NONSMOOTH CONE-CONSTRAINED OPTIMIZATION WITH APPLICATIONS TO SEMI-INFINITE PROGRAMMING[1]

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Abstract. The paper is devoted to the study of general nonsmooth problems of cone-constrained optimization (or conic programming) important for various aspects of optimization theory and applications. Based on advanced constructions and techniques of variational analysis and generalized differentiation, we derive new necessary optimality conditions (in both "exact" and "fuzzy" forms) for nonsmooth conic programs, establish characterizations of well-posedness for cone-constrained systems, and develop new applications to semi-infinite programming.

Key words: variational analysis, cone-constrained optimization, semi-infinite programming, generalized differentiation, constraint qualifications, supremum functions, metric regularity

MSC subject classifications: 49J52, 49J53, 90C31, 90C34, 90C46

1 Introduction

In this paper we consider a general class of problems belonging to *cone-constrained optimization* known also as problems of *conic programming*. Problems of this type are important and challenging from the viewpoint of optimization theory, while they are motivated by a large variety of practical applications including those in operations research, engineering and financial management, etc. To list just a few, we mention here systems control, best approximation, portfolio optimization, and antenna array weight design. Among the most remarkable special classes in cone-constrained optimization there are problems of semi-infinite programming, semidefinite programming, second-order cone programming, and copositive programming; see [1, 4, 5, 9, 18, 23, 32, 36, 37, 38] and the references therein for more details, various results and discussions on these areas, and their applications.

A general class of cone-constrained optimization problems can be written in the form

$$\begin{cases} \text{minimize } \ \vartheta(x) \ \text{ subject to} \\ f(x) \in -\Theta \subset Y, \\ x \in \Omega \subset X, \end{cases} \tag{1.1}$$

where the *characteristic constraints* are given by $f(x) \in -\Theta$ via a mapping $f\colon X \to Y$ and a closed, *convex cone* Θ in finite-dimensional or infinite-dimensional spaces. Our *standing assumptions* for the study of (1.1) are formulated at the beginning of Section 3, and some additional assumptions are imposed in Sections 6 and 7.

The specific form of the cone Θ identifies various subclasses of cone-constrained optimization problems. In particular, problems of *semi-infinite programming* (*SIP*) and *infinite*

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programming (the name depends on the dimension of the decision space X):

$$\begin{cases} \text{minimize } \vartheta(x) \text{ subject to} \\ f(x,t) \le 0 \text{ for } t \in T, \\ x \in \Omega \subset X \end{cases} \tag{1.2}$$

correspond to the cases of $\Theta = \mathcal{C}_+(T)$ or $l^\infty_+(T)$ of positive continuous or essentially bounded functions over an arbitrary (compact or noncompact) index set T.

Note that the closed and convex cone structure of the set Θ in (1.1) crucially distinguishes this class of optimization problems from other types of problems in constrained optimization. In particular, such a structure allows us by Proposition 3.1 to rewrite problem (1.1) in the form:

$$\begin{cases} \text{minimize } \vartheta(x) \text{ subject to} \\ \varphi(x) := \sup\limits_{y^* \in \Xi} \langle y^*, f(x)\rangle \le 0, \\ x \in \Omega \subset X \end{cases} \tag{1.3}$$

with $\Xi := \big\{y^* \in Y^* \big|\ \|y^*\| = 1,\ \langle y^*, y\rangle \ge 0,\ y \in \Theta\big\}$, which makes it possible to employ methods and results on generalized differentiation of *supremum functions* to the study of general cone-constrained programs and their specifications.

The main goals of this paper is to investigate a large class of *nonsmooth* and *nonconvex* cone-constrained programs (1.1) from the viewpoints of deriving verifiable *necessary optimality conditions* and characterizations of *well-posedness* (namely, metric regularity and robust Lipschitzian stability) by using specific features of cone constraints particularly reflected by form (1.3) in the case of *arbitrary Banach* constraint spaces Y. Such a generality is crucial for applications to semi-infinite and infinite programs of type (1.2) with compact and noncompact index sets T, where the most natural choices of Y are the "bad" Banach spaces $\mathcal{C}(T)$ and $l^\infty(T)$, respectively. While our methods fully work for the case of *Asplund* decision spaces X[4], the majority of the results obtained are new when the space X is finite-dimensional. Moreover, for simplicity we confine to the case of $\dim X < \infty$ our applications to well-posedness of conic systems in Section 6 and to necessary optimality condition in Section 7 considering there only problems of semi-infinite programming.

Note that when the range space Y is Asplund, some necessary optimality and well-posedness conditions for (1.1) established below can be derived from the generalized differential calculus developed in [25]. However, it is not sufficient for a number of valuable applications; in particular, those to semi-infinite programming obtained in this paper. Indeed, it is well known (see, e.g., [16]) that the space $l^\infty(T)$ is Asplund if and only if T is a finite set while $\mathcal{C}(T)$ is Asplund if and only if T is a scattered compact, which is not of any interest for applications to optimization.

To this end we observe that in the vast majority of publications on SIP of type (1.2) the index set T is assumed to be a Hausdorff compact and the constraint function $f(x)(\cdot) := f(x,\cdot)$ is an element (usually either smooth or convex) of the space $\mathcal{C}(T)$; see, e.g., [5, 18, 23, 36] and the bibliographies therein. When the functional data of (1.2) are locally Lipschitzian around the reference minimizer, a generalized Lagrange multiplier rule is established in [40] via the Clarke generalized gradient based on the separation techniques developed in [39]. As

[4]Recall that a Banach space is Asplund if each of its separable subspaces has a separable dual. This class includes, in particular, every reflexive Banach space as well as those with separable duals. We refer the reader to [16, 25] and the bibliographies therein for more details.

mentioned by the authors of [40], their approach is not suitable to derive similar Lagrange multiplier results in terms of the smaller regular/Fréchet and limiting/Mordukhovich subdifferentials since the underlying cone-constraint space $\mathcal{C}(T)$ is not Asplund. The approach developed in this paper allows us to achieve the aforementioned goals for SIP and also for infinite programs with general Asplund decision spaces X.

The rest of the paper is organized as follows. Section 2 presents some basic constructions and preliminaries from variational analysis and generalized differentiation widely used in the formulations and proofs of the main results of the paper. We also introduce here new versions of coderivatives for mappings with values in ordered Banach spaces.

Section 3 is devoted to deriving new subdifferential estimates for *supremum functions* of the special type (1.3) in the general setting of Asplund spaces X and Banach spaces Y. These results and the generalized differential calculus of variational analysis are applied in Section 4 to establish the existence of *generalized Lagrange multipliers* in first-order necessary optimality conditions obtained in the *pointbased* (i.e., expressed via generalized differential constructions defined exactly *at* optimal solutions) and *qualified* form (i.e., with nonzero Lagrange multipliers associated with cost functions) for the cone-constrained programs (1.1) under appropriate *constraint qualifications*. The qualification conditions introduced here are formulated in terms of coderivatives and reduce to the classical constraint qualifications in smooth and convex cases.

In Section 5 we derive new necessary optimality conditions of the *fuzzy* type. Conditions of this type operate not only with the reference optimal solution, as the exact/pointbased ones from Section 4, but also involve certain neighborhoods in the primal and dual spaces; see [6, 8, 25, 30, 31] on necessary optimality conditions of the fuzzy type for optimization problems with finitely many equality and inequality constraints. In contrast to the cited publications on fuzzy optimality conditions as well as to the pointbased results of Section 4, our approach leads to necessary optimality conditions in the *fuzzy qualified* form with *no constraint qualifications*. It is worth mentioning that we do *not require* that the underlying constraint cone Θ is of *nonempty interior* and thus can cover, e.g., the positive cones in the classical spaces L^p and l^p for $1 \le p < \infty$ (along with L^∞ and l^∞), which are of strong interest for applications; in particular, to economic and financial systems.

Section 6 concerns some *well-posedness* issues for cone-constrained systems of (1.1) in the setting of arbitrary Banach spaces Y. We particularly focus on *metric regularity*, which is known to be equivalent to *linear openness/covering* of set-valued mappings as well as to *Lipschitzian stability* of their inverses. Applying the results of Section 3 on subdifferentiation of supremum functions and basic tools of variational analysis allows us to estimate and precisely compute the *exact regularity bound* for cone-constrained systems by using the Fréchet and limiting coderivatives of Lipschitz continuous mapping f in (1.1).

The final Section 7 develops some applications of optimality and well-posedness results obtained in the previous sections for conic programs (1.1) with general Banach spaces Y to classes of *semi-infinite* programs (1.2) with *arbitrary* as well as with *compact* index sets. These two cases of the index set T in (1.2) correspond to the positive cones Θ in the cone-constrained scheme (1.1) with the (non-Asplund) Banach spaces $Y = l^\infty(T)$ and $Y = \mathcal{C}(T)$, respectively. In this way we derive new optimality and metric regularity/stability conditions for the aforementioned classes of semi-infinite programs. In particular, necessary optimality conditions obtained in this section essentially extend those from [40] and also the corresponding results established in our previous paper [27] by a different approach.

Our notation and terminology are basically standard and conventional in the area of variational analysis and generalized differentiation; see, e.g., [8, 25]. As usual, $\|\cdot\|$ stands for the norm of Banach space X and $\langle\cdot,\cdot\rangle$ signifies for the canonical pairing between X and its topological dual X^* with $\stackrel{w^*}{\to}$ indicating the convergence in the weak* topology of X^* and cl^* standing for the weak* topological closure of a set. For any $x\in X$ and $r>0$ the symbol $I\!B_r(x)$ stands for the closed ball centered at x with radius r, while the unit closed ball and the unit sphere in X are denoted by $I\!B_X$ and S_X, respectively. If no confusion arises, we denote by $I\!B^*$ the dual unit ball of the space in question.

Given a set $\Omega\subset X$, the notation $\mathrm{co}\,\Omega$ signifies the convex hull of Ω. Depending on the context, the symbols $x\stackrel{\Omega}{\to}\bar x$ and $x\stackrel{\varphi}{\to}\bar x$ mean that $x\to\bar x$ with $x\in\Omega$ and $x\to\bar x$ with $\varphi(x)\to\varphi(\bar x)$ respectively. Given finally a set-valued mapping $F\colon X\rightrightarrows X^*$ between X and X^*, recall that the symbol

$$\mathop{\mathrm{Lim\,sup}}_{x\to\bar x}F(x):=\Big\{x^*\in X^*\Big|\;\exists\,x_n\to\bar x,\;\exists\,x_n^*\stackrel{w^*}{\longrightarrow}x^*\;\text{ with }\;x_n^*\in F(x_n),\quad n\in I\!N\Big\}\tag{1.4}$$

stands for the *sequential Painlevé-Kuratowski outer/upper limit* of F as $x\to\bar x$ with respect to the norm topology of X and the weak* topology of X^*, where $I\!N:=\{1,2,\ldots\}$.

2 Tools of Variational Analysis

Let us begin this section with a brief description of some basic constructions of variational analysis and generalized differentiation needed in what follows. The reader is referred to the books [8, 25, 34, 35] and the bibliographies therein for more details, discussions, and additional material. Since the space X (while not Y below) under consideration is always assumed to be *Asplund*, we confine ourselves to the subdifferential constructions for functions defined on Asplund spaces; see the two-volume monograph [25] and its references for a comprehensive theory as well as for appropriate Banach space counterparts.

Given an extended-real-valued function $\varphi\colon X\to\overline{I\!R}:=(-\infty,\infty]$, denote as usual by

$$\mathrm{dom}\,\varphi:=\big\{x\in X\big|\;\varphi(x)<\infty\big\}\;\text{ and }\;\mathrm{epi}\,\varphi:=\big\{(x,r)\in X\times I\!R\big|\;\varphi(x)\le r\big\}$$

its domain and epigraph, respectively. The *regular/Fréchet subdifferential* (known also as the presubdifferential or viscosity subdifferential) of φ at $\bar x\in\mathrm{dom}\,\varphi$ is given by

$$\widehat\partial\varphi(\bar x):=\Big\{x^*\in X^*\Big|\;\liminf_{x\to\bar x}\frac{\varphi(x)-\varphi(\bar x)-\langle x^*,x-\bar x\rangle}{\|x-\bar x\|}\ge 0\Big\}.\tag{2.1}$$

with $\widehat\partial\varphi(\bar x):=\emptyset$ for $\bar x\notin\mathrm{dom}\,\varphi$. The *limiting/Mordukhovich subdifferential* (known also as the basic/general subdifferential) of φ at $\bar x$ is defined via the sequential outer limit (1.4) by

$$\partial\varphi(\bar x):=\mathop{\mathrm{Lim\,sup}}_{x\stackrel{\varphi}{\to}\bar x}\widehat\partial\varphi(x),\tag{2.2}$$

while the corresponding *singular/horizon subdifferential* of φ at $\bar x$ is

$$\partial^\infty\varphi(\bar x):=\mathop{\mathrm{Lim\,sup}}_{\substack{x\stackrel{\varphi}{\to}\bar x\\ \lambda\downarrow 0}}\lambda\widehat\partial\varphi(x).\tag{2.3}$$

It is worth mentioning that $\partial\varphi(\bar x)\ne\emptyset$ and $\partial^\infty\varphi(\bar x)=\{0\}$ provided that φ is locally Lipschitzian around $\bar x$. Furthermore, for convex functions φ both regular and limiting subdifferentials reduce to the classical subdifferentials of convex analysis.

Given further a set $\Omega \subset X$ with its indicator function $\delta(x;\Omega)$ equal to 0 for $x \in \Omega$ and to ∞ otherwise, we define the regular and limiting *normal cones* to Ω at $\bar{x}$ by, respectively,

$$\widehat{N}(\bar{x};\Omega) := \widehat{\partial}\delta(\bar{x};\Omega) \quad \text{and} \quad N(\bar{x};\Omega) := \partial\delta(\bar{x};\Omega) \tag{2.4}$$

via the corresponding subdifferentials (2.1) and (2.2). Recall that Ω is *sequentially normally compact* (SNC) at $\bar{x} \in \Omega$ if for any sequences $x_n \stackrel{\Omega}{\to} \bar{x}$ and $x_n^* \in \widehat{N}(x_n;\Omega)$ we have

$$\Big[x_n^* \stackrel{w^*}{\to} 0\Big] \Longrightarrow \Big[\|x_n^*\| \to 0\Big] \text{ as } n \to \infty.$$

This is of course automatic if X is finite-dimensional, while it also holds under certain (epi)-Lipschitzian properties of the set Ω. Respectively, a function $\varphi\colon X \to \overline{I\!R}$ is *sequentially normally epi-compact* (SNEC) at $\bar{x} \in \operatorname{dom}\varphi$ if its epigraph is SNC at $(\bar{x},\varphi(\bar{x}))$. This is the case, in particular, when either $\dim X < \infty$ or φ is locally Lipschitzian around $\bar{x}$.

Next consider a closed and convex cone $\Theta \neq \emptyset$ in a Banach space Y and a single-valued mapping $f\colon X \to Y$. The *partial order* $\le_\Theta$ on Y is defined by

$$y_1 \le_\Theta y_2 \quad \text{if and only if} \quad y_2 - y_1 \in \Theta \quad \text{for } \ y_1, y_2 \in Y$$

and the Θ-*epigraph* of f generated by the order $\le_\Theta$ is given by

$$\operatorname{epi}_\Theta f := \big\{(x,y) \in X \times Y \big|\ f(x) \le_\Theta y\big\}.$$

Recall that f is Θ-*convex* if for any $x_1, x_2 \in X$ and $t \in [0,1]$ we have

$$f\big(tx_1 + (1-t)x_2\big) \le_\Theta tf(x_1) + (1-t)f(x_2),$$

which is equivalent to the fact that the set $\operatorname{epi}_\Theta f$ is convex in $X \times Y$.

Finally in this section, we define and discuss several *coderivative* constructions for mappings with values in ordered Banach spaces that play a significant role in deriving the main results of this paper. They follow the scheme originated in [24] in the absence of ordering structures, while ordering is essential in our considerations. Although the coderivative constructions below depend on the partial order $\le_\Theta$ imposed on the range space, for simplicity we skip mentioning the cone Θ in the coderivative notation.

