# ON REGULARITY CONDITIONS FOR COMPLEMENTARITY PROBLEMS

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December 2, 2011

**ABSTRACT**

In the context of mixed complementarity problems, various concepts of solution regularity are known, each of them playing a certain role in related theoretical and algorithmic developments. In this note, we provide the complete picture of relations between the most important regularity conditions for mixed complementarity problems. A special attention is paid to the particular cases of a nonlinear complementarity problem and of a Karush–Kuhn–Tucker system.

**Key words:** mixed complementarity problem, nonlinear complementarity problem, KKT system, natural residual function, Fischer–Burmeister function, *BD*-regularity, *CD*-regularity, strong regularity, semistability, *b*-regularity, quasi-regularity.
**AMS subject classifications.** 90C33.

* This research is supported by the Russian Foundation for Basic Research Grant 10-01-00251.

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# 1 Introduction

We consider the mixed complementarity problem (MCP), which is the variational inequality on the generalized box:

$$z \in [\ell,\, u], \quad \langle \Phi(z),\, y - z\rangle \geq 0 \quad \forall\, y \in [\ell,\, u], \tag{1.1}$$

where $\Phi\colon \mathbb{R}^s \to \mathbb{R}^s$ is a given mapping, and

$$[\ell,\, u] = \{z \in \mathbb{R}^s \mid \ell_i \leq z_i \leq u_i,\, i = 1,\, \ldots,\, s\},$$

with some $\ell_i \in \mathbb{R} \cup \{-\infty\}$, $u_i \in \mathbb{R} \cup \{+\infty\}$, $\ell_i < u_i$, $i = 1,\, \ldots,\, s$. Equivalently, MCP can be stated as follows:

$$z \in [\ell,\, u], \quad \Phi_i(z) \begin{cases} \geq 0 & \text{if } z_i = \ell_i, \\ = 0 & \text{if } z_i \in (\ell_i,\, u_i),\, , \\ \leq 0 & \text{if } z_i = u_i, \end{cases} \quad i = 1,\, \ldots,\, s. \tag{1.2}$$

The MCP format covers many important applications and problem settings [12, 11, 9], and perhaps the most well-known among them are the (usual) nonlinear complementarity problem (NCP)

$$z \geq 0, \quad \Phi(z) \geq 0, \quad \langle z,\, \Phi(z)\rangle = 0, \tag{1.3}$$

corresponding to the case when $\ell_i = 0$, $u_i = +\infty$, $i = 1,\, \ldots,\, s$, and the Karush–Kuhn–Tucker (KKT) system

$$\begin{array}{c} F(x) + (h'(x))^{\mathrm{T}}\lambda + (g'(x))^{\mathrm{T}}\mu = 0, \quad h(x) = 0, \\ \mu \geq 0, \quad g(x) \leq 0, \quad \langle \mu,\, g(x)\rangle = 0, \end{array} \tag{1.4}$$

in unknown $(x,\, \lambda,\, \mu) \in \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m$, where $F\colon \mathbb{R}^n \to \mathbb{R}^n$ , $h\colon \mathbb{R}^n \to \mathbb{R}^l$ and $g\colon \mathbb{R}^n \to \mathbb{R}^m$ are given mappings, and the last two are assumed differentiable. Indeed, (1.4) can be written in the form of (1.2) by setting $s = n + l + m$,

$$\Phi(z) = (G(x,\, \lambda,\, \mu),\, h(x),\, -g(x)), \quad z = (x,\, \lambda,\, \mu)$$

with $G\colon \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m \to \mathbb{R}^n$ defined by

$$G(x,\, \lambda,\, \mu) = F(x) + (h'(x))^{\mathrm{T}}\lambda + (g'(x))^{\mathrm{T}}\mu$$

and taking

$$\begin{array}{c} \ell_i = -\infty,\, i = 1,\, \ldots,\, n + l, \quad \ell_i = 0,\, i = n + l + 1,\, \ldots,\, n + l + m, \\ u_i = +\infty,\, i = 1,\, \ldots,\, n + l + m. \end{array}$$

MCP (1.1) with $\Phi(z) = \varphi'(z)$, $z \in \mathbb{R}^s$, gives the primal first-order optimality condition for the optimization problem

$$\begin{array}{ll} \text{minimize} & \varphi(z) \\ \text{subject to} & z \in [\ell,\, u], \end{array}$$

where $\varphi : \mathbb{R}^s \to \mathbb{R}$ is a given smooth objective function. On the other hand, KKT system (1.4) with $F(x) = f'(x)$, $x \in \mathbb{R}^n$, characterizes stationary points and the associated Lagrange multipliers of the mathematical programming problem

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & h(x) = 0,\ g(x) \leq 0, \end{array}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a given smooth objective function.

Another well known fact (see, e.g., [1, 10]) is that MCP can be equivalently reformulated as a system of nonlinear equations employing a complementarity function, that is, a function $\psi\colon \mathbb{R}^2 \to \mathbb{R}$ satisfying

$$\psi(a,\, b) = 0 \quad \Longleftrightarrow \quad a \geq 0,\ b \geq 0,\ ab = 0.$$

Assuming the additional properties

$$\psi(a,\, b) < 0 \quad \forall\, a > 0,\, b < 0, \quad \psi(a,\, b) > 0 \quad \forall\, a > 0,\, b > 0, \tag{1.5}$$

from the equivalent formulation (1.2) of MCP (1.1) it can be seen that solutions of this problem coincide with solutions of the equation

$$\Psi(z) = 0, \tag{1.6}$$

where $\Psi\colon \mathbb{R}^s \to \mathbb{R}^s$ is given by

$$\Psi_i(z) = \begin{cases} \Phi_i(z) & \text{if } i \in I_\Phi, \\ \psi(z_i - l_i, \Phi_i(z)) & \text{if } i \in I_\ell, \\ -\psi(u_i - z_i, -\Phi_i(z)) & \text{if } i \in I_u, \\ \psi(z_i - l_i, -\psi(u_i - z_i, -\Phi_i(z))) & \text{if } i \in I_{\ell u}, \end{cases} \tag{1.7}$$

with

$$\begin{aligned} I_\Phi &= \{i = 1,\, \ldots,\, s \mid \ell_i = -\infty,\, u_i = +\infty\}, \\ I_\ell &= \{i = 1,\, \ldots,\, s \mid \ell_i > -\infty,\, u_i = +\infty\}, \\ I_u &= \{i = 1,\, \ldots,\, s \mid \ell_i = -\infty,\, u_i < +\infty\}, \\ I_{\ell u} &= \{i = 1,\, \ldots,\, s \mid \ell_i > -\infty,\, u_i < +\infty\}. \end{aligned}$$

In this work we consider two widely used complementarity functions, both satisfying (1.5): the natural residual (or minimum) function

$$\psi(a,\, b) = \min\{a,\, b\},$$

and the Fischer–Burmeister function

$$\psi(a,\, b) = a + b - \sqrt{a^2 + b^2}.$$

The corresponding instances of the mapping $\Psi$ in (1.7) will be denoted by $\Psi_{NR}$ and $\Psi_{FB}$, respectively.

