A Refined Gomory-Chvátal Closure for Polytopes in the Unit Cube*

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March 22, 2012

Abstract

We introduce a natural strengthening of Gomory-Chvátal cutting planes for the important class of 0/1-integer programming problems and study the properties of the elementary closure that arises from the new class of cuts. Most notably, we prove that the new closure is polyhedral, we characterize the family of all facet-defining inequalities, and we compare it to elementary closures associated with other cutting-plane procedures.

1. Introduction Cutting planes are an integral part of integer programming and, over the years, numerous types of cutting planes for integer and mixed-integer programs have been introduced and studied in the literature (see, e.g., Cornuéjols (2008) for a survey). One prominent class of cutting planes are the Gomory-Chvátal cuts (Gomory 1958; Chvátal 1973). There is an elegant theory related to Gomory-Chvátal cutting planes, and these cuts are a valuable component of most codes for solving integer programming problems. Gomory-Chvátal cuts are defined for general integer programs, but they are often used in the context of 0/1-integer programs, without taking full advantage of the additional structure gained by replacing the integer lattice with the extreme points of the unit cube. In this paper, we initiate a study of the class of cutting planes that do make use of this additional information.

A Gomory-Chvátal cutting plane for a rational polyhedron P is an inequality of the form $ax \leq \lfloor a_0 \rfloor$, where a is an integral vector and $ax \leq a_0$ is valid for P. Obviously, $ax \leq \lfloor a_0 \rfloor$ is valid for P_I, the convex hull of integer points (aka the integer hull) in P. If the components of a are relatively prime, it is not difficult to see that $ax \leq \lfloor a_0 \rfloor$ describes the integer hull H_I of the half-space $H = \{x \in \mathbb{R}^n \,|\, ax \leq a_0\}$. Geometrically, H_I arises from H by moving the hyperplane defining H inwards until it hits an integer point. Clearly, for every half-space H with the property that $P \subseteq H$, it holds that $P_I \subseteq H_I$. Because of that, the intersection of all half-spaces defined by the Gomory-Chvátal cuts of the polyhedron P contains its integer hull P_I. The resulting set is called the Gomory-Chvátal closure of the polyhedron P, and it is typically denoted by P'. Schrijver (1980) proved that P' is, again, a polyhedron, and Chvátal (1973), for polytopes, and Schrijver (1980), for polyhedra, showed that one obtains the integer hull P_I after a finite number of closure operations. Thus, the Gomory-Chvátal procedure constitutes a generic method to generate the integer hull of rational polyhedra, without requiring any knowledge of the structure of the underlying problem.

*The results of this paper were first presented at the 2010 SIAM Annual Meeting in Pittsburgh and are also part of the first author's PhD. thesis (Dunkel 2011).

Suppose that we apply the Gomory-Chvátal procedure to a polytope P for which all variables have lower and upper bounds; in particular, every integer point in P is known to be contained in some cube $[L, U]^n$, where $L \leq U$. Let $ax \leq a_0$ be a valid inequality for P. The original Gomory-Chvátal procedure guarantees that every integer point in $\mathbb{R}^n$ that satisfies the original inequality $ax \leq a_0$ also satisfies the corresponding Gomory-Chvátal cut. Yet, for the purpose of the problem considered here, it would suffice to require that every integer point in $[L, U]^n \cap \{x \mid ax \leq a_0\}$ satisfies the new inequality. This alone would guarantee that no integer point in P is cut off. From a geometric point of view, one can shift the boundary of the half-space $\{x \in \mathbb{R}^n \mid ax \leq a_0\}$ until it hits an integer point of the cube $[L, U]^n$ and ignore any integer points outside this cube. As a result, one obtains a valid inequality for the integer hull P_I that potentially dominates the corresponding Gomory-Chvátal cut.

In this paper, we formalize this idea for the important class of 0/1-integer programming problems. Many combinatorial optimization problems can be modeled as integer programs with decision variables that can take the values 0 or 1 only. The standard relaxations of these problems are optimization problems over polytopes that are contained in the unit cube $[0, 1]^n$. We introduce a refined family of Gomory-Chvátal cutting planes for this special case. More precisely, we define the M-cut associated with a valid inequality $ax \leq a_0$ for a polytope $P \subseteq [0, 1]^n$ to be the inequality $ax \leq a_0'$, which results from decreasing the right-hand side a_0 until the hyperplane $\{x \mid ax = a_0'\}$ contains a 0/1-point. Hence, the M-cut associated with a valid inequality generally dominates the corresponding Gomory-Chvátal cut. Imitating the definition of the Gomory-Chvátal closure for a polytope P, we define the M-closure of P as the intersection of all half-spaces implied by the M-cuts of P. We denote this closure by $M(P)$. As it is the case for the Gomory-Chvátal closure, the set of M-cuts of a given polytope is infinite. For that reason, it is not obvious whether the M-closure of a polytope is again a polytope. However, we show that for any rational polytope $P \subseteq [0, 1]^n$ a finite number of M-cuts suffice to describe $M(P)$. In addition, we study various other structural properties of the M-closure. For example, we present an interesting characteristic regarding facet-defining inequalities if $M(P)$ is full-dimensional. While it follows directly from the definition of the M-closure that every undominated inequality for $M(P)$ is tight at a 0/1 point, we show that any facet-defining inequality $ax \leq a_0$ for $M(P)$ corresponds to a hyperplane $\{x \in \mathbb{R}^n \mid ax = a_0\}$ that is spanned by 0/1 points. In other words, such an M-cut is tight at n affinely independent points in $\{0, 1\}^n$. This property can be seen as analogous to the fact that every undominated Gomory-Chvátal cut is associated with a hyperplane that is spanned by n affinely independent integer points. Furthermore, the M-closure shares a property with P' that is important in inductive proofs, even though the right-hand sides of M-cuts are derived in a seemingly less structured way, compared to the consistent rounding in the Gomory-Chvátal procedure. Namely, we show that, for any face F of P, it holds that $M(F) = M(P) \cap F$.

