed direction, we may further assume that v is e-normalised and lower semicontinuous, in which case v is the unique function with these properties that indirectly rationalises $X(.)$ on $\operatorname{int} D$.*

3. *In the general case (corresponding to part 1), the function $u : C \to \mathbb{R}_+$ given by* (2) *is a direct utility that rationalises $X(\cdot)$, i.e., satisfies* (3) *for $p \in \operatorname{int} D$, and that is solid g-pseudo-concave and continuous almost everywhere in* $\operatorname{int} C$.

Parts 1 and 2 of this result are given by Theorem 35 and Corollary 37 in Section 5. Part 3 is a consequence of Proposition 15 and the fact that any function u satisfying the duality (2) must be a monotonically nondecreasing functions with respect to the cone corresponding to the positive orthant. Thus the continuity is a consequence of the results of [Crouzeix et al. (1987)].

We stress the role of the relationship (3) in the subsequent proof of this result. Let Assumptions 1–4 hold and $\{f_m\}$ be any sequence of indirect utilities where each f_m rationalises $\{(x_i, p_i)\}_{i=1}^{m}$ and is normalised in a direction $e \in \operatorname{int} D$. Consider an epi-convergent subsequence of $\{f_m\}$, which exists by Theorem 34, and whose (lower semicontinuous) epi-limit is denoted v. Combining Theorem 34–35, any such v satisfies

$$N_v(p) = -\operatorname{cone} X(p) \quad \text{on } \operatorname{int} D$$

which is equivalent to (3). That is, normal cones to the level sets of asymptotic indirect utilities are independent of the particular epi-limit v. Indeed these normal cones are also independent of the demand relation sample $\{(x_i, p_i)\}_{i=1}^{\infty}$ so long as the closure of $\{p_i\}$ is D. It is easy to establish (see Lemma 36) that lower semicontinuous quasiconvex functions are completely determined by the normal cones to their level sets, and their values in a particular direction (i.e., on the ray et for $t > 0$). It follows since v is e-normalised that is it the *unique* lower semicontinuous e-normalised indirect utility that represents X.

We now link the properties of u, v above to notions from the economics literature. Consider a solid g-pseudo-concave utility u. Instead of solidity of u we can stipulate that it is *nonsatiated*, which is a standard notion from preference theory: for all $x \in \mathbb{R}^n_+$, $\delta > 0$ there exists $x' \in B_\delta(x)$ with $x' \geq x$, $x \neq x'$ and $u(x) < u(x')$. Likewise, $-v$ is nonsatiated. Indeed if we take a g-pseudo convex function v, where $-v$ is nonsatiated, the associated demand relation generated via (4) possess the properties A1–A4. For instance the necessity of GARP is given by [Eberhard et al. (2007)].

GARP is also known to be characterised by the existence of a convex utility that rationalise any finite data set [Fostel et al. (2004)]. Convexity and closure of the images $X(p)$, which follow from (4), or (3) if X is generated via a quasi-convex utility u, are also standard assumptions in revealed preference theory and have been justified in their own right [John (1998)]. A3, graph closure, is a kind of continuity that follows from g-pseudo convexity as discussed above. Axioms A2 and A3 are closely related to a maximality property for pseudo-monotone operators which are also a consequences of (4) when X is generated via a solid, g-pseudo-convex, indirect utility. Regarding A4, prices and commodity bundles are clearly positive n-tuples this essentially asserts that one should be able to probe the demand with prices within an open neighbourhood and obtain non-trivial preferences. This turns out to be closely related to the nonsatiation assumption. As $D \subseteq \mathbb{R}^n_{++}$ is bounded one can argue that the boundedness of $\text{co}\{x_i\}_1^\infty$ follows from the budget constraint and the positivity of commodity bundles.

1.1.2 Relating convergence to the work [Crouzeix et al. (2011)]

Here we begin to describe the framework that leads to Theorem 1. The asymptotic analysis that is required is closely related to recent work of [Crouzeix et al. (2011)] and which starts with a utility function u that represents the demand relation $X(\cdot)$ and is appropriately normalised[2] in a direction $d \in \text{int}\, C$. [Crouzeix et al. (2011)] constructs functions u_m^- and u_m^+ on $\mathbb{R}^n_+$ that rationalise $\{(p_i, x_i)\}_{i=1}^m$, are normalised in the direction d, and that "sandwich" the direct utility u as

$$u_m^-(x) \;\leq\; u(x) \;\leq\; u_m^+(x), \quad \text{for } u \in \mathbb{R}^n_+.$$

Moreover, u_m^- and u_m^+ sandwich any other normalised finite rationalisation and are monotonic in m: $u_m^-(x) \leq u_{m+1}^-(x) \leq u_{m+1}^+(x) \leq u_m^-(x)$ for $x \in C$.

Since the construction of u_m^- and u_m^+ is entirely symmetric in x and p, we may apply it to samples ordered as (x_i, p_i) instead of (p_i, x_i) using a normalisation direction $e \in \text{int}\, D$. Assuming there is a normalised indirect utility v that gives the demand relation via (4), this dual construction produces normalised indirect rationalisations $\check{v}_m^-$ and $\check{v}_m^+$ that sandwich $-v$. Thus by defining $v_m^- := -\check{v}_m^+$ and $v_m^+ := \check{v}_m^-(p)$, we sandwich v and any indirect utility that rationalises $\{(p_i, x_i)\}_{i=1}^m$ and is e-normalised.

Philosophically we depart from [Crouzeix et al. (2011)] in that we do not yet know whether our demand relation $X(\cdot)$ can be represented by an indirect utility v or its direct counterpart u. Nevertheless v_-^m and v_+^m sandwich any indirect utility that is a normalised finite rationalisation of $\{(p_i, x_i)\}_{i=1}^m$. For example we can always use Afriat's construction to generate an indirect rationalisation v_m and then can normalise it (as explained in section 1.2.1 to follow) to produce an indirect, normalised rationalisation f_m. Then v_-^m and v_+^m will sandwich f_m.

We summarise the properties of v_-^m and v_+^m for later use. As explained above, this result is due to [Crouzeix et al. (2011)] although we have transposed the original version from direct to indirect utilities.

Lemma 2 *Suppose the finite sample of data $\{(p_i, x_i)\}_{i=1}^m$ satisfies GARP. Then there exist real valued quasiconvex functions v_m^-, v_m^+ on $\mathbb{R}^n_+$ that*

1. *indirectly rationalise $\{(p_i, x_i)\}_{i=1}^m$,*

2. *are e-normalised,*

[2] A utility u that is normalised in a direction d satisfies $u(td) = t$ for $t > 0$. Normalising an indirect utility requires $-t$ on the right hand side.

3. *are lower and upper bounds, respectively, on any indirect utility* f_m *that rationalises* $\{(p_i, x_i)\}_{i=1}^m$ *and is e-normalised:*

$$v_m^-(p) \le f_m(p) \le v_m^+(p), \quad \text{for } p \in \mathbb{R}_+^n,$$

and

4. *are monotonic in m:* $v_m^-(p) \le v_{m+1}^-(p) \le v_{m+1}^+(p) \le v_m^-(p)$ *for* $p \in \mathbb{R}_+^n$.

See section 1.2.1 for details of the normalisation procedure which converts a convex (e.g., Afriat) rationalisation v_m into a normalised rationalisation f_m that is quasiconvex. When there is no convex indirect utility function associated with $X(\cdot)$, as in example 7 in section 2.1, we will still prove that $\{f_m\}$ converges to give existence of a quasi-convex (indeed, solid g-pseudoconvex) indirect utility function v. The normalisation process is critical in the proof because it will avoid theoretical and numerical difficulties that arise, as the sample size increases, when approximating a nonconvex function by a convex function.

As the sample size m increases, the lower minorant increases and the upper majorant decreases. If the sampling is dense then one arrives (via a monotonic limit) at an interval $[v^-(p), v^+(p)]$ which must bracket a true indirect utility value. However [Crouzeix et al. (2011)] does not consider asymptotics of this sandwiching process. For instance the question of whether or when $[v_m^-(p), v_m^+(p)]$ shrinks to a singleton is not addressed. Moreover any sequence of rationalising, normalised mappings $\{f_m\}$ are unlikely to be monotonic and therefore convergence of any kind is left open.

For any sequence of normalized quasiconvex indirect utilities $\{f_m\}_{m=1}^\infty$, where f_m rationalises $\{(p_i, x_i)\}_{i=1}^m$, we may claim the following (assembled from Corollaries 38–40 in Section 5):

Theorem 3 *Under Assumptions A1-A4: Let* v_m^- *and* v_m^+ *be the e-normalised indirect finite rationalisations as described in Lemma 2, and* $v^- = \lim_m v_m^-$ *and* $v^+ = \lim_m v_m^+$ *denote their respective pointwise (monotone) limits. Let* $\{f_m\}$ *be any sequence of mappings on* $\mathbb{R}_+^n$ *such that each* f_m *that indirectly rationalises* $\{(p_i, x_i)\}_{i=1}^m$ *and is e-normalised. Then*

1. *There is a (unique) lower semicontinuous function* $\bar{v} : \mathbb{R}_+^n \to \mathbb{R}$ *that is the lower closure of any quasiconvex function satisfying* $v(p) \in [v^-(p), v^+(p)]$ *for* $p \in \mathbb{R}_+^n$.

2. $\bar{v}$ *is the (unique) epi limit of* $\{f_m\}_{m=1}^\infty$.

3. $\bar{v}$ *rationalises the full data set* $\{(p_i, x_i)\}_{i=1}^\infty$.

4. $\bar{v}$ *is independent of the particular sample* $\{(p_i, x_i)\}_{i=1}^\infty$, *provided the closure of* $\{x_i\}_{i=1}^\infty$ *is D, and of the sequence of normalised indirect finite rationalisations* $\{v_m\}_{i=1}^\infty$ *that are fit to these data.*

5. $\bar{v}$ *rationalises the demand relation X.*

6. *For almost all* $p \in D$, v^+ *is continuous at p and, as a result,* $\bar{v}(p) = v^-(p) = v^+(p)$.

See Remark 41 for some comments on applying this convergence framework to direct rather than indirect utilities. We focus on the latter because they are more naturally related to sampling the demand relation, i.e., to probe consumption one would change the price slightly and observe the change in consumption within the same budget. More importantly this is more closely related to elasticities [Kocoska et al. (2009)].

1.2 The Anatomy of the Proof

The purpose of this section is to provide an extended abstract of the proof of the main results. Much of the difficulty encountered in proving the results stems from the need to study epi-limits of g-pseudoconvex functions which are not convex functions. Thus we wish to provide a conceptual "roadmap" of the crucial techniques that are developed in order to furnish a proof before entering into details. It is hoped that this will help the reader to understand the role of a number of technical constructions.

1.2.1 Constructing Approximate Indirect Utilities

Given $\{(x_i, p_i)\} \subset \operatorname{Graph} X$ the transitive closure of $\succeq_R$ gives a partial order $\succeq$ (of revealed preferences) that is we denotes $x \succeq y$ when there exists $x = x_0, x_1, \ldots, x_m = y$ with $x_{i+1} \succeq_R x_i$ for all i. Given $x_i \in X(p_i)$ and $x_j \in X(p_j)$, we say $x_j \succ_R x_i$ if $\langle p_j, x_j \rangle < \langle p_j, x_i \rangle$ while $x_i \notin X(p_j)$. Similarly we denote $x \succ y$ when $x \succeq y$ and there exists i with $\langle p_{i+1}, x_{i+1} - x_i \rangle > 0$ for $(x_i, p_i) \in \operatorname{Graph} X$.

The *generalised axiom of revealed preference* (GARP) says that there cannot exist a cycle $\{(x_i, p_i)\}_{i=0}^m$ with $x_0 = x_{n+1}$ and $\langle p_{i+1}, x_{i+1} - x_i \rangle \geq 0$ unless $\langle p_{i+1}, x_{i+1} - x_i \rangle = 0$. This axiom is clearly necessary for a consistent transitive closure partial order in that $\langle p_{i+1}, x_{i+1} - x_i \rangle > 0$ for some i would imply the contradiction $x_{n+1} \succ x_0$. Indeed it is known that GARP is necessary and sufficient for the existence of a preference order $\succeq_R$ such that $x \succeq_R y$ whenever $x \succeq y$, and $x \succ_R y$ whenever $x \succ y$ (see [Kanna (2004)]). It is also necessary and sufficient for the fitting of the Afriat indirect utility v_m to rationalise a finite data set $\{(x_i, p_i)\}_{i=1}^m$ [Afriat (1967)], [Fostel et al. (2004)].

Thus either via Afriat's construction or that of [Crouzeix et al. (2011)] we can obtain a set of convex, polyhedral, level curves of an indirect utility $v_m : \mathbb{R}^n_{++} \to \mathbb{R}_-$ for each data set $\{(x_i, p_i)\}_{i=1}^m$ such $-x_i \in N_{v_m}(p_i)$ for $i = 1, \ldots, m$. To make a comparison which accounts for nonuniqueness of indirect utilities that represent a demand relation, we also normalize the indirect utility. For each m define the strictly decreasing, function via the fitted indirect Afriat utility $v_m(p)$ which has been translated so that $v_m(0) = 0$ e.g. $v_m(x) \leq 0$. Begin by taking some $e \in \operatorname{int} D$, hence $e > 0$, taking m sufficiently large so that $e \in \operatorname{int} \operatorname{co} \{p_i\}_{i=1}^m$ and forming $k_m(t) := v_m(et)$ where $t > 0$. As long as we sample (x_i, p_i) with $x_i \neq 0$ then $t \mapsto k_m(t)$ is strictly decreasing. We then normalise our Afriat utilities to produce another sequence of equivalent utilities

$$f_m(p) := -k_m^{-1}(v_m(p)) \tag{5}$$

which is the composition of a convex function on $\mathbb{R}^n_{++}$ and a concave continuous increasing mapping on $\mathbb{R}$. Now e lies on the level curve $\{p \mid f_m(p) = -1\}$ for each m and also $\tau \mapsto f_m(\tau e) = -\tau$ is finite. Whenever we have $f_m(te) = -t$ we say that the indirect utility f_m has been *normalized in the direction* e, or simply *normalized* when e is already fixed.

Note that when we fit the Afriat indirect utility we have $\operatorname{dom} v_m = \operatorname{dom} f_m = \mathbb{R}^n_+$. Consequently by the chain rule of generalized gradients [Rockafellar et al. (1998)] and the fact that there exists $\mu_i^m > 0$ such that $-\mu_i^m x_i \in \partial f_m(p_i)$ for $i = 1, \ldots, m$ (as f_m rationalises $\{(p_i, x_i)\}_{i=1}^m$ as per (4)) we have

$$\partial f_m(p) = \operatorname{co}\left\{-\gamma \mu_i^m x_i \mid \gamma \in -\partial k_m^{-1}(v_m(p)) \text{ and for some } \mu_i^m > 0,\ -\mu_i^m x_i \in \partial v_m(p)\right\}. \tag{6}$$

As k_m strictly decreasing continuous then $-k_m^{-1}$ is strictly increasing and so $\gamma > 0$. Denoting the level set of f_m at $\bar{p}$ by

$$S_{f_m}(\bar{p}) \ = \ \{p \mid f_m(p) \leq f(\bar{p})\},$$

then (6) implies the normal cone to $S_{f_m}(\bar{p})$ at $\bar{p}$, denoted by $N_{f_m}(\bar{p})$ may be expresses as:

$$N_{f_m}(\bar{p}) = \operatorname{cone} \partial v_m(\bar{p}). \tag{7}$$

This is a trivial extension of the well known relationship between normal cones of level sets and subdifferentials of convex functions.

1.2.2 Epi/Quasi-convergence of a Subsequences

In order to show that a utility exists that rationalises the data set we have to establish that there is a well defined limit (in some sense) of the approximating sequence $\{f_m\}_{m=1}^\infty$. This strongly relies on the properties of epi-convergence and the properties of the underlying demand relation that we are interpolating. Epi-convergence can be directly associated with set-convergence of the level sets of the family of functions. As epi-convergent sequences always produce lower semicontinuous functions we introduce a related but weaker form convergence called quasi-convergence that does not *a priori* introduce semicontinuities but still demands an orderly stratification of limiting level sets (see section 2.2). Epi-convergence is then used as tool to study this weaker convergence.

Our convergence analysis relies on a coordinate transformation from $x \in \mathbb{R}^n$ to $(y, t) \in \mathbb{R}^{n-1} \times \mathbb{R}$ in order to represent each nonconvex function f_m, that nevertheless has convex level sets $S_{f_m}(p)$, by a family of convex functions $\{g_m(\cdot, \lambda)\}_{\lambda \in \Lambda}$ on $\mathbb{R}^{n-1}$. Here Λ is a set of scalars and each $\lambda \in \Lambda$ signifies a level of f_m, e.g., $\lambda = f(p)$ corresponds to $S_{f_m}(p)$. See section 3 for details.

Having defined the sequence $\{g_m(\cdot,\lambda)$ It will then be possible to describe normal cones to the level sets of f_m via the subdifferential of $g_m(\cdot,\lambda)$, namely

$$N_{f_m}(y,t) \;=\; \operatorname{cone}\partial g_m(\cdot,\lambda)(p).$$

Now Hausdorff set-convergence of a sequence of subdifferentials is implies by the epi-convergence of the corresponding sequence of convex functions [Attouch et al. (1991)]. Therefore this coordinate transformation from f_m to obtain the family of functions $g_m(\cdot,\lambda)$ allows us to pursue convergence of the sequence of normal cones $\{N_{f_m}(y,t)\}_{m=1}^{\infty}$ via epi-convergence of a parameterised sequence of convex functions $\{g_m(\cdot,\lambda)\}_{\lambda\in\Lambda}$ implied by the powerful epi-convergence theory for $\{f_m\}_{m=1}^{\infty}$. This results in a proposed limit, or perhaps limits, and an associated function f that is an indirect utility for X.