Given a mapping $f\colon X \to Y$ and an ordering cone $\Theta \subset Y$ always assumed to be closed and convex, we define the following Θ-coderivative constructions as positively homogeneous set-valued mappings from Y^* to X^* with the values.

- The *regular* Θ-*coderivative* of f at $\bar{x}$ is

$$\widehat{D}^*f(\bar{x})(y^*) := \Big\{x^* \in X^* \Big|\ \limsup_{(x,y)\stackrel{\operatorname{epi}_\Theta f}{\to}(\bar{x},f(\bar{x}))} \frac{\langle x^*, x-\bar{x}\rangle - \langle y^*, y - f(\bar{x})\rangle}{\|x-\bar{x}\| + \|y - f(\bar{x})\|} \le 0\Big\}. \tag{2.5}$$

- The (sequential) *normal* Θ-*coderivative* of f at $\bar{x}$ is

$$D_N^*f(\bar{x})(y^*) := \Big\{x^* \in X^* \Big|\ \exists \text{ seq. } x_n \to \bar{x}, x_n^* \in \widehat{D}^*f(x_n)(y_n^*) \text{ s.t. } (x_n^*, y_n^*) \stackrel{w^*}{\to} (x^*, y^*)\Big\}. \tag{2.6}$$

- The *topological normal* Θ-*coderivative* of f at $\bar{x}$ is

$$\overline{D}_N^*f(\bar{x})(y^*) := \Big\{x^* \in X^* \Big|\ \exists \text{ nets } x_\alpha \to \bar{x}, x_\alpha^* \in \widehat{D}^*f(x_\alpha)(y_\alpha^*) \ \text{ s.t. } (x_\alpha^*, y_\alpha^*) \stackrel{w^*}{\to} (x^*, y^*)\Big\}. \tag{2.7}$$

• The *cluster normal* Θ-*coderivative* of f at $\bar{x}$ is

$$\breve{D}^*_N f(\bar{x})(y^*) := \Big\{x^* \in X^* \Big|\ \exists \text{ seq. } x_n \to \bar{x},\ \ x_n^* \in \widehat{D}^* f(x_n)(y_n^*) \text{ s.t. } (x^*, y^*) \text{ is a weak}^* \text{ cluster point of } (x_n^*, y_n^*)\Big\}. \tag{2.8}$$

Observe that the limiting procedures employed in (2.6) and (2.7) are similar to those used for mappings with no ordering structure; see [29] for more details and comparisons (we do not consider here the "mixed" coderivative counterparts as in [25, 29]). However, the one suggested in (2.8) seems to be new even in the non-ordering setting, being important for our results on cone-constrained problems in general Banach spaces Y and their applications to SIP. Note also that constructions (2.5) and (2.6) with $\|y^*\| = 1$ reduce to the corresponding vector subdifferentials of the set-valued mapping $F(x) := f(x) + \Theta$ at $(\bar{x}, f(\bar{x}))$ introduced in [2] and largely used in [2, 3] for various issues in multiobjective optimization in case of Asplund spaces Y. The coderivatives constructions introduced here allow us to proceed efficiently in the case of arbitrary Banach spaces Y needed for our SIP applications.

Denoting $\operatorname{Dom} F := \{x \in X |\ F(x) \neq \emptyset\}$ for any set-valued mapping $F\colon X \rightrightarrows Y$, observe that $\operatorname{Dom} \widehat{D}^* f(x) \subset \Theta_+$ for any $x \in X$, where

$$\Theta_+ := \big\{y^* \in Y^* \big|\ \langle y^*, y\rangle \geq 0 \ \text{ for all } \ y \in \Theta\big\} \tag{2.9}$$

is the (positive) *polar cone* to Ω. Since Θ_+ is a weak* closed subset of Y^*, it follows from the inclusion above that the domain sets $\operatorname{Dom} D^* f(\bar{x})$, $\operatorname{Dom} \overline{D}^* f(\bar{x})$, and $\operatorname{Dom} \breve{D}^* f(\bar{x})$ are also subsets of Θ_+. It is easy to check that for mappings $f\colon X \to Y$ locally Lipschitzian around $\bar{x}$ we have the *scalarization formula*

$$\widehat{D}^* f(\bar{x})(y^*) := \widehat{\partial}\langle y^*, f\rangle(\bar{x}) \ \text{ if and only if } \ y^* \in \Theta_+, \tag{2.10}$$

where $\langle y^*, f\rangle(x) = \langle y^*, f(x)\rangle$. However, such a scalarization for the limiting coderivatives D^*_N, $\overline{D}^*_N$, and $\breve{D}^*_N$ requires stronger Lipschitzian assumptions; cf. [25, Subsection 3.1.3] for mappings with values in spaces with no ordering. In this paper we need the following limiting counterparts of scalarization that can be proved similarly to [25, Theorem 1.90]:

$$D^*_N f(\bar{x})(y^*) = \overline{D}^*_N f(\bar{x})(y^*) = \breve{D}^*_N f(\bar{x})(y^*) = \big\{\nabla f(\bar{x})^* y^*\big\} \ \text{ for all } y^* \in \Theta_+ \tag{2.11}$$

provided that f is *strictly differentiable* at $\bar{x}$, i.e.,

$$\lim_{\substack{x,u \to \bar{x} \\ x \neq u}} \frac{f(x) - f(u) - \nabla f(\bar{x})(x - u)}{\|x - u\|} = 0.$$

Furthermore, it can be derived directly from the definitions that

$$\widehat{D}^*_N f(\bar{x})(y^*) = D^*_N f(\bar{x})(y^*) = \overline{D}^*_N f(\bar{x})(y^*) = \breve{D}^*_N f(\bar{x})(y^*) = \partial\langle y^*, f\rangle(\bar{x}) \tag{2.12}$$

for all $y^* \in \Theta_+$ provided that the mapping f is Θ-*convex*.

3 Subgradients of Supremum Functions

Unless otherwise stated, throughout the whole paper we impose the following assumptions on the initial data the cone-constrained problem (1.1):

Standing Assumptions. The space X is *Asplund*, the space Y is *arbitrary Banach*, the cost function $\vartheta\colon X \to \overline{I\!R}$ is *lower semicontinuous* (l.s.c.), the set $\Omega \subset X$ is *closed*, the set $\Theta \subset Y$ is a *closed* and *convex cone*, and the mapping $f\colon X \to Y$ is *locally Lipschitzian* around the reference point $\bar{x}$ in the sense that there are constants $K, \rho > 0$ such that

$$\|f(x) - f(u)\| \le K\|x - u\| \quad \text{for all } \ x, u \in I\!B_\rho(\bar{x}). \tag{3.1}$$

The next proposition shows that problem (1.1) can be equivalently written in form (1.3).

Proposition 3.1 (cone-constrained optimization via supremum functions). *Assume that $\bar{x}$ is a feasible solution to* (1.1). *Then we have*

$$\big\{x \in X\big|\ f(x) \in -\Theta\big\} = \big\{x \in X|\ \varphi(x) \le 0\big\},$$

where φ is the supremum function defined by

$$\varphi(x) := \sup_{y^* \in \Xi} \langle y^*, f(x)\rangle \ \ \text{with} \ \ \Xi := \big\{y^* \in Y^* \big|\ \|y^*\| = 1,\ \langle y^*, y\rangle \ge 0,\ y \in \Theta\big\}. \tag{3.2}$$

Proof. Note first that the inclusion $f(x) \in -\Theta$ gives us $\langle y^*, f(x)\rangle \le 0$ for all $y^* \in \Xi$. Conversely, suppose that the latter holds and show that $f(x) \in -\Theta$. Assuming the contrary and applying the classical separation theorem, find $\bar{y}^* \in Y^* \setminus \{0\}$ and $\gamma > 0$ such that

$$\langle \bar{y}^*, f(x)\rangle > \gamma > 0 \ge \langle \bar{y}^*, y\rangle \ \ \text{for all } \ y \in -\Theta.$$

This implies that $\bar{y}^*\|\bar{y}^*\|^{-1} \in \Xi$, and hence we arrive at the contradiction

$$0 \ge \big\langle \bar{y}^*\|\bar{y}^*\|^{-1}, f(x)\big\rangle > \gamma\|\bar{y}^*\|^{-1} > 0,$$

which thus completes the proof of the proposition. $\triangle$

The main goal of this section is to study *subdifferential properties* of the supremum function (3.2) under our standing assumptions. In fact, we consider a bit more general setting of the supremum function

$$\psi(x) := \sup_{y^* \in \Lambda} \langle y^*, f(x)\rangle, \tag{3.3}$$

where Λ is an arbitrary nonempty subset of the polar cone Θ_+ in (2.9). Since $\Xi \subset \Theta_+$ for the set Ξ in (3.2), the results obtained below for the supremum function (3.3) immediately apply to the function φ in (3.2) and then are used in the subsequent sections.

Our first result provides a "fuzzy" upper estimate of limiting subgradients of the supremum function (3.3) at the reference point $\bar{x}$ via regular subgradients of the scalarized function in (2.10) at some neighboring points.

Theorem 3.2 (fuzzy estimate of limiting subgradients of supremum functions). *Suppose under the standing assumptions that $\bar{x} \in \operatorname{dom} \psi$ for the supremum function* (3.3) *and that V^* is a weak* neighborhood of the origin in X^*. Then for any $x^* \in \partial\psi(\bar{x})$ and any ε there exist $x_\varepsilon \in I\!B_\varepsilon(\bar{x})$ and $y^*_\varepsilon \in \operatorname{co}\Lambda$ with $|\langle y^*_\varepsilon, f(x_\varepsilon)\rangle - \psi(\bar{x})| < \varepsilon$ such that*

$$x^* \in \widehat{\partial}\langle y^*_\varepsilon, f\rangle(x_\varepsilon) + V^*. \tag{3.4}$$

Proof. Fix arbitrary $x^* \in \partial\psi(\bar{x})$ and $\varepsilon > 0$. It is easy to check that each function $\langle y^*, f(x)\rangle$ is locally Lipschitzian around $\bar{x}$ with same constants K and ρ as in (3.1) for all $y^* \in \Lambda$, and so is the supremum function ψ. Without loss of generality we assume that V^* is convex and that $\varepsilon \le \rho$. Then find $n \in I\!N$, $\varepsilon_n > 0$, and $x_k \in X$ for $k = 1, \ldots, n$ such that

$$\bigcap_{k=1}^{n} \Big\{ v^* \in X^* \Big| \ \langle v^*, x_k\rangle < \varepsilon_n \Big\} \subset \frac{1}{4} V^*.$$

Form further a finite-dimensional subspace $L \subset X$ by $L := \operatorname{span}\{x_1, \ldots, x_n\}$ and observe that $L^\perp := \{v^* \in X^* |\ \langle v^*, x\rangle = 0, x \in L\} \subset \frac{1}{4}V^*$. By definition of the limiting subdifferential in (2.2) there exist $\widehat{x} \in \operatorname{dom}\psi \cap I\!B_{\frac{\varepsilon}{2}}(\bar{x})$ and $u^* \in X^*$ such that $|\psi(\widehat{x}) - \psi(\bar{x})| \le \frac{\varepsilon}{2}$, $u^* \in \widehat{\partial}\psi(\widehat{x})$ and that $x^* \in u^* + \frac{V^*}{4}$. Fix $\delta > 0$ satisfying the relationships

$$4\delta \le \varepsilon, \quad \frac{12\delta}{1-2\delta} I\!B^* \subset V^*, \quad \text{and} \quad \frac{16\delta}{1-2\delta}\|u^*\| I\!B^* \subset V^*. \tag{3.5}$$

Since $u^* \in \widehat{\partial}\psi(\widehat{x})$, there is some number $\eta \in (0, \delta)$ such that

$$\psi(x) - \psi(\widehat{x}) + \delta\|x - \widehat{x}\| \ge \langle u^*, x - \widehat{x}\rangle \quad \text{for all } x \in I\!B_\eta(\widehat{x}) \subset I\!B_\rho(\bar{x}).$$

This implies that $(\widehat{x}, \psi(\widehat{x}))$ is a *local minimizer* of the following problem:

$$\begin{cases} \text{minimize } \ r + \delta\|x - \widehat{x}\| - \langle u^*, x - \widehat{x}\rangle - \psi(\widehat{x}) \ \text{ subject to} \\ \langle y^*, f(x)\rangle - r \le 0 \ \text{ for } \ y^* \in \Lambda \ \text{ and} \\ (x, r) \in I\!B_\eta(\widehat{x}) \times I\!R. \end{cases}$$

Define $A := (L \cap I\!B_\eta(\widehat{x})) \times [\psi(\widehat{x}) - 1, \psi(\widehat{x}) + 1]$, $\Psi(x, r) := r + \delta\|x - \widehat{x}\| - \langle u^*, x - \widehat{x}\rangle - \psi(\widehat{x})$, and a family of functions $\varphi_{y^*} : X \times I\!R \to \overline{I\!R}$ by $\varphi_{y^*}(x, r) := \langle y^*, f(x)\rangle - r$ for all $y^* \in \Lambda$ and $(x, r) \in X \times I\!R$. It follows from the constructions above that

$$\Big\{(x, r) \in A \Big|\ \Psi(x, r) + \eta^2 \le 0\Big\} \subset \bigcup_{y^* \in \Lambda} \Big\{(x, r) \in A \Big|\ \varphi_{y^*}(x, r) > 0\Big\}. \tag{3.6}$$

Since the set on the left-hand side of (3.6) is closed and bounded in the finite-dimensional space $L \times I\!R$, it is compact therein. Moreover, each subset $\{(x, r) \in A |\ \varphi_{y^*}(x, r) > 0\}$ is open in A due to the Lipschitz continuity of the functions φ_{y^*} on the set $I\!B_\rho(\bar{x}) \times I\!R$, which contains A. Thus we find a *finite* subset $\Pi \subset \Lambda$ satisfying

$$\Big\{(x, r) \in A \Big|\ \Psi(x, r) + \eta^2 \le 0\Big\} \subset \bigcup_{y^* \in \Pi} \Big\{(x, r) \in A \Big|\ \varphi_{y^*}(x, r) > 0\Big\}.$$

This ensures the relationships

$$\Psi(x, r) + \eta^2 \ge 0 = \Psi(\widehat{x}, \varphi(\widehat{x})) \ \text{ for all } \ (x, r) \in \widetilde{A} := \Big\{(x, r) \in A \Big|\ \varphi_{y^*}(x, r) \le 0, y^* \in \Pi\Big\},$$

where the set $\widetilde{A}$ is a closed set in $I\!B_\rho(\bar{x}) \times I\!R$. Using now the *Ekeland variational principle* gives us $(\widetilde{x}, \widetilde{r}) \in \widetilde{A}$ such that $\|\widetilde{x} - \widehat{x}\| + |\widetilde{r} - \varphi(\widehat{x})| \le \frac{\eta}{2}$ and

$$\Psi(x, r) + 2\eta(\|x - \widetilde{x}\| + |r - \widetilde{r}|) \ge \Psi(\widetilde{x}, \widetilde{r}) \quad \text{for all } \ (x, r) \in \widetilde{A}.$$

The latter means that $(\widetilde{x},\widetilde{r})$ is a *local optimal solution* to the following optimization problem:

$$\begin{cases} \text{minimize } \widetilde{\Psi}(x,r):=\Psi(x,r)+2\eta(\|x-\widetilde{x}\|+|r-\widetilde{r}|) \ \text{ subject to} \\ \varphi_{y^*}(x,r)\le 0 \ \text{ for } \ y^*\in\Pi \ \text{ and } \ (x,r)\in A. \end{cases} \tag{3.7}$$

It is obvious that the functions $\widetilde{\Psi}(\cdot,\cdot)$ and $\varphi_{y^*}(\cdot,\cdot)$ are Lipschitz continuous around $(\widetilde{x},\widetilde{r})$ for all $y^*\in\Pi$. Applying the necessary optimality conditions from [25, Theorem 5.17] to problem (3.7), we find multipliers $\lambda_0,\lambda_1,\ldots,\lambda_m\ge 0$, not equal to zero simultaneously, and dual elements $y_1^*,y_2^*,\ldots,y_m^*\in\Pi(\widetilde{x},\widetilde{r}):=\{y^*\in\Pi|\ \varphi_{y^*}(\widetilde{x},\widetilde{r})=0\}$ such that

$$(0,0)\in\partial\Big(\lambda_0\widetilde{\Psi}+\sum_{k=1}^m\lambda_k\varphi_{y_k^*}\Big)(\widetilde{x},\widetilde{r})+N((\widetilde{x},\widetilde{r});A).$$