For a given solution $\bar{z} \in \mathbb{R}^s$ of MCP (1.1) (equivalently, of (1.2)), define the index sets

$$\begin{aligned} I_+ = I_+(\bar{z}) &= \{i = 1, \ldots, s \mid \Phi_i(\bar{z}) = 0, \ \bar{z}_i \in (\ell_i, u_i)\}, \\ I_0 = I_0(\bar{z}) &= \{i = 1, \ldots, s \mid \Phi_i(\bar{z}) = 0, \ \bar{z}_i \in \{\ell_i, u_i\}\}, \\ N = N(\bar{z}) &= \{i = 1, \ldots, s \mid \Phi_i(\bar{z}) \neq 0\}. \end{aligned}$$

Observe that no matter how smooth the underlying mapping $\Phi$ is, if the strict complementarity condition $I_0 = \emptyset$ is violated then both mappings $\Psi = \Psi_{NR}$ and $\Psi = \Psi_{FB}$ are not necessarily differentiable at $\bar{z}$. At the same time, they are locally Lipschitz-continuous at $\bar{z}$ provided $\Phi$ possesses this property. The relevant generalized differential objects for these mappings are therefore the $B$-differential

$$\partial_B \Psi(\bar{z}) = \{J \in \mathbb{R}^{s \times s} \mid \exists \{z^k\} \subset \mathcal{S}_\Psi \text{ such that } \{z^k\} \to \bar{z}, \ \{\Psi'(z^k)\} \to J\},$$

where $\mathcal{S}_\Psi$ is the set of points at which $\Psi$ is differentiable, and Clarke's generalized Jacobian

$$\partial \Phi(\bar{z}) = \operatorname{conv} \partial_B \Phi(\bar{z}),$$

where conv stands for the convex hull (see [3, Section 2.6.1], [9, Section 7.1]).

Assume that $\Phi$ is differentiable near $\bar{z}$, with its derivative being continuous at $\bar{z}$ (implying local Lipschitz continuity of $\Phi$ at $\bar{z}$). The following upper estimate of $\partial_B \Psi_{NR}(\bar{z})$ (see, for example, [14]) can be derived immediately by the definition of $\Psi_{NR}$: the rows of any matrix $J \in \partial_B \Psi_{NR}(\bar{z})$ satisfy

$$J_i \begin{cases} = \Phi_i'(\bar{z}) & \text{if } i \in I_+, \\ \in \{\Phi_i'(\bar{z}), \, e^i\} & \text{if } i \in I_0, \\ = e^i & \text{if } i \in N, \end{cases} \tag{1.8}$$

where $e^i$ is the $i$-th row of the $s \times s$ unit matrix, $i = 1, \ldots, s$. The set of matrices in $\mathbb{R}^{s \times s}$ with rows satisfying (1.8) will be denoted by $\Delta_{NR}(\bar{z})$. Therefore, $\partial_B \Psi_{NR}(\bar{z}) \subset \Delta_{NR}(\bar{z})$.

The upper estimate of $\partial \Psi_{FB}(\bar{z})$ has been obtained in [1]. We are not aware of any reference for the proof of the corresponding upper estimate for $\partial_B \Psi_{FB}(\bar{z})$, but it can be easily derived directly from the definitions. Specifically, the rows of any matrix $J \in \partial_B \Psi_{FB}(\bar{z})$ satisfy

$$J_i = \begin{cases} \Phi_i'(\bar{z}) & \text{if } i \in I_+, \\ \alpha_i \Phi_i'(\bar{z}) + \beta_i e^i & \text{if } i \in I_0, \\ e^i & \text{if } i \in N, \end{cases} \tag{1.9}$$

where $(\alpha_i, \beta_i) \in \mathbb{R} \times \mathbb{R}$ belongs to the set

$$S = \{(a, b) \in \mathbb{R}^2 \mid (a-1)^2 + (b-1)^2 = 1\} \tag{1.10}$$

for each $i \in I_0$. The set of matrices with rows satisfying (1.9) with some $(\alpha_i, \beta_i) \in S$, $i \in I_0$, will be denoted by $\Delta_{FB}(\bar{z})$. With this notation, $\partial_B \Psi_{FB}(\bar{z}) \subset \Delta_{FB}(\bar{z})$.

The rest of the paper is organized as follows. In Section 2 we recall the most important and widely used regularity conditions for MCP, including those related to the generalized differential objects defined above. In Section 3 we establish the complete picture of relations between these regularity conditions for general MCP and for NCP. Section 4 is concerned with the specificities of KKT systems.

## 2 Regularity conditions for MCP

We will be saying that a set of square matrices is nonsingular if every matrix in this set is nonsingular. The following regularity conditions play a central role in justification of local superlinear convergence of semismooth Newton methods applied to equation (1.6) with a locally Lipschitzian mapping in the left-hand side (see [16, 17, 21, 22] and [9, Section 7.5]).

**Definition 2.1** A mapping $\Psi\colon \mathbb{R}^s \to \mathbb{R}^s$ is said to be *BD-regular* (*CD-regular*) at $\bar{z} \in \mathbb{R}^s$ if the set $\partial_B \Psi(\bar{z})$ ($\partial \Psi(\bar{z})$) is nonsingular.

Assuming again that $\bar{z} \in \mathbb{R}^s$ is a solution of MCP (1.1), and employing the upper estimates of $\partial_B \Psi_{NR}(\bar{z})$ and $\partial_B \Psi_{FB}(\bar{z})$ discussed in the previous section, we conclude that $BD$-regularity of $\Psi_{NR}$ (of $\Psi_{FB}$) at $\bar{z}$ is implied by nonsingularity of $\Delta_{NR}(\bar{z})$ (of $\Delta_{FB}(\bar{z})$), while $CD$-regularity of $\Psi_{NR}$ (of $\Psi_{FB}$) at $\bar{z}$ is implied by nonsingularity of $\operatorname{conv} \Delta_{NR}(\bar{z})$ (of $\operatorname{conv} \Delta_{FB}(\bar{z})$).