A technical but critical aspect of convergence comes from the fact that although the analysis described above involves convergence of cones via convergence of epigraphs, our interest is in convergence of bounded sets that generate cones and likewise of subdifferentials. This can be inferred from (4), where the demand relation is given as the "base" of a the normal cone $-N_v(p)$ where v is an indirect utility that we will show exists as an epi-limit. We therefore explore, in section 2.4, the relationship between set-convergence of families of cones and set-convergence of their bases in a somewhat abstract setting.

We've mentioned section 3 for the transformation from f_m to g_m and we further note that convergence of $\{f_m\}$ and $\{g_m\}$ is given in section 4. We give the milestones of this convergence analysis next.

A number of steps must be taken before convergence of the sequence $\{f_m\}$ can be established. First we use the compactness property of epi-convergence to extract an epi-convergent subsequence of the family functions $\{f_m(\cdot,\cdot)\}_{m=1}^{\infty}$ and sets Λ_m for which the associated level curve functions $p\mapsto g_m(p,\lambda)$ is convex for $\lambda\in\Lambda_m$. We then note that there exists a sequence $\lambda_{m_k}\downarrow\lambda\in\Lambda=\lim_k\Lambda_{m_k}$ such that $\{g_{m_k}(\cdot,\lambda_{m_k})\}_{k=1}^{\infty}$ actually epi-converges to a convex family of functions $g\left(\cdot,\lambda\right)$ for all $\lambda\in\Lambda$. One may then associate a set functions $\{g_m(\cdot,\lambda_{m_k})\}_{k=1}^{\infty}$ to a sequence of lower semicontinuous, solid, g-pseudo-convex functions $\{f_{m_k}\}_{k=1}^{\infty}$ which also epi-converge (as its level sets set-converge) to an f while retaining the normalization property $f\left(te\right)=-t$. The study of the correspondence between $\{g(\cdot,\lambda)\}_{\lambda\in\Lambda}$ and f allow one the establish the various properties of $\{g(\cdot,\lambda)\}_{\lambda\in\Lambda}$ that force the limiting function f to be solid, g-pseudo-convex and nonsatiated. The associated convergence of subdifferentials along with the rationalisation property $-x_i\in N_{f_m}\left(p_i\right)^{\dagger}=\partial g_m(\cdot,\lambda_i)\left(p_i\right)$ (for $\lambda_i=f_m\left(p_i\right)$) allows us to observe that the limiting function f's subdifferential interpolates a countably dense set of points $x_i\in X(p_i)$. Once the problem of recovery of X from such a dense selection has been studied we may deduce that

$$\begin{aligned} X(p) &\subseteq -\operatorname{cone}\{x\mid x\in\partial g(\cdot,\lambda)(p) \text{ for } \lambda=f(p)\}\cap\{x\mid\langle x,p\rangle=1\} &(8)\\ &= -N_f(p)\cap\{x\mid\langle x,p\rangle=1\}. &(9)\end{aligned}$$

This later point turns out to be critically important to this study. Many multi-valued operators, such a monotone operators, possess this recoverability property where one only needs to form the smallest similar object that interpolates a dense selection in order to recover the original operator. This is not surprising once one is able to connect X to maximal quasi-monotone operators, such as those studies in [Aussel et al. (2011)], that retain many of the properties of derivative like objects. The properties A1-A4 are sufficient to show this is true for both $\operatorname{cone}X(p)$ and $-N_f(p)$. In fact N_f must be the minimal operator of this type and $-\operatorname{cone}X$ being also from this same family forces equality in (8). The desired direct utility is then given by $u(x)=\inf\{v(p)\mid\langle p,x\rangle=1\}$ and can also be shown to be solid, g-pseudo-concave and nonsatiated. The proof of inclusion (8) is under taken in section 4 while the properties of *maximal pseudo-monotone operators* (such as N_f and $-\operatorname{cone}X$) is undertaken in section 2.4 (see definitions there).

1.2.3 Convergence\Uniqueness of any Rationalizing Approximation

In section 5 we make the observation that the reconstructive properties of X means that the equality (9) holds irrespective of the sampling made (as long as it is a dense selection) nor the means via which one constructs a normalized sequence of approximate indirect utilities.

This along with the normalization condition $f\left(te\right)=-t$ forces uniqueness of the epi-limit f that can be constructed via this program. There a many ramifications of this observation such as epi-convergence of the normalized Afriat approximate utilities to the (common) lower closure of any other indirect, normalised utility v that rationalises X and the observation that $v^{-}(p) = v^{+}(p)$ almost everywhere with respect to p at points of continuity of v^{+}.

2 Preliminaries

2.1 Sets and functions

We begin with notation for sets related to functions, such as epigraphs and level sets, and various classes of functions that are described through these sets. Denote the closure of a set $S \subset \mathbb{R}^n$ by $\overline{S}$.

The set of natural numbers is denoted $\mathbb{N}$. Denote by $\mathbb{R}$ the set of real numbers and $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$.[3] $\mathbb{R}^n$ denotes n-dimensional Euclidean space, $\mathbb{R}^n_+$ and $\mathbb{R}^n_{++}$ denote the set of n-dimensional vectors with nonnegative and strictly positive components, respectively.

Given $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ denote by $\operatorname{epi} f := \{(x, \alpha) \mid \alpha \geq f(x)\}$ and $\operatorname{epi}_s f := \{(x, \alpha) \mid \alpha > f(x)\}$. Next, given $\lambda \in \mathbb{R}$, let us define the level and strict level sets of f by

$$\begin{array}{rclrcl} S_\lambda(f) & := & \{x \mid f(x) \leq \lambda\}, & \widetilde{S}_\lambda(f) & := & \{x \mid f(x) < \lambda\}, \\ S_f(x) & := & \{x' \mid f(x') \leq f(x)\}, & \widetilde{S}_f(x) & := & \{x' \mid f(x') \leq f(x)\}. \end{array}$$

Then,

$$S_\lambda(f) = \bigcap_{\mu > \lambda} S_\mu(f) = \bigcap_{\mu > \lambda} \widetilde{S}_\mu(f),$$

$$f(x) = \inf \{ \lambda \mid (x, \lambda) \in \operatorname{epi} f \} = \inf \{ \lambda \mid x \in S_\lambda(f) \} = \inf \{ \lambda \mid x \in \widetilde{S}_\lambda(f) \}.$$

Then f is said to be convex if $\operatorname{epi} f$ is convex and quasiconvex if $S_\lambda(f)$ is convex for each $\lambda \in \mathbb{R}$. The function f is said to be lower semicontinuous at a if for any $\lambda < f(a)$ there is a neighborhood V of a such that $f(x) > \lambda$ for all $x \in V$. The function f is said to be lower semicontinuous if lower semicontinuous at any $a \in \mathbb{R}^n$. It is known that f is lower semicontinuous if and only if $\operatorname{epi} f$ is closed and, also, if and only if $S_\lambda(f)$ is closed for any $\lambda \in \mathbb{R}$.

Given $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, let us consider the function g be defined by $g(x) = \inf \{ \lambda \mid (x, \lambda) \in \overline{\operatorname{epi} f} \}$. It is easily seen that $\operatorname{epi} g = \overline{\operatorname{epi} f}$. Hence g is lower semicontinuous and is the supremum of all lower semicontinuous functions bounded from above by f. It is called the lower closure, or lower semicontinuous hull and is denoted by $\bar{f}$.

Next, let the function h be defined by $h(x) = \inf\{ \mu \mid x \in \overline{S_\mu(f)} \}$. Then, $S_\lambda(h) = \bigcap_{\mu > \lambda} \overline{S_\mu(f)}$. Then, h is lower semicontinuous and is the supremum of all lower semicontinuous functions bounded from above by f and therefore coincides with $\bar{f}$. We summarize these results below.

Proposition 4 *Given $f : \mathbb{R}^n \to \overline{\mathbb{R}}$,*

$$\bar{f}(x) = \inf\{\lambda \mid (x, \lambda) \in \overline{\operatorname{epi} f} \} = \inf\{ \lambda \mid x \in \overline{S_\lambda(f)} \},$$

$$\operatorname{epi} \bar{f} = \overline{\operatorname{epi} f}, \qquad S_\lambda(\bar{f}) = \bigcap_{\mu > \lambda} \overline{S_\mu(f)} = \bigcap_{\mu > \lambda} \overline{\widetilde{S}_\mu(f)}.$$

It follows that $\bar{f}$ is always lower semicontinuous.

As immediate consequences one has:

Corollary 5 *If f is convex (quasiconvex) so is $\bar{f}$.*

Corollary 6 *f is lower semicontinuous at a if and only if $f(a) = \overline{f}(a)$.*

Proof. By construction $f(a) \geq \overline{f}(a)$. Assume that there is λ such that $f(a) > \lambda > \overline{f}(a)$. Then $a \in \overline{S_\lambda(f)}$. Any neighbourhood of a contains points p with $f(p) \leq \lambda$. Therefore f is not lower semicontinuous at a. Conversely, assume that $f(a) = \overline{f}(a)$. Since $f \geq \overline{f}$ and $\overline{f}$ is lower semicontinuous, for any $\lambda < f(a)$ there exists a neighbourhood V of a such that $f(p) \geq \overline{f}(p) > \lambda$. Then f is lower semicontinuous at a. ■

[3] We do not directly deal with functions that take the values $\pm\infty$. We allow for this possibility when taking limits of finite rationalisations, but then show limiting functions take finite values.

Since $\bar{f} \leq f$ and $\bar{f}$ is lower semicontinuous one has $\overline{S_\lambda(\bar{f})} = S_\lambda(\bar{f}) \supset S_\lambda(f)$ and therefore $S_\lambda(\bar{f}) \supset \overline{S_\lambda(f)}$. The equality does not hold in general as seen in the following example: $f(x_1, x_2) = x_2$ if $x_1 = 0$, $x_2 < 0$ and $f(x_1, x_2) = 0$ otherwise. Then f is quasiconvex and $S_0(\bar{f}) = \mathbb{R}^2 \neq \{0\} \times (-\infty, 0] = \overline{S_0(f)}$. We shall see in Appendix 3 that the equality holds under some condition.

Define the normal cones to the level and strict level sets of f by

$$N_f(x) := \{x^* \mid \langle x' - x, x^* \rangle \leq 0 \text{ for all } x' \in S_f(x)\} \quad \text{if } x \in \operatorname{dom} f, \tag{10}$$

$$\tilde{N}_f(x) := \left\{x^* \mid \langle x' - x, x^* \rangle \leq 0 \text{ for all } x' \in \tilde{S}_f(x)\right\} \quad \text{if } x \in \overline{\tilde{S}_f(x)}, \tag{11}$$

(when $x \notin \overline{\tilde{S}_f(x)}$ we define $\tilde{N}_f(x) = \{0\}$).

We say f is g-pseudo-convex function when it quasiconvex and $\overline{S_\lambda(f)} = \overline{\tilde{S}_\lambda(f)}$ for $\lambda > \inf f := \inf\{f(x) \mid x \in \mathbb{R}^n\}$. We say f is *solid* when $\operatorname{int} \tilde{S}_\lambda(f) \neq \emptyset$ for $\lambda > \inf f$. Similarly we say g is pseudoconcave (resp. g-pseudo-concave) if $f = -g$ is pseudoconvex (resp. g-pseudo-convex). For pseudoconcave function one must study the upper level sets $S_\lambda(-g)$ and $\tilde{S}_\lambda(-g)$. This is a broader class than convex functions as the next example shows.

Example 7

$$u(x) := \min\{u_1(x), u_2(x)\}, \quad x \geq 0$$

where $u_1(x) = 4\sqrt{x_1 x_2}$ *and*

$$u_2(x) = \begin{cases} \frac{x_1 + 3x_2}{1 + x_2} & \text{if } x_1 + x_2 \leq 2 \\ x_1 + x_2 & \text{if } x_1 + x_2 \geq 2 \end{cases}.$$

This function u is pseudoconcave and increasing (also solid g-pseudo-concave) but no strictly increasing function k exists such that $x \mapsto k(u(x))$ *is concave.*

2.2 Convergence of sequences of sets and functions

It is well known that epi-convergence characterises convergence of level sets (see [Rockafellar et al. (1998), Proposition 7.7]) so we turn our attention to this type of convergence in our construction of a limiting utility. We give some standard definitions, see [Rockafellar et al. (1998)] for example.

Definition 8 *1. For a family of sets* $\{C^v \mid v \in \mathbb{N}\}$ *in* $\mathbb{R}^n$*, denote the upper Kuratowski-Painlevé limit by* $\limsup_{v\to\infty} C^v$ *and the lower Kuratowski-Painlevé limit by* $\liminf_{v\to\infty} C^v$*. Clearly* $\liminf_{v\to\infty} C^v \subseteq \limsup_{v\to\infty} C^v$*. When these coincide we say that* $\{C^v \mid v \in \mathbb{N}\}$ *set-converges, or simply converges, to* C *(in the Kuratowski-Painlevé sense) and write* $C = \lim_{v\to\infty} C^v$*.*

Let $\{g, g^v \mid \mathbb{R}^n \to \mathbb{R}_\infty, v \in \mathbb{N}\}$ *be a family of proper extended-real-valued functions. Then the lower epi-limit* $\text{e-li}_{v\to\infty} g^v$ *is the function having as its epi-graph the outer limit of the sequence of sets* $\operatorname{epi} g^v$*:*

$$\operatorname{epi}(e\text{-}\operatorname{li}_{v\to\infty} g^v) := \limsup_v (\operatorname{epi} g^v).$$

The upper epi-limit $\text{e-ls}_{v\to\infty} g^v$ *is the function having as its epigraph the inner limit of sets* $\operatorname{epi} f^v$*:*

$$\operatorname{epi}(e\text{-}\operatorname{ls}_{v\to\infty} g^v) := \liminf_v (\operatorname{epi} g^v).$$

When these two functions are equal, they are jointly called the epi-limit of $\{g^v\}$ *and denoted* $\text{e-lim}_v g^v$*. In this case the sequence* g^v *is said to epi-converge (to its epi-limit as* $v \to \infty$*).*

In particular it is possible to express the epi limit infimum (supremum) as follows

$$e\text{-}\operatorname{ls}_{v\to\infty} g^v(p) = \min_{\{p^v \to p\}} \liminf_{v\to\infty} g^v(p^v)$$

$$e\text{-}\operatorname{li}_{v\to\infty} g^v(x) = \min_{\{p^v \to p\}} \limsup_{v\to\infty} g^v(p^v),$$

where the "min" signifies that there exist sequences that attain this infimum.

Any epi-limit is a lower semicontinuous function. This may be viewed as undesirable because lower semicontinuity is not a fundamental notion when studying quasiconvex functions. To this end we make the following definition of a limiting function which does not imply lower semicontinuity in the limit, but, like epi-convergence, demands an orderly inclusion of associated level sets in the limit.

Definition 9 *Given a family of extended-real valued functions* $\{g^v\}_{v\in\mathbb{N}}$ *we say this family quasi-convergent to* g *as* $v\to\infty$ *(or in short q-converge) if and only if for all* λ *we have*

1. *For all* $\lambda_v \to \lambda$ *and all* $\mu > \lambda$ *we have*
$$\limsup_v S_{\lambda_v}(g^v) \subseteq \overline{S_\mu(g)}.$$

2. *For some* $\mu_v \to \mu$ *and all* $\lambda < \mu$ *we have*
$$\liminf_v \tilde{S}_{\mu_v}(g^v) \supseteq \tilde{S}_\lambda(g);$$

Denote by $Q\left(\{g^v\}_{v\in\mathbb{N}}\right)$ *the equivalence class of functions to which* $\{g^v\}_{v\in\mathbb{N}}$ *quasi-converges.*

This is about the weakest (non-Hausdorff) convergence one could demand and still obtain an orderly convergence of level sets. This concept appears strictly weaker than epi-convergence (as we shall shortly show).

Example 10 *Consider the piecewise linear function*

$$g^v(x) := \begin{cases} 1 & \text{for } \frac{1}{v} \le x \\ vx & \text{for } -\frac{1}{v} < x < \frac{1}{v} \\ -1 & \text{for } x \le -\frac{1}{v} \end{cases}$$

Then $\{g^v\}_{v\in\mathbb{N}}$ *clearly quasi-converges and* $Q\left(\{g^v\}_{v\in\mathbb{N}}\right)$ *contains all functions with* $g(x) = -1$ *if* $x < 0$, $g(x) = +1$ *if* $x > 0$ *and* $g(0) \in [-1,1]$.

This phenomenon, namely lack of uniqueness of prospective limits is not academic in that it arises in utility approximation. The following demonstrates that epi-convergence is a stronger convergence than q-convergence.

Proposition 11 ([Rockafellar et al. (1998)] Proposition 7.7) *For functions* $\{f, f^v \mid \mathbb{R}^n \to \mathbb{R}_\infty, v \in \mathbb{N}\}$ *we have*
$$f = e\text{-}\lim_v f^v$$
if and only if both the following hold.