Since $(\widetilde{x},\widetilde{r})\in\text{int}\,(B_\eta(\widehat{x})\times[\varphi(\widehat{x})-1,\varphi(\widehat{x})+1])$, it follows from the above inclusion that

$$\begin{aligned}(0,0)\ &\in\partial\Big(\lambda_0\widetilde{\Psi}+\sum_{k=1}^m\lambda_k\varphi_{y_k^*}\Big)(\widetilde{x},\widetilde{r})+N\big((\widetilde{x},\widetilde{r});(L\cap I\!\!B_\eta(\widehat{x}))\times[\psi(\widehat{x})-1,\psi(\widehat{x})+1]\big)\\ &=\partial\Big(\lambda_0\widetilde{\Psi}+\sum_{k=1}^m\lambda_k\varphi_{y_k^*}\Big)(\widetilde{x},\widetilde{r})+N(\widetilde{x};L)\times\{0\}\\ &\subset\partial\Big(\lambda_0\widetilde{\Psi}+\sum_{k=1}^m\lambda_k\varphi_{y_k^*}\Big)(\widetilde{x},\widetilde{r})+L^\perp\times\{0\}.\end{aligned}\tag{3.8}$$

If $\lambda_0=0$, we get from (3.8) the inclusion

$$(0,0)\in\partial\Big(\sum_{k=1}^m\lambda_k\langle y_k^*,f\rangle\Big)(\widetilde{x})\times\Big\{-\sum_{k=1}^m\lambda_k\Big\}+L^\perp\times\{0\},$$

which implies in turn that $\sum_{k=1}^m\lambda_k=0$, i.e., $\lambda_k=0$ for all $k=0,\ldots,m$. This contradiction shows that $\lambda_0\ne 0$. We can make $\lambda_0=1$ and then get from (3.8) that

$$(u^*,0)\in\partial\Big(\sum_{k=1}^m\lambda_k\langle y_k^*,f\rangle\Big)(\widetilde{x})\times\Big\{1-\sum_{k=1}^n\lambda_k\Big\}+(\delta+2\eta)I\!\!B_{X^*}\times 2[-\eta,\eta]+L^\perp\times\{0\}. \tag{3.9}$$

Define $\widetilde{\lambda}:=\sum_{k=1}^m\lambda_k$, $\widetilde{\lambda}_k:=\widetilde{\lambda}^{-1}\lambda_k$ for $k=1,\ldots,m$, and $\widetilde{u}^*:=\widetilde{\lambda}^{-1}u^*$. Then (3.9) gives us that $|1-\widetilde{\lambda}|\le 2\eta<2\delta$. Dividing both sides of (3.9) by $\widetilde{\lambda}$, we obtain

$$\begin{aligned}\widetilde{u}^*\ &\in\ \partial\Big(\sum_{k=1}^n\widetilde{\lambda}_k\langle y_k^*,f\rangle\Big)(\widetilde{x})+\frac{\delta+2\eta}{\widetilde{\lambda}}I\!\!B^*+\frac{L^\perp}{\widetilde{\lambda}}\subset\partial\Big(\sum_{k=1}^n\langle\widetilde{\lambda}_k y_k^*,f\rangle\Big)(\widetilde{x})+\frac{3\delta}{1-2\delta}I\!\!B^*+L^\perp\\ &\subset\ \partial\Big(\sum_{k=1}^n\langle\widetilde{\lambda}_k y_k^*,f\rangle\Big)(\widetilde{x})+\frac{V^*}{4}+\frac{V^*}{4}\subset\partial\langle y_\varepsilon^*,f\rangle(\widetilde{x})+\frac{V^*}{2}\end{aligned}$$

with $y_\varepsilon^*:=\sum_{k=1}^m\widetilde{\lambda}_k y_k^*\in\text{co}\,\Pi\subset\text{co}\,\Lambda$, the fact $L^\perp\subset\frac{1}{4}V^*$ and (3.3) are used in the above inclusion. Thus there is $v^*\in\partial\langle y_\varepsilon^*,f\rangle(\widetilde{x})$ satisfying $\widetilde{u}^*\in v^*+\frac{V^*}{2}$. By definition (2.2) of the limiting subdifferential we find $x_\varepsilon\in I\!\!B_\delta(\widetilde{x})$ and $w^*\in\widehat{\partial}\langle y_\varepsilon^*,f\rangle(x_\varepsilon)$ such that $|\langle y_\varepsilon^*,f(x_\varepsilon)\rangle-\langle y_\varepsilon^*,f(\widetilde{x})\rangle|\le\delta$ and $v^*\in w^*+\frac{V^*}{8}$. Observe that

$$\|x_\varepsilon-\bar{x}\|\le\|x_\varepsilon-\widetilde{x}\|+\|\widetilde{x}-\widehat{x}\|+\|\widehat{x}-\bar{x}\|\le\delta+\delta+\frac{\varepsilon}{2}\le\varepsilon. \tag{3.10}$$

Furthermore, we have the estimates

$$\begin{aligned} |\langle y_\varepsilon^*, f(x_\varepsilon)\rangle - \psi(\bar{x})| &\leq |\langle y_\varepsilon^*, f(x_\varepsilon) - f(\widetilde{x})\rangle| + |\langle y_\varepsilon^*, f(\widetilde{x})\rangle - \widetilde{r}| + |\widetilde{r} - \psi(\widehat{x})| + |\psi(\widehat{x}) - \psi(\bar{x})| \\ &\leq \delta + \Big|\sum_{k=1}^m \widetilde{\lambda}_k \langle y_k^*, f(\widetilde{x})\rangle - \widetilde{r}\Big| + \frac{\eta}{2} + \frac{\varepsilon}{2} = \delta + \frac{\eta}{2} + \frac{\varepsilon}{2} \leq \varepsilon \end{aligned} \tag{3.11}$$

by taking into account that $\langle y_k^*, f(\widetilde{x})\rangle = \widetilde{r}$ for all $k = 1, \ldots, m$. Note further that

$$\|u^* - \widetilde{u}^*\| = \frac{1 - \widetilde{\lambda}}{\widetilde{\lambda}}\|u^*\| \leq \frac{2\eta}{1 - 2\eta}\|u^*\| \leq \frac{2\delta}{1 - 2\delta}\|u^*\|,$$

which implies the following inclusions:

$$\begin{aligned} x^* &\in u^* + \frac{V^*}{4} \subset \widetilde{u}^* + \frac{2\delta}{1 - 2\delta}\|u^*\| I\!B^* + \frac{V^*}{4} \subset v^* + \frac{V^*}{2} + \frac{V^*}{8} + \frac{V^*}{4} \\ &\subset w^* + \frac{V^*}{8} + \frac{V^*}{2} + \frac{V^*}{8} + \frac{V^*}{4} \subset \widehat{\partial}\langle y_\varepsilon^*, f\rangle(x_\varepsilon) + V^*. \end{aligned}$$

Combining this with (3.10) and (3.11) completes the proof of the theorem. △

We refer the reader to [7, Theorem 3.18] for fuzzy estimates of *regular* subgradients (2.1) of supremum functions in reflexive spaces and to our recent paper [27, Theorem 3.1] for more elaborated estimates of such subgradients in Asplund spaces. However, applying these estimates to the function ψ in (3.3) gives us weaker results in comparison with the one obtained in Theorem 3.2. Based on this theorem, we now derive *pointbased* (i.e., involving the reference point $\bar{x}$) upper estimates of the limiting subdifferential of the function ψ via the corresponding limiting coderivatives of f depending on the assumptions imposed on the spaces X and Y in question.

Theorem 3.3 (pointbased estimates of limiting subgradient of supremum functions via coderivatives). *In the setting of Theorem* 3.2 *assume that the set* Λ *is bounded in* Y^*. *Then the limiting subdifferential of* ψ *at* $\bar{x}$ *is estimated by*

$$\partial\psi(\bar{x}) \subset \big\{x^* \in \overline{D}_N^* f(\bar{x})(y^*)\big|\ y^* \in \mathrm{cl}^*\mathrm{co}\,\Lambda,\ \langle y^*, f(\bar{x})\rangle = \psi(\bar{x})\big\} \tag{3.12}$$

via the topological Θ*-coderivative* (2.7) *of* f *at* $\bar{x}$. *If* $\dim X < \infty$, *we have the estimate*

$$\partial\psi(\bar{x}) \subset \big\{x^* \in \breve{D}_N^* f(\bar{x})(y^*)\big|\ y^* \in \mathrm{cl}^*\mathrm{co}\,\Lambda,\ \langle y^*, f(\bar{x})\rangle = \psi(\bar{x})\big\} \tag{3.13}$$

via the cluster Θ*-coderivative* (2.8). *If in addition the dual unit ball* $I\!B_{Y^*}$ *is weak** *sequentially compact in* Y^*, *then the cluster* Θ*-coderivative can be replaced in* (3.13) *by its normal counterpart* $D_N^* f(\bar{x})(y^*)$ *from* (2.6).

Proof. To justify estimate (3.12), we first construct a *filter* $\{V_\alpha^*\}_{\alpha\in A}$ of neighborhoods of the origin in X^* and a *net* $\{\varepsilon_\alpha\}_{\alpha\in A} \subset I\!R_+$ such that $\varepsilon_\alpha \to 0^+$. Let $\mathcal{N}_{X^*}$ be the set of all weak* neighborhoods of the origin in X^*, and let A be the set that is bijective with $\mathcal{N}_{X^*}$. Denote the bijective correspondence by subscript labeling $\mathcal{N}_{X^*} = \{V_\alpha^*|\ \alpha \in A\}$. Then A is a *directed set*, where the direction is given by $\alpha \succeq \beta$ if and only if V_α^* is contained in V_β^*. Fix any $v^* \in S_{X^*}$ and define

$$\varepsilon_\alpha := \sup\big\{r \in [0, \rho)\big|\ rv^* \in V_\alpha^*\big\} \quad \text{for all } \alpha \in A,$$

where ρ is taken from (3.1). Observe that $\varepsilon_\alpha > 0$ for all $\alpha \in A$ and that $\varepsilon_\alpha \to 0$. Indeed, for any $\alpha \in A$ there is $\delta \in (0, \rho)$ sufficiently small such that $\delta I\!B^* \subset V_\alpha^*$. It is obvious that $\varepsilon_\alpha > \delta$. Furthermore, for any $\varepsilon > 0$ the existence of some $\alpha_0 \in A$ with $\varepsilon_{\alpha_0} < \varepsilon$ implies that $\varepsilon_\alpha < \varepsilon$ for all $\alpha \succeq \alpha_0$ by definition of the set A. Hence if the net $\{\varepsilon_\alpha\}$ does not converge to 0, there is some $\varepsilon > 0$ such that $\varepsilon_\alpha > \varepsilon$ for all $\alpha \in A$, which yields that $\varepsilon v^* \in V_\alpha^*$ for all $\alpha \in A$. This contradiction justifies that $\varepsilon_\alpha \to 0^+$.

Now pick an arbitrary limiting subgradient $x^* \in \partial\psi(\bar{x})$. Employing Theorem 3.2 for any $\alpha \in A$ allows us to find $x_\alpha \in I\!B_{\varepsilon_\alpha}(\bar{x})$ and $y_\alpha^* \in \operatorname{co}\Lambda$ such that

$$x^* \in \widehat{\partial}\langle y_\alpha^*, f\rangle(x_\alpha) + V_\alpha^* \quad \text{and} \quad |\langle y_\alpha^*, f(x_\alpha)\rangle - \psi(\bar{x})| \le \varepsilon_\alpha.$$

By using the scalarization formula (2.10) we get $u_\alpha^* \in \widehat{\partial}\langle y_\alpha^*, f\rangle(x_\alpha) = \widehat{D}^* f(x_\alpha)(y_\alpha^*)$ and $v_\alpha^* \in V_\alpha^*$ with $x^* = u_\alpha^* + v_\alpha^*$. Since the filter $\{V_\alpha^*\}_{\alpha \in A}$ weak* converges to 0, the *derived net* $\{v_\alpha^*\}_{\alpha\in A}$ also weak* converges to 0. This implies that $u_\alpha^* \xrightarrow{w^*} x^*$. Since the set $\operatorname{co}\Lambda$ is bounded in Y^*, the classical Alaoglu-Bourbaki theorem allows us to find a subnet of $\{y_\alpha^*\}_{\alpha\in A}$ (no relabeling) weak* converging to some $y^* \in \operatorname{cl}^*\operatorname{co}\Lambda$. This yields that $x^* \in \overline{D}_N^* f(\bar{x})(y^*)$. Moreover, by $\varepsilon_\alpha \to 0, x_\alpha \to \bar{x}$, and $y_\alpha^* \xrightarrow{w^*} y^*$ we have

$$0 = \lim \varepsilon_\alpha = \lim\langle y_\alpha^*, f(x_\alpha)\rangle - \psi(\bar{x}) = \langle y^*, f(\bar{x})\rangle - \psi(\bar{x}),$$

which thus justifies the validity of estimate (3.12) via the topological coderivative of f at $\bar{x}$.

When the space X is finite-dimensional, we can choose $\widetilde{\mathcal{N}}_{X^*} := \{I\!B(0, \frac{1}{n})|\ n \in I\!N\}$ instead of $\mathcal{N}_{X^*}$ in the proof above and then find $A = I\!N$ and a *sequence* $\varepsilon_n \in (0, \rho)$ such that $\varepsilon_n \to 0$ as $n \to \infty$. Following the similar arguments, we arrive at estimate (3.13) via the cluster coderivative $\breve{D}^* f(\bar{x})(y^*)$.

Finally, assuming the weak* sequential compactness of the dual unit ball $I\!B_{Y^*}$ implies in the arguments above that all the limiting elements of $\breve{D}_N^* f(\bar{x})(y^*)$ belongs actually to $D_N^* f(\bar{x})(y^*)$. This completes the proof of the theorem. $\triangle$

Regarding the weak* sequential compactness assumptions imposed on $I\!B_{Y^*}$ in the last part of Theorem 3.3, recall that it holds, in particular, for Banach spaces admitting an equivalent norm Gâteaux differentiable at nonzero points, for weak Asplund spaces (including every Asplund space and every weakly compactly generated space, and hence every reflexive and every separable space), etc. We refer the reader to [16] for more information on the aforementioned classes of Banach spaces.

4 Pointbased Optimality and Qualification Conditions for Cone-Constrained Programs

In this section we use the supremum-type representation (1.3) of the original cone-constrained optimization problem (1.1), the subdifferential estimates of the supremum function obtained in Theorem 3.3, and generalized differential calculus of variational analysis to derive *pointbased necessary conditions* for optimal solutions of (1.1) via the *limiting* constructions of Section 2 under appropriate *constraint qualifications.*

The following theorem presents the main results of this section.

Theorem 4.1 (necessary optimality conditions for cone-constrained programs). *Let $\bar{x}$ be an optimal solution to problem* (1.1) *under the standing assumptions of Section* 3.

Suppose also that either ϑ is SNEC at $\bar{x}$ or Ω is SNC at $\bar{x}$ and that the qualification condition

$$\partial^\infty\vartheta(\bar{x})\cap\big(-N(\bar{x};\Omega)\big)=\{0\} \tag{4.1}$$

is satisfied; both the SNEC property and the qualification condition (4.1) are automatic when ϑ is locally Lipschitzian around $\bar{x}$. Then one of the following assertions holds:

(i) *There exists $y^*\in\Theta_+$ such that*

$$0\in\partial\vartheta(\bar{x})+\overline{D}^*_N f(\bar{x})(y^*)+N(\bar{x};\Omega)\quad\textit{and}\quad\langle y^*,f(\bar{x})\rangle=0. \tag{4.2}$$

(ii) *There exists $y^*\in\mathrm{cl}^*\mathrm{co}\,\Xi$ such that*

$$0\in\partial^\infty\vartheta(\bar{x})+\overline{D}^*_N f(\bar{x})(y^*)+N(\bar{x};\Omega)\quad\textit{and}\quad\langle y^*,f(\bar{x})\rangle=0. \tag{4.3}$$

If the space X is finite-dimensional, the above conclusion holds with replacing $\overline{D}^*_N f(\bar{x})$ by $\breve{D}^*_N f(\bar{x})$. If furthermore $I\!B_{Y^*}$ is weak* sequentially compact in Y^*, then the topological coderivative $\overline{D}^*_N f(\bar{x})$ can be replaced in (4.2) and (4.3) by the sequential one $D^*_N f(\bar{x})$.