Define the sets

$$\Sigma = \{(a,\, b) \in \mathbb{R}^2 \mid a + b = 1,\ a \geq 0,\ b \geq 0\},$$

and

$$B = \operatorname{conv} S = \{(a,\, b) \in \mathbb{R}^2 \mid (a-1)^2 + (b-1)^2 \leq 1\}.$$

From the definition of $\Delta_{NR}(\bar{z})$ by the standard tools of convex analysis it readily follows that $\operatorname{conv} \Delta_{NR}(\bar{z})$ consists of all matrices $J \in \mathbb{R}^{s\times s}$ with rows satisfying (1.9) with some $(\alpha_i,\, \beta_i) \in \Sigma$, $i \in I_0$. Similarly, since $J_i$ in (1.9) depends linearly on $(\alpha_i,\, \beta_i)$, by the standard tools of convex analysis it immediately follows that $\operatorname{conv} \Delta_{FB}(\bar{z})$ consists of all matrices $J \in \mathbb{R}^{s\times s}$ with rows satisfying (1.9) with some $(\alpha_i,\, \beta_i) \in B$, $i \in I_0$.

In the case of NCP (1.3), nonsingularity of $\Delta_{NR}(\bar{z})$ is known under the name of *b-regularity* of solution $\bar{z}$ [20]. This condition amounts to the following: the matrix

$$\begin{pmatrix} (\Phi'(\bar{z}))_{I_+ I_+} & (\Phi'(\bar{z}))_{I_+ K} \\ (\Phi'(\bar{z}))_{K I_+} & (\Phi'(\bar{z}))_{KK} \end{pmatrix}$$

is nonsingular for any index set $K \subset I_0$. Here and throughout by $M_{K_1 K_2}$ we denote the submatrix of a matrix $M$ corresponding to row numbers $i \in K_1$ and column numbers $j \in K_2$.

In the case of KKT system (1.4), nonsingularity of $\Delta_{NR}(\bar{z})$ is known as *quasi-regularity* of solution $\bar{z} = (\bar{x},\, \bar{\lambda},\, \bar{\mu}) \in \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^m$ [8]. Assume that $F$ is differentiable near $\bar{x}$, with its derivative being continuous at $\bar{x}$, and $h$ and $g$ are twice differentiable near $\bar{x}$, with their second derivatives being continuous at $\bar{x}$, and define the index sets

$$A_+ = A_+(\bar{x},\, \bar{\mu}) = \{i = 1,\, \ldots,\, m \mid \bar{\mu}_i > g_i(\bar{x}) = 0\},$$
$$A_0 = A_0(\bar{x},\, \bar{\mu}) = \{i = 1,\, \ldots,\, m \mid \bar{\mu}_i = g_i(\bar{x}) = 0\},$$

Quasi-regularity amounts to saying that the matrix

$$\begin{pmatrix} \dfrac{\partial G}{\partial x}(\bar{x},\, \bar{\lambda},\, \bar{\mu}) & (h'(\bar{x}))^{\mathrm{T}} & (g'_{A_+\cup K}(\bar{x}))^{\mathrm{T}} \\ h'(x) & 0 & 0 \\ g'_{A_+\cup K}(\bar{x}) & 0 & 0 \end{pmatrix}$$

is nonsingular for any index set $K \subset A_0$, where by $y_K$ we mean the subvector of $y$ with components $y_i$, $i \in K$.

Getting back to general MCP, in addition to the regularity conditions mentioned above, we will consider the following two properties.

**Definition 2.2** A solution $\bar{z}$ of MCP (1.1) is referred to as *strongly regular* if for each $r \in \mathbb{R}^s$ close enough to 0 the perturbed linearized MCP

$$z \in [\ell,\, u], \quad \langle \Phi(\bar{z}) + \Phi'(\bar{z})(z - \bar{z}) - r,\, y - z\rangle \geq 0 \quad \forall\, y \in [\ell,\, u],$$

has near $\bar{z}$ the unique solution $z(r)$ and the mapping $z(\cdot)$ is locally Lipschitz-continuous at 0.

The concept of strong regularity is due to [23], and it keeps playing an important role in modern variational analysis (see, e.g., [9, Chapter 5], [6, Chapter 2], and bibliographical comments therein).

A simple algebraic characterization of strong regularity for NCP was obtained in [23], and was extended to MCP in [7]. Recall that a square matrix $M$ is referred to as a $P$-matrix if all its principal minors are positive. Solution $\bar{z}$ is strongly regular if, and only if, $(\Phi'(\bar{z}))_{I_+I_+}$ is a nonsingular matrix, and its Schur complement

$$(\Phi'(\bar{z}))_{I_0I_0} - (\Phi'(\bar{z}))_{I_0I_+}(\Phi'(\bar{z}))^{-1}_{I_+I_+}(\Phi'(\bar{z}))_{I_+I_0}$$

in the matrix

$$\begin{pmatrix} (\Phi'(\bar{z}))_{I_+I_+} & (\Phi'(\bar{z}))_{I_+I_0} \\ (\Phi'(\bar{z}))_{I_0I_+} & (\Phi'(\bar{z}))_{I_0I_0} \end{pmatrix}$$

is a P-matrix.

The following weaker regularity concept was introduced in [2] as the main ingredient of sharp local convergence analysis of Newton-type methods for variational problems. Other applications of this property are concerned with sensitivity and error bounds [9, Sections 5.3, 6.2].

**Definition 2.3** A solution $\bar{z}$ of MCP (1.1) is referred to as *semistable* if for any $r \in \mathbb{R}^s$ close enough to 0, any solution $z(r)$ of the perturbed MCP

$$z \in [\ell,\, u], \quad \langle \Phi(z) - r,\, y - z\rangle \geq 0 \quad \forall\, y \in [\ell,\, u],$$

close enough to $\bar{z}$, satisfies the estimate

$$\|z(r) - \bar{z}\| = O(\|r\|).$$

# 3 Relations between regularity conditions

## 3.1 Known relations

In this section we recall the relations between the regularity properties stated above, which we consider to be known, or at least well-understood.

We start with some relations between $BD$-regularity and $CD$-regularity. It is evident that $\Delta_{NR}(\bar{z}) \subset \Delta_{FB}(\bar{z})$, implying also the inclusion for the convex hulls $\mathrm{conv}\,\Delta_{NR}(\bar{z}) \subset \mathrm{conv}\,\Delta_{FB}(\bar{z})$. In particular, nonsingularity of $\Delta_{FB}(\bar{z})$ implies nonsingularity of $\Delta_{NR}(\bar{z})$.