Cornuéjols and Li (2000) gave a detailed overview and comparison of the elementary closures associated with known families of cutting planes for 0/1 integer programs. In particular, they established all inclusion relationships between these closures. We study the relationship between the M-closure of a polytope and these well-known other closures, complementing the picture drawn by Cornuéjols and Li. We also investigate the sequence of successively tighter approximations of the integer hull of a polytope arising from the repeated application of all M-cuts. Clearly, as the M-closure of a polytope P in the unit cube $[0, 1]^n$ is contained in P', any such sequence will result in P_I after polynomially many steps (Bockmayr et al. 1999; Eisenbrand and Schulz 2003). We show that if the integer hull of a polytope is empty, then the M-procedure requires, in the worst case, as many iterations as the Gomory-Chvátal procedure to generate an empty polytope. Finally, we study complexity questions related to the M-closure. We show that, in fixed dimension, the M-closure can be computed in polynomial time. In general, however, the membership problem is

NP-hard.

The paper is structured as follows: In Section 2, we formally introduce M-cuts and the M-closure of a polytope. In Section 2.1, we exhibit various structural properties of the M-closure. A detailed comparison of the M-closure with other known elementary closures from the literature follows in Section 3. In Section 4, we study the cutting-plane procedure associated with M-cuts. More precisely, we analyze how quickly an iterative application of all M-cuts will generate the integer hull of a given polytope. Section 5 is concerned with the computational complexity of the M-closure, and Section 6 contains our concluding remarks.

2. The M-Closure of a Polytope We consider rational polytopes $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ whose integer points are contained in the n-dimensional unit cube $[0,1]^n$. As in this case the inequalities $0 \leq x_i \leq 1$, for $i = 1, \ldots, n$, are known to be valid for the integer hull P_I, we assume, w.l.o.g., that

$$P \subseteq [0,1]^n \ . \tag{1}$$

For any vector $a \in \mathbb{Z}^n$, we define a_P to be the smallest number such that $ax \leq a_P$ is a valid inequality for P, that is,

$$a_P := \max\{ax \mid x \in P\} \ .$$

Consider an arbitrary inequality $ax \leq a_0$ such that the polytope P is contained in the half-space $H = \{x \in \mathbb{R}^n \mid ax \leq a_0\}$.[1] If $a \in \mathbb{Z}^n$ and $\gcd(a_1, \ldots, a_n) = 1$, then $(ax = a_0) \cap \mathbb{Z}^n \neq \emptyset$ if and only if $a_0 \in \mathbb{Z}$. In particular, $(ax \leq \lfloor a_0 \rfloor)$ is the integer hull H_I of the half-space H. Clearly, as P_I is contained in H_I, the inequality $ax \leq \lfloor a_0 \rfloor$ is a cutting plane for P. In fact, it is the Gomory-Chvátal cut associated with the valid inquality $ax \leq a_0$. Now suppose that $(ax = \lfloor a_0 \rfloor) \cap \{0,1\}^n = \emptyset$. Then assumption (1) implies that $P_I \subseteq (ax \leq \lfloor a_0 \rfloor - 1)$. In other words, we can further decrease the right-hand side $\lfloor a_0 \rfloor$ and still guarantee that the resulting inequality does not cut off any integer points in P. Let us define the *knapsack value* for any pair $(a, a_0) \in \mathbb{Q}^n \times \mathbb{R}$ as

$$KV(a, a_0) := \max\left\{ax \mid x \in \{0,1\}^n, ax \leq a_0\right\} \ ,$$

where we set $KV(a, a_0) := -\infty$ if $(ax \leq a_0) \cap \{0,1\}^n = \emptyset$. Then it holds that

$$P_I \subseteq \big(ax \leq KV(a, a_0)\big) \ ,$$

and the inequality $ax \leq KV(a, a_0)$ is a cutting plane for P. Geometrically speaking, this cut is obtained by moving the hyperplane $(ax = a_0)$ towards the integer hull until it hits some 0/1-point (see Figure 1 for an illustration). The set of 0/1-points satisfying the inequality $ax \leq a_0$ is the same as the set of 0/1-points satisfying the cut $ax \leq KV(a, a_0)$.

For any $a \in \mathbb{Q}^n$, the inequality $ax \leq KV(a, a_P)$ is called the M-cut associated with the valid inequality $ax \leq a_P$. The simultaneous application of all M-cuts to a polytope gives rise to the M-closure $M(P)$ of a polytope $P \subseteq [0,1]^n$, which we define as

$$M(P) := \bigcap_{a \in \mathbb{Z}^n} \Big(ax \leq KV(a, a_P)\Big) \ . \tag{2}$$

[1]From now on, we denote a half-space $\{x \in \mathbb{R}^n \mid ax \leq a_0\}$ by $(ax \leq a_0)$ and a hyperplane $\{x \in \mathbb{R}^n \mid ax = a_0\}$ by $(ax = a_0)$.

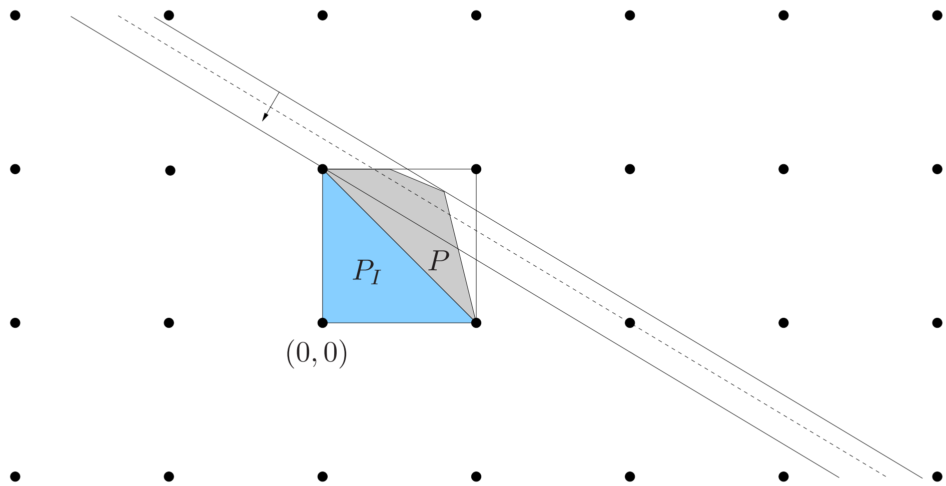

Figure 1: The inequality $3x_1 + 5x_2 \leq 7 - \varepsilon$ is valid for $P \subseteq [0,1]^n$, for some $\varepsilon > 0$. The hyperplane $3x_1 + 5x_2 = 6$ does not contain a 0/1-point, but the hyperplane $3x_1 + 5x_2 = 5$ does. The inequality $3x_1 + 5x_2 \leq 5$ is valid for P_I and dominates the corresponding Gomory-Chvátal cut $3x_1 + 5x_2 \leq 6$.