Proof. Observe first that the validity of both the SNEC property of ϑ at $\bar{x}$ and the qualification condition (4.1) for local Lipschitzian cost functions ϑ follows from the discussions in Section 2 after (2.3) and (2.4). Further, it is easy to see that $\bar{x}$ is a local optimal solution of the following *minimax problem* of *unconstrained optimization*:

$$\text{minimize }\;\Psi(x):=\max\big\{\big(\vartheta+\delta(\cdot;\Omega)\big)(x)-\vartheta(\bar{x}),\varphi(x)\big\}\;\text{ subject to }\;x\in X, \tag{4.4}$$

where $\varphi(x)$ is defined in (3.2), and where Ψ is obviously l.s.c. around $\bar{x}$. If $\varphi(\bar{x})<0$, then there is a neighborhood U of $\bar{x}$ such that $\Psi(x)-\varphi(x)>0$ for $x\in U$, which implies that $\Psi(x)=(\vartheta+\delta(\cdot;\Omega))(x)$ for $x\in U$. Since $\bar{x}$ is a local optimal solution to problem (4.4), we have by the generalized Fermat rule that

$$0\in\partial\Psi(\bar{x})=\partial\big(\vartheta+\delta(\cdot;\Omega)\big)(\bar{x}).$$

It follows from the assumptions imposed on ϑ and Ω and the sum rules for the limiting and singular subdifferentials from [25, Theorem 3.36] that

$$\partial\big(\vartheta+\delta(\cdot;\Omega)\big)(\bar{x})\subset\partial\vartheta(\bar{x})+N(\bar{x};\Omega)\;\text{ and }\;\partial^\infty\big(\vartheta+\delta(\cdot;\Omega)\big)(\bar{x})\subset\partial^\infty\vartheta(\bar{x})+N(\bar{x};\Omega). \tag{4.5}$$

Thus we have $0\in\partial\vartheta(\bar{x})+N(\bar{x};\Omega)$, which ensures the validity of the necessary optimality conditions in (4.2) with $y^*=0$ in this case.

Next we consider the case of $\varphi(\bar{x})=0$. Since φ is locally Lipschitzian around $\bar{x}$, it follows from [25, Theorem 3.36] that

$$\begin{array}{l}\partial^\infty\Psi(\bar{x})\subset\partial^\infty\big(\vartheta+\delta(\cdot;\Omega)\big)(\bar{x})+\partial^\infty\varphi(\bar{x})=\partial^\infty\big(\vartheta+\delta(\cdot;\Omega)\big)(\bar{x})\;\text{ and}\\ \partial\Psi(\bar{x})\subset\bigcup\Big\{\lambda_1\circ\partial\big(\vartheta+\delta(\cdot;\Omega)\big)(\bar{x})+\lambda_2\partial\varphi(\bar{x})\Big|\;(\lambda_1,\lambda_2)\in I\!R^2_+,\;\lambda_1+\lambda_2=1\Big\},\end{array} \tag{4.6}$$

where $\lambda\circ\partial\vartheta(\bar{x})$ denotes $\lambda\partial\vartheta(\bar{x})$ when $\lambda>0$ and $\partial^\infty\vartheta(\bar{x})$ when $\lambda=0$. Since $0\in\partial\Psi(\bar{x})$ we get from (4.5) and (4.6) that there exist $x^*\in N(\bar{x};\Omega)$ and $(\lambda_1,\lambda_2)\in I\!R^2_+$ such that $\lambda_1+\lambda_2=1$ and that

$$0\in\lambda_1\circ\partial\vartheta(\bar{x})+\lambda_2\partial\varphi(\bar{x})+x^*. \tag{4.7}$$

If $\lambda_1 \neq 0$ in (4.7), it follows that there is $u^* \in \partial\vartheta(\bar{x})$ with $-x^* - \lambda_1 u^* \in \lambda_2 \partial\varphi(\bar{x})$. If $\lambda_2 = 0$ in (4.7) and thus $\lambda_1 = 1$, we obtain (4.2) with $y^* = 0$ due to

$$0 = u^* + x^* \in \partial\vartheta(\bar{x}) + \overline{D}^*_N f(\bar{x})(0) + N(\bar{x};\Omega).$$

Otherwise Theorem 3.3 with $\Lambda = \Xi$ allows us to find $y^* \in \mathrm{cl}^*\mathrm{co}\,\Xi$ satisfying

$$\frac{-x^* - \lambda_1 u^*}{\lambda_2} \in \overline{D}^*_N f(\bar{x})(y^*) \quad \text{and} \quad \langle y^*, f(\bar{x})\rangle = \varphi(\bar{x}) = 0.$$

Hence we arrive at the inclusions

$$0 \in u^* + \frac{\lambda_2}{\lambda_1}\overline{D}^*_N f(\bar{x})(y^*) + \frac{x^*}{\lambda_1} \subset \partial\vartheta(\bar{x}) + \overline{D}^*_N f(\bar{x})\Big(\frac{\lambda_2 y^*}{\lambda_1}\Big) + N(\bar{x};\Omega),$$

which justify the conditions of (4.2) in this case.

Supposing then that $\lambda_1 = 0$, we get from (4.7) the existence of $v^* \in \partial^\infty\vartheta(\bar{x})$ such that $-v^* - x^* \in \partial\varphi(\bar{x})$. Applying Theorem 3.3 again gives us $z^* \in \mathrm{cl}^*\mathrm{co}\,\Xi$ satisfying the conditions $-v^* - x^* \in \overline{D}^*_N f(\bar{x})(z^*)$ and $\langle z^*, f(\bar{x})\rangle = 0$, which readily yield (4.3). The rest of the theorem, which deals with the particular structures of the spaces X and Y, follows by the above arguments from the corresponding results of Theorem 3.3. △

Note that assertion (ii) of Theorem 4.1 holds trivially if $0 \in \mathrm{cl}^*\mathrm{co}\,\Xi$. Indeed, in this case we always have $0 \in \overline{D}^*_N f(\bar{x})(0) \cap \partial^\infty\vartheta(\bar{x}) \cap N(\bar{x};\Omega)$. The next proposition shows that 0 is never an element of $\mathrm{cl}^*\mathrm{co}\,\Xi$ if, in particular, the interior of the cone Θ is nonempty.

Proposition 4.2 (solid cone constraints). *The following assertions are equivalent:*

(i) $0 \notin \mathrm{cl}^*\mathrm{co}\,\Xi$.

(ii) *There are $r > 0$ and $y_0 \in Y$ such that $\langle y^*, y_0\rangle > r$ for all $y^* \in \Xi$.*

(iii) $\mathrm{int}\,\Theta \neq \emptyset$.

Proof. Implication (i)$\Longrightarrow$(ii) follows directly from the classical separation theorem. To prove (ii)$\Longrightarrow$(iii), assume that (ii) holds and get for any $y \in I\!\!B_r(y_0)$ that

$$\langle y^*, y\rangle = \langle y^*, y_0\rangle + \langle y^*, y - y_0\rangle \geq r - \|y^*\| \cdot \|y - y_0\| > r - r = 0$$

whenever $y^* \in \Xi$. This yields that $y \in \Theta$ and so ensures (iii). Finally, suppose that (iii) holds and then find $y_1 \in \Theta$ and $s > 0$ such that $I\!\!B_s(y_1) \subset \Theta$. For any $y^* \in \Xi$ we have

$$\langle y^*, y_1\rangle = \langle y^*, y_1\rangle - s\|y^*\| + s \geq \langle y^*, y_1\rangle - \sup_{y \in I\!\!B_s(0)} \langle y^*, y\rangle + s = \inf_{y \in I\!\!B_s(y_1)} \langle y^*, y\rangle + s \geq s > 0.$$

This clearly implies that $\langle y^*, y_1\rangle > s$ whenever $y^* \in \mathrm{co}\,\Xi$. Thus (i) is satisfied, which completes the proof of the proposition. △

We can observe from the proof of Theorem 4.1 with taking Proposition 4.2 into account that in the case of solid cone constraints the necessary optimality conditions in (4.2) hold under an enhanced constraint qualification.

Corollary 4.3 (necessary optimality conditions under enhanced qualifications for solid cone constraints). *Assume in the setting of Theorem* 4.1 *that* $\mathrm{int}\,\Theta \neq \emptyset$ *and that the following qualification condition*

$$\big(\partial^\infty\vartheta(\bar{x}) + N(\bar{x};\Omega)\big) \cap \big(-\overline{D}^*_N f(\bar{x})(\Xi_0)\big) = \emptyset \tag{4.8}$$

holds with $\Xi_0 := \{y^* \in \Xi |\ \langle y^*, f(\bar{x})\rangle = 0\}$*. Then there is* $y^* \in \Theta_+$ *such that the optimality conditions* (4.2) *are satisfied. If* $\dim X < \infty$*, then* $\overline{D}^*_N f(\bar{x})$ *can be replaced by* $\breve{D}^*_N f(\bar{x})$ *in* (4.2)*. Furthermore,* $\overline{D}^*_N f(\bar{x})$ *can be replaced by* $D^*_N f(\bar{x})$ *in* (4.2) *if in addition the dual unit ball* $I\!B_{Y^*}$ *is weak** *sequentially compact in* Y^**.*

Proof. Following the proof of Theorem 4.1, it is sufficient to show that $\lambda_1 \neq 0$ under the assumptions made. Arguing by contradiction, suppose that $\lambda_1 = 0$ and then find dual elements $x^* \in N(\bar{x}; \Omega)$, $v^* \in \partial^\infty \vartheta(\bar{x})$, and $z^* \in \mathrm{cl}^* \mathrm{co}\, \Xi$ such that

$$-v^* - x^* \in \overline{D}^*_N f(\bar{x})(z^*) \quad \text{and} \quad \langle z^*, f(\bar{x})\rangle = 0.$$

It follows from Proposition 4.2 that $z^* \neq 0$. Hence we have

$$\partial^\infty \vartheta(\bar{x}) + N(\bar{x}; \Omega) \ni \frac{v^*}{\|z^*\|} + \frac{x^*}{\|z^*\|} = -\frac{-v^* - x^*}{\|z^*\|} \in -\overline{D}^*_N f(\bar{x})\Big(\frac{z^*}{\|z^*\|}\Big),$$

which contradicts the imposed qualification condition (4.8) due to $\frac{z^*}{\|z^*\|} \in \Xi_0$ and thus completes the proof of the corollary. $\triangle$

Our last result in this section specifies a consequence of Theorem 4.1 for the case of a locally Lipschitzian cost function (when the SNEC property of ϑ and the qualification condition (4.1) are automatic) and either the strictly differentiable or Θ-convex structures of the cone-constraint mapping f in (1.1). We can see that in such settings the qualification condition (4.8) of Corollary 4.3 is equivalent to Robinson's constraint qualification [33] and the classical Slater condition, respectively.

Corollary 4.4 (cone-constrained problems in special settings). *Assume in the framework of Corollary* 4.3 *that* ϑ *is locally Lipschitzian around* $\bar{x}$ *and the constraint set* $\Omega \subset X$ *is convex. The following assertions hold:*

(i) *If* f *is strictly differentiable at* $\bar{x}$*, then the qualification condition* (4.8) *is equivalent to Robinson's constraint qualification:*

$$0 \in \mathrm{int}\big\{f(\bar{x}) + \nabla f(\bar{x})(\Omega - \bar{x}) + \Theta\big\} \tag{4.9}$$

and the optimality condition (4.2) *reduces to the existence of* $y^* \in \Theta_+$ *with* $\langle y^*, f(\bar{x})\rangle = 0$ *and* $x^* \in \partial\vartheta(\bar{x})$ *satisfying*

$$\langle x^* + \nabla f(\bar{x})^* y^*, x - \bar{x}\rangle \geq 0 \ \textit{ for all } \ x \in \Omega. \tag{4.10}$$

(ii) *If* f *is* Θ*-convex, then the qualification condition* (4.8) *is equivalent to Slater's constraint qualification:*

$$\textit{there is } \ x_0 \in \Omega \ \textit{ with } \ f(x_0) \in -\mathrm{int}\, \Theta \tag{4.11}$$

while the optimality condition (4.2) *reduces to the existence of* $y^* \in \Theta_+$ *with* $\langle y^*, f(\bar{x})\rangle = 0$*,* $u^* \in \partial\langle y^*, f\rangle(\bar{x})$*, and* $x^* \in \partial\vartheta(\bar{x})$ *satisfying*

$$\langle x^* + u^*, x - \bar{x}\rangle \geq 0 \ \textit{ for all } \ x \in \Omega. \tag{4.12}$$

Proof. Since $\partial^\infty\vartheta(\bar{x}) = \{0\}$ for locally Lipschitzian functions and due to the convexity of Ω the qualification condition (4.8) has the form

$$\nexists x^* \in -\overline{D}^*_N f(\bar{x})(\Xi_0) \;\text{ with }\; \langle x^*, x-\bar{x}\rangle \le 0 \;\text{ for all }\; x \in \Omega.$$

To justify (i), assume that f is strictly differentiable at $\bar{x}$ and observe by applying the classical supporting hyperplane theorem that condition (4.9) is equivalent to

$$N\big(0; f(\bar{x}) + \nabla f(\bar{x})(\Omega - \bar{x}) + \Theta\big) = \{0\}. \tag{4.13}$$

Suppose that condition (4.8) holds and show that (4.13) is satisfied. Indeed, if on the contrary there is $y^* \in N\big(0; f(\bar{x}) + \nabla f(\bar{x})(\Omega - \bar{x}) + \Theta\big)$ with $\|y^*\| = 1$, then

$$\langle y^*, f(\bar{x}) + \nabla f(\bar{x})(x-\bar{x}) + z\rangle \le 0 \quad \text{for all }\; x \in \Omega \;\text{ and }\; z \in \Theta,$$

which implies that $y^* \in -\Theta_+$ with $\langle y^*, f(\bar{x})\rangle \le 0$. Moreover, note that $\langle y^*, f(\bar{x})\rangle \ge 0$, since $y^* \in -\Theta_+$ and $f(\bar{x}) \in -\Theta$. It follows that $y^* \in -\Xi_0$ and that $\nabla f(\bar{x})^* y^* \in N(\bar{x}; \Omega)$. By scalarization (2.11) we arrive at a contradiction with (4.8).

Conversely, suppose that Robinson's constraint qualification (4.9) is satisfied. If there is some $z^* \in \Xi_0$ such that $N(\bar{x};\Omega) \cap \big(-\overline{D}^*_N f(\bar{x})(z^*)\big) \ne \emptyset$, we easily get from (2.11) that $-z^* \in N\big(0; f(\bar{x}) + \nabla f(\bar{x})(\Omega - \bar{x}) + \Theta\big)$, which implies that $z^* = 0$. This is a contradiction, which justifies the equivalence between (4.8) and (4.9) in assertion (i). The equivalence between the necessary optimality conditions (4.2) and (4.10) in this case follows from the structure of the normal cone to convex sets and the coderivative scalarization (2.11), which completes the proof of assertion (i).