The converse implication is not true, in general; moreover, nonsingularity of $\Delta_{NR}(\bar{z})$ does not necessarily imply neither $CD$-regularity $\Psi_{NR}$ at $\bar{z}$, nor $BD$-regularity $\Psi_{FB}$ at $\bar{z}$, as demonstrated by the following example taken from [18, Example 2.1].

**Example 3.1** Let $s = 2$ and consider NCP (1.3) with $\Phi(z) = (-z_1 + z_2, -z_2)$. The point $\bar{z} = 0$ is the unique solution of this NCP.

It can be directly checked that

$$\partial_B \Psi_{NR}(\bar{z}) = \Delta_{NR}(\bar{z}) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \right\},$$

and therefore, $\Delta_{NR}(\bar{z})$ is nonsingular. On the other hand,

$$\frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1/2 \\ 0 & 1 \end{pmatrix}$$

is a singular matrix, and hence, $\Psi_{NR}$ is not $CD$-regular at $\bar{z}$.

For each $k = 1, 2, \ldots$, set $z^k = (1/k, 2/k)$. Then $\Psi_{FB}$ is differentiable at $z^k$,

$$\Psi'_{FB}(z^k) = \begin{pmatrix} 0 & 1 - 1/\sqrt{2} \\ 0 & -\sqrt{2} \end{pmatrix},$$

and the sequence $\{z^k\}$ converges to $\bar{z}$. Therefore, the singular matrix in the right-hand side of the last relation belongs to $\partial_B \Psi_{FB}(\bar{z})$, and hence, $\Psi_{FB}$ is not $BD$-regular at $\bar{z}$.

The next example, taken from [5, Example 2], demonstrates that $\partial_B \Psi_{FB}(\bar{z})$ can be smaller than $\Delta_{FB}(\bar{z})$. Moreover, $\Psi_{NR}$ and $\Psi_{FB}$ can be both $CD$-regular at $\bar{z}$ when both $\Delta_{NR}(\bar{x})$ and $\Delta_{FB}(\bar{x})$ contain singular matrices.

**Example 3.2** Let $s = 2$, $\Phi(z) = ((z_1 + z_2)/2, (z_1 + z_2)/2)$. Then $\bar{z} = 0$ is the unique solution of NCP (1.3).

It can be directly checked that

$$\partial_B \Psi_{NR}(\bar{z}) = \left\{ \begin{pmatrix} 1 & 0 \\ 1/2 & 1/2 \end{pmatrix}, \begin{pmatrix} 1/2 & 1/2 \\ 0 & 1 \end{pmatrix} \right\},$$

and hence,

$$\partial \Psi_{NR}(\bar{z}) = \left\{ \begin{pmatrix} t + (1-t)/2 & (1-t)/2 \\ t/2 & (1-t) + t/2 \end{pmatrix} \,\middle|\, t \in [0, 1] \right\}.$$

By direct computation, $\det J = 1/2$ for all $J \in \partial \Psi_{NR}(\bar{z})$, implying $CD$-regularity (and hence $BD$-regularity) of $\Psi_{NR}$ at $\bar{z}$. At the same time,

$$\Delta_{NR}(\bar{z}) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1/2 & 1/2 \end{pmatrix}, \begin{pmatrix} 1/2 & 1/2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \right\},$$

where the matrix

$$J_0 = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \tag{3.1}$$

is singular.

Furthermore, $\Delta_{FB}(\bar{z})$ consists of matrices of the form

$$J = J(\alpha,\, \beta) = \begin{pmatrix} \alpha_1/2 + \beta_1 & \alpha_1/2 \\ \alpha_2/2 & \alpha_2/2 + \beta_2 \end{pmatrix} \tag{3.2}$$

for all $(\alpha_i,\, \beta_i) \in S$, $i = 1,\, 2$. By direct computation,

$$\det J(\alpha,\, \beta) = \frac{1}{2}(\alpha_1\beta_2 + \alpha_2\beta_1 + 2\beta_1\beta_2).$$

Since the inclusion $(\alpha_i,\, \beta_i) \in S$ implies that $\alpha_i \geq 0$, $\beta_i \geq 0$ for $i = 1,\, 2$, this determinant equals zero only provided

$$\alpha_1\beta_2 = 0, \quad \alpha_2\beta_1 = 0, \quad \beta_1\beta_2 = 0,$$

and hence, $\alpha_1 = \alpha_2 = 1$, $\beta_1 = \beta_2 = 0$, implying that $J_0$ defined in (3.1) is the only singular matrix which belongs $\Delta_{FB}(\bar{z})$. Moreover, employing the above characterization of the structure of $\operatorname{conv} \Delta_{FB}(\bar{z})$, it is evident that $J_0$ is the only singular matrix in this set. However, it can be directly checked that this matrix does not belong neither to $\partial_B \Psi_{FB}(\bar{z})$ nor even to $\partial \Psi_{FB}(\bar{z})$, and therefore, $\Psi_{FB}$ is $CD$-regular (and hence $BD$-regular) at $\bar{x}$.

One could conjecture that $BD$-regularity (or at least $CD$-regularity) of $\Psi_{FB}$ at $\bar{z}$ implies $CD$-regularity (or at least $BD$-regularity) of $\Psi_{NR}$ at $\bar{z}$, but this is also not the case as demonstrated by the following example taken from [4].

**Example 3.3** Let $s = 2$, $\Phi(z) = (z_2,\, -z_1 + z_2)$. Then $\bar{z} = 0$ is the unique solution of NCP (1.3), and it can be seen that $\partial_B \Psi_{NR}(\bar{x})$ contains the singular matrix

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix},$$

while $\Psi_{FB}$ is $CD$-regular at $\bar{z}$.

It is also known that $BD$-regularity of $\Psi_{FB}$ at $\bar{z}$ does not imply $CD$-regularity of $\Psi_{FB}$ at $\bar{z}$. This can be seen from the following simple example.

**Example 3.4** Let $s = 1$, $\Phi(z) = -z$. Then $\bar{z} = 0$ is the unique solution of NCP (1.3).

Obviously, $\Psi_{FB}(z) = -\sqrt{2}|z|$, $\partial_B \Psi_{FB}(\bar{z}) = \{-\sqrt{2},\, \sqrt{2}\}$, implying $BD$-regularity of $\Psi_{FB}$ at $\bar{z}$. At the same time, $\partial \Psi_{FB}(\bar{z}) = \Delta_{FB}(\bar{z}) = [-\sqrt{2},\, \sqrt{2}]$ contains 0.

Finally, we summarize the known relations concerning semistability and strong regularity.

In [5] it has been shown that $BD$-regularity of either $\Psi_{NR}$ or $\Psi_{FB}$ at $\bar{z}$ implies semistability. The converse implication is not true. This can be seen from Examples 3.1 and 3.2.