In analogy to the Gomory-Chvátal procedure, we define the M-procedure as the iterative application of the M-closure operation to a polytope. More precisely, we set $M^{(0)}(P) := P$ and $M^{(k+1)}(P) := M(M^{(k)}(P))$, for every integer $k \geq 0$. As a result, we obtain a sequence

$$P \supseteq M(P) \supseteq M^{(2)}(P) \supseteq \ldots \supseteq P_I$$

of successively tighter approximations of P_I. We call the smallest number $t \in \mathbb{N}$ such that $M^{(t)}(P) = P_I$ the *M-rank* of P.

2.1 Structural Properties In the following lemma, we compile some basic properties that the M-closure shares with several other elementary closures (Pokutta and Schulz 2010). The proof is straightforward.

LEMMA 2.1 *Let P and Q be polytopes in $[0,1]^n$. Then the following properties hold: The M-closure*

(1) *approximates the integer hull: $P_I \subseteq M(P) \subseteq P$.*

(2) *preserves inclusion: If $P \subseteq Q$, then $M(P) \subseteq M(Q)$.*

(3) *rounds coordinates: If $x_i \leq \varepsilon$ (or $x_i \geq \varepsilon$) is valid for P for some $0 < \varepsilon < 1$, then $x_i \leq 0$ (resp. $x_i \geq 1$) is valid for $M(P)$.*

(4) *commutes with coordinate flips: Let $\tau_i : [0,1]^n \to [0,1]^n$ with $x_i \mapsto (1 - x_i)$ be a coordinate flip. Then $\tau_i(M(P)) = M(\tau_i(P))$.*

The next property, which is useful in inductive proofs and also holds for many other elementary closures, is harder to prove.

LEMMA 2.2 *Let $P \subseteq [0,1]^n$ be a polytope and let F be a face of P. Then*

$$M(F) = M(P) \cap F \ .$$

Proof. Suppose that $P = \{x \in \mathbb{R}^n \mid Ax \leq b\} \neq \emptyset$, where $A \in \mathbb{Z}^{m\times n}$ and $b \in \mathbb{Z}^m$. Furthermore, assume that $F = P \cap (ax = a_P) \neq \emptyset$, where $a \in \mathbb{Z}^n$, $\gcd(a) = 1$, and $P \subseteq (ax \leq a_P)$. The inclusion $M(F) \subseteq M(P) \cap F$ is obviously true, since any inequality that is valid for P is also valid for F and because $M(F) \subseteq F$.

In the remainder of the proof, we show that $M(P) \cap F \subseteq M(F)$. If $(ax = a_P) \cap \{0,1\}^n = \emptyset$, then $K(a, a_P) < a_P$ and the M-cut associated with $ax \leq a_P$ strictly dominates $ax \leq a_P$. Consequently, $M(P) \cap F = \emptyset \subseteq M(F)$. Therefore, let us assume in the following that

$$(ax = a_P) \cap \{0,1\}^n \neq \emptyset \ . \tag{3}$$

Take an arbitrary M-cut $cx \leq KV(c, c_F)$ for F that is associated with the valid inequality $cx \leq c_F$ for F. There exist a vector $\lambda \in \mathbb{Q}^m_+$ and a number $\mu \in \mathbb{Q}$ such that $c = \lambda A + \mu a$ and $c_F \geq \lambda b + \mu a_P$. Consider the set

$$S = \left\{x \in \{0,1\}^n \,\middle|\, ax = a_P,\ \lambda Ax \leq \lambda b\right\} \ .$$

We will distinguish two cases. First, assume $S = \emptyset$. Let

$$\begin{aligned} M_1 &:= \min\left\{\lambda Ax \,\middle|\, x \in [0,1]^n,\ ax = a_P\right\} \ , \\ M_2 &:= \min\left\{\lambda Ax \,\middle|\, x \in \{0,1\}^n,\ ax \geq a_P + 1\right\} \ , \\ M_3 &:= \max\left\{\lambda Ax \,\middle|\, x \in \{0,1\}^n,\ ax \leq a_P - 1\right\} \ . \end{aligned}$$

Note that M_1 is finite, since $F \neq \emptyset$. Furthermore, $M_2 \in (-\infty, \infty]$ and $M_3 \in [-\infty, \infty)$. Therefore, we can find a finite number ν that satisfies

$$\nu > \max\{\lambda b - M_2, M_3 - M_1, 0\} \ . \tag{4}$$

The inequality $\bar{c}x \leq \bar{c}_0$ with $\bar{c} := \lambda A + \nu a$ and $\bar{c}_0 := \lambda b + \nu a_P$ is valid for P. If $M_2 = \infty$, then $\{0,1\}^n \cap (ax \geq a_P + 1) = \emptyset$. Otherwise, (4) implies for every $x \in \{0,1\}^n \cap (ax \geq a_P + 1)$ that

$$\bar{c}x = \lambda Ax + \nu ax \geq M_2 + \nu + \nu a_P > \lambda b + \nu a_P = \bar{c}_0 \ .$$

That is, every such point violates the inequality $\bar{c}x \leq \bar{c}_0$. Consequently, and together with the assumption $S = \emptyset$, we obtain

$$\{0,1\}^n \cap (ax \geq a_P) \cap (\bar{c}x \leq \bar{c}_0) = \emptyset \ . \tag{5}$$

Hence, the right-hand side of the M-cut for P associated with $\bar{c}x \leq \bar{c}_0$ satisfies

$$\begin{aligned} KV(\bar{c}, \bar{c}_0) &= \max\left\{\bar{c}x \,\middle|\, x \in \{0,1\}^n,\ \bar{c}x \leq \bar{c}_0\right\} \\ &= \max\left\{(\lambda A + \nu a)\,x \,\middle|\, x \in \{0,1\}^n,\ ax \leq a_P - 1,\ (\lambda A + \nu a)\,x \leq \lambda b + \nu a_P\right\} \ . \end{aligned}$$