Next we prove assertion (ii), where the constraint mapping f is Θ-convex in (1.1). Assume first that the Slater condition (4.11) does not hold, i.e., $f(\Omega) \cap (-\text{int}\,\Theta) = \emptyset$. Then it is easy to check that $A \cap (-\text{int}\,\Theta) = \emptyset$, where $A := \{f(x) + \Theta \mid x \in \Omega\}$ is a convex set in Y. Applying the separation theorem to these two sets gives us $w^* \in S_{Y^*}$ such that

$$\langle w^*, f(x)\rangle \ge \langle w^*, -z\rangle \quad \text{for all }\; x \in \Omega,\; z \in \Theta.$$

It follows that $w^* \in \Theta_+$ and $\langle w^*, f(x)\rangle \ge 0$ for all $x \in \Omega$. Since $f(\bar{x}) \in -\Theta$, we get that $\langle w^*, f(\bar{x})\rangle = 0$ and $\langle w^*, f(x)\rangle - \langle w^*, f(\bar{x})\rangle \ge 0$ for all $x \in \Omega$. This implies that

$$0 \in \partial(\langle w^*, f\rangle + \delta(\cdot;\Omega))(\bar{x}) \subset \partial\langle w^*, f\rangle(\bar{x}) + N(\bar{x};\Omega).$$

Thus we arrive at $N(\bar{x};\Omega) \cap \big(-\overline{D}^*_N f(\bar{x})(w^*)\big) \ne \emptyset$ due to the scalarization formula in (2.12), which means that condition (4.8) is violated.

Conversely, assume that the Slater condition (4.11) holds and then find $x_0 \in \Omega$ with $f(x_0) \in -\text{int}\,\Theta$. Supposing that there is $u^* \in \Xi_0$ with $N(\bar{x};\Omega) \cap \big(-\overline{D}^*_N f(\bar{x})(u^*)\big) \ne \emptyset$, we get from the coderivative scalarization (2.12) that $0 \in \partial\langle u^*, f\rangle(\bar{x}) + N(\bar{x};\Omega)$. This implies that $0 \le \langle u^*, f(x_0)\rangle - \langle u^*, f(\bar{x})\rangle = \langle u^*, f(x_0)\rangle$. Since $-f(x_0) \in \text{int}\Theta$, it follows from the proof of the implication [(iii)$\Longrightarrow$(i)] in Proposition 4.2 that $\langle u^*, -f(x_0)\rangle > 0$, which is a contradiction. Thus we justify the equivalence between the qualification conditions (4.8) and (4.11) in the convex setting under consideration. Finally, the necessary optimality conditions in (4.2) reduce to those in (4.12) in this setting due to the convexity of the set Ω and the scalarization formula (2.12). △

5 Qualified Fuzzy Optimality Conditions for Cone-Constrained Programs with No Constraint Qualifications

In this section we derive necessary optimality conditions of the new type for cone-constrained problems (1.1). These results are essentially different from those obtained in Section 4 in the following two major points:

(i) The results below are given in a *qualified* form (i.e., with nonzero multipliers associated with cost functions), while they are established *without any constrained qualification.*

(ii) The results below are given in a *fuzzy form*, i.e., they involve neighborhoods of the reference optimal solution.

The results of the fuzzy type have been obtained in the literature for nonlinear programs *under* some qualification conditions; see Section 1 and more discussions below.

Let us start with a useful proposition, which gives a fuzzy estimate of limiting normals to inverse images of sets under Lipschitzian mappings.

Proposition 5.1 (fuzzy estimates of normals to inverse images). *Under the standing assumptions of Section* 3 *let* $\bar{x} \in f^{-1}(-\Theta)$ *for* $f\colon X \to Y$*, and let* V^* *be a weak* neighborhood of the origin in* X^**. Then for any limiting normal* $x^* \in N(\bar{x}; f^{-1}(-\Theta))$ *and any positive number* ε*, there exist* $x_\varepsilon \in I\!B_\varepsilon(\bar{x})$ *and* $y^*_\varepsilon \in \Theta_+$ *such that*

$$x^* \in \widehat{\partial}\langle y^*_\varepsilon, f\rangle(x_\varepsilon) + V^* \quad \text{with} \quad |\langle y^*_\varepsilon, f(x_\varepsilon)\rangle| \le \varepsilon. \tag{5.1}$$

Proof. It follows from the convex separation theorem that

$$\delta\big(x; f^{-1}(-\Theta)\big) = \sup_{y^* \in \Theta_+} \langle y^*, f(x)\rangle \quad \text{for all } x \in X,$$

which means that the indicator of inverse images can be represented as the supremum of a family of Lipschitzian functions. Applying Theorem 3.2 to the case of $\Lambda := \Theta_+$ ensures the existence of $y^*_\varepsilon \in \operatorname{co}\Lambda = \Theta_+$ and $x_\varepsilon \in I\!B_\varepsilon(\bar{x})$ such that

$$|\langle y^*_\varepsilon, f(x_\varepsilon)\rangle| = |\langle y^*_\varepsilon, f(x_\varepsilon)\rangle - \delta\big(\bar{x}; f^{-1}(-\Theta)\big)| \le \varepsilon \ \text{ and } \ x^* \in \widehat{\partial}\langle y^*_\varepsilon, f\rangle(x_\varepsilon) + V^*,$$

which justifies (5.1) and completes the proof of the proposition. △

By using Proposition 5.1 and the "weak fuzzy sum rule" from [15, Theorem 2] we derive the main result of this section.

Theorem 5.2 (fuzzy optimality conditions for cone-constrained programs). *Let* $\bar{x}$ *be a local optimal solution to problem* (1.1) *under the standing assumptions made. Then for any weak* neighborhood* V^* *of the origin in* X^* *and any* $\varepsilon > 0$ *there exist* $x_0, x_1, x_\varepsilon \in I\!B_\varepsilon(\bar{x})$ *and* $y^*_\varepsilon \in \Theta_+$ *such that* $|\vartheta(x_0) - \vartheta(\bar{x})| \le \varepsilon$*,* $x_1 \in \Omega$*, and*

$$0 \in \widehat{\partial}\vartheta(x_0) + \widehat{\partial}\langle y^*_\varepsilon, f\rangle(x_\varepsilon) + \widehat{N}(x_1; \Omega) + V^* \quad \textit{with} \quad |\langle y^*_\varepsilon, f(x_\varepsilon)\rangle| \le \varepsilon. \tag{5.2}$$

Proof. Assume without loss of generality that V^* is convex in X^*. Since $\bar{x}$ is an optimal solution to (1.1), we have by the generalized Fermat rule that

$$0 \in \widehat{\partial}\big(\vartheta + \delta(\cdot; \Omega) + \delta(\cdot; f^{-1}(-\Theta))\big)(\bar{x}).$$

Employing there the weak fuzzy sum rule from [15, Theorem 2] gives us $x_0 \in I\!B_\varepsilon(\bar{x})$ with $|\vartheta(x_0) - \vartheta(\bar{x})| \le \varepsilon$, $x_1 \in \Omega \cap I\!B_\varepsilon(\bar{x})$, and $x_2 \in f^{-1}(-\Theta) \cap I\!B_{\frac{\varepsilon}{2}}(\bar{x})$ such that

$$0 \in \widehat{\partial}\vartheta(x_0) + \widehat{N}(x_1; \Omega) + \widehat{N}(x_2; f^{-1}(-\Theta)) + \frac{V^*}{2}.$$

Thus there is $x^* \in \widehat{N}(x_2; f^{-1}(-\Theta)) \subset N(x_2; f^{-1}(-\Theta))$ satisfying

$$0 \in x^* + \widehat{\partial}\vartheta(x_0) + \widehat{N}(x_1; \Omega) + \frac{V^*}{2}.$$

By Proposition 5.1 we find $x_\varepsilon \in I\!B_{\frac{\varepsilon}{2}}(x_2)$ and $y_\varepsilon^* \in \Theta_+$ such that

$$x^* \in \widehat{\partial}\langle y_\varepsilon^*, f\rangle(x_\varepsilon) + \frac{V^*}{2} \quad \text{with} \quad |\langle y_\varepsilon^*, f(x_\varepsilon)\rangle| \le \varepsilon.$$

This yields the inclusions

$$0 \in \widehat{\partial}\vartheta(x_0) + \widehat{N}(x_1; \Omega) + \frac{V^*}{2} + \widehat{\partial}\langle y_\varepsilon^*, f\rangle(x_\varepsilon) + \frac{V^*}{2} \subset \widehat{\partial}\vartheta(x_0) + \widehat{\partial}\langle y_\varepsilon^*, f\rangle(x_\varepsilon) + \widehat{N}(x_1; \Omega) + V^*,$$

which imply in turn the optimality conditions in (5.2) by taking into account the obvious estimates $\|x_\varepsilon - \bar{x}\| \le \|x_\varepsilon - x_2\| + \|x_2 - \bar{x}\| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. $\triangle$

As a consequence of the fuzzy optimality conditions of Theorem 5.2 we derive the following *sequential* KKT necessary optimality conditions for a particular setting of cone-constrained programs (1.1) with *no constraint qualifications*.

Corollary 5.3 (sequential optimality conditions for cone-constrained programs). *Assume in the framework of Theorem* 5.2 *that* $\dim X < \infty$*,* $\Omega = X$*, and the cost function* ϑ *is Lipschitz continuous around* $\bar{x}$*. Then there exist a subgradient* $x^* \in \partial\vartheta(\bar{x})$ *and sequences* $\{x_n\} \subset X$*,* $\{x_n^*\} \subset X^*$*, and* $\{y_n^*\} \subset \Theta_+$ *with* $x_n^* \in \widehat{\partial}\langle y_n^*, f\rangle(x_n)$ *for all* $n \in I\!N$ *such that*

$$x_n \to \bar{x}, \;\; x_n^* \to -x^*, \;\; \textit{and} \;\; \langle y_n^*, f(x_n)\rangle \to 0 \;\; \textit{as} \;\; n \to \infty. \tag{5.3}$$

Proof. Since X is finite-dimensional and $\Omega = X$, we can select $V^* = \frac{1}{n}I\!B^*$, $\varepsilon = \frac{1}{n}$ and then find from (5.2) vectors $u_n, x_n \to \bar{x}$ as well as dual elements $u_n^* \in \widehat{\partial}\vartheta(u_n)$, $y_n^* \in \Theta_+$, and $x_n^* \in \widehat{\partial}\langle y_n^*, f\rangle(x_n)$ such that

$$-u_n^* \in x_n^* + \frac{1}{n}I\!B^* \quad \text{with} \quad |\langle y_n^*, f(x_n)\rangle| \le \frac{1}{n} \;\; \text{as} \;\; n \to \infty. \tag{5.4}$$

It follows from the local Lipschitz continuity of ϑ around $\bar{x}$ that the sequence $\{u_n^*\}$ is bounded, and hence it converges (without loss of generality) to some limiting subgradient $x^* \in \partial\vartheta(\bar{x})$ by definition (2.2). This implies due to the inclusion in (5.4) that $x_n^* \to -x^*$, which justifies (5.3) and thus completes the proof of the corollary. $\triangle$

Observe that the proof of Theorem 5.2 holds true with no change if the local Lipschitz continuity of f therein is replaced by that of the scalarized function $x \mapsto \langle y^*, f(x)\rangle$ for all $y^* \in \Theta_+$. This is always the case when $f\colon X \to Y$ is a continuous Θ-convex mapping. Under such convexity assumptions the sequential necessary optimality conditions from (5.3) are established in [21] for reflexive spaces X.

The final result of this section presents an enhanced version of Theorem 5.2 for problems of nondifferentiable programming with finitely many equality and inequality constraints.

Theorem 5.4 (fuzzy optimality conditions in nondifferentiable programming). *Let the standing assumptions on X, ϑ, and Ω be satisfied, and let $\bar{x}$ be a local optimal solution to the nondifferentiable program*

$$\begin{cases} \text{minimize } \ \vartheta(x) \ \text{ subject to} \\ \varphi_i(x) \le 0, \ \ i = 1, \ldots, m, \\ \varphi_i(x) = 0, \ \ i = m+1, \ldots, m+r, \\ x \in \Omega, \end{cases} \tag{5.5}$$

where the functions $\varphi_i \colon X \to I\!R$ are Lipschitz continuous around $\bar{x}$ under the validity of the standing assumptions on the other data. Then for any weak* neighborhood V^* of the origin in X^* and any $\varepsilon > 0$ there exist vectors $x_0, x_1, \ldots, x_{m+r}, \widehat{x} \in I\!B_\varepsilon(\bar{x})$ and multipliers $(\lambda_1, \ldots, \lambda_{m+r}) \in I\!R^m_+ \times I\!R^r$ such that

$$0 \in \widehat{\partial}\vartheta(x_0) + \sum_{i=1}^{m} \lambda_i \widehat{\partial}\varphi_i(x_i) + \sum_{i=m+1}^{m+r} \widehat{\partial}(\lambda_i\varphi_i)(x_i) + \widehat{N}(\widehat{x};\Omega) + V^* \tag{5.6}$$

with $\widehat{x} \in \Omega$, $\big|\sum_{i=1}^{m+r} \lambda_i\varphi_i(x_i)\big| \le \varepsilon$, and $|\vartheta(x_0) - \vartheta(\bar{x})| \le \varepsilon$.

Proof. Employing Theorem 5.2 in the case of $Y := I\!R^{m+r}$, $f := (\varphi_1, \ldots, \varphi_{m+r})$, and $\Theta := I\!R^m_+ \times 0_r \subset Y$ gives us $x_0, x_\varepsilon, \widehat{x} \in I\!B_{\frac{\varepsilon}{2}}(\bar{x})$, and $(\lambda_1, \ldots, \lambda_{m+r}) \in \Theta_+ = I\!R^m_+ \times I\!R^r$ such that $|\vartheta(x_0) - \vartheta(\bar{x})| \le \varepsilon$, $\widehat{x} \in \Omega$, and

$$0 \in \widehat{\partial}\vartheta(x_0) + \widehat{\partial}\Big(\sum_{i=1}^{m+r} \lambda_i\varphi_i\Big)(x_\varepsilon) + \widehat{N}(\widehat{x};\Omega) + \frac{V^*}{2} \ \text{ with } \ \Big|\sum_{i=1}^{m+r} \lambda_i\varphi_i(x_\varepsilon)\Big| \le \frac{\varepsilon}{2}. \tag{5.7}$$

Thus there is $x^* \in \widehat{\partial}\Big(\sum_{i=1}^{m+r} \lambda_i\varphi_i\Big)(x_\varepsilon)$ satisfying

$$0 \in x^* + \widehat{\partial}\vartheta(x_0) + \widehat{N}(\widehat{x};\Omega) + \frac{V^*}{2}.$$

Then we apply to x^* the weak fuzzy sum rule from [15, Theorem 2] and find $x^*_1, \ldots, x^*_{m+r}$ together with $x_1, \ldots, x_{m+r} \in I\!B_{\frac{\varepsilon}{2}}(x_\varepsilon)$ such that

$$x^*_i \in \widehat{\partial}(\lambda_i\varphi_i)(x_i), \quad |\lambda_i\varphi_i(x_i) - \lambda_i\varphi_i(x_\varepsilon)| \le \frac{\varepsilon}{2(m+r)} \ \text{ for } \ i = 1, \ldots, m+r, \ \text{ and}$$

$$x^* \in \sum_{i=1}^{m+r} x^*_i + \frac{V^*}{2}.$$

It follows from the above that $\|x_i - \bar{x}\| \le \|x_i - x_\varepsilon\| + \|x_\varepsilon - \bar{x}\| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ for all $i = 1, \ldots, m+r$ and that the inclusions

$$\begin{aligned} 0 &\in \widehat{\partial}\vartheta(x_0) + \widehat{N}(\widehat{x};\Omega) + \frac{V^*}{2} + \sum_{i=1}^{m+r} \widehat{\partial}(\lambda_i\varphi_i)(x_i) + \frac{V^*}{2} \\ &\in \widehat{\partial}\vartheta(x_0) + \sum_{i=1}^{m} \lambda_i \widehat{\partial}\varphi_i(x_i) + \sum_{i=m+1}^{m+r} \widehat{\partial}(\lambda_i\varphi_i)(x_i) + \widehat{N}(\widehat{x};\Omega) + V^* \end{aligned} \tag{5.8}$$

hold. Moreover, we get from (5.7) that

$$\Big|\sum_{i=1}^{m+r}\lambda_i\varphi_i(x_i)\Big|\le\sum_{i=1}^{m+r}\Big|\lambda_i\varphi_i(x_i)-\lambda_i\varphi_i(x_\varepsilon)\Big|+\Big|\sum_{i=1}^{m+r}\lambda_i\varphi_i(x_\varepsilon)\Big|\le(m+r)\frac{\varepsilon}{2(m+r)}+\frac{\varepsilon}{2}=\varepsilon.$$

This together (5.8) implies (5.6) and then completes the proof of the theorem. $\triangle$

The study of fuzzy necessary optimality conditions for nondifferentiable programming, including those with non-Lipschitzian data, goes back to [6, Theorem 2.1] in the case of reflexive spaces X. Some extensions of the results in [6] are derived in [30, 31] in the case of Asplund spaces; see also [25, Subsection 5.1.3] and the commentaries therein for more details. However, all the aforementioned results are given in the Fritz John form, which does not guarantee that the coefficient of $\widehat{\partial}\vartheta(x_0)$ is not zero as in (5.6). The qualified/KKT form obtained in Theorem 5.4 follows from the Fritz John one only under some constraint qualifications; see, e.g., [8, Theorems 3.3.7 and 3.3.13]. Observe furthermore that the necessary optimality conditions of Theorem 5.4 obtained for Lipschitzian functional constraints provide the additional information $\big|\sum_{i=1}^{m+r}\lambda_i\varphi_i(x_i)\big|\le\varepsilon$ on Lagrange multipliers, which is new in comparison with known results in this direction even under qualification conditions.