As for strong regularity, the proof in [10, Theorem 1] implies the following: if $\bar{z}$ is a strongly regular solution of MCP (1.1) then $\operatorname{conv} \Delta_{FB}(\bar{x})$ (and hence, $\operatorname{conv} \Delta_{NR}(\bar{x})$) is nonsingular. In particular, strong regularity cannot be implied by any of the conditions not implying nonsingularity of $\operatorname{conv} \Delta_{FB}(\bar{x})$.

Table 1 summarizes the relations discussed in this section. Plus (minus) in each cell means that the property in the title of the row implies (does not imply) the property in the title of the column. Question mark means that to the best of our knowledge, the presence or the absence of the corresponding implications has been unknown so far.

Table 1: Regularity conditions for MCP: known relations

| Property | Semistability | $BD$-regularity of $\Psi_{NR}$ | $BD$-regularity of $\Psi_{FB}$ | $CD$-regularity of $\Psi_{NR}$ | $CD$-regularity of $\Psi_{FB}$ | Nonsingularity of $\Delta_{NR}$ | Nonsingularity of $\Delta_{FB}$ | Nonsingularity of $\operatorname{conv} \Delta_{NR}$ | Nonsingularity of $\operatorname{conv} \Delta_{FB}$ | Strong regularity |
|---|---|---|---|---|---|---|---|---|---|---|
| Semistability | + | − | − | − | − | − | − | − | − | − |
| $BD$-regularity of $\Psi_{NR}$ | + | + | − | − | − | − | − | − | − | − |
| $BD$-regularity of $\Psi_{FB}$ | + | − | + | − | − | − | − | − | − | − |
| $CD$-regularity of $\Psi_{NR}$ | + | + | ? | + | ? | − | − | − | − | − |
| $CD$-regularity of $\Psi_{FB}$ | + | − | + | − | + | − | − | − | − | − |
| Nonsingularity of $\Delta_{NR}$ | + | + | − | − | − | + | − | − | − | − |
| Nonsingularity of $\Delta_{FB}$ | + | + | + | ? | ? | + | + | ? | ? | ? |
| Nonsingularity of $\operatorname{conv} \Delta_{NR}$ | + | + | ? | + | ? | + | ? | + | ? | ? |
| Nonsingularity of $\operatorname{conv} \Delta_{FB}$ | + | + | + | + | + | + | + | + | + | ? |
| Strong regularity | + | + | + | + | + | + | + | + | + | + |

### 3.2 Remaining relations

We now fill the gaps in Table 1. We begin with the following fact.

**Proposition 3.1** *For a solution $\bar{z}$ of MCP* (1.1) *the following properties are equivalent:*

$(a)$ $\operatorname{conv} \Delta_{NR}(\bar{z})$ *is nonsingular.*

$(b)$ $\Delta_{FB}(\bar{z})$ *is nonsingular.*

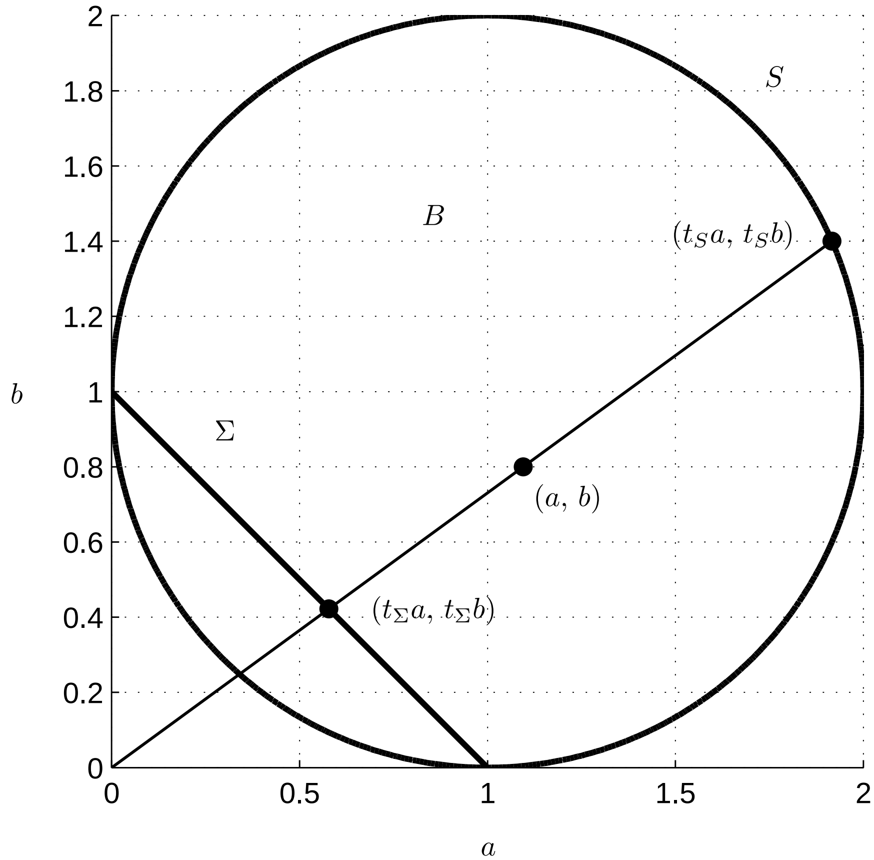



Figure 1: Sets $\Sigma$, $S$ and $B$.

(*c*) $\operatorname{conv} \Delta_{FB}(\bar{z})$ *is nonsingular.*

**Proof.** A key observation is the following: for any point $(a,\, b) \in B$ there exists $t_\Sigma > 0$ such that $(t_\Sigma a,\, t_\Sigma b) \in \Sigma$, and there exists $t_S > 0$ such that $(t_S a,\, t_S b) \in S$ (see Figure 1). This implies that any matrix $J$ defined in (1.9) with $(\alpha_i,\, \beta_i) \in B$ for all $i \in I_0$, can be transformed into matrices of the same form but with $(\alpha_i,\, \beta_i) \in \Sigma$ and $(\alpha_i,\, \beta_i) \in S$, respectively, for all $i \in I_0$, and this transformation can be achieved by multiplication of some rows of $J$ by appropriate positive numbers. In particular, such matrices are nonsingular only simultaneously, which gives the needed equivalence. ■

Furthermore, any of the equivalent properties (a)–(c) in Proposition 3.1 implies strong regularity of $\bar{z}$, as we show next. We first prove the following Lemma closely related to [15, Proposition 2.7] (see also [10, Proposition 4]).