1. *For all* $\lambda^v \to \lambda$ *we have*
$$\limsup_v S_{\lambda^v}(f^v) \subseteq S_\lambda f.$$

2. *For some* $\lambda^v \downarrow \lambda$ *we have*
$$\liminf_v S_{\lambda^v}(f^v) \supseteq S_\lambda f.$$

Remark 12 *The two inclusions imply the existence of a sequence* $\lambda^v \downarrow \lambda$ *(the one given in 2) for which*
$$\lim_v S_{\lambda^v} f^v = S_\lambda f.$$
Also the theory of epi-convergence ensure that monotonic sequences epi-converge. Thus monotonic sequences also are q-convergent sequences.

As shown in [Martinez-Legaz (1991)] the level sets of a quasiconvex functions that enjoys a symmetric duality are most naturally assumed to be evenly convex. That is, the (convex) level sets are intersections of open half spaces. How can one reconcile limiting processes with quasiconvexity?

Theorem 13 *If we have a family of extended-real valued, functions* $\{g^v\}_{v\in\mathbb{N}}$ *that quasi-converges to* g *as* $v\to\infty$ *then* $\{g^v\}_{v\in\mathbb{N}}$ *actually epi-converges to* $\overline{g}$ *(and hence also quasi-converges to* $\overline{g}$ *as well).*

Proof. See Appendix 1. for proof. ■

Thus a quasiconvex function constructed via this kind of limiting process can only be unique up to lower closure. In particular, for any given family of quasiconvex functions the lower closure of any quasi-limit or, equivalently, the epi-limit is a representatives of the larger class of all quasi-limits.

2.3 A Suitable Class of Utilities

In order to determine a necessary set of assumption to impose on $X(\cdot)$ we will discuss the class of solid pseudo-convex\concave functions, see section 2.1. This class attracts our attention because it is, first, reasonably broad in the context of utility theory and, second, admits a self consistent and self contained theory including complete duality between direct and indirect utilities as described in [Eberhard et al. (2007)].

When presented with a utility function $u(\cdot)$ a consumer is deemed to consume a commodity bundle $x \in X(p)$ at a price p and so gain $u(x)$ utility. The indirect utility is the maximum utility that can be gained at price p within a unit budget, denoted $v(p)$, see (4). The indirect utility is quasiconvex under mild assumptions and we have that u can be defined from v via the dual formula (2) (first shown by [Diewert (1974)] but for the most general result see [Martinez-Legaz (1991)]). Indeed for (2) to hold we may assume v is non-increasing (i.e. for $p_1 \geq p_2$ we have $v(p_1) \leq v(p_2)$), evenly quasi–convex (i.e., its level sets may be obtained via the intersection of open half spaces), and satisfy $v(p) \leq \lim_{\alpha \uparrow 1} v(\alpha p)$ for all $p \in \mathbb{R}^n_+ \backslash \mathbb{R}^n_{++}$. In this event we deduce that u evenly quasi-concave, non-decreasing and satisfies $u(x) \geq \lim_{\alpha \uparrow 1} u(\alpha x)$ for all $x \in \mathbb{R}^n_+ \backslash \mathbb{R}^n_{++}$. As open and closed convex sets are evenly convex, both upper and lower semicontinuous quasi–convex are evenly quasi–convex. Clearly the properties above placed on v hold when v is lower semicontinuous on $\mathbb{R}^n_+$ and decreasing on $\mathbb{R}^n_+ \backslash \mathbb{R}^n_{++}$ (which is the case for the indirect utility we construct).

The graph of the demand function is the set

$$\operatorname{Graph} X \ := \ \{(p,x) \mid u(x) = v(p)\} \tag{12}$$

When this symmetric duality (2) holds we have the corresponding optimal solution set $P(x)$ is defined by $\{p \mid u(x) = v(p)\}$. On comparison with (12) we see that the graph of $\operatorname{Graph} P$ corresponds to $\operatorname{Graph} X^{-1}$. That is, $p \in P(x)$ if and only if $x \in X(p)$.

Remark 14 *When the duality formula (2) holds then we must have u is nondecreasing and v is non-increasing. Indeed when $x_1 \geq x_2$ we have*

$$\begin{aligned} &\left\{p \in \mathbb{R}^n_+ \mid \langle p, x_1 \rangle \leq 1\right\} \subseteq \left\{p \in \mathbb{R}^n_+ \mid \langle p, x_2 \rangle \leq 1\right\} \\ &\quad \text{and so} \quad u(x_1) = \inf\{v(p) \mid \langle p, x_1 \rangle \leq 1\} \geq \inf\{v(p) \mid \langle p, x_2 \rangle \leq 1\} = u(x_2). \end{aligned}$$

We say u is nonsatiated if within any neighbourhood V of any given point x_2 there exists $x_1 \in V$ with $u(x_1) > u(x_2)$.

Proposition 15 *Suppose the indirect utility $v : \mathbb{R}^n_+ \to \mathbb{R}_\infty$ is a proper, solid and g-pseudo-convex function that admits the duality formula (2). Then the utility $u : \mathbb{R}^n_+ \to \mathbb{R}_\infty$ is a proper, solid and $-u$ is g-pseudo-convex. In particular we must have u nonsatiated.*

Proof. See the Appendix 2 for a proof. ■

We could have framed the last results with the roles of u and v (x and p) interchanged to obtain the following.

Proposition 16 *Suppose the direct utility $u : \mathbb{R}^n_+ \to -\mathbb{R}_\infty$ is a proper, solid and $-u$ is pseudo-convex. Then the indirect utility $v : \mathbb{R}^n_+ \to \mathbb{R}_\infty$ is a proper, solid and pseudo-convex. In particular we must have v nonsatiated.*

Remark 17 *The last two propositions ensure that $v : \mathbb{R}^n_+ \to \mathbb{R}_\infty$ that is a proper, solid and g-pseudo-convex function that admits the duality formula (2) is nonsatiated.*

Proposition 18 *Denote the indirect utility by v and suppose that it is a proper, solid g- pseudo-convex function that admits the duality formula (2). Then for $p \in \operatorname{dom} v$ we have*

$$[-N_v(p)] \cap \{x \mid \langle x, p \rangle = 1\} = X(p) \tag{13}$$

where the normal cone to the level set $S_v(p)$ at p is given by

$$N_v(p) := \{y \mid \langle p' - p, y \rangle \leq 0 \quad \text{for all } p' \in S_v(p)\}. \tag{14}$$

Proof. We have from Proposition 15 that u is nonsatiated. Thus the optimal value x of (1) satisfies $\langle x, p\rangle = 1$ and if $u(x) = v(p)$ we have from (1) that this is equivalent to saying $\langle p', x\rangle \leq 1 \Longrightarrow v(p') \geq v(p)$. As u is nonsatiated we can also claim that when $x \in \mathbb{R}^n_+$, $\langle x, p\rangle = 1$ and $\langle p' - p, x\rangle < 0$ implies $v(p') > v(p)$. This is because $\langle p', x\rangle < 1$ and so x is strictly in budget. Thus it is possible to improve the utility obtained from x at price p' (due to nonsatiation). That is there must exist x' with $\langle p', x'\rangle < 1$ and $u(x') > u(x)$ and so $v(p') > u(x) = v(p)$. Thus we may write

$$\begin{aligned} X(p) &= \left\{x \in \mathbb{R}^n_+ \mid \langle x, p\rangle = 1 \text{ and } \langle p' - p, x\rangle \leq 0 \text{ implies } v(p') \geq v(p)\right\} \\ &= \left\{x \in \mathbb{R}^n_+ \mid \langle x, p\rangle = 1 \text{ and } \langle p' - p, x\rangle < 0 \text{ implies } v(p') > v(p)\right\} \end{aligned}$$

Alternatively, using the contrapositive

$$\begin{aligned} x \in X(p) \quad &\text{iff} \quad \langle x, p\rangle = 1 \text{ and } \forall p' \in \tilde{S}_v(p) \Longrightarrow \langle p' - p, -x\rangle < 0 \\ &\text{iff} \quad \langle x, p\rangle = 1 \text{ and } \forall p' \in S_v(p) \Longrightarrow \langle p' - p, -x\rangle \leq 0 \end{aligned} \tag{15}$$

there establishing the identity (13). ■

2.4 Minimal cccvC and Maximal Pseudo-Monotonicity

As we have seen in the last section, normal cone operators are central to the description of the demand relation, e.g., Proposition 18. Indeed GARP is equivalent to the cyclical pseudo-monotonicity of $-N_v$ (see [Eberhard et al. (2007)]). Clearly $\tilde{N}_v(x)$ as defined in (11) has closed convex conical images. When v is solid, g-pseudo-convex then we also have $p \in \overline{\tilde{S}_v(p)} = \overline{S_v(p)}$ and hence $N_v(p) = \tilde{N}_v(p)$. When, in addition, v is lower semicontinuous then N_v is upper semicontinuous (see Theorem 20 below). In the proof of our main results we will need to deal with such operators (in the abstract) prior to establishing they actually coincide with a normal cone operator to a level set of a solid, g-pseudo-convex indirect utility. A general study of a related class of abstract operators has recently been undertaken in [Aussel et al. (2011)].

In [Eberhard et al. (2007)] it is shown that a maximal cyclically pseudo–monotone relation (see below for definitions) or a maximal pseudo monotone relation $\Gamma : \mathbb{R}^n \to \mathbb{R}^n$ has convex, conic images. If $x \mapsto \Gamma(x)$ also have a closed graph then these images must also be closed. To allow the origin to be included we must define pseudo-monotonicity carefully.

Definition 19 *[Eberhard et al. (2007)] Let $\Gamma : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set valued mapping.*

1. *Γ is pseudo-monotone (PM) on $D \subset \mathbb{R}^n$ if for all $p, q \in D$, the existence of $x \in \Gamma(p) \setminus \{0\}$ with $\langle x, p - q\rangle > 0$ implies $\langle y, p - q\rangle > 0$ for all $y \in \Gamma(q) \setminus \{0\}$.*

2. *Γ is cyclically pseudo-monotone (CPM) if for any natural number m and (p_i, x_i) with $x_i \in \Gamma(p_i) \setminus \{0\}$ for $i = 0, \ldots, m-1$, we have*

$$\langle x_i, p_{i+1} - p_i\rangle \leq 0 \quad \text{implies} \quad \langle x_m, p_0 - p_m\rangle \leq 0, \quad \forall x_m \in \Gamma(p_m).$$

The relation Γ that we will use later is the conic extension of the demand relation $X(\cdot)$. Recall $\operatorname{cone} X := \bigcup_{\lambda \geq 0} \lambda X$. The following theorem justifies our interest in such operators.

Theorem 20 ([Eberhard et al. (2007)]) *Suppose $v : D \to \mathbb{R}_\infty$ is lower semicontinuous, solid and g-pseudo convex on $V \subseteq D$. Then $p' \mapsto N_v(p')$ is both maximally pseudo–monotone on $\operatorname{int} V$ and also (maximally) cyclically pseudo–monotone with a closed graph.*

The closed graph property is not immediate for all pseudo-monotone operators but holds for the ones we construct here under the assumptions A1-A4.

Lemma 21 *Let $X : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set valued mapping and D be a compact set in $\mathbb{R}^n$. If $\{(p, x) : p \in D, x \in X(p)\}$ is compact and contains no pairs of the form $(p, 0)$ then the set mapping $\Gamma : D \rightrightarrows \mathbb{R}^n$ defined by*

$$\Gamma(p) = -\operatorname{cone} X(p)$$

has a closed graph.

Proof. Let $(x_n, p_n) \in \operatorname{Graph} \Gamma \backslash \{0\}$ with $x_n \in D$ and $(x_n, p_n) \to (x, p)$. If $x = 0$ then $x \in \Gamma(p)$. Now suppose $x \neq 0$. Then there exists $\gamma_n > 0$ and x_n such that $-\gamma_n x_n \in X(p_n)$. As $\{(p, x) : p \in D, x \in X(p)\}$ is compact there exists a convergent subsequence of $\{\gamma_n\}$ denoted by γ_{n_k}. The $\gamma_{n_k} \to \gamma \geq 0$ and $\lim_k (p_{n_k}, -\gamma_{n_k} x_{n_k}) = (p, -\bar{x})$ with $-\bar{x} \in X(p)$ and $p \in D$. By assumption $\bar{x} \neq 0$ and so $\bar{x} = \gamma x$ implies $\gamma \neq 0$ and so $x \in \frac{1}{\gamma}\bar{x} \in -\operatorname{cone} X(p)$. ■

Definition 22 *Let $\Gamma : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set valued mapping with conic images.*

1. *The **effective domain** of Γ, $\operatorname{Edom} \Gamma$, is the set of $p \in \mathbb{R}^n$ such that $\Gamma(p)$ contains a nonzero vector.*
2. *We say Γ is **orientable in the direction** $e \in \mathbb{R}^n$ **on a set** $D \subset \operatorname{Edom}\Gamma$, or **orientable on** D for short, if $\langle x, e\rangle < 0$ for all $x \in \Gamma(p)\backslash\{0\}$ and $p \in D$.*
3. *We say Γ is $C-$**upper semicontinuous on a set** $D \subset \operatorname{Edom} \Gamma$ if for each $p \in D$ and for each open, cone K such that $\Gamma(p) \subseteq K \cup \{0\}$, there is an open neighbourhood V (relative to D) of p such that $\Gamma(p') \subseteq K \cup \{0\}$ for all $p' \in V$.*
4. *We say Γ is **cccv** if its images (or set values) are closed, convex and conic.*

   *We say Γ is **cccvc** if it is cccv and, in addition, has a closed graph.*

   *We say Γ is **cccvC** if it is cccv and, in addition, C–upper semicontinuous.*
5. *We say Γ is **maximal (cyclically) pseudo-monotone** on a set $D \subset \operatorname{Edom} \Gamma$ if it is (cyclically) pseudo-monotone and for any (cyclically) pseudo-monotone operator $G : D \rightrightarrows \mathbb{R}^n$ for which $D \subseteq \operatorname{Edom} G$ and $\Gamma(p) \subseteq \operatorname{cone} G(p)$ for all $p \in D$ then $\operatorname{cone} \Gamma(p) = \operatorname{cone} G(p)$ for all $p \in D$.*

Lemma 23 *Suppose D is a nonempty compact set in $\mathbb{R}^n$ and $\Gamma : D \rightrightarrows \mathbb{R}^n$ is cccvc and orientable in the direction e. Denote p by (y, t) under a rotation of coordinates setting $e/\|e\|$ as the nth canonical basis vector and $D^\#$ as the set of points corresponding to D in the new coordinate system. Then*

$$\Gamma^\dagger(y, t) \ := \ \left\{\frac{z}{\tau} \mid (z, -\tau) \in \Gamma(p)\right\} \tag{16}$$

is upper semicontinuous and uniformly bounded in diameter on the domain $\operatorname{dom} \Gamma^\dagger \subseteq D^\#$ (in fact, under the coordinate transformation, $\operatorname{Edom} \Gamma$ becomes $\operatorname{dom} \Gamma^\dagger$). In particular $\operatorname{Edom} \Gamma$ is a relatively closed subset of D and Γ is cccvC.

Proof. See Appendix 1 for proof. ■

Now it is clear that when a relation Γ is simultaneously maximally pseudo–monotone and cyclically pseudo–monotone then its must also be maximally cyclically pseudo–monotone. In [Eberhard et al. (2007)] it is shown that the normal cones to the level sets of solid, g-pseudo-convex functions is indeed simultaneously maximally pseudo–monotone and cyclically pseudo–monotone, with a closed graph. We show that such multi-functions are closely related to the minimality of the following class of mappings.

Theorem 24 *Suppose $\Gamma : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is (cyclically) pseudo-monotone and cccvC. If $\emptyset \neq D \subseteq \operatorname{int}(\operatorname{Edom} \Gamma)$ then Γ is simultaneously maximally (cyclically) pseudo–monotone on D and the minimal cccvC relation whose effective domain contains D.*

Proof. See Appendix 3 for proof. ■

A **cusco** is an upper semicontinuous convex, compact valued relation which have been widely studied, see [Borwein et al. (1997)].

Corollary 25 *Suppose $\Gamma : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is pseudo-monotone, cccvc and orientable in the direction e. Suppose further that $D \subseteq \operatorname{int}(\operatorname{Edom} \Gamma)$ is nonempty. Let $D^\#$ and $\Gamma^\dagger$ be as in Lemma 23. Then $\Gamma^\dagger$ is minimal in the class of cusco's whose domains contain $D^\#$.*

Proof. Lemma 23 says that Γ is cccvC, hence Theorem 24 says it is minimal in the class of cccvC relations whose effective domains contain $D^{\#}$. Lemma 23 also says that $\Gamma^{\dagger}$ is a cusco whose domain contains $D^{\#}$.