6 Well-Posedness of Cone-Constrained Systems

This section is devoted to some fundamental *well-posedness* properties of the cone-constrained systems in (1.1) parameterized by elements in Y. This means that we study a certain stability of feasible solution sets for cone-constrained programs under parameter perturbations. To specify the issue, form a set-valued mapping $F\colon X\rightrightarrows Y$ by

$$F(x):=f(x)+\Theta=\big\{y\in Y\big|\ f(x)-y\in-\Theta\big\} \tag{6.1}$$

with $f(\bar{x})\in-\Theta$ and focus on deriving verifiable conditions for its metric regularity around the point $(\bar{x},0)$. As mentioned in Section 1, this property is equivalent to other fundamental well-posedness properties of set-valued mappings related to linear openness/covering of F and robust Lipschitzian stability of inverse mappings. Recall that mapping (6.1) with $\operatorname{gph}F=\operatorname{epi}_\Theta f$ is *metrically regular* around $(\bar{x},0)\in\operatorname{gph}F$ if there exist $\mu>0$ and neighborhoods U of $\bar{x}$ and V of 0 such that we have the estimate

$$d\big(x;F^{-1}(y)\big)\le\mu\,d\big(y;F(x)\big)\quad\text{for any}\quad x\in U\ \text{ and }\ y\in V, \tag{6.2}$$

where $d(\cdot;\Omega)$ stands for the usual distance function associated with the set in question. The infimum of all such moduli $\mu>0$ over (μ,U,V) from (6.2) is called the *exact regularity bound* of F around $(\bar{x},0)$ and is denoted by $\operatorname{reg}F(\bar{x},0)$. We refer the reader to [8, 19, 25, 34] for details on metric regularity and related properties and various applications.

The main goal of this section is to derive *sufficient* as well as *necessary and sufficient* conditions for metric regularity of cone-constrained systems (6.1), with *evaluating* the exact regularity bound, for general nonsmooth and nonconvex mappings $f\colon X\to Y$ in (6.1) that take values in *arbitrary Banach* spaces Y. Note that in the Asplund space setting the corresponding results can be derived from those in [25, Sections 4.1 and 4.2] and more elaborated in [17] via the calculus rules therein for regular and limiting coderivative constructions. Furthermore, upper estimates and precise formulas for the exact regularity bound are obtained in [17, 25] only in the case of finite-dimensional spaces Y. Note also that, since general

Banach spaces are not "trustworthy" for the Fréchet type subdifferential/coderivative constructions used, the corresponding results of [19, 22] seem not to be applicable in the setting under consideration. As mentioned in Section 1, a major motivation for our study is to cover, in particular, general nonconvex models of semi-infinite programming, which unavoidably require to consider the non-Asplund and not Fréchet trustworthy Banach spaces $Y = \mathcal{C}(T)$ and $Y = l^\infty(T)$; see Section 7 for more details.

In what follows we keep our standing assumptions on the initial data of (6.1) formulated at the beginning of Section 3 requiring for simplicity that the *domain/decision* space X is *finite-dimensional*, which corresponds to semi-infinite programs considered in Section 7. The proofs below can be readily extended to the case of general *Asplund* decision spaces.

The first theorem below provides an upper estimate with the case of equality therein for the exact regularity bound of F at $(\bar{x}, 0)$ via the regular coderivative (2.5) of f at neighboring points. The obtained estimate and equality clearly imply a sufficient as well as a necessary and sufficient condition for metric regularity, respectively. Note that $\widehat{D}^* f(x)(y^*)$ in (6.3) can be replaced by $\widehat{\partial}\langle y^*, f\rangle(x)$ with $y^* \in \Theta_+$ due to the scalarization formula (2.10).

Theorem 6.1 (neighborhood evaluation of the exact regularity bound for cone-constrained systems). *In addition to the standing assumptions of Section* 3 *let $\bar{x}$ be such that $f(\bar{x}) \in -\Theta$ for the cone-constrained system* (6.1)*, and let the set Ξ be defined in* (1.3)*. Then we have the upper estimate*

$$\operatorname{reg} F(\bar{x}, 0) \le \inf_{\eta>0} \sup \Big\{ \frac{1}{\|x^*\|} \Big| \; x^* \in \widehat{D}^* f(x)(y^*), \; x \in I\!B_\eta(\bar{x}), \; y^* \in \Xi, |\langle y^*, f(\bar{x})\rangle| < \eta \Big\}, \tag{6.3}$$

which holds as equality if $f(\bar{x}) = 0$.

Proof. Denote by $a(\bar{x})$ the right-hand side of (6.3) and consider the nontrivial case in (6.3) when $a(\bar{x}) < \infty$. Arguing by contradiction, suppose that $\operatorname{reg} F(\bar{x}, 0) > a(\bar{x})$ and thus $x^* \neq 0$ in (6.3). Hence there are sequences $(x_n, y_n) \to (\bar{x}, 0)$ and $k < \alpha_n < k+1$ for some number $k > a(\bar{x})$ such that we have

$$d\big(x_n; F^{-1}(y_n)\big) > \alpha_n d\big(y_n; F(x_n)\big) > 0. \tag{6.4}$$

Define $\psi_n(x) := d(y_n; F(x))$ and then $\varepsilon_n := \psi_n(x_n) > 0$. Since the set $F(x) = f(x) + \Theta$ is convex for all $x \in X$, we apply the classical Fenchel-Rockafellar duality theorem to get

$$\begin{aligned} \psi_n(x) \;&= \inf_{y \in Y} \Big\{ \|y - y_n\| + \delta(y; F(x)) \Big\} \\ &= \max_{y^* \in Y^*} \Big\{ -\sup_{y \in Y}(\langle y^*, y\rangle - \|y - y_n\|) - \sup_{v \in Y}(\langle -y^*, v\rangle - \delta(v; f(x) + \Theta)) \Big\} \\ &= \max_{y^* \in Y^*} \Big\{ -\sup_{y \in Y}(\langle y^*, y + y_n\rangle - \|y\|) - \sup_{v \in \Theta}\langle -y^*, f(x) + v\rangle \Big\} \\ &= \max_{y^* \in Y^*} \Big\{ -\langle y^*, y_n\rangle - \delta(y^*; I\!B_{Y^*}) + \langle y^*, f(x)\rangle - \delta(y^*; \Theta_+) \Big\} \\ &= \max_{y^* \in \widetilde{\Xi}} \langle y^*, f(x) - y_n\rangle, \end{aligned} \tag{6.5}$$

where $\widetilde{\Xi} := \Theta_+ \cap I\!B_{Y^*}$. Thus the distance function ψ_n can be represented as the supremum of Lipschitzian functions as in Theorem 3.2. This function is Lipschitz continuous on $I\!B_\rho(\bar{x})$

with rank K, where K and ρ are defined in (3.1). Without loss of generality, suppose that $x_n \in I\!B_\rho(\bar{x})$ for all $n \in I\!N$ and get therefore the estimates

$$\begin{aligned} \varepsilon_n = \psi_n(x_n) &\leq \psi_n(\bar{x}) + K\|x_n - \bar{x}\| = \max_{y^* \in \widetilde{\Xi}} \langle y^*, f(\bar{x}) - y_n\rangle + K\|x_n - \bar{x}\| \\ &\leq \max_{y^* \in \widetilde{\Xi}} \langle y^*, -y_n\rangle + K\|x_n - \bar{x}\| \leq \|y_n\| + K\|x_n - \bar{x}\|, \end{aligned}$$

which imply that $\varepsilon_n \to 0$ as $n \to \infty$. Since ψ_n is nonnegative, we have while recalling the definition of ε_n that

$$\psi_n(x) + \varepsilon_n \geq \psi_n(x_n) \quad \text{for all } x \in I\!B_\rho(\bar{x}).$$

Applying now the Ekeland variational principle gives us $\widehat{x}_n \in I\!B_\rho(\bar{x})$ satisfying

$$\|\widehat{x}_n - x_n\| \leq \alpha_n\varepsilon_n < (k+1)\varepsilon_n \ \text{ and } \ \psi_n(x) + \alpha_n^{-1}\|x - \widehat{x}_n\| \geq \psi(\widehat{x}_n) \text{ on } I\!B_\rho(\bar{x}). \tag{6.6}$$

It follows from (6.4) and (6.6) that $\|\widehat{x}_n - x_n\| < d(x_n; F^{-1}(y_n))$, which yields $\widehat{x}_n \notin F^{-1}(y_n)$, i.e., $y_n \notin F(\widehat{x}_n)$. Thus $\psi_n(\widehat{x}_n) = d(y_n; F(\widehat{x}_n)) > 0$. Moreover, by (6.6) we have

$$0 \in \partial(\psi_n + \alpha_n^{-1}\|\cdot - \widehat{x}_n\|)(\widehat{x}_n) \subset \partial\psi_n(\widehat{x}_n) + \alpha_n^{-1} I\!B_{X^*},$$

hence there is $x_n^* \in \alpha_n^{-1} I\!B_{X^*}$ with $x_n^* \in \partial\psi_n(\widehat{x}_n)$. By the representation of ψ_n in (6.5) and Theorem 3.2 for the setting under consideration ($V^* = \delta_n I\!B^*$), for any $\delta_n \in (0, \psi_n(\widehat{x}_n))$ sufficiently small we find $\widetilde{x}_n \in I\!B_{\delta_n}(\widehat{x}_n)$ and $y_n^* \in \text{co}\,\widetilde{\Xi} = \widetilde{\Xi}$ such that

$$x_n^* \in \widehat{\partial}\langle y_n^*, f\rangle(\widetilde{x}_n) + \delta_n I\!B^* \quad \text{and} \quad |\langle y_n^*, f(\widetilde{x}_n) - y_n\rangle - \psi_n(\widehat{x}_n)| < \delta_n. \tag{6.7}$$

Due to the obvious estimates

$$\|\bar{x} - \widetilde{x}_n\| \leq \|\bar{x} - x_n\| + \|x_n - \widehat{x}_n\| + \|\widehat{x}_n - \widetilde{x}_n\| \leq \|\bar{x} - x_n\| + (k+1)\varepsilon_n + \delta_n, \tag{6.8}$$

it follows from (6.5) and (6.7) that $y_n^* \neq 0$ and that

$$\begin{aligned} \psi_n(\widehat{x}_n) &\leq \langle y_n^*, f(\widetilde{x}_n) - y_n\rangle + \delta_n = \langle y_n^*, f(\widetilde{x}_n) - f(\widehat{x}_n)\rangle + \langle y_n^*, f(\widehat{x}_n) - y_n\rangle + \delta_n \\ &\leq \|y_n^*\| \cdot \|f(\widetilde{x}_n) - f(\widehat{x}_n)\| + \|y_n^*\| \Big\langle \frac{y_n^*}{\|y_n^*\|}, f(\widehat{x}_n) - y_n \Big\rangle + \delta_n \\ &\leq \|y_n^*\| K \|\widetilde{x}_n - \widehat{x}_n\| + \|y_n^*\|\psi_n(\widehat{x}_n) + \delta_n \\ &\leq K\delta_n + \|y_n^*\|\psi_n(\widehat{x}_n) + \delta_n, \end{aligned}$$

which implies in turn that

$$1 \geq \|y_n^*\| \geq 1 - \frac{(K+1)\delta_n}{\psi_n(\widehat{x}_n)}. \tag{6.9}$$

Observe further from (6.7) that

$$\begin{aligned} |\langle y_n^*, f(\bar{x})\rangle| &\leq |\langle y_n^*, f(\bar{x}) - f(\widetilde{x}_n)\rangle| + |\langle y_n^*, f(\widetilde{x}_n) - y_n\rangle - \psi_n(\widehat{x}_n)| + \psi_n(\widehat{x}_n) + |\langle y_n^*, y_n\rangle| \\ &\leq \|y_n^*\| K \|\bar{x} - \widetilde{x}_n\| + \delta_n + \psi_n(x_n) + K\|\widehat{x}_n - x_n\| + \|y_n^*\| \cdot \|y_n\| \\ &\leq K\|\bar{x} - \widetilde{x}_n\| + \delta_n + \varepsilon_n + K(k+1)\varepsilon_n + \|y_n\|. \end{aligned}$$

This ensures together with (6.9) that

$$|\langle \widehat{y}_n^*, f(\bar{x})\rangle| \leq \Big(K\|\bar{x} - \widetilde{x}_n\| + \delta_n + \varepsilon_n + K(k+1)\varepsilon_n + \|y_n\|\Big)\Big(1 - \frac{(K+1)\delta_n}{\psi_n(\widehat{x}_n)}\Big)^{-1}, \tag{6.10}$$

where $\widehat{y}_n^* := \|y_n^*\|^{-1}y_n^* \in \Xi$. Moreover, it follows from (6.7) and the scalarization formula (2.10) that there is $u_n^* \in \widehat{\partial}\langle y_n^*, f\rangle(\widetilde{x}_n) = \widehat{D}^*f(\widetilde{x}_n)(y_n^*)$ satisfying $\|x_n^* - u_n^*\| \le \delta_n$. Combining this with (6.9) and the fact $x_n^* \in \alpha_n^{-1}I\!B_{X^*}$ gives us the relationships

$$\widehat{u}_n^* := \|y_n^*\|^{-1}u_n^* \in \widehat{D}^*f(\widetilde{x}_n)(\widehat{y}_n^*) \quad \text{and} \quad \|\widehat{u}_n^*\| \le \frac{\|x_n^*\| + \delta_n}{\|y_n^*\|} \le (\alpha_n^{-1} + \delta_n)\Big(1 - \frac{(K+1)\delta_n}{\psi_n(\widehat{x}_n)}\Big)^{-1}.$$

Since $\alpha_n > k > a(\bar{x})$, we may choose δ_n sufficiently small so that the right-hand side of the last estimate above is strictly smaller than $k^{-1} < a(\bar{x})^{-1}$ and that $\max\{\|\widetilde{x}_n - \bar{x}\|, |\langle \widehat{y}_n^*, f(\bar{x})\rangle|\} \to 0$ as $n \to \infty$ due to (6.8) and (6.10). Therefore, for $\eta > 0$ small enough we have for all n large enough that $\widetilde{x}_n \in I\!B_\eta(\bar{x})$, $\langle \widehat{y}_n^*, f(\bar{x})\rangle < \eta$ with $\widehat{y}_n^* \in \Xi$, and $\|\widehat{u}_n^*\| < k^{-1} < a(\bar{x})^{-1}$. This contradicts the definition of $a(\bar{x})$ and thus justifies the regularity estimate (6.3).