**Lemma 3.1** *If $M \in \mathbb{R}^{p\times p}$ is not a $P$-matrix then there exist $\alpha,\, \beta \in \mathbb{R}^p$ such that $(\alpha_i,\, \beta_i) \in S$ for all $i = 1,\, \ldots,\, p$, where $S$ is defined in* (1.10)*, and the matrix*

$$M(\alpha,\, \beta) = \mathrm{diag}(\alpha)M + \mathrm{diag}(\beta) \tag{3.3}$$

*is singular.*

Here and in the sequel $\mathrm{diag}(\alpha)$ is the diagonal matrix with diagonal elements equal to the components of the vector $\alpha$.

**Proof.** We argue by induction. If $m = 1$ then the assumption that $M$ is not a $P$-matrix means that $M$ is a nonpositive scalar. It follows that the circle $S$ in $(\alpha,\, \beta)$-plane always has a nonempty intersection with the straight line given by the equation $\mathrm{diag}(\alpha)M + \mathrm{diag}(\beta) = 0$. The points in this intersection are the needed pairs $(\alpha,\, \beta)$.

Suppose now that the assertion is valid for any matrix in $\mathbb{R}^{(p-1)\times(p-1)}$, and suppose that the matrix $M \in \mathbb{R}^{p\times p}$ with elements $m_{i,j}$, $i,\, j = 1,\, \ldots,\, p$, has a nonpositive principal minor. If the only such minor is $\det M$ then set $\alpha_i = 1$, $\beta_i = 0$, $i = 2,\, \ldots,\, p$, and compute

$$\begin{aligned} \det M(\alpha,\, \beta) &= \det \begin{pmatrix} \alpha_1 m_{11} + \beta_1 & \alpha_1 m_{12} & \ldots & \alpha_1 m_{1p} \\ m_{21} & m_{22} & \ldots & m_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ m_{p1} & m_{p2} & \ldots & m_{pp} \end{pmatrix} \\ &= \alpha_1 \det M + \beta_1 \det M_{\{2,\ldots,p\}\{2,\ldots,p\}}, \end{aligned}$$

where the last equality can be obtained by expanding the determinant by the first row. Since $\det M \leq 0$ and $\det M_{\{2,\ldots,m\}\{2,\ldots,m\}} > 0$, we again obtain that the circle $S$ in $(\alpha_1,\, \beta_1)$-plane always has a nonempty intersection with the straight line given by the equation $\det M(\alpha,\, \beta) = 0$ with respect to $(\alpha_1,\, \beta_1)$, and we again get the needed $\alpha$ and $\beta$.

It remains to consider the case of existence of an index set $K \subset \{1,\, \ldots,\, p\}$ such that $\det M_{KK} \leq 0$, and there exists $k \in \{1,\, \ldots,\, m\} \setminus K$. Removing the $k$-th row and column from $M$, we then get the matrix $\tilde{M} = \mathbb{R}^{(p-1)\times(p-1)}$ with a nonpositive principal minor. By the hypothesis of the induction there exist $\tilde{\alpha} = (\alpha_1,\, \ldots,\, \alpha_{k-1},\, \alpha_{k+1},\, \ldots,\, \alpha_p) \in \mathbb{R}^{p-1}$, $\tilde{\beta} = (\beta_1,\, \ldots,\, \beta_{k-1},\, \beta_{k+1},\, \ldots,\, \beta_p) \in \mathbb{R}^{p-1}$ such that $(\alpha_i,\, \beta_i) \in S$ for all $i = 1,\, \ldots,\, p$, $i \neq k$, and the matrix $\mathrm{diag}(\tilde{\alpha})\tilde{M} + \mathrm{diag}(\tilde{\beta})$ is singular. Setting $\alpha_k = 0$, $\beta_k = 1$, we again obtain the needed $\alpha$ and $\beta$. ∎

**Proposition 3.2** *For a solution $\bar{z}$ of MCP* (1.1)*, if $\Delta_{FB}(\bar{z})$ is nonsingular then $\bar{z}$ is strongly regular.*

**Proof.** Nonsingularity of all matrices in $\Delta_{FB}(\bar{x})$ is equivalent to saying that the matrix

$$D(\alpha,\, \beta) = \begin{pmatrix} (\Phi'(\bar{x}))_{I_+I_+} & (\Phi'(\bar{x}))_{I_+I_0} \\ \mathrm{diag}(\alpha)(\Phi'(\bar{x}))_{I_0I_+} & \mathrm{diag}(\alpha)(\Phi'(\bar{x}))_{I_0I_0} + \mathrm{diag}(\beta) \end{pmatrix} \tag{3.4}$$

is nonsingular for all $\alpha = (\alpha_i,\ i \in I_0)$ and $\beta = (\beta_i,\ i \in I_0)$ satisfying $(\alpha_i,\ \beta_i) \in S,\ i \in I_0$. Taking $\alpha_i = 0,\ \beta_i = 1,\ i \in I_0$, we immediately obtain that $(\Phi'(\bar{x}))_{I_+I_+}$ is nonsingular.

Suppose that $\bar{x}$ is not strongly regular. Then nonsingularity of $(\Phi'(\bar{x}))_{I_+I_+}$ implies that

$$(\Phi'(\bar{x}))_{I_0I_0} - (\Phi'(\bar{x}))_{I_0I_+}((\Phi'(\bar{x}))_{I_+I_+})^{-1}(\Phi'(\bar{x}))_{I_+I_0}$$

is not a $P$-matrix. Therefore, by Lemma 3.1 we obtain the existence of $\alpha = (\alpha_i,\ i \in I_0)$ and $\beta = (\beta_i,\ i \in I_0)$ satisfying $(\alpha_i,\ \beta_i) \in S,\ i \in I_0$, and such that the matrix

$$\mathrm{diag}(\alpha)((\Phi'(\bar{x}))_{I_0I_0} - (\Phi'(\bar{x}))_{I_0I_+}((\Phi'(\bar{x}))_{I_+I_+})^{-1}(\Phi'(\bar{x}))_{I_+I_0}) + \mathrm{diag}(\beta)$$

is singular. But this matrix is the Schur complement of the nonsingular matrix $(\Phi'(\bar{x}))_{I_+I_+}$ in $D(\alpha,\ \beta)$, and therefore, we conclude that $D(\alpha,\ \beta)$ is also singular (see, e.g., [19, Theorem 2.1]). ∎

Combining Propositions 3.1 and 3.2 with the known fact that strong regularity of $\bar{z}$ implies nonsingularity of $\mathrm{conv}\,\Delta_{FB}(\bar{x})$, we finally obtain that any of the equivalent properties (a)–(c) in Proposition 3.1 is further equivalent to strong regularity.