Let ϕ be a cusco on $D^{\#}$ with $\emptyset \neq \phi(y,t) \subset \Gamma^{\dagger}(y,t)$ for each $(y,t) \in D^{\#}$. Then the conical lifting of ϕ given by $\Phi(y,t) := \{(tz, -t) \,:\, z \in \phi(y,t)\}$ is cccvC such that $\Phi(y,t) \subset \Gamma(y,t)$ for each $(y,t) \in D^{\#}$, and $\operatorname{Edom} \Phi = D^{\#}$. Thus Φ is pseudo-monotone being a subset of such an operator. Hence by Theorem 24 Φ is maximal pseudo-monotone and thus coincides with Γ on $D^{\dagger}$. This implies in turn that ϕ coincides with $\Gamma^{\dagger}$ on $D^{\#}$ giving the promised minimality of $\Gamma^{\dagger}$. ■

Theorem 26 *Suppose $\Gamma : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is pseudo-monotone, cccvc and orientable in the direction e. Suppose further that $\emptyset \neq D \subseteq \operatorname{int}(\operatorname{Edom}\Gamma)$. Let $D^{\#}$ and $\Gamma^{\dagger}$ be as in Lemma 23. For any countably dense subset $\{p_i\}$ of D, its coordinate transformation to $\{(y_i,t_i)\}$ is such that by taking any $(z_i,\tau_i) \in \Gamma(y_i,t_i)$ we recover $\Gamma^{\dagger}$ via*

$$\Gamma^{\dagger}(y,t) = \lim_{0<\delta\to 0} \operatorname{co}\left\{ \frac{-z_i}{\tau_i} \,:\, (y_i,t_i) \in B_{\delta}(y,t) \right\} \quad \textit{for all } (y,t) \in D^{\#}.$$

Proof. Corollary 25 shows that $\Gamma^{\dagger}$ is a minimal usco on $D^{\dagger}$. Hence using [Borwein et al. (1997), Proposition 1.4] and the minimality of $\Gamma^{\dagger}$ the result follows. ■

3 Level Curves of g-Pseudo-Convex Functions

Recall that a solid, g-pseudo convex function f has convex level sets whose closures coincide with the closures of its strict level sets, and that have interior when the level is above the minimum value of f. This section describes a coordinate transformation from such a function f to a family of convex "level curve" functions $g(\cdot,\lambda)$ on $\mathbb{R}^{n-1}$, indexed by the levels $\lambda \in \mathbb{R}$, and back again. This a kind of duality between g-pseudo-convex functions and (families) of lower dimensional convex functions.

We start with an approximate indirect utility f_m, a rationalisation of the demand sample $\{(p_i,x_i)\}_{i=1}^m$ that is a normalised so that $f_m(te) = -t$ for fixed $e \in \operatorname{int} D$. This rationalisation is quasiconvex but not necessarily convex. Section 3.1 converts f_m to the family convex functions $g_m(\cdot,\lambda)$ using the approach of [Borde et al. (1990)]. We also relate normal cones of level sets of f_m to the subdifferentials of $g_m(\cdot,\lambda)$.

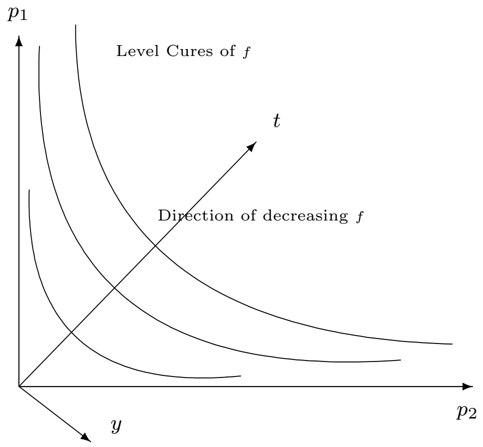
Recall that the normalisations f_m are generally only g-pseudo-convex even though they are derived from convex rationalisations v_m. Section 3.2 gives conditions under which a family of convex functions $g(\cdot,\lambda)$ can be used to derive a solid, g-pseudo-convex function f by defining its level sets. This explains how to recover f_m from the family $g_m(\cdot,\lambda)$. More importantly, later, in Section 4 when we take epi-limits as $m \to \infty$ of the latter to give a family of convex functions $g(\cdot,\lambda)$, we will need this duality to derive a g-pseudo-convex function f that will be our indirect utility for the demand relation X.

3.1 A Coordinate Transformation

In the following analysis we assume we have at hand a sequence of approximate indirect utilities $\{f_m\}$ with $f_m(tp) \le -t$, that rationalise each of the finite data sets that these functions are built on (we do not care what construction is used). The constructions used in [Afriat (1967)] and [Crouzeix et al. (2011)] produce approximating utilities that are ultimately defined on any bounded region D inside $\mathbb{R}^n_{++}$. Following [Borde et al. (1990)] we will see that it is useful to make the following change of basis of the local coordinate system around $e \in \operatorname{int} D \subseteq \mathbb{R}^n_{++}$, so that $p \mapsto (y,t)$ where $y \in \mathbb{R}^{n-1}$ and t is the scalar that gives projection of p on $\{te/\|e\| \mid t \in \mathbb{R}\}$.

Throughout this and later sections, we will abuse notation by writing $f(p)$ as $f(y,t)$ where (y,t) is the point $p \in D$ in the new coordinates.

Now a neighbourhood of e may be taken to have the form $D^{\#} = Y \times T$ in the new coordinate system, where Y and T are closed convex neighbourhoods in $\mathbb{R}^{n-1}$ and $\mathbb{R}$ respectively and the resultant function we will denote by $t \mapsto f_m(y,t)$ is decreasing.



Set $\bar{\lambda} = f_m(e)$ and for simplicity of notation, translate e to the origin in Y. Let $\lambda_0 = \inf\{f_m(y,t) \mid (y,t) \in Y \times T\}$. For $\lambda > \lambda_0$ define

$$g_m(y,\lambda) = \inf\{t \mid f_m(y,t) \leq \lambda\}, \qquad \lambda \in (\lambda_0, +\infty). \tag{17}$$

Although we won't analyse convergence of $\{f_m\}$, or actually of $\{g_m\}$, until section 4, we note a result to decouple the domain and range of f_m from the index m.

Lemma 27 *Posit the axioms A1 to A4 of section 1.1. Suppose* $e \in \operatorname{int} D \neq \emptyset$. *We will make that change of basis of the local coordinate system as described above: let* $e^{\#}$ *denote* e *in the new coordinates, and likewise* $D^{\#}$ *and* $C^{\#}$ *denote the regions* D *and* C*, respectively, in the new basis. Suppose in addition that we have at hand a sequence of approximate indirect utilities* $\{f_m\}$ *with* $f_m(te) \leq -t$ *that rationalises the data* $\{(x_i,p_i)\}_{i=1}^m$. *Then for* m *sufficiently large there exist compact regions* $Y \subseteq \mathbb{R}^{n-1}$, $T \subseteq \mathbb{R}$ *such that* $Y \times T \subseteq D^{\#}$, $e^{\#} \in \operatorname{int} Y \times \operatorname{int} T$ *and*

$$T \ \subseteq \ f_m(Y \times T) \ \subseteq \ \Lambda_m \ := \ \{\lambda \mid \lambda \geq f_m(y,t)\,, \ (y,t) \in Y \times \ T\} \tag{18}$$

In addition for all $(y,t) \in Y \times T$ *we have*

$$N^{\dagger}_{f_m}(y,t) = \operatorname{cone} \partial_y g_m(y,\lambda) \quad \textit{for} \quad \lambda = f_m(y,t) \in T.$$

Proof. First note that as $\{p_i\}_{i=1}^{\infty}$ is dense in $\operatorname{int} D$,

$$D_m := \operatorname{int} \operatorname{co}\{(p_i)\}_{i=1}^m \neq \emptyset$$

and $D_m \subseteq D_{m+1}$. Thus for sufficiently large m we have $Y \times T \subseteq D^{\#}_m$. By construction $\operatorname{dom} f_k \supseteq D_m$ and so we may use the same sets Y and T for $k \geq m$. Clearly for all $\lambda \in \Lambda_m(Y,T)$ we have $S_f(y,t) \neq \emptyset$ and as $f_m(te) \leq -t$ and $e/\|e\|$ corresponds to the coordinate t and choice for T satisfies

$$\begin{aligned} \Lambda_m(Y,T) &\supseteq \left\{\lambda \mid \lambda = f_m(y,t)\,, \ (y,t) \in \operatorname{int} D^{\#}_m\right\} \\ &\supseteq \left\{\lambda \mid \lambda = f_m(0,t)\,, \ (0,t) \in \operatorname{int} D^{\#}_m\right\} \supseteq T. \end{aligned}$$

As f_m is a quasiconvex function the remaining property follow from the analysis of [Borde et al. (1990)]. ■

3.2 Duality between convex level sets and lower dimensional convex functions

A family of g-pseudo-convex functions can always be fitted to samples that satisfy GARP, in fact we can rationalise finitely many data with a convex indirect utility function via Afriat's construction. Symmetrically we have observed that when we do have our demand relation generated by a solid, lower semicontinuous, g-pseudo-convex indirect utility then any such finite sample must satisfy GARP due to the fact that such samples are from a cyclically pseudo-monotone normal cone operator (see Theorem 20). As we will generate our utility from level curves an important tool that we must develop in this sections is a duality between the level curve functions g_m and the corresponding solid, quasiconvex function f_m which in our case will correspond to the indirect utility. Our proof of existence of a limiting utility revolves around the construction of a convergent family of level curve functions described by $\{g_m(\cdot,\lambda)\}_{\lambda\in\Lambda_m}$. We need to establish when an appropriate limit g of $\{g_m\}$ provides a suitable nested family of level curves $\{g(\cdot,\lambda)\}_{\lambda\in\Lambda}$ so as to define a desirable limiting indirect utility function f. Thus it is critical to understand the properties that enable a solid, g-pseudo-convex, utility f to be defined via a family of level curve functions g. Given f one may define the level curve function as:

$$g(y,\lambda) = \inf\{t \mid f(y,t) \le \lambda\}. \tag{19}$$

We state the following result for completeness but do not use it directly in the analysis that follows.

Proposition 28 ([Eberhard et al. (2007)]) *Suppose $f : Y \times T \to \mathbb{R}$ is lower semicontinuous in $t \in T$ for each fixed $y \in Y$ and g defined as in (19) is monotonic non-increasing and lower semicontinuous in λ. Then for $(y,t) \in Y \times T$, the functions f and g satisfy*

$$f(y,t) = \sup\{\lambda \mid g(y,\lambda) > t\}.$$

It turns out that the lower semicontinuity in λ of g is the crucial property around which our analysis revolves. It suffices to restrict our analysis to a neighbourhood when analyzing the normal cone operator. We need to catalog the properties of the limiting family $\{g(\cdot,\lambda)\}_{\lambda\in\Lambda}$. We need sets T, Y that are neighbourhoods of the origin in $\mathbb{R}$ and $\mathbb{R}^n$ respectively and an interval Λ (with interior in $\mathbb{R}$) such that

$$\begin{aligned} &y \mapsto g(y,\lambda) \text{ is defined on } Y \text{ for all } \lambda \in \Lambda \text{ and} \\ &T \subseteq g(y,\Lambda) := \cup_{\lambda\in\Lambda} g(y,\lambda) \text{ for all } y \in Y. \end{aligned} \tag{20}$$

We say f is *g-pseudo-convex on $Y \times T$* if the function given by f on $Y \times T$ and ∞ outside $Y \times T$ is g-pseudo-convex. The usual convention of $\sup\emptyset = -\infty$ is used. We recall essential notation: The normal cone of the level sets and strict level sets of g are given by $N_g(t,y)$ and $\tilde{N}_g(t,y)$ respectively, see (10) and (11). The bases of each of these normal cones, described in Lemma 23 by (16), are denoted $N_g^\dagger(t,y)$ and $\tilde{N}_g^\dagger(t,y)$ respectively.

Theorem 29 *Consider the following properties of $g(y,\lambda)$ defined on $Y \times \Lambda$ where $\Lambda := [\lambda_0, \hat{\lambda})$ (g not necessarily obtained via (19)).*

B1 $g(y,\lambda_1) \ge g(y,\lambda_2)$ whenever $y \in Y$, $\lambda_0 < \lambda_1 < \lambda_2 < \hat{\lambda}$;

B2 for all $\lambda > \lambda_0$, $g(\cdot,\lambda)$ is convex;

B3 $g(\cdot,\lambda)$ is finite on Y for all $\lambda \in \Lambda$;

Extend g outside Λ by placing $g(y,\lambda) = g(y,\lambda_0)$ for $\lambda < \lambda_0$ and $g(y,\lambda) = g(y,\hat{\lambda})$ for $\lambda > \hat{\lambda}$. Let T be an open set around the origin in $\mathbb{R}$ that satisfies (20)*. If we assume properties B1 to B3 then*

$$f(y,t) = \sup\{\lambda \mid g(y,\lambda) > t\} \tag{21}$$

defines a solid, quasi–convex function, monotonically non–increasing in t and lower semicontinuous in (y,t) with $S_f(y,t) \cap (Y \times T) \supseteq \operatorname{epi}_s g(\cdot,\lambda) \cap (Y \times T)$ and $\tilde{S}_f(y,t) \cap (Y \times T) \subseteq \operatorname{epi} g(\cdot,\lambda) \cap (Y \times T)$ for $\lambda = f(y,t)$. We also have the following.

1. *If we assume $\lambda \mapsto g(y,\lambda)$ is lower semicontinuous then for $\lambda = f(y,t)$ we have*

$$\overline{S_f(y,t) \cap (Y \times T)} = \operatorname{epi} g(\cdot,\lambda) \cap (Y \times T) \quad \textit{and} \tag{22}$$

$$N_f^\dagger(y,t) = \partial_y g(y,\lambda). \tag{23}$$

In particular $\lambda = f(y,t)$ implies $t \ge g(y,\lambda)$ (or $t \ge g(y, f(y,t))$). When $y \in \operatorname{int} Y \ne \emptyset$, $t \in \operatorname{int} T \ne \emptyset$ we have $\overline{S_f(y,t) \cap (Y \times T)} = S_f(y,t) \cap (Y \times T)$ in which case $t = g(y,\lambda)$ implies $f(y,t) = \lambda$.

2. *If we assume $\lambda \mapsto g(y,\lambda)$ is upper semicontinuous then for $\lambda = f(y,t)$ we have*

$$\overline{\tilde{S}_f(y,t) \cap (Y \times T)} = \operatorname{epi} g(\cdot,\lambda) \cap \overline{(Y \times T)} \quad \text{and} \tag{24}$$

$$\tilde{N}_f^{\dagger}(y,t) = \partial_y g(y,\lambda). \tag{25}$$

In particular $\lambda = f(y,t)$ implies $t \leq g(y,\lambda)$ (or $t \leq g(y, f(y,t))$).

3. *If $t \mapsto f(y,t)$ is strictly decreasing and finite then $\lambda \mapsto g(y,\lambda)$ is continuous and $g(y,\lambda) = \inf\{t \mid f(y,t) \leq \lambda\}$. In particular $\overline{S_f(y,t) \cap (Y \times T)} = \overline{\tilde{S}_f(y,t) \cap (Y \times T)}$ and so f is g-pseudo-convex on $Y \times T$.*

Proof. See Appendix 3 for proof. ■

Next note that if $f(y,t) > \lambda$ for all $\lambda \in \Lambda$ then $f(y,t) = +\infty$ (as $g(y,\lambda)$ is constant for $\lambda > \hat{\lambda}$ and so $g(y,\lambda) > t$ for all $\lambda \geq \hat{\lambda}$). We single out the following fact for later use.

Corollary 30 *Assume B1 to B3 in Theorem 29 then (21) defines a strictly decreasing function of t when f is finite if and only if g is continuous in λ and nonconstant.*

4 Convergence of Approximations

Section 4.1 summarises the continuity properties of quasiconvex functions such as the normalised indirect utilities v_m that rationalise $\{(p_i, x_i)\}_{i=1}^m$. Our primary contribution is in Section 4.2, where we investigate conditions under which the sequence of normalised indirect utility functions $\{f_m\}$ has an epi-limit $\bar{f}$ and how this relates to the underlying demand relation X.

4.1 Continuity along lines

Proposition 31 (Corollary 3.2, [Crouzeix (2005)]) *Assume that f is quasiconvex, $\operatorname{int}(S_\lambda(f)) \neq \emptyset$ and $a \notin \overline{S_\lambda(f)}$. Take some $b \in \operatorname{int}(S_\lambda(f))$. Then*

1. *f is lower semicontinuous at a if and only if the function $\tau(t) = f(a + t(a-b))$ is lower semi-continuous at $t = 0$;*

2. *f is upper semicontinuous at a if and only if the function $\tau(t) = f(a + t(a-b))$ is upper semi-continuous at $t = 0$.*

Corollary 32 *Suppose $v : \mathbb{R}^n \to \mathbb{R}_{\_}$ is a solid, quasiconvex function such that $v(td) = -t$. Then v is continuous along the line $\{td \mid t \geq 0\}$ and so $\bar{v}(te) = -t$.*

Proof. Take $a = \tau e \in \{te \mid t \in (0,+\infty)\}$ with $\tau < \lambda$ such that $b = a + \lambda e \in \operatorname{int} S_{-\lambda}(v)$ (because $v(a + \lambda e) = v((\tau+\lambda)e) = -(\tau+\lambda) < -\lambda$) and with $a \notin \overline{S_{-\lambda}(v)}$ (because $v(\tau e) = -\tau > -\lambda$). Consequently Proposition 31 applied and as we have $t \mapsto v(a + te)$ is continuous at $t = 0$ it follows that v is continuous at $x = a$. Thus its lower closure coincide with v along the diagonal. ■

As pointed out in the introduction, we may fit a normalised indirect utility f_m to the demand data $\{(x_i, p_i)\}_{i=1}^m$ such that

$$f_m(y,t) \in \left[v_m^-(y,t),\ v_m^+(y,t)\right] \tag{26}$$

where v_m^- and v_m^+ are indirect utilities that are constructed via the algorithms of [Crouzeix et al. (2011)]. After some rescaling, translations and rotations obtain a family of level curve functions $\{g_m(y,\lambda)\}_{m=1}^\infty$ for $(y,\lambda) \in V :=\ Y \times \Lambda_m$ where Y is a fixed neighbourhood of the origin and Λ_m contains a fixed neighbourhood of the origin. Each function $y \mapsto g_m(y,\lambda)$ is finite, continuous, convex and for $\lambda_1 \leq \lambda_2$ we have $g_m(y,\lambda_1) \geq g_m(y,\lambda_2)$.