To complete the proof of the theorem, it remains to show that the equality holds in (6.3) when $f(\bar{x}) = 0$. Indeed, it follows from (6.2) and the definition of $\operatorname{reg} F(\bar{x}, 0)$ that for any $\varepsilon > 0$ there are neighborhoods U of $\bar{x}$ and V of $f(\bar{x}) = 0$ with

$$d\big(x; F^{-1}(y)\big) \le \big(\operatorname{reg} F(\bar{x}, 0) + \varepsilon\big)\|y - f(x)\| \ \text{ for } \ x \in U \ \text{ and } \ y \in V. \tag{6.11}$$

Picking $y^* \in \Xi$ and $x^* \in \widehat{D}^*f(x)(y^*)$ for some x with $x \in U$ and $f(x) \in V$, by (2.5) we find $\delta > 0$ ensuring the inequality

$$\langle x^*, u - x\rangle - \langle y^*, f(u) - f(x)\rangle \le \varepsilon(\|u - x\| + \|f(u) - f(x)\|) \quad \text{for } u \in I\!B_\delta(x). \tag{6.12}$$

It follows from (6.11) that for any $y \in Y$ close to $f(x)$ there is $u \in F^{-1}(y)$ near x such that

$$\|x - u\| \le (\operatorname{reg} F(\bar{x}, 0) + 2\varepsilon)\|y - f(x)\| \ \text{ with } \ y - f(u) \in \Theta.$$

Combining this with (6.12) gives us the estimates

$$\begin{aligned}
\langle -y^*, y - f(x)\rangle &\le \langle -y^*, f(u) - f(x)\rangle \le \varepsilon(\|u - x\| + \|f(u) - f(x)\|) - \langle x^*, u - x\rangle \\
&\le (\varepsilon(1 + K) + \|x^*\|)\|u - x\| \\
&\le (\varepsilon(1 + K) + \|x^*\|)(\operatorname{reg} F(\bar{x}, 0) + 2\varepsilon)\|y - f(x)\|
\end{aligned}$$

for y near $f(x)$. Thus we find $\nu > 0$ with $I\!B_\nu(f(x)) \subset V$ and get from the above that

$$1 = \|y^*\| = \sup_{y \in I\!B_\nu(f(x))\setminus f(x)} \frac{\langle -y^*, y - f(x)\rangle}{\|y - f(x)\|} \le (\varepsilon(1 + K) + \|x^*\|)(\operatorname{reg} F(\bar{x}, 0) + 2\varepsilon),$$

which implies in turn that

$$\|x^*\|^{-1} \le \big[(\operatorname{reg} F(\bar{x}, 0) + 2\varepsilon)^{-1} - \varepsilon(1 + K)\big]^{-1}.$$

Letting finally $\varepsilon \to 0$, we arrive at that $a(\bar{x}) \le \operatorname{reg} F(\bar{x}, 0)$. This justifies the equality in (6.3) and completes the proof of the theorem. △

Note that in the case of smooth functions f in (6.1) the metric regularity (6.2) of such cone-constrained systems was first established in the seminal paper by Robinson [33] under his constraint qualification (4.9). As shown in Corollary 4.4, condition (4.9) is equivalent under the imposed smoothness of f to our qualification condition (4.8), which can be written in the general setting of this section as

$$\big(\ker \breve{D}_N^* f(\bar{x})\big) \cap \Xi_0 = \emptyset \ \text{ with } \ \Xi_0 = \big\{y^* \in \Xi \big|\ \langle y^*, f(\bar{x})\rangle = 0\big\}. \tag{6.13}$$

The next theorem proves the sufficiency of the *pointbased* condition (6.13) for metric regularity of (6.1) under the standing assumptions above, provides a verifiable upper estimate of the exact regularity bound $\operatorname{reg} F(\bar{x},0)$ calculated at $\bar{x}$, and justifies the equality therein when f is strictly differentiable at $\bar{x}$. It seems that the obtained calculations of the exact regularity bound are new even in the case of smooth mappings f in (6.1).

Theorem 6.2 (pointbased conditions for metric regularity of cone-constrained systems). *Let $f(\bar{x}) \in -\Theta$ and $\operatorname{int}\Theta \neq \emptyset$ in the setting of Theorem 6.1. Then the constrained qualification (6.13) is sufficient for the metric regularity of F around $(\bar{x},0)$ with the exact regularity bound of F at $(\bar{x},0)$ estimated by*

$$\operatorname{reg} F(\bar{x},0) \le b(\bar{x}) := \sup\Big\{\frac{1}{\|x^*\|}\;\Big|\; x^* \in \breve{D}^*_N f(\bar{x})(y^*),\ y^* \in \operatorname{cl}^*\Xi,\ \langle y^*, f(\bar{x})\rangle = 0\Big\}, \tag{6.14}$$

where $x^* \neq 0$ due to the qualification condition (6.13). If furthermore Ξ is weak* closed in Y^* and if f is either Θ-convex, or strictly differentiable at $\bar{x}$, then we have the equality $\operatorname{reg} F(\bar{x},0) = b(\bar{x})$ in (6.14), where $b(\bar{x})$ is calculated by

$$b(\bar{x}) = \sup\big\{\|y^*\| \;\big|\; (y^*, -x^*) \in N\big((0,\bar{x}); \operatorname{gph} F^{-1}\big),\ \|x^*\| = 1\big\}, \tag{6.15}$$

which reduces to the formulas

$$b(\bar{x}) = \big\{\|y^*\| \big|\ \langle y^*, y\rangle \le \langle x^*, x-\bar{x}\rangle \ \text{ for all } \ y \in F(x),\ \|x^*\| = 1\big\} \tag{6.16}$$

in the case of Θ-convex mappings f and to

$$b(\bar{x}) = \sup\Big\{\frac{1}{\|\nabla f(\bar{x})^* y^*\|}\;\Big|\; y^* \in \Xi \ \text{ with } \ \langle y^*, f(\bar{x})\rangle = 0\Big\} \tag{6.17}$$

when f is strictly differentiable at $\bar{x}$.

Proof. First we show that the qualification condition (6.13) guarantees that the number $a(\bar{x})$, the right-hand side of (6.3), is finite. Indeed, the contrary means the existence of a sequence $(x_n, x_n^*, y_n^*) \in X \times X^* \times Y^*$ such that

$$x_n \to \bar{x},\ \|x_n^*\| \to 0,\ y_n^* \in \Xi,\ x_n^* \in \widehat{D}^* f(\bar{x})(y_n^*), \ \text{ and } \ \langle y_n^*, f(x_n)\rangle \to 0 \tag{6.18}$$

as $n \to \infty$. By $\|y_n^*\| = 1$ for all $n \in I\!\!N$ we find a *subnet* of $\{y_n^*\}$ weak* converging to some $y^* \in \operatorname{cl}^*\Xi$. Then it follows from (6.18) and the cluster coderivative construction (2.8) that $0 \in \breve{D}^*_N f(\bar{x})(y^*)$ with $\langle y^*, f(\bar{x})\rangle = 0$. Proposition 4.2 ensures that $y^* \neq 0$ and therefore

$$\frac{y^*}{\|y^*\|} \in \big(\ker \breve{D}^*_N f(\bar{x})\big) \cap \Xi_0.$$

This contradicts the qualification condition (6.13) and thus justifies that the number $a(\bar{x})$ is finite. By Theorem 6.1 we have that F is metrically regular around $(\bar{x},0)$.

Since $a(\bar{x})$ is finite, it follows from the regularity bound estimate in (6.3) that there is a sequence $(x_n, x_n^*, y_n^*) \in X \times X^* \times Y^*$ such that

$$x_n \to \bar{x},\ \frac{1}{\|x_n^*\|} \to a(\bar{x}),\ y_n^* \in \Xi,\ x_n^* \in \widehat{D}^* f(\bar{x})(y_n^*), \ \text{ and } \ \langle y_n^*, f(x_n)\rangle \to 0 \tag{6.19}$$

as $n \to \infty$. Again we find a subnet of $\{(x_n^*, y_n^*)\}$ weak* converging to some $(x^*, y^*) \in X^* \times \text{cl}^*\Xi$ and conclude from (6.19) that $x^* \in \breve{D}_N^* f(\bar{x})(y^*)$ and $y^* \in \text{cl}^*\Xi$ with $\langle y^*, f(\bar{x})\rangle = 0$. This gives us $a(\bar{x}) = \|x^*\|^{-1}$ and thus derives the upper estimate (6.14) from that in (6.3).

To justify the equality in (6.14) with the corresponding representations of $b(\bar{x})$, observe that the weak* closedness of Ξ yields the formula

$$\Xi_0 = \big\{y^* \in \text{cl}^*\Xi \big|\ \langle y^*, f(\bar{x})\rangle = 0\big\}.$$

If f is Θ-convex, we easily get from (2.12) that $x^* \in \breve{D}_N^* f(\bar{x})(y^*)$ with $y^* \in \Xi_0$ if and only if $(x^*, -y^*) \in N((\bar{x}, 0); \text{gph}\, F)$ with $y^* \in S_{Y^*}$. Thus

$$\text{reg}\, F(\bar{x}, 0) \le b(\bar{x}) = \sup\Big\{\frac{1}{\|x^*\|}\ \Big|\ (x^*, -y^*) \in N\big((\bar{x}, 0); \text{gph}\, F\big),\ \|y^*\| = 1\Big\}$$

by (6.14). On the other hand, we have from [25, Theorem 1.54] that

$$\text{reg}\, F(\bar{x}, 0) \ge \sup\big\{\|y^*\|\ \big|\ (y^*, -x^*) \in N\big((0, \bar{x}); \text{gph}\, F^{-1}\big),\ \|x^*\| = 1\big\}, \tag{6.20}$$

which implies the equality in (6.14) with $b(\bar{x})$ calculated by (6.15). The explicit formula (6.16) for calculating $b(\bar{x})$ in the case of Θ-convex mappings follows from the classical form of the normal cone in convex analysis.

To complete the proof of the theorem, it remains to justify the equality case for mappings f strictly differentiable at $\bar{x}$. In this case we have from [25, Theorem 1.38] and the coderivative formulas in (2.11) that

$$\begin{aligned}\breve{D}_N^* f(\bar{x})(y^*) &= \big\{\nabla f(\bar{x})^* y^*\big\} = \big\{x^* \in X^* \big|\ (x^*, -y^*) \in \widehat{N}\big((\bar{x}, 0); \text{gph}\, F\big)\big\}\\ &= \big\{x^* \in X^* \big|\ (x^*, -y^*) \in N\big((\bar{x}, 0); \text{gph}\, F\big)\big\} \quad \text{for any } \ y^* \in \Xi_0.\end{aligned}$$

Combining this with (6.14) and the lower estimate of the regularity bound $\text{reg}\, F(\bar{x}, 0)$ in (6.20) gives us the relationships

$$\begin{aligned}\text{reg}\, F(\bar{x}, 0) &\le b(\bar{x}) \le \sup\Big\{\frac{1}{\|x^*\|}\Big|\ (x^*, -y^*) \in \widehat{N}\big((\bar{x}, 0); \text{gph}\, F\big),\ \|y^*\| = 1\Big\}\\ &= \sup\Big\{\frac{1}{\|x^*\|}\Big|\ (x^*, -y^*) \in N\big((\bar{x}, 0); \text{gph}\, F\big),\ \|y^*\| = 1\Big\}\\ &\le \sup\Big\{\|y^*\|\Big|\ (y^*, -x^*) \in \widehat{N}\big((0, \bar{x}); \text{gph}\, F^{-1}\big),\ \|x^*\| = 1\Big\} \le \text{reg}\, F(\bar{x}, 0),\end{aligned}$$

which imply the equality in (6.14) and formula (6.15) for representing $b(\bar{x})$ in this case. The explicit calculation of $b(\bar{x})$ by (6.17) follows from (6.14) with $\breve{D}_N^* f(\bar{x})(y^*) = \{\nabla f(\bar{x})^* y^*\}$ for strictly differentiable mappings, which thus ends the proof of the theorem. △

Note that, in the case of Θ-convex mappings f in (6.1), the equality in (6.14) with the representation of $b(\bar{x})$ by the second formula in (6.15) can be also derived from [20, Theorem 3] and [28, Theorem 3.4] by using somewhat different approaches. Though the condition "$\langle y^*, f(\bar{x})\rangle = 0$" is not in (6.15) and (6.16) as in (6.14), it is implicitly contained in the condition $(y^*, -x^*) \in N\big((0, \bar{x}); \text{gph}\, F^{-1}\big)$. Finally in this section, observe the weak* closedness assumption imposed on $\Xi \subset Y^*$ for ensuring the equality in Theorem 6.2 seems to be restrictive in infinite dimensions, since Ξ is a part of the unit sphere S_{Y^*}, which is never weak* closed in infinite-dimensional Banach spaces by the classical Josefson-Nissenzweig theorem; see, e.g., [13, Chapter 12]. However, we show in Section 7 that the weak* closed assumption on Ξ is satisfied for the space $Y = l_\infty(T)$ with $\Theta = l_+^\infty(T)$ when T is an arbitrary index set as well as for the space $Y = \mathcal{C}(T)$ with $\Theta = \mathcal{C}_+(T)$ when T is a compact set. Both of these spaces appear in applications to the corresponding models of semi-infinite programming considered below.

7 Applications to Semi-Infinite Optimization

The final section of this paper is devoted to applications of the results obtained above to the models of *semi-infinite programming* formulated as

$$\begin{cases} \text{minimize } \ \vartheta(x) \ \text{ subject to} \\ f(x,t) \le 0 \ \text{ with } t \in T, \\ x \in \Omega \subset X \ \text{ with } \ \dim X < \infty, \end{cases} \tag{7.1}$$

where $\vartheta : X \to \overline{I\!R}$ and $f : X \times T \to \overline{I\!R}$ are extended-real-valued functions, and where the index set T is *arbitrary* (possibly infinite and noncompact). Given a local optimal solution $\bar{x}$ to problem (7.1), we suppose that the standing assumptions of Section 3 hold for ϑ and Ω, while the function $f(x,t)$ is *locally Lipschitzian* with respect to x around $\bar{x}$ *uniformly* in $t \in T$, i.e., there exist positive constants K and ρ such that

$$|f(x,t) - f(y,t)| \le K\|x - y\| \quad \text{for all } \ x, y \in I\!B_\rho(\bar{x}) \ \text{ and } \ t \in T. \tag{7.2}$$

Define the ε*-active set*

$$T_\varepsilon(\bar{x}) := \big\{t \in T \big|\ f(\bar{x},t) \ge -\varepsilon\big\} \text{ for all } \ \varepsilon \ge 0$$

and denote $T(\bar{x}) := T_0(\bar{x})$. It follows from the uniform Lipschitz property of f in (7.2) that for any $\varepsilon > 0$ there is $\delta > 0$ sufficiently small such that $f(x,t) < 0$ whenever $x \in I\!B_\delta(\bar{x})$ and $t \notin T_\varepsilon(\bar{x})$. This observation allows us to restrict the inequality constraints in problem (1.2) to the set $T_\varepsilon(\bar{x})$ with keeping all the local properties assumed around $\bar{x}$. Observe further from (7.2) that the function $f(x,\cdot)$ is bounded on $T_\varepsilon(\bar{x})$ for each x around $\bar{x}$. These discussions show that there is no restriction to suppose that $f(x,\cdot)$ for $x \in X$ are elements of $l^\infty(T)$, the space of all bounded real-valued functions on T with the supremum norm:

$$\|p\| := \sup_{t\in T} |p_t| = \sup\big\{|p(t)| \,\big|\, t \in T\big\} \quad \text{for } \ p \in l^\infty(T).$$

Using the function $f(x,t)$ of two variables in (7.1), define the mapping $f\colon X \to l^\infty(T)$ by $f(x)(\cdot) := f(x,\cdot) \in l^\infty(T)$ for all $x \in X$. It follows from (7.2) that this mapping is locally Lipschitzian around $\bar{x}$ as in (1.3). Further, it is easy to see that f is $l^\infty_+(T)$-convex if and only if all the functions $f(\cdot,t)$ as $t \in T$ are convex with respect to the variable x. Moreover, the mapping $f\colon X \to l^\infty(T)$ is strictly differentiable at $\bar{x}$ only if the functions $f(\cdot,t)\colon X \to I\!R$ are *uniformly strictly differentiable* at $\bar{x}$ for all $t \in T$ in the sense of [26], i.e., they are Fréchet differentiable at $\bar{x}$ with

$$\sup_{t\in T} \sup_{\substack{x,u\in I\!B_\eta(\bar{x})\\ x\ne u}} \frac{|f(x,t) - f(u,t) - \langle \nabla_x f(\bar{x},t), x - u\rangle|}{\|x - u\|} \longrightarrow 0 \ \text{ as } \ \eta \downarrow 0.$$

When the index set T is a compact Hausdorff space and the functions $p(\cdot) \in l^\infty(T)$ are restricted to be continuous on T, $l^\infty(T)$ reduces to the space of continuous functions $\mathcal{C}(T)$ with the maximum norm. As discussed in Section 1, both spaces $l^\infty(T)$ and $\mathcal{C}(T)$ are Banach but not Asplund. Furthermore, it is well known that $l^\infty(T)$ is never separable unless T is finite, while the space $\mathcal{C}(T)$ is separable provided that T is a compact metric space.