Note that

$$\{0,1\}^n \cap (\bar{c}x \leq \bar{c}_0) \neq \emptyset \ ,$$

since, otherwise, the fact that $\bar{c}x \leq \bar{c}_0$ is valid for P would imply $P = \emptyset$. Hence, we have

$$\{0,1\}^n \cap (\bar{c}x \leq \bar{c}_0) = \{0,1\}^n \cap (ax \leq a_P - 1) \cap \big((\lambda A + \nu a)\,x \leq \lambda b + \nu a_P\big) \neq \emptyset \ ,$$

which implies that M_3 is finite. We obtain

$$\begin{aligned} KV(\bar{c}, \bar{c}_0) &\leq \max\left\{(\lambda A + \nu a)\,x \,\middle|\, x \in \{0,1\}^n,\ ax \leq a_P - 1\right\} \\ &\leq \max\left\{\lambda Ax \,|\, x \in \{0,1\}^n,\ ax \leq a_P - 1\right\} + \nu(a_P - 1) \\ &= M_3 + \nu(a_P - 1) \ . \end{aligned}$$

As a result, $(\lambda A+\nu a)\, x \leq M_3+\nu(a_P-1)$ is valid for $M(P)$. For an arbitrary point $x \in [0,1]^n \cap (ax = a_P)$, (4) implies that

$$\bar{c}x = (\lambda A + \nu a)\, x = \lambda A x + \nu a_P \geq (M_1 + \nu) + \nu(a_P - 1) > M_3 + \nu(a_P - 1) \geq KV(\bar{c}, \bar{c}_0) \ ,$$

that is, any point in F violates the M-cut $\bar{c}x \leq KV(\bar{c}, \bar{c}_0)$ for P. This implies that

$$M(P) \cap F = \emptyset \subseteq M(F) \ .$$

Now consider the second case: $S \neq \emptyset$. We define two more constants

$$\begin{aligned} M_4 &:= \min\left\{\lambda A x \,\middle|\, x \in \{0,1\}^n\right\} \ , \\ M_5 &:= \max\left\{\lambda A x \,\middle|\, x \in \{0,1\}^n,\ ax = a_P,\ \lambda A x \leq \lambda b\right\} \ . \end{aligned}$$

Both M_4 and M_5 are finite because of the assumption $S \neq \emptyset$. Now we can choose ν to be finite and such that

$$\nu > \max\{\lambda b - M_4, M_3 - M_5, 0\} \ . \tag{6}$$

The inequality $\bar{c}x \leq \bar{c}_0$ with $\bar{c} := \lambda A + \nu a$ and $\bar{c}_0 := \lambda b + \nu a_P$ is valid for P. With the definition of M_4 and condition (6), we get for any point $x \in \{0,1\}^n \cap (ax \geq a_P + 1)$,

$$\bar{c}x = \lambda A x + \nu a x \geq M_4 + \nu + \nu a_P > \lambda b + \nu a_P = \bar{c}_0 \ .$$

It follows that

$$\{0,1\}^n \cap (ax \geq a_P + 1) \cap (\bar{c}x \leq \bar{c}_0) = \emptyset \ . \tag{7}$$

Also

$$\{0,1\}^n \cap (ax = a_P) \cap (\bar{c}x \leq \bar{c}_0) = S \neq \emptyset \ . \tag{8}$$

If

$$\{0,1\}^n \cap (ax \leq a_P - 1) \cap (\bar{c}x \leq \bar{c}_0) \neq \emptyset \ ,$$

then M_3 is finite and we get with (6),

$$\begin{aligned} & \max\left\{\bar{c}x \,\middle|\, x \in \{0,1\}^n,\ ax \leq a_P - 1,\ \bar{c}\,x \leq \bar{c}_0\right\} \\ = \ & \max\left\{(\lambda A + \nu a)\, x \,\middle|\, x \in \{0,1\}^n,\ ax \leq a_P - 1,\ (\lambda A + \nu a)\, x \leq \lambda b + \nu a_P\right\} \\ \leq \ & \max\left\{(\lambda A + \nu a)\, x \,\middle|\, x \in \{0,1\}^n,\ ax \leq a_P - 1\right\} \\ \leq \ & \max\left\{\lambda A x \,\middle|\, x \in \{0,1\}^n,\ ax \leq a_P - 1\right\} + \nu(a_P - 1) \\ = \ & (M_3 - \nu) + \nu a_P \\ < \ & M_5 + \nu a_P \\ = \ & \max\left\{\lambda A x \,\middle|\, x \in \{0,1\}^n,\ ax = a_P,\ \lambda A x \leq \lambda b\right\} + \nu a_P \\ = \ & \max\left\{(\lambda A + \nu a)\, x \,\middle|\, x \in \{0,1\}^n,\ ax = a_P,\ (\lambda A + \nu a)\, x \leq \lambda b + \nu a_P\right\} \\ = \ & \max\left\{\bar{c}x \,\middle|\, x \in \{0,1\}^n,\ ax = a_P,\ \bar{c}x \leq \bar{c}_0\right\} \ . \end{aligned}$$

Because of (7), (8), and the last observation, the maximum in the computation of $KV(\bar{c}, \bar{c}_0)$ is achieved by a 0/1-point in $(ax = a_P)$. In particular,

$$\begin{aligned} KV(\bar{c}, \bar{c}_0) &= \max\{\bar{c}x \,|\, x \in \{0,1\}^n,\ \bar{c}x \leq \bar{c}_0\} \\ &= \max\{(\lambda A + \nu a)\, x \,|\, x \in \{0,1\}^n,\ ax = a_P,\ (\lambda A + \nu a)\, x \leq \lambda b + \nu a_P\} \\ &= \max\{\lambda A x \,|\, x \in \{0,1\}^n,\ ax = a_P,\ \lambda A x \leq \lambda b\} + \nu a_P \\ &= M_5 + \nu a_P \ . \end{aligned}$$

On the other hand, the right-hand side of the M-cut $cx \leq K(c, c_F)$ for F satisfies

$$\begin{aligned}
KV(c, c_F) &= \max\{cx \,|\, x \in \{0,1\}^n,\ cx \leq c_F\} \\
&\geq \max\left\{(\lambda A + \mu a)\, x \,\middle|\, x \in \{0,1\}^n,\ (\lambda A + \mu a)\, x \leq \lambda b + \mu a_P\right\} \\
&\geq \max\left\{(\lambda A + \mu a)\, x \,\middle|\, x \in \{0,1\}^n,\ ax = a_P,\ (\lambda A + \mu a)\, x \leq \lambda b + \mu a_P\right\} \\
&= \max\left\{\lambda A x \,\middle|\, x \in \{0,1\}^n,\ ax = a_P,\ \lambda A x \leq \lambda b\right\} + \mu a_P \\
&= M_5 + \mu a_P \ .
\end{aligned}$$