Remark 33 *Consider any selection f_m as in (26). If we have an epi-convergent subsequence with epi-limit $f = e\text{-}\lim_k f_{m_k}$ then*

$$f(0,t) \in \left[e\text{-}\limsup_k v_{m_k}^-(0,t),\ e\text{-}\liminf_k v_{m_k}^+(0,t)\right].$$

As $\{v^-_{m_k}\}_{k=1}^{\infty}$ *is monotonic non-decreasing we have*

$$e\text{-}\limsup_k v^-_{m_k}(0,t) = \overline{\sup_k v^-_{m_k}(0,t)} := \bar{v}^-(0,t).$$

By Theorem 5.1 of [Crouzeix et al. (2011)] and the dense sampling we must have the pointwise limit $v^-(0,t) := \lim_{k\to\infty} v^-_{m_k}(0,t) = -t$. *Now by Corollary 32 we have continuity following so* $\bar{v}^-(0,t) = v^-(0,t) = -t$ *giving* $f(0,t) = e\text{-}\lim_k f_{m_k}(0,t) \geq -t$. *On the other hand by construction* $v^+_{m_k}(0,t) = -t$ *for all* k *and so* $f(0,t) \leq e\text{-}\liminf_k v^+_{m_k}(0,t) \leq -t$ *ensuring the equality* $f(0,t) = -t$. *This observation is used the main theorem 34.*

4.2 Convergence Analysis of $\{f_m\}$

Fix $e \in \mathrm{int}(D)$ and recall the coordinate transformation in section 3.1 that places the final (nth) canonical basis vector as $e/\|e\|$. Having constructed a sequence of approximating normalized, indirect utilities $\{f_m\}_{m=1}^{\infty}$ we may use the new coordinate system to construct the corresponding family of level curve functions via (17), i.e., $g_m(y,\lambda) = \inf\{t \mid f_m(y,t) \leq \lambda\}$.

Recall from Lemma 27 that for large m, $e^{\#}$ is contained in interior of a compact set $Y \times T \subset D^{\#}$ such that for some fixed interval Λ we have

$$T \subset \Lambda \subseteq \Lambda_m := \{\lambda \mid \lambda \geq f_m(y,t), \ (y,t) \in Y \times T\}.$$

Now use a compactness property of epi-convergence (see [Beer (1993), Theorem 5.2.12]) which yields a Kuratowski-Painlevé convergent subsequence of any sequence of sets. In terms of epiconvergence this means that we can extract from $\{g_m(\cdot,\lambda)\}_{m=1}^{\infty}$ and associated sets $\{\Lambda_m\}_{m=1}^{\infty}$ an epi-convergent subsequence $\{g_{m_k}(\cdot)\}_{k=1}^{\infty}$ for which $\{\Lambda_{m_k}\}_{k=1}^{\infty}$ also set-converges. This gives

$$g(y,\lambda) = e\text{-}\lim_k g_{m_k}(y,\lambda), \tag{27}$$

and the limit $\Lambda := \lim_k \Lambda_{m_k}$ contains a fixed open interval of the origin according to Lemma 27. We can thus define

$$f(y,t) := \sup\{\lambda \mid g(y,\lambda) > t\},$$

a (possibly extended-real-valued) function on $Y \times T \subseteq D$ that only depends on the initial subsequence chosen. Thus f and g are well defined but we do not know as yet if they have the properties we desire. We will need to appeal to Theorem 29 to establish the required properties.

Theorem 34 *Suppose our demand correspondence* $X : \mathrm{Edom}\, X \rightrightarrows \mathbb{R}^n$ *satisfies GARP and has an effective domain that is orientable in the direction* $e \in \mathrm{int}\,\mathrm{Edom}\, X$. *After a coordinate transformation where* $e/\|e\|$ *is the* n*th canonical basis vector, suppose we have a closed bounded set* $Y \times T \subseteq \mathrm{int}\,\mathrm{Edom}\, X$. *Let* g *an epi-limit of* $\{g_m\}$ *and* Λ *be a corresponding set-limit of* $\{\Lambda_m\}$ *as detailed above.Then the function* $\lambda \mapsto g(y,\lambda)$ *is continuous, on* Λ *and* $y \mapsto g(y,\lambda)$ *is convex on* Y *and finite on* $\mathrm{int}\, Y$ *for* $\lambda \in \Lambda$ *(and hence Lipschitz continuous there). In addition* $g(y,\lambda_1) \geq g(y,\lambda_2)$ *for* $y \in Y$ *and* $\lambda_i \in \Lambda$ *with* $\lambda_1 < \lambda_2$ *and hence we may define*

$$f(y,t) := \sup\{\lambda \mid g(y,\lambda) > t\} \tag{28}$$

with f *being a real valued, solid,* g*-pseudo-convex function on* $Y \times T$, *strictly decreasing in* t, *with level curves defined by* $y \mapsto g(y,\lambda)$. *In particular there is an epi-convergent subsequence giving* $f = e\text{-}\lim_k f_{m_k}$ *such that* $f(0,t) = -t \in T$ *and the base* $N_f^{\dagger}$ *of the normal cone mapping* N_f *is upper semicontinuous on* $Y \times T$, *and yields the following two identities for* $(y,t) \in Y \times T$:

$$N_f^{\dagger}(y,t) = \{z \mid z \in \partial_y g(y,\lambda) \text{ for } \lambda = f(y,t)\} \tag{29}$$

$$X(y,t) \subseteq -\,\mathrm{cone}\left\{(z,-1) \mid z \in N_f^{\dagger}(y,t)\right\} \cap (z,\tau) \mid \langle (z,\tau),(y,t)\rangle = 1. \tag{30}$$

Proof. We take for granted the existence of an epi-limit of g of a subsequence g_{m_k} and corresponding epi-limit Λ of Λ_{m_k}, as discussed above.

Step 1: (We show that subsequence $\{f_{m_k}\}_{k=1}^{\infty}$ epi-converges to f on $Y \times T$ to a solid, quasiconvex function).

As all upper epi-limits produce lower semicontinuous functions we have immediately that

$$(y, \lambda) \mapsto g(y, \lambda)$$

is jointly lower semicontinuous and in particular $\lambda \mapsto g(y, \lambda)$ is lower semicontinuous for all $y \in Y$. For a given $\lambda \in \Lambda$ take $(y_k, \lambda_k) \to (y, \lambda)$ attaining the minimum in the composite expression for the epi-limit (27) and we see that

$$\begin{aligned} g(y, \lambda) = & \lim_k g_{m_k}(y_k, \lambda_k) = \limsup_k g_{m_k}(y_k, \lambda_k) \\ & \geq \min_{\{y'_k \to y\}} \limsup_k g_{m_k}(y'_k, \lambda_k) = e\text{-}\operatorname{ls}_k g_{m_k}(y, \lambda_k) \end{aligned}$$

and clearly

$$\begin{aligned} g(y, \lambda) = & \min_{\{y'_k \to y, \lambda'_k \to \lambda\}} \lim_k g_{m_k}(y'_k, \lambda'_k) \\ & \leq \min_{\{y'_k \to y\}} \liminf_k g_{m_k}(y'_k, \lambda_k) \; = \; e\text{-}\lim_k g_{m_k}(y, \lambda_k). \end{aligned}$$

Thus $g(y, \lambda) = e\text{-}\lim_k g_{m_k}(y, \lambda_k)$ for all $y \in Y$ and as

$$\operatorname{epi} g(\cdot, \lambda) = \lim_k (\operatorname{epi} g_{m_k}(\cdot, \lambda_k))$$

with each $\operatorname{epi} g_{m_k}(\cdot, \lambda_k)$ convex we conclude that $\operatorname{epi} g(\cdot, \lambda)$ and hence $y \mapsto g(y, \lambda)$ is convex. Suppose that $\bar{Y} \subseteq Y$ has $\operatorname{dom} g(\cdot, \lambda) \cap \operatorname{int} \bar{Y} \neq \emptyset$ then via the sum theorem for epi-convergence of convex functions ([Beer (1993)], Theorem 7.4.5) as applied to the sum $\delta_{\operatorname{epi} g_{m_k}(\cdot, \lambda_k)} + \delta_{\bar{Y} \times T}$ it follows that

$$\operatorname{epi} g(\cdot, \lambda) \cap \left(\bar{Y} \times T\right) = \lim_k (\operatorname{epi} g_{m_k}(\cdot, \lambda_k) \cap \left(\bar{Y} \times T\right)) \tag{31}$$

As $g_{m_k}(y, \lambda_1) \geq g_{m_k}(y, \lambda_2)$ for all $y \in Y$ and $\lambda_i \in \operatorname{int} \Lambda$ with $\lambda_1 < \lambda_2$ we have for any $\lambda_1^k \to \lambda_1$ and $\lambda_2^k \to \lambda_2$ eventually $\lambda_1^k \leq \lambda_1 < \lambda_2^k$ and so using the definition of epi-convergence

$$\begin{aligned} g(y, \lambda_1) \geq & \min_{\{y'_k \to y, \; \lambda_1^k \to \lambda_1\}} \limsup_k g_{m_k}\left(y'_k, \lambda_1^k\right) \\ & \geq \min_{\{y'_k \to y, \; \lambda_2^k \to \lambda_2\}} \liminf \; g_{m_k}\left(y'_k, \lambda_2^k\right) \geq g(y, \lambda_2). \end{aligned}$$

Now define f via (28) and apply Theorem 29 part (1) to deduce that on any $\bar{Y} \subseteq Y$ with $\operatorname{int} \bar{Y} \neq \emptyset$ on which $y \mapsto g(y, \lambda)$ is finite for $\lambda \in \Lambda$ (i.e. $\operatorname{dom} g(\cdot, \lambda) \cap \operatorname{int} \bar{Y} \neq \emptyset$) we have f defining a solid, quasi–convex function, monotonically non–increasing in t and lower semicontinuous in (y, t) with

$$S_\lambda(f) \cap \left(\bar{Y} \times T\right) = S_f(y, t) \cap \left(\bar{Y} \times T\right) = \operatorname{epi} g(\cdot, \lambda) \cap \left(\bar{Y} \times T\right). \tag{32}$$

Next observe that because $g(y, \lambda) = e\text{-}\lim_k g_{m_k}(y, \lambda)$ we have for any sequence $\lambda_1^k \to \lambda$ that

$$g(y, \lambda) \leq \; e\text{-}\liminf_k g_{m_k}\left(y, \lambda_1^k\right)$$

and so

$$\operatorname{epi} g(\cdot, \lambda) \supseteq \limsup_k \operatorname{epi} g_{m_k}\left(\cdot, \lambda_1^k\right). \tag{33}$$

Combining (33) with the identity $S_{\lambda_1^k}(f_{m_k}) = \operatorname{epi} g_{m_k}\left(\cdot, \lambda_1^k\right)$ we have for any sequence $\lambda_1^k \to \lambda$ that

$$S_f(y, t) \cap \left(\bar{Y} \times T\right) \supseteq \left(\limsup_k S_{\lambda_1^k} f_{m_k}\right) \cap \left(\bar{Y} \times T\right) \supseteq \limsup_k \left(S_{\lambda_1^k}(f_{m_k}) \cap \left(\bar{Y} \times T\right)\right).$$

Combining (31) with $S_{\lambda^k}(f_{m_k}) = \operatorname{epi} g_{m_k}\left(\cdot, \lambda^k\right)$ we have for our particular $\lambda^k \to \lambda$ used to define g

$$S_\lambda\left(e\text{-}\lim_k f_{m_k}\right) \cap \left(\bar{Y} \times T\right) = S_\lambda(f) \cap \left(\bar{Y} \times T\right) = \lim_k \left(S_{\lambda^k}(f_{m_k}) \cap \left(\bar{Y} \times T\right)\right).$$

It follows from 12 that $f|_{\bar{Y}\times T} = e\text{-}\lim_k f_{m_k}|_{\bar{Y}\times T}$.

It remains to show that $y \mapsto g(y,\lambda)$ is finite on Y for $\lambda \in \Lambda$ which we will accomplish by showing that the partial subdifferential of g with respect to y is nonempty. Then, in subsequent steps of the proof, we may take $\bar{Y} = Y$ above and so clearly $\operatorname{dom} g(\cdot,\lambda) \cap \operatorname{int} Y \neq \emptyset$.

As, in finite dimensions, all the myriad of variational convergences coincide we have (in particular) Mosco-convergence of $\left\{g_{m_k}\left(\cdot,\lambda^k\right)\right\}_{k=1}^{\infty}$ on Y and hence may invoke Attouch's theorem (see [Beer (1993)], Theorem 8.3.9). Thus we have the graphical convergence of the subdifferential

$$\partial_y g(\cdot,\lambda)(y) = g\text{-}\lim_{k\to\infty} \partial_y g_{m_k}\left(\cdot,\lambda^k\right)(y).$$

By Theorem 29, 1 and the approximating construction it follows that for all $y \in Y$ and $t \in T$ we have (irrespective of finiteness of g) for $\lambda = f(y,t)$ that

$$\partial_y g(\cdot,\lambda)(y) = g\text{-}\lim_{k\to\infty} N^{\dagger}_{f_{m_k}}(y,t). \tag{34}$$

Because $Y \times T \subseteq \operatorname{int} \operatorname{Edom} X$ we have a dense set of points $\{(y_i,t_i)\}_{i=1}^{\infty}$ in $Y \times T$ such that $0 \neq (z_i,\tau_i) \in X(y_i,t_i) \neq \emptyset$. As f_{m_k} rationalises the sample $\{(z_i,\tau_i)\}_{i=1}^{m_k}$ from the demand X we have for all i eventually (for m_k large) that

$$-\frac{z_i}{\tau_i} \in N^{\dagger}_{f_{m_k}}(y_i,t_i) \quad \text{for} \quad (y_i,t_i) \in Y \times T.$$

Thus by (23) of Theorem 29 part (1)

$$\emptyset \neq \partial g_{m_k}\left(\cdot,\lambda^k\right)(y) = N^{\dagger}_{f_{m_k}}(y_i,t_i) \text{ for } \lambda^k = f_{m_k}(y_i,t_i). \tag{35}$$

By our assumption A1-A3 and Lemmas 21 and 23 we have $X^{\dagger}$ is upper semicontinuous and $Y \times T$ are closed bounded sets with interior. Thus by Lemma 23 we have $X^{\dagger}$ uniformly bounded in diameter on $Y \times T$ and hence each $\left\{\frac{z_i}{\tau_i}\right\}_{i=1}^{\infty}$ also lying in a compact set. Using the graphical convergence in (34) we have

$$\begin{aligned}\emptyset \neq g\text{-} & \limsup_{\substack{(y_i^k,t_i^k)\to(y,t)\\ f_{m_k}(y_i^k,t^k)=\lambda^k\to\lambda}} \operatorname{co}\left\{-\frac{z_i}{\tau_i} \mid -\frac{z_i}{\tau_i} \in -X^{\dagger}\left(y_i^k,t_i^k\right) \subseteq N^{\dagger}_{f_{m_k}}\left(y_i^k,t_i^k\right)\right\} \\ \subseteq\ & \partial_y g(\cdot,\lambda)(y)\end{aligned} \tag{36}$$

Thus $\partial_y g(\cdot,\lambda)(y) \neq \emptyset$ for all $\lambda \in \Lambda$ and $y \in Y$ which implies $g(\cdot,\lambda)$ is finite valued on Y for each $\lambda \in \Lambda$. As $y \mapsto g(y,\lambda)$ is finite and convex on Y we have $y \mapsto g(y,\lambda)$ Lipschitz continuous on Y for $\lambda \in \Lambda$.

Step 2: (Showing $f(0,t) = -t$)

Now as $f = e\text{-}\lim_k f_{m_k}$ on $Y \times T$ we have f lower semicontinuous and proper. Via remark 33 we have $f(0,t) = -t$ for $t \in T$ and hence f is nonconstant (where $(0,1)$ corresponds to the original $e/\|e\|$ internal to the original D).

Step 3: (Showing $t \mapsto f(y,t)$ is strictly decreasing and g-pseudo-convex (or equivalently we have upper semicontinuity of $\lambda \mapsto g(y,\lambda)$).) If $t \mapsto f(y,t)$ was not strictly decreasing then by Proposition 1 of [Eberhard et al. (2007)] we would have f constant on some neighbourhood V within D (the level set construction rules out a local minimum). Let $(y,t) \in \operatorname{int} V$ then by definition $N_f(y,t) = \{0\}$. This implies $\operatorname{Edom} N_f \cap V = \emptyset$ which contradicts $D \subseteq \operatorname{Edom} X$ in conjunction with (36) and (37). Applying Theorem 29 and the last observation we deduce that f is g-pseudo-convex on $Y \times T$.