Following [14], recall next some facts about the dual spaces to $l^\infty(T)$ and $\mathcal{C}(T)$ needed below. The dual space $l^\infty(T)^*$ is isomorphic to the space $ba(T)$ of *bounded and additive measures* μ on T satisfying the relationship

$$\langle \mu, p\rangle = \int_T p(t)\mu(dt) \quad \text{for any} \quad \mu \in ba(T) \ \text{ and } \ p \in l^\infty(T)$$

with the *dual norm* on $ba(T)$ defined as the *total variation* of μ on the index set T by

$$\|\mu\| := \sup_{A\subset T} \mu(A) - \inf_{B\subset T} \mu(B).$$

In what follows we always identify the measure space $ba(T)$ with the dual space $l^\infty(T)^*$. Denote by $ba_+(T)$ the set of *positive* (nonnegative) bounded and additive measures on T, i.e., $ba_+(T) := \{\mu \in ba(T)|\ \mu(A) \geq 0,\ A \subset T\}$. It is easy to check that

$$ba_+(T) = \Big\{\mu \in ba(T)\Big|\ \int_T p(t)\mu(dt) \geq 0,\ p \in l^\infty_+(T)\Big\},$$

where $l^\infty_+(T) := \{p \in l^\infty(T)\big|\ p_t \geq 0,\ t \in T\}$ is the *positive cone* in $l^\infty(T)$.

When T is a compact topological space, denote by $\mathcal{B}(T)$ the σ-algebra of all Borel sets on T. As well known, the dual space to $\mathcal{C}(T)$ is the space $rca(T)$ of all *regular* finite real-valued Borel measures on T equipped with the total variation norm $\|\mu\|$. We define the nonnegative regular Borel measures by

$$rca_+(T) := \big\{\mu \in rca(T)\big|\ \mu(A) \geq 0 \ \text{ for all } \ A \in \mathcal{B}(T)\big\},$$

which is equivalent to the representation

$$rca_+(T) = \Big\{\mu \in rca(T)\Big|\ \int_T p(t)\mu(dt) \geq 0 \ \text{ for all } \ p \in \mathcal{C}_+(T)\Big\},$$

where $\mathcal{C}_+(T)$ is the set of all nonnegative continuous functions on T. Recall that a Borel measure μ is said to be *supported* on $A \in \mathcal{B}(T)$ if $\mu(B) = 0$ for all $B \in \mathcal{B}(T)$ with $B \cap A = \emptyset$ and then observe the following simple while useful proposition.

Proposition 7.1 (supported measures). *Let T be a compact Hausdorff space, and let $p \in \mathcal{C}_+(T)$. If the measure $\mu \in rca_+(T)$ satisfies the relationship $\int_T p(t)\mu(dt) = 0$, then it is supported on the set $\{t \in T|\ p(t) = 0\}$.*

Proof. Define $A := \{t \in T|\ p(t) = 0\}$ and pick any $B \in \mathcal{B}(T)$ such that $B \cap A = \emptyset$. Since μ is a regular measure, we have

$$\mu(B) = \sup\big\{\mu(C)\big|\ C \subset B,\ C \text{ compact}\big\}.$$

To justify that $\mu(B) = 0$, we only need to prove that $\mu(C) = 0$ for all compact sets C contained in B. To proceed, define $\delta := \max\{p(t)|\ t \in C\} \geq 0$ and observe that $\delta > 0$ since $C \cap A = \emptyset$. It follows that

$$0 = \int_T p(t)\mu(dt) = \int_{T\setminus C} p(t)\mu(dt) + \int_C p(t)\mu(dt) \geq \int_C p(t)\mu(dt) \geq \delta\mu(C) \geq 0,$$

which implies that $\mu(C) = 0$ and thus completes the proof of the proposition. $\triangle$

As discussed above, the SIP problem (7.1) can be formulated as a cone-constrained program (1.1) with $Y = l^\infty(T)$ and $\Theta = l^\infty_+(T)$. Applying Theorem 4.1 to this setting gives us the following necessary optimality conditions for nonsmooth SIP problems.

Theorem 7.2 (necessary optimality conditions for nonsmooth semi-infinite programs with arbitrary index sets). *Let $\bar{x}$ be a local optimal solution to the SIP problem* (7.1) *under the standing assumptions of this section. For the constraint function $f(x,t)$ in* (7.1) *define the measure set*

$$ba_+(T)(f) := \Big\{\mu \in ba_+(T)\big|\ \mu(T) = 1,\ \int_T f(\bar{x},t)\mu(dt) = 0\Big\}$$

and assume that the qualification conditions (4.1) *and*

$$(\partial^\infty \vartheta(\bar{x}) + N(\bar{x};\Omega)) \cap \big(-\breve{D}^*_N f(\bar{x})(ba_+(T)(f))\big) = \emptyset \tag{7.3}$$

are satisfied. Then there is a measure $\mu \in ba_+(T)$ such that

$$0 \in \partial\vartheta(\bar{x}) + \breve{D}^*_N f(\bar{x})(\mu) + N(\bar{x};\Omega) \quad \textit{with} \quad \int_T f(\bar{x},t)\mu(dt) = 0. \tag{7.4}$$

Proof. To derive this result from Theorem 4.1, recall the remarkable fact from the geometry of Banach spaces that $\text{int}\, l^\infty_+(T) \neq \emptyset$. It follows from the above discussions that in the notation of Corollary 4.3 specified to problem (7.1) we get

$$\text{int}\,\Theta \neq \emptyset \ \text{ and } \ \Theta_+ = ba_+(T).$$

Furthermore, let us check that $\Xi_0 = ba_+(T)(f)$. Indeed, it readily follows from

$$\mu(T) \geq \|\mu\| \geq \langle \mu, e\rangle = \int_T \mu(dt) = \mu(T) \quad \text{for all } \ \mu \in ba_+(T),$$

where e is the unit function in $l^\infty(T)$, i.e., $e(t) = 1$ for all $t \in T$. Hence the qualification condition (4.8) of Corollary 4.3 reduces to (7.3) for the SIP problem (7.1). Then following the proof of Corollary 4.3 in the setting under consideration, we arrive at the necessary optimality condition (7.4) and thus complete the proof of the theorem. $\triangle$

Note that the limiting coderivative form (7.4) of the qualified necessary optimality conditions in Theorem 7.2 is different from the subdifferential form obtained in our recent paper [27] for SIP and infinite programming problems. But now we are able to cover a general class of uniformly Lipschitz functions $f(x,t)$ in contrast to its "equicontinuously subdifferentiable" subclass considered in [27].

When T is a *compact* metric space, the underlying space $Y = \mathcal{C}(T)$ is separable, and thus the unit ball of the dual space $\mathcal{C}^*(T) = rca(T)$ is *sequentially* weak* compact. This allows us to use the (sequential) normal coderivative (2.6) to derive the corresponding necessary optimality conditions for the SIP problem (7.1).

Corollary 7.3 (necessary optimality conditions for nonsmooth semi-infinite programs with compact index sets). *In the setting of Theorem* 7.2*, suppose that the index set T is a compact metric space and that the function $t \mapsto f(x,t)$ is continuous on T for each $x \in X$. Assume further that the qualification conditions* (4.1) *and*

$$\Big(\partial^\infty \vartheta(\bar{x}) + N(\bar{x};\Omega)\Big) \cap \Big(-D^*_N f(\bar{x})\big(rca_+(T)(f)\big)\Big) = \emptyset \tag{7.5}$$

are satisfied, where $rca_+(T)(f) := \{\mu \in rca_+(T)|\ \mu(T) = 1,\ \mu$ is supported on $T(\bar{x})\}$. Then there is a measure $\mu \in rca_+(T)$ supported on $T(\bar{x})$ such that

$$0 \in \partial\vartheta(\bar{x}) + D^*_N f(\bar{x})(\mu) + N(\bar{x};\Omega). \tag{7.6}$$

Proof. Since the unit ball of $\mathcal{C}^*(T)$ is sequentially weak* compact, combining the last part in Corollary 4.3 with Proposition 7.1 gives us the existence of the measure μ in (7.6) under the assumptions imposed and thus completes the proof of the corollary. $\triangle$

Note that the SIP model (7.1) with a compact index set T has been studied in [40] from the viewpoint of necessary optimality conditions, without addressing the noncompactness of T therein as in Theorem 7.2 above. Our results of Corollary 7.3, obtained in the same compact setting by using an approach completely different from [40], significantly improve those in [40] from both viewpoints of deriving *stronger* necessary optimality conditions under *weaker* constraint qualifications. The principal difference between the results of Corollary 7.3 and the corresponding ones in [40] is that the latter employ Clarke's generalized differential constructions that are usually essentially larger than our nonconvex limiting constructions of Section 2, being in fact their *convexifications*; see, e.g., [25, Section 3.2.3] for precise results and comparison. Observe, in particular, that our normal coderivative appeared in the qualification (7.5) and optimality conditions (7.6) of Corollary 7.3 is always smaller (significantly smaller as a rule) than the so-called "Clarke epi-coderivative" $D^*_C f(\bar{x})(\mu)$ of f at $\bar{x}$ defined in [40] to describe the corresponding qualification and optimality conditions therein. Let us present just a simple example to illustrate the situation.

Example 7.4 (illustration of qualification and optimality conditions for SIP over compact index sets). Consider the following one-dimensional SIP (with $x \in I\!\!R$):

$$\text{minimize } \vartheta(x) := x^2 \quad \text{subject to } \ f(x,t) := -|x| - t \le 0, \ t \in T := [0,1] \subset I\!\!R.$$

It is obvious that $\bar{x} = 0$ is the only optimal solution to this problem and that $T(\bar{x}) = \{0\}$. The Clarke epi-coderivative [40] of f at $\bar{x}$ is easily calculated by

$$D^*_C f(\bar{x})(\mu) = \big[-\mu(T), \mu(T)\big] \quad \text{for all } \ \mu \in rca_+(T). \tag{7.7}$$

Furthermore, we can directly calculate the regular normal cone of (2.4) in this setting by

$$\widehat{N}\big((x, f(x)); \mathrm{epi}_\Theta f\big) = \left\{ \begin{array}{ll} \big\{(r, -\mu) \in I\!\!R \times rca_+(T) \big| \ r = -\mu(T)\big\} & \text{if } \ x > 0 \\ \big\{(r, -\mu) \in I\!\!R \times rca_+(T) \big| \ r = \mu(T)\big\} & \text{if } \ x < 0, \end{array} \right.$$

which implies that $D^*_N f(\bar{x})(\mu) = \{-\mu(T), \mu(T)\}$ for all $\mu \in rca_+(T)$. Thus the qualification condition (7.5) of Corollary 7.3

$$\Big(\partial^\infty \vartheta(\bar{x})\Big) \cap \Big(-D^*_N f(\bar{x})\big(rca_+(T)(f)\big)\Big) = \{0\} \cap \{-1, 1\} = \emptyset$$

holds and allows us to confirm the optimality of $\bar{x} = 0$ by the necessary optimality condition (7.6) of this corollary. On the other hand, the corresponding "generalized constraint qualification" of [40, Theorem 3.1$'$] reduces to

$$\Big(\mathrm{co}\, \partial^\infty \vartheta(\bar{x})\Big) \cap \Big(-D^*_C f(\bar{x})\big(rca_+(T)(f)\big)\Big) = \{0\} \cap [-1, 1] = \{0\} \ne \emptyset,$$

i.e., it fails, and the optimality conditions of [40] are not applicable in this example.

The concluding result of this section presents applications of the metric regularity conditions for cone-constrained systems obtained in Theorem 6.2 to the case of infinite inequality constraints from (7.1) under parameter perturbations.

Theorem 7.5 (pointbased characterizations of metric regularity of infinite inequality systems). *Let in the setting of Theorem* 6.2 *we have the SIP inequality system* $F: X \rightrightarrows l^\infty(T)$ *given by*

$$F(x) := \big\{p \in l^\infty(T)\big|\ f(x,t) \le p(t),\ t \in T\big\} \quad \textit{for all} \quad x \in X, \tag{7.8}$$

where T is an arbitrary index set. Pick $\bar{x} \in \ker F$ such that the qualification condition

$$\big(\ker \breve{D}^*_N f(\bar{x})\big) \cap \big(ba_+(T)(f)\big) = \emptyset \tag{7.9}$$

is satisfied. Then F is metrically regular around $(\bar{x}, 0)$ and its exact regularity bound at $(\bar{x}, 0)$ is upper estimated by

$$\mathrm{reg}\, F(\bar{x}, 0) \le \sup\Big\{\frac{1}{\|x^*\|}\Big|\ x^* \in \breve{D}^*_N f(\bar{x})(\mu),\ \mu \in ba_+(T)(f)\Big\}. \tag{7.10}$$

Moreover, if for all $t \in T$ the functions $x \mapsto f(x,t)$ are either convex or uniformly strictly differentiable at $\bar{x}$, then the equality holds in (7.10) *and we have*

$$\mathrm{reg}\, F(\bar{x}, 0) = \sup\big\{\|\mu\|\ \big|\ (\mu, -x^*) \in N\big((0,\bar{x}); \mathrm{gph}\, F^{-1}\big),\ \|x^*\| = 1\big\}, \tag{7.11}$$

where the exact regularity bound in (7.11) *is further specified similarly to* (6.16) *and* (6.17) *in convex and smooth cases, respectively.*

Proof. Recall that $\mathrm{int}\, l^+_\infty(T) \ne \emptyset$. By Theorem 6.2 and the discussions above it is sufficient to check that the set $\Xi = \{\mu \in ba_+(T)|\ \|\mu\| = 1\}$ is weak* closed. To proceed, take any net $\{\mu_\nu\}_{\nu \in \mathcal{N}} \subset \Xi$ weak* converging to μ and show that $\mu \in \Xi$. Indeed, it follows that

$$1 = \lim_\nu \|\mu_\nu\| = \lim_\nu \mu_\nu(T) = \lim_\nu \langle \mu_\nu, e\rangle = \langle \mu, e\rangle = \mu(T) = \|\mu\|,$$

where e is the unit function in $l^\infty(T)$. This readily implies that Ξ is weak* closed in $ba(T)$ and thus completes the proof of the theorem. △

Note that metric regularity of the mapping F in (7.8) is equivalent to *robust Lipschitzian stability* (formalized via the Lipschitz-like or Aubin property) of the inverse mapping F^{-1} with respect to parameter perturbations of $p \in l^\infty(T)$. Such Lipschitzian stability has been intensively studied in recent publications in the case of *linear* and *convex* inequality systems in semi-infinite and infinite programming with *arbitrary* index sets; see, e.g., [11, 12, 28] and the references therein. The equality in (7.11) for convex systems can be derived from the results of these papers. However, the exact bound estimate (7.10) under the qualification condition (7.9) in the general uniformly Lipschitzian case for f and the equality therein for uniformly strictly differentiable functions seem to be new in the SIP literature.

If the index set T is *compact* in (7.8), arguing as in the proof of Corollary 7.3 leads us to an appropriate counterpart of Theorem 7.5 with replacing the coderivative $\breve{D}^*_N$ by D^*_N and the set $ba_+(T)(f)$ by $rca_+(T)(f)$ in conditions (7.9) and (7.10). In this way we extend the corresponding result of [10] obtained for linear semi-infinite systems.

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