As a result, we obtain

$$\begin{aligned}
\big(\bar{c}x \leq KV(\bar{c}, \bar{c}_0)\big) \cap (ax = a_P) &= \big((\lambda A + \nu a)\, x \leq M_5 + \nu a_P\big) \cap (ax = a_P) \\
&= (\lambda A x \leq M_5) \cap (ax = a_P) \\
&= \big((\lambda A + \mu a)\, x \leq M_5 + \mu a_P\big) \cap (ax = a_P) \\
&\subseteq \big(cx \leq KV(c, c_F)\big) \cap (ax = a_P) \ ,
\end{aligned}$$

and the M-cut $\bar{c}x \leq KV(\bar{c}, \bar{c}_0)$ for P dominates the M-cut $cx \leq KV(c, c_F)$ for F. This completes the proof. □

2.2 Polyhedrality Similar to the Gomory-Chvátal closure, the M-closure of a polytope P is defined as the intersection of an infinite number of half-spaces. In the case of the Gomory-Chvátal closure, it is well known that, instead of looking at every valid inequality for P, it suffices to consider only those normal vectors which are associated with the Hilbert bases of the basic feasible cones of P, the number of which is finite. Equivalently, the elementary closure P' can be derived from a totally dual integral system $Ax \leq b$ describing P, such that A is an integral matrix. In that case, P' is obtained by rounding down the right-hand side vector, that is, $P' = \{x \in \mathbb{R}^n \,|\, Ax \leq \lfloor b \rfloor\}$ (Schrijver 1980). Naturally, the question arises as to whether the M-closure of a polytope can also be described by a finite system of inequalities, that is, whether $M(P)$ is a polytope itself.

The way in which the M-procedure decreases the right-hand sides of valid inequalities is much less structured than the rounding operation for Gomory-Chvátal cuts. While for some inequalities the right-hand side is simply rounded down, as is the case for the Gomory-Chvátal procedure, there are M-cuts that strictly dominate the corresponding Gomory-Chvátal cut. Hence, the reasoning behind the proof that the Gomory-Chvátal closure of a polytope is also a polytope cannot be applied for the M-closure. Yet, we show that $M(P)$ is indeed a polytope. The general idea of our proof is to represent the M-cuts of P as feasible points of a finite collection of polyhedra. Since every M-cut induces a partition of the set of 0/1-points into three sets — the set of 0/1-points that satisfy the cut with strict inequality, the set of 0/1-points at which the cut is tight, and the set of 0/1-points that violate the M-cut — it is possible to partition the infinite set of M-cuts into a finite number of subsets that are associated with these partitions. For each of these sets, we will construct a single polyhedron such that every feasible point of that polyhedron corresponds to an M-cut in the set, and vice versa. We then use the decomposition theorem for polyhedra to show that every M-cut is dominated by a finite set of M-cuts that correspond to the basic feasible solutions and extreme rays of the polyhedron that the M-cut is associated with.

First, let us introduce some notation. Let (L, E, G) with $L \,\dot{\cup}\, E \,\dot{\cup}\, G = \{0,1\}^n$ be a partition of the set of 0/1-points in $\mathbb{R}^n$. We will denote by $M_{(L,E,G)}(P)$ the set of all pairs $(a, a_0) \in \mathbb{Q}^{n+1}$ that correspond to M-cuts $ax \le a_0$ for P for which

$$\begin{aligned} ax &< a_0 && \text{for all } x \in L \ , \\ ax &= a_0 && \text{for all } x \in E \ , \\ ax &> a_0 && \text{for all } x \in G \ . \end{aligned}$$

Thus, every M-cut associated with the set $M_{(L,E,G)}(P)$ partitions the set of 0/1-points into the three sets L, E, and G. Every point in L is strictly contained in the half-space ($ax \le a_0$), every point in E is on its boundary ($ax = a_0$), and every point in G violates the inequality. As every half-space has an infinite number of representations, $M_{(L,E,G)}(P)$ contains an infinite number of representatives of the same half-space.

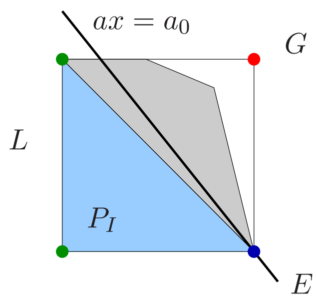

Figure 2: Classification of M-cuts according to the implied partition of 0/1-points into three sets L, E, and G.

Since every M-cut of P is associated with a certain partition of the 0/1-points in $\mathbb{R}^n$, the union of the sets $M_{(L,E,G)}(P)$ over all possible partitions (L, E, G) of $\{0,1\}^n$ completely describes $M(P)$. In other words, $M(P)$ is the intersection of all inequalities associated with the finite number of sets $M_{(L,E,G)}(P)$. We will first show that for every partition (L, E, G), the set $M_{(L,E,G)}(P)$ can be represented by a polyhedron.

Lemma 2.3 *Let $P \subseteq [0,1]^n$ be a non-empty polytope and let $\mathcal{V}(P)$ denote its set of extreme points. Furthermore, let (L, E, G) be an arbitrary partition of $\{0,1\}^n$ with $L, E, G \neq \emptyset$, and let $0 < \delta \in \mathbb{Q}$ be a small constant. If (a, a_0) is a pair in $M_{(L,E,G)}(P)$, then there exists $\lambda > 0$ such that $(\lambda a, \lambda a_0)$ is a feasible point of the polyhedron $Q_{(L,E,G)}(P)$ that is defined by the following constraints in variables $(\alpha, \alpha_0) \in \mathbb{R}^{n+1}$:*

$$\begin{aligned} &\alpha x = \alpha_0 && \textit{for all } x \in E && (Q_{(L,E,G)}(P)) \\ &\alpha x \ge \alpha_0 + \delta && \textit{for all } x \in G \\ &\alpha x \le \alpha_0 - \delta && \textit{for all } x \in L \\ &\alpha z \le \alpha x - \delta && \textit{for all } x \in G \textit{ and for all } z \in \mathcal{V}(P) \ . \end{aligned}$$

Conversely, every rational point (α, α_0) in $Q_{(L,E,G)}(P)$ is a pair in $M_{(L,E,G)}(P)$.