Step 4: (Showing $N_f^{\dagger}(y,t)$ is upper semicontinuous on $Y \times T$)

We have verified all the assumption of Theorem 29 and hence we may deduce from (23) that

$$N_f(y,t) = \operatorname{cone}\left\{(z,-1) \mid z \in N_f^{\dagger}(y,t) = \partial_y g(y,\lambda) \text{ for } \lambda = f(y,t)\right\}. \tag{37}$$

Consequently (37) gives $\langle (z,\tau),(0,1)\rangle < 0$ for all $(z,\tau) \in N_f(y,t)$ and so N_f is orientable by $(0,1)$. As f is g-pseudo-convex we have $N_f = \tilde{N}_f$ and so we may apply [Borde et al. (1990)] (or see [Eberhard et al. (2007), Lemma 5]) where it is shown that for g-pseudo-convex functions $\tilde{N}_f$ has a closed graph (outside of

$\arg\min f$, which is empty in this case) and hence $N_f = \tilde{N}_f$ is **cccvc**. Now apply Lemma 23 we have $N_f^{\dagger}(y,t)$ upper semicontinuous.

Step 5: (Establishing the containment (30))

Finally we note that (37), the C–upper semicontinuity of N_f, along with $(z_i, \tau_i) \in X(y_i, t_i)$ and $(z_i, \tau_i) \subseteq N_f\,(y_i, t_i)$, (13) and Theorem 26 as applied to the reconstruction of X from a dense selection implies (30). ■

To ensure we have equality in (30) we need to discuss minimality of closed convex, conic valued C-upper semicontinuous multifunction. This is done in the next section.

5 The Main Results

We are now in a position to prove the existence of a solid g-pseudo–convex utility rationalising the underlying preference structure, thus resolving the *problem of revealed preference* under reasonable assumptions.

We review the effects of the axioms A1–A4 that we will assume throughout. These say or imply that $-X : D \rightrightarrows \mathbb{R}^n$ is cyclically pseudo–monotone (i.e. GARP holds for X) with closed graph and convex images on a closed, bounded set D that has interior, and for each $p \in D$, $X(p)$ contains a nonzero member $x(\in C)$. Thus D is contained in $\operatorname{Edom} X$, the effective domain defined in section 2.4. We also fix the direction $e \in \operatorname{int} D$ along which we will normalise our rationalisations. That is, we take an indirect utility v_m that rationalises $\{(p_i, x_i)\}_{i=1}^m$, such as Afriat's construction, and then normalise (see section 1.2.1) to produce a rationalisation f_m such that $f(te) = -t$ for $t > 0$. We then apply a coordinate transformation (see section 3.1) which places the nth coordinate in the direction $e/\|e\|$, and maps a generic point $p \in \mathbb{R}^n$ to $p^{\#} \equiv (y,t)$ where $y \in \mathbb{R}^{n-1}$, $t \in \mathbb{R}$. Let $D^{\#}$ as the transformation of D. We rewrite $f_m(p)$ as $f_m(y,t)$ (admitting this abuse of notation), which is well defined on any fixed domain $Y \times T \subseteq D^{\#}$.

Theorem 35 *Under Assumptions A1–A4: For any $e \in \operatorname{int}(D)$ there exists a lower semicontinuous, solid, g-pseudo-convex indirect utility function $v : D \to \mathbb{R}$ such that v is e-normalised, $p_i \in \arg\min\{v(p) \mid \langle x_i, p\rangle \le 1\}$ for all i, and*

$$X(p) \;=\; -N_v(p) \cap \{x \mid \langle x, p\rangle \le 1\} \quad \text{for all } p \in \operatorname{int} D.$$

Defining $u(x) := \min\{v(p) \mid \langle x, p\rangle \le 1\}$ for $x \in C$ gives a solid, g-pseudo-concave and nonsatiated utility that rationalises the demand relation $X(\cdot)$ for $p \in \operatorname{int} D$.

Proof. Denote the underlying conic valued mapping $\Gamma(p) := -\operatorname{cone} X(p)$ generated by the demand correspondence. First we note for $p \in D$ that as $\Gamma(p) \subseteq -\mathbb{R}^n_+$ it is orientable in any direction $e \in \operatorname{int}(D)$ (i.e., since $D, -\Gamma(p) \subseteq \mathbb{R}^n_+$, then $e > 0$ and $\langle x, e\rangle < 0$ for all $x \in \Gamma(p)\backslash\{0\}$). Apply Lemma 23 and Theorem 24 to deduce that Γ is simultaneously maximally cyclically pseudo–monotone, maximally pseudo–monotone and the minimal **cccvC**.

We proceed by taking a sequence $\{p_i\}$ that is dense in D and sampling the demand relation $x_i \in X(p_i)$ for each i. Next we fit a sequence of normalized, indirect utilities $\{f_m\}_{m=1}^{\infty}$ that rationalises this increasing (and ultimately dense) sample. As described immediately prior to Theorem 34 we may extract an epi-convergent subsequence of the sequence $\{f_m\}_{m=1}^{\infty}$ that converges. Furthermore, as described by Theorem 34, this converges on any open rectangular region $Y \times T \subseteq D^{\#}$. Now any $p \in D$ may be contained a closed, rectangular region $G^{\#}$ within $D^{\#}$ so that $p^{\#} \in \operatorname{int} G^{\#} := Y \times T \subseteq D^{\#}$. Theorem 34 as applied to the restriction to G of the epi-convergent subsequence $\{f_{m_k}\}_{k=1}^{\infty}$ and the associated level curve family $\{g^{\#}_{m_k}\}_{k=1}^{\infty}$ (restricted to $G^{\#}$) gives a solid, g-pseudo-convex function $v : G \to \mathbb{R}$ with $p_i \in \arg\min\{v(p) \mid \langle x_i, p\rangle \le 1\}$ for all i and

$$X(p) \subseteq -N_v(p) \cap \{x \mid \langle x, p\rangle \le 1\} \quad \text{for all } p \in \operatorname{int} D. \tag{38}$$

Suppose we apply the same construction on two over lapping rectangular region G and G' with $G \cap G'$ corresponding to another such rectangular region to which Theorem 34 applies. As the construction of v is dependent only a fixed epi-convergent subsequence of the functions $\{f_m\}_{m=1}^{\infty}$ which are defined universally over D the indirect utilities so constructed on G and G' must coincide within $G \cap G'$. Thus we may deduce that v is solid, g-pseudo-convex on the whole of D.

Moreover, by Theorem 34, $N_v^\dagger$ is upper semicontinuous and hence, by Lemma 23, N_v is C-upper semicontinuous. As v is solid g-pseudo–convex Theorem 20 says that $N_v = \widetilde{N}_v$ has a closed graph and maximally cyclically pseudo–monotone operator. Indeed all the assumptions of Theorem 24 are satisfied for N_v and hence $N_v^\dagger$ is a minimal **cusco**. By construction of $\Gamma(p)$ and (38) then $\Gamma^\dagger(p) \subseteq N_v^\dagger(p)$ and we must have equality (due to minimality). Hence cone $X(p) = -N_{\bar{v}}(p)$ and as $X(p) = \operatorname{cone} X(p) \cap \{x \mid \langle x, p\rangle \leq 1\}$ we have equality in (38). By Proposition 15 we have the associated direct utility u is solid, g-pseudo-concave and nonsatiated. ■

The proof is constructive in that we obtain the indirect utility via a limiting process as described in the statement and proof of Theorem 34. Note that this indirect utility is independent of the demand relation sample $\{(x_i, p_i)\}_{i=1}^\infty$ so long as the closure of $\{p_i\}$ is D. This has some surprising consequences, starting with uniqueness.

Lemma 36 *Suppose f_1 and f_2 are finite valued, lower semi-continuous, solid, g-pseudo-convex functions on the domain $\operatorname{int} D$. If f_1 and f_2 are e-normalised and*

$$N_{f_1}(p) = N_{f_2}(p) \quad \text{for all } p \in \operatorname{int} D \tag{39}$$

then $f_1 = f_2$ on $\operatorname{int} D$.

Proof. Let $g_i(\cdot, \lambda)$ be the level curve function of f_i $(i = 1, 2)$ obtained via the constructions outlined in section 3.1 i.e. $f(p)$ is written as $f(y, t)$ where (y, t) is the point $p \in D$ in the new coordinates after a rotation that places e along the coordinate axis denoted by t. Then $g_i(\cdot, \lambda)$ is a lower semi-continuous convex function of y for each fixed λ. By assumption, f_i and hence $g_i(\cdot, \lambda)$ are finite valued and so $g_i(\cdot, \lambda)$ is a Lipschitz continuous convex function. It follows from Theorem 29 that

$$N_{f_i}^\dagger(y, t) = \partial_y g_i(\cdot, \lambda)(y) \quad \text{for } t = g(y, \lambda).$$

Consequently via (39) we have

$$\partial_y g_1(\cdot, \lambda) \equiv \partial_y g_2(\cdot, \lambda)$$

for all fixed λ. Consequently (using any one of many results to which this applies i.e. Theorem 12.25 of [Rockafellar et al. (1998)]) $g_1 = g_2 + c$ where c is a scalar (potentially dependent on λ). Now we use the normalisation i.e.

$$g_i(0, \lambda) = -\lambda$$

to deduce that

$$g_1(\cdot, \lambda) = -\lambda = g_2(\cdot, \lambda) + c = -\lambda + c \quad \Longrightarrow \quad c = 0.$$

Using the formula (21) it follows that $f_1 \equiv f_2$ on $\operatorname{int} D$. ■

We state an immediate corollary of Theorem 35 and Lemma 36:

Corollary 37 *Under Assumptions A1–A4: For any $e \in \operatorname{int}(D)$, the function v (constructed in Theorem 34 and) presented in Theorem 35 is the unique lower semicontinuous, e-normalised indirect utility that represents X on $\operatorname{int} D$; denote this by $\bar{v}$.*

We conclude with implications for the integration work of [Crouzeix et al. (2011)]. From Lemma 2 in section 1.1.2 the indirect utilities v_m^-, v_m^+ have the following four properties that we assume tacitly for the remainder of this section: They rationalise $\{(p_i, x_i)\}_{i=1}^m$, are normalised in the direction e, are lower and upper bounds, respectively, on any indirect utility that rationalises these data and are monotonic in m: $v_m^- \leq v_{m+1}^- \leq v_{m+1}^+ \leq v_m^+$ on D.

Corollary 38 *Under assumptions A1–A4: Suppose $e \in \operatorname{int}(D)$ and $\{f_m\}_{m=1}^\infty$ is any sequence of functions such that each f_m rationalises $\{(x_i, p_i)\}_{i=1}^m$ and satisfies*

$$f_m(p) \in [v_m^-(p), v_m^+(p)] \quad \text{for } p \in \operatorname{int}(D). \tag{40}$$

Then $\{f_m\}_{m=1}^\infty$ actually epi-converges to $\bar{v}$ given in Corollary 37.

Recall from Definition 9 that $Q(\{f_m\}_{m=1}^\infty)$ denotes the equivalence class of functions to which $\{f_m\}_{m=1}^\infty$ quasi-converges.

Corollary 39 *Under Assumptions A1–A4 and given $e \in \operatorname{int} D$: Consider all samples $\{(x_i, p_i)\}_{i=1}^{\infty}$ of the demand relation $X(\cdot)$ such that the closure of $\{p_i\}$ is D. Denote by $\mathcal{IA}pprox$ the corresponding family of sequences $\{f_m\}_{m=1}^{\infty}$ that are sandwiched as in (40) where v_m^- and v_m^+, described above, depend on the sample of the demand relation. Then the lower closure of $v \in Q(\{f_m\}_{m=1}^{\infty})$ is independent of $\{f_m\}_{m=1}^{\infty} \in \mathcal{IA}pprox$; all these lower closures coincide with $\bar{v}$ given in Corollary 37. That is, lower closures of quasi-limits of e-normalised, indirect finite rationalisations are independent of the sample $\{(x_i, p_i)\}_{i=1}^{\infty}$.*

Proof. All $\{v^m\}_{m=1}^{\infty} \in \mathcal{IA}pprox$ are subject to Corollary 38 and so by Theorem 13 all $v \in Q(\{v^m\}_{m=1}^{\infty})$ epi-converge to $\bar{v}$ which is unique (again by Corollary 38). ■

[Crouzeix et al. (2011)] gives pointwise smallest and largest indirect utilities that represent $X(\cdot)$ on D and are e-normalised; denote these v^- and v^+ respectively. The previous corollary says that that the quasi-limit of the largest finite indirect rationalisation v_m^+ has the same lower closure as all other normalized, representative functions $v(p)$ that select from the interval $[v^-(p), v^+(p)]$. We show that this forces a the limit to be unique almost everywhere:

Corollary 40 *Under assumptions A1–A4, suppose $e \in \operatorname{int}(D)$ and assume all indirect utilities below are e-normalised:*

1. *Let v^- (resp. v^+) denote the smallest (resp. largest) indirect utilities that rationalise any given dense sample that is consistent with A1-A4. Then v^+ coincides with v^- (and with $\bar{v}$ given in Corollary 37) at all points of continuity of v^+. Moreover this set of points of continuity is of full measure within $\operatorname{int}(D)$.*

2. *When v^+ is continuous the q-limit v is unique (and coincides with $v^- = v^+ = \bar{v}$).*

Proof. First note that for any given dense sample $\{(x_i, p_i)\}_{i=1}^{\infty}$ the associated finite rationalisations $\{v_m^-\}_{m=1}^{\infty}$, $\{v_m^+\}_{m=1}^{\infty} \in \mathcal{IA}pprox$ and so both epi-converge to $\bar{v}$, which is the unique lower semicontinuous, normalized, solid, g-pseudo-convex indirect utility that rationalises the data. The pointwise (monotone) limits v^-, v^+ or are clearly quasi-limits of $\{v_m^-\}$ and $\{v_m^+\}$, respectively which (pointwise) bound all other quasi-limits of any other sequence of finitely rationalising, normalised indirect utilities. Consequently by Theorem 13, taking lower closures gives $\bar{v} = \overline{v^-} = \overline{v^+}$. This argument applies irrespective of the sample taken. Thus we deduce that the lower closures of both v^- and v^+ are the same and independent of the dense sample take. According to [Crouzeix et al. (1987)], v^+ is continuous almost everywhere in D hence the results follows. ■

Of course the above developments apply to the any specific indirect utility construction. For example suppose for each m that f_m is the normalised Afriat rationalisation of $\{(x_i, p_i)\}_{i=1}^{m}$, as described in section 1.2.1. Then $\{f_m\}_{m=1}^{\infty}$ epi-converges to $\bar{v}$ according to in Corollaries 39 and 40.

Remark 41 *The above results, which were developed for indirect utilities, can, dually, be applied to direct utilities. In particular we may sample from the multifunction X^{-1} and the fit direct utilities $\{u_m\}_{m=1}^{\infty}$ that rationalise finite samples $\{(x_i, p_i)\}_{i=1}^{m}$ with $p_i \in X^{-1}(x_i)$. Properties like cyclical pseudo-monotonicity are symmetric properties of a graph and so we can reframe the analysis in this way. Thus we may claim that there exists an interval $[u^-(x), u^+(x)]$ that contains any normalised direct utility $u(x)$ with $u(td) = t$ for some $d \in \operatorname{int} C$ and all $t > 0$. Taking into account a reversal of sign (used to convert g-quasi-convex functions into g-quasi-concave functions) we may find an upper semicontinuous utility function $\underline{u}$ that coincides with the upper closures of both u^- and u^+ such that all three functions coincide at points of continuity of $\underline{u}$ which is almost everywhere in C. Furthermore $\underline{u}$ is uniquely defined up to upper semicontinuity and normalisation.*

6 Appendix: 1. Quasi-convergence and Lower Closures

Proof of Theorem 13

Proof. Suppose for all $\lambda_v \to \lambda < \mu$ we have $\limsup_v S_{\lambda_v}(g^v) \subseteq \overline{S_\mu(g)}$ then as $\bigcap_{u>\lambda} \overline{S_\mu(g)} = S_\lambda(\bar{g})$ it follows that

$$\limsup_v S_{\lambda_v}(g^v) \subseteq S_\lambda(\bar{g}).$$

For the limit infimum one can argue as in [Rockafellar et al. (1998)] using graphical limits. Denote the strict profile function

$$E^s_{g^v}(x) := \{\alpha \in \mathbb{R} \mid g^v(x) < \alpha\}$$

and so we have $\mathrm{Graph} E^s_{g^v} = \mathrm{epi}_s\, g^v := \{(x,\alpha) \mid g^v(x) < \alpha\}$. Also note that

$$\left(E^s_{g^v}\right)^{-1}(\alpha) = \{x \in \mathbb{R}^n \mid g^v(x) < \alpha\} = \widetilde{S}_\alpha(g^v).$$

Now (see [Rockafellar et al. (1998)], Chapter 5 section E) we have

$$g\text{-}\limsup E^s_{g^v}(x) \supseteq E^s_g(x) \quad \text{iff} \quad g\text{-}\limsup \left(E^s_{g^v}\right)^{-1}(\alpha) \supseteq \left(E^s_g\right)^{-1}(\alpha)$$

which translated to saying (using Proposition 5.33 of [Rockafellar et al. (1998)]) and the graph closedness of graphical limits

$$\begin{aligned} &\limsup_v \mathrm{epi}_s\, g^v \supseteq \overline{\mathrm{epi}_s\, g} \\ \text{iff} \quad &g\text{-}\limsup \widetilde{S}_\alpha(g^v) = \bigcup_{\{\alpha^v \to \alpha\}} \liminf_v \widetilde{S}_{\alpha^v}(g^v) \supseteq \overline{\widetilde{S}_\alpha(g^v)}. \end{aligned} \tag{41}$$

Now the left hand side of (41) can be reworked as follows: as $\limsup_v \mathrm{epi}_s\, g^v$ is always closed it is clear that as all $\alpha^v = g^v(x^v)$ can be approach via $\beta^v > g^v(x^v)$ with $|\beta^v - \alpha^v| \to 0$ that $\limsup_v \mathrm{epi}_s\, g^v = \limsup_v \mathrm{epi}\, g^v$. Hence

$$\limsup_v \mathrm{epi}_s\, g^v \supseteq \mathrm{epi}_s\, g \quad \text{iff} \quad \limsup_v \mathrm{epi}\, g^v \supseteq \overline{\mathrm{epi}_s\, g} = \mathrm{epi}\, \overline{g}.$$

Consequently (41) implies

$$e\text{-}\liminf g^v(x) \supseteq \overline{g}(x) \quad \text{iff} \quad \bigcup_{\{\alpha^v \to \alpha\}} \liminf_v \widetilde{S}_{\alpha^v}(g^v) \supseteq \overline{\widetilde{S}_\alpha(g^v)}. \tag{42}$$

From the right hand side of (42) we need to deduce that there exists $\alpha^v \downarrow \alpha$ such that

$$\liminf_v \widetilde{S}_{\alpha^v}(g^v) \supseteq \overline{\widetilde{S}_\alpha(g^v)}.$$

This follows from the same argument supplied in Proposition 7.7 of [Rockafellar et al. (1998)] and so direct the reader to this proof for details. Once again using the fact that $\liminf_v \widetilde{S}_{\alpha^v}(g^v)$ is closed we have deduced

$$e\text{-}\liminf g^v(x) \supseteq \overline{g}(x) \quad \text{iff} \quad \forall \alpha, \exists \alpha^v \to \alpha, \quad \liminf_v \widetilde{S}_{\alpha^v}(g^v) \supseteq \widetilde{S}_\alpha(g^v).$$

Now consider the second part in the definition of the a quasi-limit, namely there exists $\mu^v \to \mu$ such that for all $\lambda < \mu$ we have

$$\liminf_v \widetilde{S}_{\mu^v}(g^v) \supseteq \widetilde{S}_\lambda(g^v). \tag{43}$$

As we have $\bigcup_{\lambda<\mu} \widetilde{S}_\lambda(g^v) = \widetilde{S}_\mu(g^v)$ we have (43) implying e-$\liminf g^v(x) \supseteq \overline{g}(x)$.