Now the only thing to clarify is whether $CD$-regularity of $\Psi_{NR}$ at $\bar{z}$ implies $CD$-regularity or (at least $BD$-regularity) of $\Psi_{FB}$ at $\bar{z}$. The next example demonstrates that this is not the case; moreover, even a combination of $CD$-regularity of $\Psi_{NR}$ at $\bar{z}$ and nonsingularity of $\Delta_{NR}(\bar{z})$ does not imply $BD$-regularity (and even less so $CD$-regularity) of $\Psi_{FB}$ at $\bar{z}$.

**Example 3.5** Let $n = 2$, $\Phi(z) = (-z_1 + 3z_2/(2\sqrt{2}),\ 2z_1 + (1 - 3/(2\sqrt{2}))z_2)$. Then $\bar{z} = 0$ is the unique solution of the corresponding NCP.

Consider any sequence $\{z^k\} \subset \mathbb{R}^2$ such that $z_1^k < 0$, $z_2^k = 0$ for all $k$, and $z_1^k$ tends to 0 as $k \to \infty$. Then for all $k$ it holds that $\Phi$ is differentiable at $z^k$, and

$$\begin{aligned}\Phi'(z^k) &= \begin{pmatrix} -\dfrac{2z_1^k}{\sqrt{2(z_1^k)^2}} & \dfrac{3}{2\sqrt{2}}\left(1 + \dfrac{z_1^k}{\sqrt{2(z_1^k)^2}}\right) \\ 2\left(1 - \dfrac{2z_1^k}{\sqrt{(2z_1^k)^2}}\right) & 1 + \left(1 - \dfrac{3}{2\sqrt{2}}\right)\left(1 - \dfrac{2z_1^k}{\sqrt{(2z_1^k)^2}}\right) \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{2} & \dfrac{3}{2\sqrt{2}}\left(1 - \dfrac{1}{\sqrt{2}}\right) \\ 4 & 1 + 2\left(1 - \dfrac{3}{2\sqrt{2}}\right) \end{pmatrix},\end{aligned}$$

and therefore, the singular matrix in the right-hand side belongs to $\partial_B \Psi_{FB}(\bar{z})$.

At the same time, it can be directly checked that

$$\partial_B \Psi_{NR}(\bar{z}) = \left\{ \begin{pmatrix} 1 & 0 \\ 2 & 1 - \dfrac{3}{2\sqrt{2}} \end{pmatrix},\ \begin{pmatrix} -1 & \dfrac{3}{2\sqrt{2}} \\ 0 & 1 \end{pmatrix} \right\},$$

and hence,

$$\partial\Psi_{NR}(\bar{z}) = \left\{ \left( \begin{array}{cc} t-(1-t) & (1-t)\dfrac{3}{2\sqrt{2}} \\ 2t & t\left(1-\frac{3}{2\sqrt{2}}\right)+(1-t) \end{array} \right) \middle| \; t \in [0,\, 1] \right\}.$$

By direct computation, for any matrix $J(t)$ in the right-hand side of the last relation we have

$$\det J(t) = \left(2 - \frac{3}{2\sqrt{2}}\right)t - 1 \leq 1 - \frac{3}{2\sqrt{2}} < 0 \quad \forall\, t \in [0,\, 1],$$

implying $CD$-regularity of $\Psi_{NR}$ at $\bar{z}$.

Observe also that

$$\Delta_{NR}(\bar{z}) = \left\{ \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right),\, \left(\begin{array}{cc} 1 & 0 \\ 2 & 1-\dfrac{3}{2\sqrt{2}} \end{array}\right),\, \left(\begin{array}{cc} -1 & \dfrac{3}{2\sqrt{2}} \\ 0 & 1 \end{array}\right),\, \left(\begin{array}{cc} -1 & \dfrac{3}{2\sqrt{2}} \\ 2 & 1-\dfrac{3}{2\sqrt{2}} \end{array}\right) \right\},$$

and it is evident that $\Delta_{NR}(\bar{z})$ is nonsingular, that is, $\bar{z}$ is a $b$-regular solution.

Example 3.5 shows that replacing $\partial_B\Psi_{NR}(\bar{z})$ by its convexification $\partial\Psi_{NR}(\bar{z})$, or by its different enlargement $\Delta_{NR}(\bar{z})$, and assuming nonsingularity of all matrices in the resulting set does not imply even $BD$-regularity of $\Psi_{FB}$ at $\bar{z}$. However, by Propositions 3.1 and 3.2, applying both these enlargements *together* (that is, assuming nonsingularity of all matrices in $\operatorname{conv}\Delta_{NR}(\bar{z})$) implies strong regularity of $\bar{z}$.

# 4 KKT systems

Some implications that are not valid for a general MCP turn out to be true for the special case of a KKT system.

The key observation is that unlike the case of NCP, for KKT systems formula (1.8) gives not only an outer estimate of the $B$-differential of $\Psi_{NR}$ but its exact characterization: for any solution $\bar{z} = (\bar{x},\, \bar{\lambda},\, \bar{\mu})$ of the KKT system (1.4) it holds that $\partial_B\Psi_{NR}(\bar{z}) = \Delta_{NR}(\bar{z})$, implying also the equality $\partial\Psi_{NR}(\bar{z}) = \operatorname{conv}\Delta_{NR}(\bar{z})$. This is due to the fact that the primal variable $x$ and the dual variable $\mu$ are "decoupled" in the arguments of the natural residual complementarity function, and for any $J \in \Delta_{NR}(\bar{z})$, one can readily construct a sequence $\{z^k\} \subset \mathcal{S}_{\Psi_{NR}}$ such that $\{z^k\} \to \bar{z}$ and $\{\Psi'_{NR}(z^k)\} \to J$.

Therefore, in the case of the KKT system $BD$-regularity of $\Psi_{NR}$ at $\bar{z}$ is equivalent to nonsingularity of $\Delta_{NR}(\bar{z})$ (that is, to quasi-regularity of this solution), while $CD$-regularity of $\Psi_{NR}$ at $\bar{z}$ is equivalent to nonsingularity of $\operatorname{conv}\Delta_{NR}(\bar{z})$.

As for $\Psi_{FB}$, it can be seen that formula (1.9) gives the exact characterization of the $B$-differential of this mapping at $\bar{z}$ provided that the gradients $g'_i(\bar{x})$, $i \in A_0$, are linearly independent. We next show that the latter condition is automatically satisfied at any solution $\bar{z}$ of the KKT system such that $\Psi_{FB}$ is $BD$-regular at this solution.