Proof. For the first part of the lemma, assume that (a, a_0) is a pair in $M_{(L,E,G)}(P)$. That is, $ax \le a_0$ is an M-cut for P that is tight at all 0/1-points in E, violated by all points in G and satisfied with strict inequality by the points in L. Then there exist positive numbers Δ_1, Δ_2, and Δ_3 such that

$$\begin{aligned} K_L &:= \max\{ax \,|\, x \in L\} \le a_0 - \Delta_1 < a_0 + \Delta_2 \le \min\{ax \,|\, x \in G\} =: K_G \ , \\ K_z &:= \max\{az \,|\, z \in \mathcal{V}(P)\} \le \min\{ax \,|\, x \in G\} - \Delta_3 = K_G - \Delta_3 \ . \end{aligned}$$

Note that K_L, K_G, and K_z are finite (see Figure 3 for an illustration). For every $\lambda > 0$ satisfy-

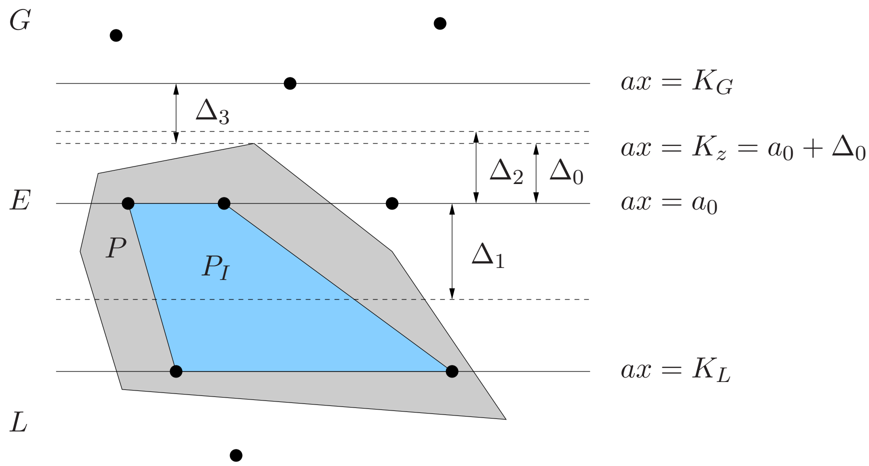

Figure 3: The inequality $ax \leq a_0$ is an M-cut for P that is associated with the valid inequality $ax \leq a_0+\Delta_0$ of P, where $\Delta_0 \geq 0$. The set of 0/1-points in $(ax = a_0)$ is precisely E, all points in G lie strictly above $(ax = a_0 + \Delta_0)$, all points in L lie strictly below $(ax = a_0)$. (Note: Only 0/1-points are illustrated and no other integer points.)

ing $\lambda\Delta_i \geq \max\{\alpha_0, \delta\}$ for $i = 1, 2, 3$, it holds that

$$\begin{aligned} a_0 \leq \quad \lambda\Delta_1 \quad &\leq \quad \lambda(a_0 - K_L) \ , \\ a_0 \leq \quad \lambda\Delta_2 \quad &\leq \quad \lambda(K_G - a_0) \ , \\ a_0 \leq \quad \lambda\Delta_3 \quad &\leq \quad \lambda(K_G - K_z) \ , \end{aligned}$$

and therefore

$$\begin{aligned} \delta \quad \leq \quad \lambda(a_0 - K_L) \quad &= \quad \lambda a_0 - \max\{\lambda ax \,|\, x \in L\} \ , \\ \delta \quad \leq \quad \lambda(K_G - a_0) \quad &= \quad \min\{\lambda ax \,|\, x \in G\} - \lambda a_0 \ , \\ \delta \quad \leq \quad \lambda(K_G - K_z) \quad &= \quad \min\{\lambda ax \,|\, x \in G\} - \max\{\lambda az \,|\, z \in \mathcal{V}(P)\} \ . \end{aligned}$$

Consequently, $(\lambda a, \lambda a_0)$ is a point in $Q_{(L,E,G)}(P)$.

Now consider the second part of the lemma. If (α, α_0) is a rational point in $Q_{(L,E,G)}(P)$, then for $a := \alpha$ and $a_0 := \alpha_0$, $ax \leq a_0$ is the M-cut associated with the valid inequality $ax \leq a_0 + \Delta_0$, where $\Delta_0 := \max\{ax \,|\, x \in P\} - a_0$. This is true because for every 0/1-point $x \in G$, we have

$$ax \geq \max\{az \,|\, z \in \mathcal{V}(P)\} + \delta = a_0 + \Delta_0 + \delta > a_0 + \Delta_0 \ ,$$

and thus,

$$\begin{aligned} a_0 \quad &= \quad \max\left\{ax \,\middle|\, x \in \{0,1\}^n,\ x \in (L \cup E),\ ax \leq a_0 + \Delta_0\right\} \\ &= \quad \max\left\{ax \,\middle|\, x \in \{0,1\}^n,\ ax \leq a_0 + \Delta_0\right\} \\ &= \quad KV(a, a_0 + \Delta_0) \ . \end{aligned}$$

□

Note that $Q_{(L,E,G)}(P)$ is either empty or unbounded, since any feasible solution can be scaled by an arbitrary positive constant greater than 1 to obtain another feasible solution. More precisely,

$Q_{(L,E,G)}(P)$ is non-empty if and only if there exists a hyperplane that contains every point in E and that strictly separates $\text{conv}(G)$ from $\text{conv}(L)$.

In order to show that $M(P)$ can be described by a finite set of inequalities, we want to make use of the fact that any polyhedron can be represented as the Minkowski sum of the convex hull of its vertices plus the cone generated by its extreme rays. Lemma 2.3 implies that any rational point in $Q_{(L,E,G)}(P)$ and, in particular, any vertex of $Q_{(L,E,G)}(P)$, defines a valid inequality for $M(P)$. In the following lemma, we show that also every extreme ray of $Q_{(L,E,G)}(P)$ is associated with a valid inequality for $M(P)$.