Now apply Proposition 11 to obtain epi-convergence of $\{g^v\}$ to $\overline{g}$ and hence q-converges to $\overline{g}$ as well. ■

7 Appendix: 2. Minimality and Maximality Cone Valued Operators

Proof of Lemma 23.

Proof. Note that when $(z,-\tau) \in \Gamma(y,t)$ and $z \neq 0$ then $\tau \neq 0$ otherwise we would have $(z,-\tau) \neq 0$ and $\langle (z,-\tau),(0,1)\rangle = 0$ contrary to assumption (since $d \equiv (0,1)$ in the new coordinate system). This implies that Γ is generated by $\Gamma^\dagger$ in that $\Gamma(y,-t) = \mathrm{cone}\{(w,-1) \,|\, w \in \Gamma^\dagger(y/t)\} \cup \{0\}$ for $(y,-t) \in \mathrm{dom}\,\Gamma$. Next note that when $(y,t) \in \mathrm{dom}\,\Gamma$ then this implies $(\frac{z}{\tau},-1) \in \Gamma^\dagger(y,t)$ is well defined for all nonzero $(z,-\tau) \in \Gamma(y,t)$ (i.e. $\mathrm{dom}\,\Gamma \subseteq \mathrm{dom}\,\Gamma^\dagger$). When $(y,t) \notin \mathrm{dom}\,\Gamma$ then $\Gamma^\dagger(y,t) = \emptyset$ and so $\mathrm{dom}\,\Gamma = \mathrm{dom}\,\Gamma^\dagger$.

We now show that $\Gamma^{\dagger}(y,t)$ is uniformly bounded in diameter on $D^{\#}$. To show this by contradiction, suppose $\left\|\frac{z_n}{\tau_n}\right\| \to \infty$ for some $(z_n, -\tau_n) \in \Gamma(y_n, t_n)$ with $(y_n, t_n) \in \operatorname{dom}\Gamma^{\dagger} = D^{\#}$. As $D^{\#}$ is compact we may assume that $(y_n, t_n) \to (y,t)$ by extracting a convergent subsequence. Now $\frac{1}{\|z_n\|}(z_n, -\tau_n) \in \Gamma(y_n, t_n)$ as $\Gamma(y_n, t_n)$ is a closed convex cone and so by taking subsequences and renumbering we have $\frac{z_n}{\|z_n\|} \to \hat{z} \neq 0$ and $\frac{1}{\|z_n\|}(z_n, -\tau_n) \to (\hat{z}, 0) \in \Gamma(y,t)$ by the closed graph property. But then $\langle (0,1), (\hat{z}, 0)\rangle = 0$ a contradiction to assumption.

Next we show that $\Gamma^{\dagger}(y,t)$ has a closed graph. Take $\{(y_n, t_n)\} \subseteq \operatorname{dom}\Gamma$ and $w_n \in \Gamma^{\dagger}(y_n, t_n)$ with $(y_n, t_n) \to (y,t)$ then $w_n = \frac{z_n}{\tau_n}$ where $(z_n, -\tau_n) \in \Gamma(y_n, t_n)$ and so $(\frac{z_n}{\tau_n}, -1) \in \Gamma(y_n, t_n)$ (as Γ is conic valued). Thus when $w_n \to w$ we have $(w, -1) \in \Gamma(y,t)$ implying $w \in \Gamma^{\dagger}(y,t)$. The closed graph of $\Gamma^{\dagger}$ in conjunction with the uniform bound implies (Hausdorff) upper semicontinuity (see [Berge (1963)]). In particular we have also shown that $\operatorname{dom}\Gamma$ is closed.

Now take an open cone K and $(y,t) \in D^{\#} \cap \operatorname{dom}\Gamma^{\dagger}$ such that

$$\Gamma(y,t) \subseteq K \cup \{0\}.$$

Let $K^{\dagger} := \{y'/t' \,:\, (y', -t') \in K\}$, an open set such that $K \supset \operatorname{cone}\left\{(w,-1) \,:\, w \in K^{\dagger}\right\} \cup \{0\}$.

We have $\Gamma^{\dagger}(y,t) \subseteq K^{\dagger} \cup \{0\}$ and by the upper semicontinuity of $\Gamma^{\dagger}$ there exists a neighborhood $\tilde{V}$ of (y,t) such that for all $(y', t') \in \tilde{V} \cap \operatorname{dom}\Gamma^{\dagger}$, $\Gamma^{\dagger}(y', t') \subseteq K^{\dagger} \cup \{0\}$ implying

$$\{(w,-1) \mid w \in \Gamma^{\dagger}(y', t')\} \subseteq \{(w,-1) \mid w \in K^{\dagger}\} \cup \{0\}.$$

Taking cones on each side of this inclusion gives

$$\Gamma(y', t') \subseteq K \cup \{0\} \text{ for all } (y', t') \in \tilde{V} \cap \operatorname{dom}\Gamma,$$

verifying C-upper semicontinuity relative to $D \cap \operatorname{dom}\Gamma$. When $(y', t') \in \hat{V} \cap (\operatorname{dom}\Gamma)^c$ then $\Gamma(y', t') = \{0\} \subseteq K \cup \{0\}$ and hence Γ is C–upper semicontinuous on all of D. ■

Proof of Theorem 24.

Proof. Consider first the case of pseudo-monotonicity. Let $p \in D$. We proceed in a similar way to [Phelps (1992), Lemma 7.7] by showing that if (p,x) is pseudo monotonically related to $\operatorname{Graph}\Gamma$ then $x \in \Gamma(p)$. This yields maximal monotonicity. We may assume $x \neq 0$ because $\Gamma(p)$, being a nonempty closed cone, always contains 0.

For a contradiction, assume there exists a (p,x) with $x \neq 0$ such that

$$\langle x, p' - p\rangle > 0 \Longrightarrow \langle x', p' - p\rangle > 0 \quad \forall x' \in \Gamma(p') \setminus \{0\}\ ; \qquad p' \in D \tag{44}$$

and $x \notin \Gamma(p)$. We can separate the nonempty, closed, convex, cone $\Gamma(p)$ from x by a hyperplane passing through the origin i.e. for some $\bar{p}$ we have

$$\Gamma(p) \subseteq \{z \mid \langle \bar{p}, z\rangle < 0\} \cup \{0\} := K \text{ and } \ 0 < \langle x, \bar{p}\rangle.$$

By the definition of C–upper semicontinuity we have a neighbourhood $V \subseteq \operatorname{int} D$ of p for which $\Gamma(V) \subseteq K$. Thus for $t > 0$ sufficiently small, we have $p' = p + t\bar{p} \in V \subseteq \operatorname{int} D$ such that $\Gamma(p + t\bar{p}) \subseteq K$. By assumption $p' \in \operatorname{Edom}\Gamma$ hence there exists $x' \in \Gamma(p') \setminus \{0\}$ and as $\langle x, p' - p\rangle = t\langle x, \bar{p}\rangle > 0$ then, by (44) we must have $t\langle x', \bar{p}\rangle > 0$ implying $x' \notin K$, a contradiction.

Now consider any cccvC relation $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ with $\operatorname{Edom}\Phi \supset D$ and $\Phi(p) \subset \Gamma(p)$ for $p \in D$. Since Φ inherits pseudo-monotonicity from Γ, the analysis above applies to Φ, showing that it is maximal pseudo-monotone on D. Thus $\Phi(p) \supset \Gamma(p)$ for $p \in D$, i.e., these relations coincide on D. By the same token, Γ, as a mapping on D, must be a minimal as a cccvC relation whose effective domain contains D.

When Γ is cyclically pseudo-monotone it must also be pseudo-monotone. Apply the first part of the argument we deduce that Γ is maximally pseudo-monotone. As the maximal cyclical pseudo-monotone extension with the same effective domain cannot be larger than the maximal pseudo-monotone extension they both must coincide. ■

8 Appendix: 3. Solid, g-Pseudo-Convex Functions

We wish to first establish that taking a lower closure does not destroy the property of being solid, g-pseudo-convex.

Lemma 42 *Assume that v is quasiconvex and $\operatorname{int}(S_\lambda(v)) \neq \emptyset$. Then $\overline{S_\lambda(v)} = S_\lambda(\bar{v})$. Hence, the convex sets $S_\lambda(\bar{v})$ and $S_\lambda(v)$ have the same interior, the same closure and the same boundary.*

Proof. Take some $a \in \operatorname{int}(S_\lambda)(v)$. Then $a \in \operatorname{int}(S_\mu(v))$ for all $\mu > \lambda$. Assume that $b \in S_\lambda(\bar{v})$. Then $b \in \overline{S_\mu(v)}$ for all $\mu > \lambda$. For any $t \in (0,1)$, the point $a + t(b-a)$ belongs to $\operatorname{int}(S_\mu)(v)$ for any $\mu > \lambda$. Thus

$$
\begin{aligned}
a + t(b-a) &\in \bigcap_{\mu > \lambda} S_\mu(v) \\
&= \bigcap_{\mu > \lambda} \{p' \mid v(p') \leq \mu\} = \{p' \mid v(p') \leq \lambda\} = S_\lambda(v).
\end{aligned}
$$

It follows that b belongs to $\overline{S_\lambda(v)}$. ■

We note the following fact shown in [Eberhard et al. (2007)], where D is any closed convex subset of $\mathbb{R}^n$. In our case we will take $D = \mathbb{R}^n_+$. This notion is closely related to the nonsatiation associated with utilities.

Definition 43 *We say a function $v : D \to \mathbb{R}_\infty$ is inwardly nonconstant if and only if for all $v(\bar{p}) > \inf v$ there exists a $z \in D$ with $v(\bar{p}) > v(z) > \inf v$.*

Proposition 44 ([Eberhard et al. (2007)]) *Suppose $v : D \to \overline{\mathbb{R}}$ is lower semi-continuous, solid and g-pseudo–convex then v is inwardly nonconstant.*

Proposition 45 ([Eberhard et al. (2007)]) *Suppose $v : D \to \overline{\mathbb{R}}$ is lower semicontinuous, solid and quasi–convex as well as inwardly nonconstant. Let $\bar{p}$ be such that $v(\bar{p}) > \inf v$. Then there exists a λ such that $\inf v < \lambda < v(\bar{p})$ with $\operatorname{int} \tilde{S}_\lambda(v) \neq \emptyset$ and $\bar{p} \notin \overline{\tilde{S}_\lambda(v)}$.*

Denote by $\bar{v}$ the lower semicontinuous hull of v .

Corollary 46 *Suppose a function $v : D \to \overline{\mathbb{R}}$ is a proper, solid g-pseudo-convex function then so is its lower semicontinuous hull $\bar{v}$. In particular both v and $\bar{v}$ are non-nonsatiated and*

$$
\overline{\tilde{S}_{\bar{\lambda}}(\bar{v})} = \overline{\tilde{S}_{\bar{\lambda}}(v)} = \overline{S_{\bar{\lambda}}(v)} = S_{\bar{\lambda}}(\bar{v}) = \overline{S_{\bar{\lambda}}(\bar{v})}. \tag{45}
$$

Proof. The identities (45) follow from our prior discussion. It follows that $\bar{v}$ is g-pseudo-convex and is also lower semicontinuous. It is also solid because

$$
\emptyset \neq \operatorname{int} S_{\bar{\lambda}}(v) = \operatorname{int} \overline{S_{\bar{\lambda}}(v)} = \operatorname{int} \overline{S_{\bar{\lambda}}(\bar{v})} = \operatorname{int} S_{\bar{\lambda}}(\bar{v}).
$$

Consequently $\bar{v}$ is inwardly nonconstant so there exists z such that $\bar{v}(\bar{p}) > \bar{v}(z)$ and using Proposition 45 we have for appropriately chosen λ that $z \in \operatorname{int}\{p \mid \bar{v}(p) < \lambda < \bar{v}(\bar{p})\}$. Now take any $\delta > 0$ and consider the neighbourhood $B_\delta(\bar{p})$. Now as $\bar{v}$ is solid, g-pseudo-convex we have $\bar{p} \in \overline{\{p \mid \bar{v}(p) < \bar{v}(\bar{p})\}} = \{p \mid \bar{v}(p) \leq \bar{v}(\bar{p})\}$ and so $\bar{p} \in \operatorname{bd}\{p \mid \bar{v}(p) < \bar{v}(\bar{p})\}$. By convexity (see [Rockafellar et al. (1998)] Theorem 2.33) we have

$$
p' := \bar{p} + t(z - \bar{p}) \in \operatorname{int}\{p \mid \bar{v}(p) < \bar{v}(\bar{p})\}
$$

and hence of $t > 0$ sufficiently small $p' := \bar{p} + t(z - \bar{p}) \in \{p \mid \bar{v}(p) < \bar{v}(\bar{p})\} \cap B_\delta(\bar{p})$ implying $\bar{v}(p') < \bar{v}(\bar{p})$ and so $\bar{v}$ is nonsatiated.

Now take a sequence $p_m \to p'$ that achieves the limit infimum

$$
\lim_m v(p_m) = \liminf_{y \to p'} v(y) = \bar{v}(p').
$$

Then for m large we have $p_m \in B_\delta(\bar{p})$ and

$$
v(p_m) \leq \bar{v}(p') + \frac{1}{2}(\bar{v}(\bar{p}) - \bar{v}(p')) < \bar{v}(\bar{p}) \leq v(\bar{p}),
$$

proving v is nonsatiated. ■

By the results of [Eberhard et al. (2007)] it follows that $v : D \to \mathbb{R}_\infty$ being lower semicontinuous, proper, solid and g-pseudo-convex implies v is classically pseudo–convex in the sense that

$$v(x') < v(x) \quad \text{implies} \quad \langle z, x' - x \rangle < 0 \tag{46}$$

for all $z \in N_v(x)\setminus\{0\}$. Thus being solid, g-pseudo-convex is stronger than the classical notion of pseudo-convexity.

We now need to address how these properties are transformed via the duality formula (2).

The proof of Proposition 15.

Proof. First note that by Corollary 46 $\bar{v}$ is also proper, solid g-pseudo-convex function. Using the duality formula (2) we have

$$\lambda > u(x) \quad \text{iff} \quad \exists p \text{ such that } v(p) < \lambda \text{ and } \langle p, x \rangle \leq 1.$$

It follows that

$$\inf \left\{ \lambda \mid \tilde{S}_\lambda(v) \cap \{p \mid \langle p, x \rangle \leq 1\} \neq \emptyset \right\} = u(x). \tag{47}$$

Suppose $u(x) > \lambda$ then we claim that $S_\lambda(v) \cap \{p \mid \langle p, x \rangle \leq 1\} = \emptyset$. If not then $S_\lambda(v) \cap \{p \mid \langle p, x \rangle \leq 1\} \neq \emptyset$ and as $S_\lambda(v) = \cap_{\varepsilon > 0} \tilde{S}_{\lambda+\varepsilon}(v)$ we have $\tilde{S}_{\lambda+\varepsilon}(v) \cap \{p \mid \langle p, x \rangle \leq 1\} \neq \emptyset$ for all $\varepsilon > 0$ implying (using (47))

$$\lambda + \varepsilon \geq \inf\{v(p) \mid \langle p, x \rangle \leq 1\} = u(x),$$

giving the contradiction $\lambda \geq u(x)$.