**Proposition 4.1** *For a solution $\bar{z} = (\bar{x}, \bar{\lambda}, \bar{\mu})$ of KKT system* (1.4)*, if $\Psi_{FB}$ is BD-regular at $\bar{z}$ then $\bar{x}$ satisfies the linear independence constraint qualification: the gradients $h'_j(\bar{x})$, $j = 1, \ldots, l$, $g'_i(\bar{x})$, $i \in A = A_+ \cup A_0$, are linearly independent.*

**Proof.** Fix any sequence $\{\mu^k\} \subset \mathbb{R}^m$ such that $\mu^k_A > 0$, $\mu^k_{\{1,\ldots,m\}\setminus A} = 0$ for all $k$, and $\{\mu^k_A\} \to \bar{\mu}_A$. Then the sequence $\{(\bar{x}, \bar{\lambda}, \mu^k)\}$ converges to $(\bar{x}, \bar{\lambda}, \bar{\mu})$, and it can be easily seen that for any $k$ the mapping $\Psi_{FB}$ is differentiable at $(\bar{x}, \bar{\lambda}, \mu^k)$, and the sequence of its Jacobians converges to the matrix

$$\begin{pmatrix} \frac{\partial G}{\partial x}(\bar{x}, \bar{\lambda}, \bar{\mu}) & (h'(\bar{x}))^{\mathrm{T}} & (g'_A(\bar{x}))^{\mathrm{T}} & (g'_{\{1,\ldots,m\}\setminus A}(\bar{x}))^{\mathrm{T}} \\ h'(x) & 0 & 0 & 0 \\ -g'_A(\bar{x}) & 0 & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$$

(perhaps after an appropriate re-ordering of rows and columns), where $I$ stands for the identity matrix of the appropriate size. The needed result is now evident. ■

From this proposition and the preceding discussion it follows that for a KKT system $BD$-regularity of $\Psi_{FB}$ at $\bar{z}$ implies nonsingularity of $\Delta_{FB}(\bar{z})$, and hence, by Proposition 3.1, nonsingularity of $\operatorname{conv} \Delta_{FB}(\bar{z})$ (which in its turn implies $CD$-regularity of $\Psi_{FB}$ at $\bar{z}$). Also, it now becomes evident, that $CD$-regularity of $\Psi_{FB}$ at a solution $\bar{z}$ of the KKT system implies nonsingularity of $\Delta_{FB}(\bar{z})$.

Therefore, for KKT systems, we have three groups of equivalent conditions. The first group consists of $BD$-regularity of $\Psi_{FB}$ at $\bar{z}$, $CD$-regularity of $\Psi_{NR}$ at $\bar{z}$, $CD$-regularity of $\Psi_{FB}$ at $\bar{z}$, nonsingularity of $\Delta_{FB}(\bar{z})$, nonsingularity of $\operatorname{conv} \Delta_{NR}(\bar{z})$, nonsingularity of $\operatorname{conv} \Delta_{FB}(\bar{z})$, and strong regularity of $\bar{z}$. The second group consists of $BD$-regularity of $\Psi_{NR}$ at $\bar{z}$, and of nonsingularity of $\Delta_{NR}(\bar{z})$. The last group consists of semistability. Conditions in the first group imply conditions in the second, and conditions in the second imply semistability.

The absence of converse implications is well-understood. The NCP in Example 3.4 corresponds to the primal first-order optimality conditions for the optimization problem

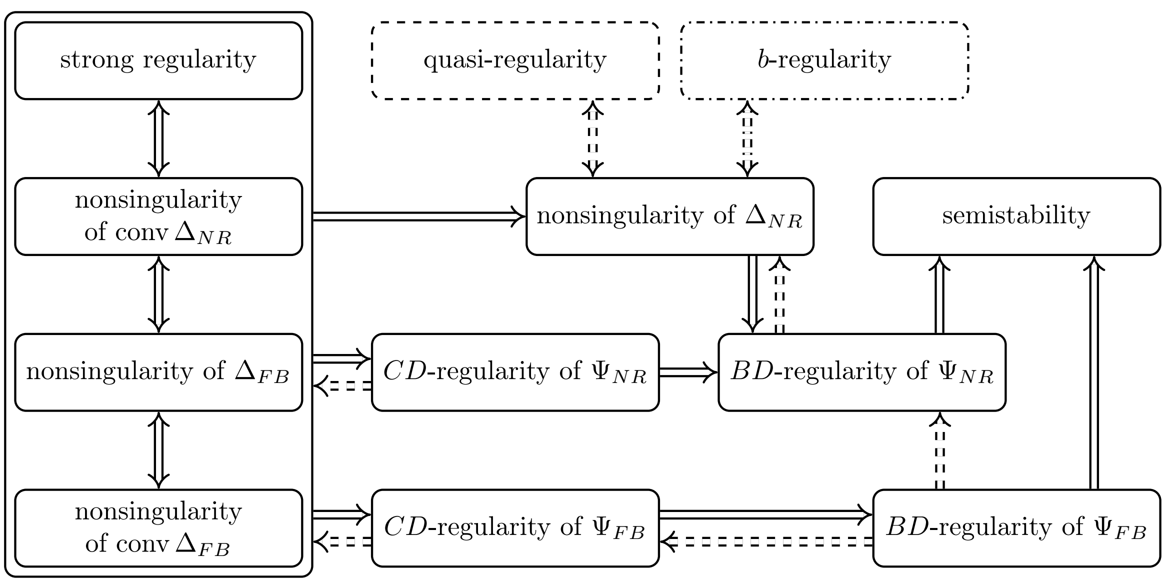
$$\begin{array}{ll} \text{minimize} & -\frac{1}{2}x^2 \\ \text{subject to} & x \geq 0. \end{array}$$

The unique solution of the corresponding KKT system is $\bar{z} = (\bar{x}, \bar{\mu}) = (0, 0)$, and it can be readily seen that this solution is quasi-regular, but $\Delta_{FB}(\bar{z})$ contains a singular matrix.

The fact that semistability does not necessarily imply $BD$-regularity of $\Psi_{NR}$ is demonstrated. e.g., by [13, Example 1].

In conclusion we present all the relations between regularity conditions in question in the form of a flowchart in Figure 2. The solid arrows correspond to implications valid for a general MCP, while the dashed arrows show the additional implications valid for KKT systems. The diagram is complete: if two blocks in it are not connected by (a sequence of) arrows, it means the absence of the corresponding implication, even for the particular cases of NCP or KKT system (observe that all counterexamples presented in Section 3 are NCPs).

Figure 2: Regularity conditions: complete flowchart of relations.



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