LEMMA 2.4 *If $Q_{(L,E,G)}(P)$ is nonempty and (r, r_0) is an extreme ray of $Q_{(L,E,G)}(P)$, then $rx \leq r_0$ defines a valid inequality for $M(P)$.*

PROOF. Let (r, r_0) be an extreme ray of $Q_{(L,E,G)}(P)$ and suppose that there exists some $\bar{x} \in M(P)$ with $r\,\bar{x} > r_0$. Define $\gamma := r\bar{x} - r_0 > 0$. Now consider an arbitrary rational point (α, α_0) in $Q_{(L,E,G)}(P)$. Then $(\alpha, \alpha_0) + \lambda(r, r_0)$ is in $Q_{(L,E,G)}(P)$, for any $\lambda \geq 0$. By Lemma 2.3, it follows that $(\alpha + \lambda r)\, x \leq \alpha_0 + \lambda r_0$ is a valid inequality for $M(P)$ for every rational $\lambda \geq 0$. In particular, $(\alpha + \lambda r)\, \bar{x} \leq \alpha_0 + \lambda r_0$. We obtain $\lambda\gamma \leq \alpha_0 - \alpha\bar{x}$ for any $\lambda \geq 0$. However, since the right-hand side of the last inequality is finite, there exists some rational λ such that $\lambda\gamma > a_0 - a\bar{x}$. This is a contradiction to the fact that $\bar{x} \in M(P)$ and, therefore, $r\,\bar{x} \leq r_0$ must hold. □

Now we can show that the M-closure of any polytope can be described by a finite set of inequalities.

THEOREM 2.1 *If $P \subseteq [0,1]^n$ is a rational polytope, then $M(P)$ is a polytope.*

PROOF. If $P = \emptyset$, there is nothing to show. Therefore, assume that $P \neq \emptyset$. First, observe that $P \neq \emptyset$ implies that there is no M-cut for P that belongs to a set $M_{(L,E,G)}(P)$ with $E = \emptyset$. Now, let $M_{(L=\emptyset)}(P)$ denote the union of all sets $M_{(L,E,G)}(P)$ for which $L = \emptyset$. For any cut $ax \leq a_0$ in $M_{(L=\emptyset)}(P)$, it holds that $[0,1]^n \subseteq (ax \geq a_0)$. We claim that the set $(ax \leq a_0) \cap [0,1]^n$ is a face of the unit cube: For this, assume w.l.o.g. that $(0,\ldots,0) \in E$ and therefore $a_0 = 0$ and $a_i \geq 0$, for $i = 1,\ldots,n$. Let I denote the set of indices i such that $a_i > 0$. Then

$$(ax = 0) \cap \{0,1\}^n = \{0,1\}^n \cap \left(\bigcap_{i \in I} (x_i = 0) \right)$$

implies that $(ax = 0) \cap [0,1]^n$ is, indeed, a face of the unit cube. Since the unit cube has only finitely many faces, there exists a finite set of M-cuts that dominate every M-cut in $M_{(L=\emptyset)}(P)$. Next, consider M-cuts $ax \leq a_0$ in sets $M_{(L,E,G)}(P)$ such that $G = \emptyset$. Then $[0,1]^n \subseteq (ax \leq a_0)$ and, hence, any such cut is dominated by the cube constraints. We define M_0 to be the intersection of the unit cube $[0,1]^n$ with the M-cuts in $M_{(L=\emptyset)}(P)$.

Let $\mathcal{P}$ denote the set of all partitions of $\{0,1\}^n$ into disjoint sets L, E, and G such that $L, E, G \neq \emptyset$. Then with the above observations and Lemma 2.3, it holds that

$$M(P) = M_0 \cap \bigcap_{(L,E,G) \in \mathcal{P}} \left\{ (\alpha x \leq \alpha_0) \,\middle|\, (\alpha, \alpha_0) \in Q_{(L,E,G)}(P) \cap \mathbb{Q}^{n+1} \right\} \;. \tag{9}$$

For any partition $(L, E, G) \in \mathcal{P}$, the polyhedron $Q_{(L,E,G)}(P)$ has a representation in terms of its finite set of vertices $\mathcal{V}_{(L,E,G)}$ and its finite set of extreme rays $\mathcal{R}_{(L,E,G)}$, that is,

$$Q_{(L,E,G)}(P) = \text{conv}\left(\mathcal{V}_{(L,E,G)}\right) + \text{cone}\left(\mathcal{R}_{(L,E,G)}\right) \;. \tag{10}$$

Since P is rational, $Q_{(L,E,G)}(P)$ is a rational polyhedron and, hence, the points in $\mathcal{V}_{(L,E,G)}$ and $\mathcal{R}_{(L,E,G)}$ are rational. We claim that $M(P) = M^*(P) \cap M_0$, where

$$M^*(P) := \bigcap_{(L,E,G)\in\mathcal{P}} \left(\left\{ (vx \le v_0) \,\middle|\, (v,v_0) \in \mathcal{V}_{(L,E,G)} \right\} \;\cap\; \left\{ (rx \le r_0) \,\middle|\, (r,r_0) \in \mathcal{R}_{(L,E,G)} \right\} \right) \ . \quad (11)$$

Clearly, because of Lemma 2.3 and 2.4,

$$M(P) \subseteq M^*(P) \cap M_0 \ .$$

For the other inclusion, take any inequality $\alpha x \le \alpha_0$ for $M(P)$ from representation (9) that is not valid for M_0, that is, $(\alpha, \alpha_0) \in Q_{(L,E,G)}(P) \cap \mathbb{Q}^{n+1}$. By (10), the point (α, α_0) can be written as a convex combination of solutions in $\mathcal{V}_{(L,E,G)}$ and nonnegative combinations of extreme rays in $\mathcal{R}_{(L,E,G)}$. Suppose that $\mathcal{V}_{(L,E,G)} = \{(v^1, v_0^1), \ldots, (v^s, v_0^s)\}$ and $\mathcal{R}_{(L,E,G)} = \{(r^1, r_0^1), \ldots, (r^t, r_0^t)\}$. Then there exist $\lambda_i \ge 0$, for $i = 1, \ldots, s$, and $\mu_j \ge 0$, for $j = 1, \ldots, t$, such that $\sum_{i=1}^s \lambda_i = 1$ and

$$(\alpha, \alpha_0) = \sum_{i=1}^{s} \lambda_i (v^i, v_0^i) + \sum_{j=1}^{t} \mu_j (r^j, r_0^j) \ .$$