Denote the lower semicontinuous hull of v by $\bar{v}$. Because $\cap_{\varepsilon>0} \operatorname{int} \tilde{S}_{\lambda+\varepsilon}(v) \neq \emptyset$ we have

$$S_\lambda(\bar{v}) = \overline{S_\lambda(v)} = \overline{\cap_{\varepsilon>0} \tilde{S}_{\lambda+\varepsilon}(v)} = \cap_{\varepsilon>0} \overline{\tilde{S}_{\lambda+\varepsilon}(v)} = \cap_{\varepsilon>0} \tilde{S}_{\lambda+\varepsilon}(\bar{v}), \tag{48}$$

the same argument as before implies $\bar{u}(x) := \inf\{\bar{v}(p) \mid \langle p, x \rangle \leq 1\} > \lambda$ implies

$$S_\lambda(\bar{v}) \cap \{p \mid \langle p, x \rangle \leq 1\} = \overline{S_\lambda(v)} \cap \{p \mid \langle p, x \rangle \leq 1\} = \emptyset.$$

Now suppose $x \in \mathbb{R}^n_{++}$. As $\overline{S_\lambda(v)}$ (is a closed convex set) there exists a positive gap between $\overline{S_\lambda(v)}$ and the closed, bounded set $\left\{p \in \mathbb{R}^n_+ \mid \langle p, x \rangle \leq 1\right\}$. Using (48) again there exists $\varepsilon > 0$ such that $\tilde{S}_{\lambda+\varepsilon}(\bar{v}) \cap \{p \mid \langle p, x \rangle \leq 1\} = \emptyset$ and hence $\bar{u}(x) \geq \lambda + \varepsilon$ implying

$$u(x) = \inf\{v(p) \mid \langle p, x \rangle \leq 1\} \geq \inf\{\bar{v}(p) \mid \langle p, x \rangle \leq 1\} = \bar{u}(x) \geq \lambda + \varepsilon > \lambda$$

Consequently if $x \in \mathbb{R}^n_{++}$ then $u(x) > \lambda$ when $\overline{S_\lambda(v)} \cap \{p \mid \langle p, x \rangle \leq 1\} = \emptyset$. First note that this implies the existence of $\delta > 0$ such that for all $x' \in B_\delta(x) \subseteq \mathbb{R}^n_{++}$ we have $\overline{S_\lambda(v)} \cap \{p \mid \langle p, x' \rangle \leq 1\} = \emptyset$. Thus $\operatorname{int}\left\{x \in \mathbb{R}^n_+ \mid u(x) > \lambda\right\} \neq \emptyset$ and u has been shown to be solid.

To show u is g-pseudoconcave first take $\bar{x} \in \left\{x \in \mathbb{R}^n_{++} \mid u(x) = \lambda\right\}$ then $\tilde{S}_\mu(v) \cap \{p \mid \langle p, \bar{x} \rangle \leq 1\} = \emptyset$ for all $\mu < \lambda$. Thus $\tilde{S}_\mu(v) \subseteq \{p \mid \langle p, \bar{x} \rangle \geq 1\}$ for all $\mu < \lambda$ and so $\cup_{\mu<\lambda} \tilde{S}_\mu(v) = \tilde{S}_\lambda(v) \subseteq \{p \mid \langle p, \bar{x} \rangle \geq 1\}$. Consequently $\overline{\tilde{S}_\lambda(v)} \subseteq \{p \mid \langle p, \bar{x} \rangle \geq 1\}$. Using the fact that $\overline{\tilde{S}_\lambda(v)} = \overline{S_\lambda(v)}$ we have for all $t > 1$ that

$$\begin{aligned} \overline{S_\lambda(v)} \cap \{p \mid \langle p, \bar{x} \rangle \leq 1/t\} &\subseteq \{p \mid \langle p, \bar{x} \rangle \geq 1\} \cap \{p \mid \langle p, \bar{x} \rangle \leq 1/t\} = \emptyset \\ \text{implying} \quad \overline{S_\lambda(v)} \cap \{p \mid \langle p, t\bar{x} \rangle \leq 1\} &= \emptyset \end{aligned}$$

or $u(t\bar{x}) > \lambda$. Taking $t_n \downarrow 1$ we find $x_n := t_n \bar{x} \in \left\{x \in \mathbb{R}^n_{++} \mid u(x) > \lambda\right\}$ with $x_n \to \bar{x}$ giving

$$\overline{\left\{x \in \mathbb{R}^n_{++} \mid u(x) > \lambda\right\}} \supseteq \left\{x \in \mathbb{R}^n_{++} \mid u(x) \geq \lambda\right\}.$$

Finally we invoke Corollary 46. Now use [Rockafellar et al. (1998)] Theorem 2.33

$$\begin{aligned} \overline{\left\{x \in \mathbb{R}^n_+ \mid u(x) > \lambda\right\}} &= \overline{\operatorname{int}\left\{x \in \mathbb{R}^n_+ \mid u(x) > \lambda\right\}} \\ &= \overline{\operatorname{int}\left\{x \in \mathbb{R}^n_{++} \mid u(x) > \lambda\right\}} \\ &= \overline{\left\{x \in \mathbb{R}^n_{++} \mid u(x) > \lambda\right\}} = \overline{\left\{x \in \mathbb{R}^n_{++} \mid u(x) \geq \lambda\right\}}. \end{aligned}$$

Now consider $\bar{x} \in \left\{x \in \mathbb{R}^n_+ \backslash \mathbb{R}^n_{++} \mid u(x) \geq \lambda\right\}$ then as $\bar{x} + te \geq \bar{x}$ where $e = (1, 1, \ldots, 1)$, we have $u(\bar{x} + te) \geq u(\bar{x}) \geq \lambda$ and hence for $t_n \downarrow 0$ we have $x_n = \bar{x} + t_n e \in \left\{x \in \mathbb{R}^n_{++} \mid u(x) \geq \lambda\right\}$ with $x_n \to \bar{x}$ and so

$$\overline{\left\{x \in \mathbb{R}^n_+ \mid u(x) > \lambda\right\}} \supseteq \left\{x \in \mathbb{R}^n_+ \mid u(x) \geq \lambda\right\}$$

$$\text{giving} \qquad \overline{\left\{x \in \mathbb{R}^n_+ \mid u(x) > \lambda\right\}} = \overline{\left\{x \in \mathbb{R}^n_+ \mid u(x) \geq \lambda\right\}}$$

and g-pseudoconcavity of u. ■

9 Appendix: 4. Proof of Theorem 29

The proof of Theorem 29.

Proof. To show f is quasi–convex we need to show that $S_K(f) := \{(y,t) \mid f(y,t) \leq K\}$ is convex for arbitrary $K \geq \inf f$. Take $(y', t'), (y,t) \in S_K(f)$ and we want to show that

$$\mu(y', t') + (1-\mu)(y,t) \in S_K(f) \quad \text{for } \mu \in [0,1]$$

which is equivalent to showing that

$$g(\mu y' + (1-\mu)y, \lambda) > \mu t' + (1-\mu)t \tag{49}$$

implies $K \geq \lambda$. Suppose contrary to the assertion that $\lambda > K$ for some λ such that (49) holds. As $(y', t'), (y,t) \in S_K(f) \subseteq S_\lambda(f)$ we immediately have $g(y', \lambda) \leq t'$ and $g(y, \lambda) \leq t$. Now using convexity of g we have

$$\begin{aligned} g(\mu y' + (1-\mu)y, \lambda) &\leq \mu g(y', \lambda) + (1-\mu) g(y, \lambda) \\ &\leq \mu t' + (1-\mu)t \end{aligned}$$

a contradiction to (49). Thus $S_K(f)$ is convex. It is clear from definitions that $t' > g(y', \lambda)$ implies $\lambda \geq f(y', t')$ and so $\operatorname{epi}_s g(\cdot, \lambda)$ is locally contained in $S_f(y,t)$ when $\lambda = f(y,t)$. Similarly $f(y', t') < \lambda$ implies $g(y', \lambda) \leq t'$ giving $\tilde{S}_f(y,t)$ locally contained in $\operatorname{epi} g(\cdot, \lambda)$.

As $(Y \times T) \cap \operatorname{epi}_s g(\cdot, \lambda) \subseteq S_f(y,t)$ then as $y \mapsto g(y, \lambda)$ is finite and convex on Y it is continuous and hence $\operatorname{int} \operatorname{epi}\ _s g(\cdot, \lambda) \neq \emptyset$ and so

$$\emptyset \neq \operatorname{int}\left((Y \times T) \cap \operatorname{epi}_s g(\cdot, \lambda)\right) \subseteq \operatorname{int} S_f(y,t)$$

and f is solid quasiconvex.

Note that from the definitions we have $f(y, t_1) \geq f(y, t_2)$ for $t_1 \leq t_2$. To demonstrate that $(y,t) \to f(y,t)$ is lower semicontinuous first note that if $f(y,t) = -\infty$ it is clearly lower semicontinuous there. Now assume $f(y,t) > -\infty$ and take an arbitrary λ strictly less that $f(y,t)$ (so $g(y, \lambda) > t$) and $(y_n, t_n) \to (y, t)$. Choose $\delta > 0$ such that $g(y, \lambda) \geq t + \delta > t$. For n large we have $t + \frac{\delta}{2} > t_n$. Also as $y \mapsto g(y, \lambda)$ finite and $y \in \operatorname{int} Y$ it is continuous, due to convexity. Thus for n large $g(y_n, \lambda) > t + \frac{\delta}{2}$ and so $g(y_n, \lambda) > t_n$ and $f(y_n, t_n) > \lambda$. Thus

$$\liminf_{n \to \infty} f(y_n, t_n) > \lambda$$

and as this holds for all sequences converging to t we have

$$\liminf_{(y', t') \to (y,t)} f(y', t') = \min_{\{(y_n, t_n) \to (y,t)\}} \liminf_{n \to \infty} f(y_n, t_n) \geq \lambda \tag{50}$$

for arbitrary $f(y,t) > \lambda$. Lower semicontinuity thus follows from (50).

Consider now (1), the case when $\lambda \mapsto g(y, \lambda)$ is lower semicontinuous. Suppose that $(y', t') \notin \operatorname{epi} g(\cdot, \lambda)$ where $\lambda = f(y,t)$ then $g(y', \lambda) > t'$. By lower semicontinuity we have $\{\lambda' \mid g(y', \lambda') > t'\}$ is open and so there is a $\delta > 0$ such that $g(y', \lambda') > t'$ for all $\lambda' \in (\lambda - \delta, \lambda + \delta)$ and hence $f(y', t') \geq \lambda + \delta > \lambda = f(y,t)$. Thus $(y', t') \notin S_f(y,t)$ and so $S_f(y,t) \cap (Y \times T) \subseteq \operatorname{epi} g(\cdot, \lambda) \cap (Y \times T)$. We have already shown $(Y \times T) \cap \operatorname{epi}_s g(\cdot, \lambda) \subseteq S_f(y,t)$ and as

$$\overline{(Y \times T) \cap \operatorname{epi}_s g(\cdot, \lambda)} = \overline{(Y \times T)} \cap \operatorname{epi} g(\cdot, \lambda)$$

by convexity and the fact that $\operatorname{int} \operatorname{epi}\ g(\cdot,\lambda) = \operatorname{epi}_s g(\cdot,\lambda) \neq \emptyset$ the first part (22) of the result 1. follows on taking the closure. When $y \in \operatorname{int} Y \neq \emptyset$, $t \in \operatorname{int} T \neq \emptyset$ standard convex analysis results gives $\overline{S_f(y,t) \cap (Y \times T)} = S_f(y,t) \cap (Y \times T)$. In this case the graph of $g(\cdot,\lambda)$ corresponds to the boundary of $S_f(y,t)$ and so if $(y,t) \in \operatorname{Graph} g(\cdot,\lambda)$ then $\lambda = f(y,t)$.

The second part (23) follows on using the usual relationship between epigraphs and normal cones to convex sets. That is, $\operatorname{cone} \partial_y g(y,\lambda) \times \{1\} = N_f((y,t))$ and so $N_f^{\dagger}((y,t)) = \partial_y g(y,\lambda)$ whenever $\lambda = f(y,t)$ or $t = g(y,\lambda)$.

Finally we note that $\lambda = f(y,t)$ implies (by definitions) that

$$\lambda = \sup\{\lambda' \mid g(y,\lambda') > t\}.$$

As $\lambda' \mapsto g(y,\lambda')$ is lower semicontinuous we have $\{\lambda' \mid g(y,\lambda') > t\}$ an open interval and so $\lambda \notin \{\lambda' \mid g(y,\lambda') > t\}$ implying $t \geq g(y,\lambda)$. When $t > g(y,\lambda)$ then $N^{\dagger}((y,t)) = \{0\}$.

Consider 2. When $\lambda \mapsto g(y,\lambda)$ is upper semicontinuous take $(y',t') \in \operatorname{epi}_s g(\cdot,\lambda)$ where $\lambda = f(y,t)$. Then $t' > g(y',\lambda)$ implying $\lambda \notin \{\lambda' \mid g(y',\lambda') \geq t'\}$. By the upper semicontinuity assumption this set is closed and so there exists a $\delta > 0$ such that

$$(\lambda-\delta,\lambda+\delta) \cap \{\lambda' \mid g(y',\lambda') > t'\} \subseteq (\lambda-\delta,\lambda+\delta) \cap \{\lambda' \mid g(y',\lambda') \geq t'\} = \emptyset.$$

Thus $\lambda - \delta \geq f(y',t')$ giving $(y',t') \in \tilde{S}_f(y,t)$. Thus

$$(Y \times T) \cap \operatorname{epi}_s g(\cdot,\lambda) \subseteq (Y \times T) \cap \tilde{S}_f(y,t) \subseteq (Y \times T) \cap \operatorname{epi} g(\cdot,\lambda).$$

On taking a closure we arrive at the desired conclusion for (24). The second part (25) follows again on using the usual relationship between epigraphs and normal cones to convex sets. That is, $\operatorname{cone} \partial_y g(y,\lambda) \times \{1\} = \tilde{N}_f((y,t))$ and so $\tilde{N}_f^{\dagger}((y,t)) = \partial_y g(y,\lambda)$ when $t = g(y,\lambda)$ and $\tilde{N}_f^{\dagger}((y,t)) = \{0\}$ otherwise, completing 2.

Finally we note that by the monotonicity and upper semicontinuity of $\lambda' \mapsto g(y,\lambda')$ we have $\lambda \in \{\lambda' \mid g(y,\lambda') \geq t\}$ a closed interval. Now either there is a discontinuity of $g(y,\cdot)$ at $\lambda = f(y,t)$ in which case $\{\lambda' \mid g(y,\lambda') > t\} = \{\lambda' \mid g(y,\lambda') \geq t\}$ or there is a point of continuity of $g(y,\cdot)$ at λ implying $g(y,\lambda) = t$. In the first case $f(y,t) = \lambda$ is attained at the end point of the closed interval $\{\lambda' \mid g(y,\lambda') \geq t\}$ implying $g(y,\lambda) \geq t$. When $g(y,\lambda) > t$ then $\tilde{K}(y,t) = \emptyset$ and so $\tilde{N}_f^{\dagger}((y,t)) = \{0\}$.

Consider now 3. Suppose $t \mapsto f(y,t)$ is strictly decreasing and $\lambda \mapsto g(y,\lambda)$ is discontinuous at λ. That is for some fixed $\varepsilon > 0$ and $\delta > 0$ sufficiently small $g(y,\lambda+\delta) + \varepsilon \leq g(y,\lambda)$ (recall that g is decreasing in λ). Then choose $t \in (g(y,\lambda) - \frac{3\varepsilon}{4}, g(y,\lambda) - \frac{\varepsilon}{4}) := I$ and note that

$$\lambda + \delta \geq f(y,t) = \sup\{\lambda' \mid g(y,\lambda') > t\} \geq \lambda.$$

Letting $\delta \downarrow 0$ we obtain $f(y,t) = \lambda$ for all $t \in I$ a contradiction to the strict decreasing assumption thus $\lambda \mapsto g(y,\lambda)$ is continuous.

We now show that $g(y,\lambda) = \inf\{t \mid f(y,t) \leq \lambda\}$ holds. Suppose $f(y,t) < \lambda$ then $g(y,\lambda) \leq t$ and hence

$$\inf\{t \mid f(y,t) < \lambda\} \geq g(y,\lambda) \quad \text{and so}$$
$$\inf\{t \mid f(y,t) \leq \lambda\} \geq \inf\{t \mid f(y,t) < \lambda'\} \geq g(y,\lambda') \text{ for all } \lambda < \lambda'.$$

Letting $\lambda' \downarrow \lambda$ and using the continuity of g in λ we obtain

$$\inf\{t \mid f(y,t) \leq \lambda\} \geq g(y,\lambda) > -\infty. \tag{51}$$

Now suppose that $\inf\{t \mid f(y,t) \leq \lambda\} > g(y,\lambda)$ then there exists $\bar{t}$ such that for all $t > \bar{t} > g(y,\lambda)$ we have $f(y,t) \leq \lambda$ and either $f(y,\bar{t}) = \lambda$ or $f(y,\bar{t}) > \lambda$. In the latter case of $f(y,\bar{t}) > \lambda$ we have by the definition of f that

$$g(y,\lambda) > \bar{t}$$

a contradiction. In the former case for any $\lambda' < \lambda$ we have again from the definition of f that

$$g(y,\lambda') > \bar{t} > g(y,\lambda)$$

and on letting $\lambda' \uparrow \lambda$ the continuity of g gives rise a contradiction again. Thus equality holds in (51) as claimed. ■

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