For an an arbitrary point $x \in M^*(P)$, we have

$$\alpha\, x = \sum_{i=1}^{s} \lambda_i\, v^i x + \sum_{j=1}^{t} \mu_j\, r^j x \;\le\; \sum_{i=1}^{s} \lambda_i\, v_0^i + \sum_{j=1}^{t} \mu_j\, r_0^j = \alpha_0 \ ,$$

that is, x satisfies the inequality $\alpha x \le \alpha_0$. Hence, $M^*(P) \cap M_0 \subseteq M(P)$. Since the number of partitions of the set of 0/1 points in $\mathbb{R}^n$ is finite, that is, $|\mathcal{P}|$ is finite, $M^*(P)$ is a polyhedron. As a result, $M^*(P) \cap M_0$ is a polytope, implying that the same is true for $M(P)$. □

2.3 Facet Characterization By definition (2), every facet-defining inequality for the M-closure of a non-empty polytope is tight at a 0/1-point. This is why in the proof of Theorem 2.1, we could restrict ourselves to partitions (L, E, G) of the set of 0/1-points with $E \neq \emptyset$. It turns out that the partitions that imply undominated M-cuts satisfy a stronger property: If $M(P)$ is a k-dimensional polytope and $ax \le a_0$ is a facet-defining M-cut for $M(P)$, the hyperplane $(ax = a_0)$ contains at least k affinely independent 0/1-points. Furthermore, the corresponding facet $F = P \cap (ax = a_0)$ is contained in the affine subspace spanned by the 0/1-points in $(ax = a_0)$. In the special case that P_I has full dimension, every facet-defining inequality for $M(P)$ is associated with a hyperplane that is spanned by n affinely independent 0/1-points and, therefore, corresponds to a facet of some 0/1-polytope. Our proof of this property relies on the fact that the M-closure of a polytope is a polytope itself.

THEOREM 2.2 *Let $P \subseteq [0,1]^n$ be a polytope and assume that its closure $M(P)$ has dimension $k > 0$. If $ax \le a_0$ is a facet-defining M-cut for $M(P)$, then $(ax = a_0)$ contains at least k affinely independent points in $\{0,1\}^n$. Furthermore,*

$$M(P) \cap (ax = a_0) \subseteq \operatorname{conv}\big((ax = a_0) \cap \{0,1\}^n\big) \ .$$

Proof. Let $ax \leq a_0$ be an M-cut for P that is facet-defining for $M(P)$. Let $F = M(P) \cap (ax = a_0)$ be the corresponding facet. Define

$$\begin{aligned} L &:= \left\{x \in \{0,1\}^n \,\middle|\, ax < a_0\right\} \ , \\ E &:= \left\{x \in \{0,1\}^n \,\middle|\, ax = a_0\right\} \ , \\ G &:= \left\{x \in \{0,1\}^n \,\middle|\, ax > a_0\right\} \ . \end{aligned}$$

Furthermore, let $S := \text{conv}(E)$.

We will show that for any point $z \in (ax = a_0) \setminus S$, there exists an inequality $cx \leq c_0$ that is valid for $M(P)$ and violated by z, thereby proving that $F \subseteq S$. If we know that $F \subseteq S$, then $\dim(M(P)) = k$ implies that $k - 1 = \dim(F) \leq \dim(S)$. In particular, $(ax = a_0)$ must contain at least k affinely independent 0/1 points.

Consider an arbitrary point $z \in (ax = a_0) \setminus S$. The set S is a (lower-dimensional) integral polytope and can therefore be described by a finite system of rational inequalities. Since $z \notin S$, there exists some rational inequality $fx \leq f_0$ that is valid for S, but is violated by z; that is, $fz > f_0$. Suppose that we can find $\lambda > 0$ and $\varepsilon > 0$ such that $c := a + \lambda f$ and $c_0 := a_0 + \lambda f_0$ satisfy

(i) $cx \leq c_0$ for all $x \in L \cup E$,

(ii) $cx \leq \min\{cx \,|\, x \in G\} - \varepsilon$ is valid for P.

Then the inequality $cx \leq c_0$ is valid for $M(P)$, since the right-hand side of the M-cut for P associated with the valid inequality $cx \leq \min\{cx \,|\, x \in G\} - \varepsilon$ satisfies

$$\begin{aligned} KV\big(c, \min\{cx \,|\, x \in G\} - \varepsilon\big) &= \max\left\{cx \,\middle|\, x \in \{0,1\}^n,\ cx \leq \min\{cx \,|\, x \in G\} - \varepsilon\right\} \\ &= \max\left\{cx \,\middle|\, x \in (L \cup E)\ cx \leq \min\{cx \,|\, x \in G\} - \varepsilon\right\} \\ &\leq \max\left\{cx \,\middle|\, x \in (L \cup E)\right\} \leq c_0 \ . \end{aligned}$$

Furthermore, $cz = az + \lambda fz > a_0 + \lambda f_0 = c_0$, that is, $z \notin M(P)$.

In the remainder of the proof we will show that such λ and ε indeed exist. First, observe that property (i) holds for every point $x \in E \subseteq S$ for any choice of λ, since then $x \in (ax = a_0)$ and $x \in S$ imply that $cx = ax + \lambda fx \leq a_0 + \lambda f_0 = c_0$. Regarding property (i) for the set L, let us define $a_L := \max\{ax \,|\, x \in L\}$ and $f_L := \max\{fx \,|\, x \in L\}$. If λ satisfies

$$\lambda(f_L - f_0) \leq a_0 - a_L \ , \tag{12}$$

then $cx = ax + \lambda fx \leq a_L + \lambda f_L \leq a_0 + \lambda f_0 = c_0$ for all $x \in L$. Note that $a_0 - a_L > 0$. Now consider property (ii). Since $ax \leq a_0$ is an M-cut for P, we know that there exists some $\varepsilon_a > 0$ such that the inequality

$$ax \leq \min\{ax \,|\, x \in G\} - \varepsilon_a$$

is valid for P. If we choose λ and ε such that

$$\lambda \max\{fx \,|\, x \in P\} - \varepsilon_a \leq \lambda \min\{fx \,|\, x \in G\} - \varepsilon \ , \tag{13}$$

then for any $x \in P$,

$$\begin{aligned} cx = ax + \lambda fx &\leq \min\{ax \,|\, x \in G\} - \varepsilon_a + \lambda \max\{fx \,|\, x \in P\} \\ &\leq \min\{ax \,|\, x \in G\} + \lambda \min\{fx \,|\, x \in G\} - \varepsilon \\ &\leq \min\{cx \,|\, x \in G\} - \varepsilon \ . \end{aligned}$$

Define $\varepsilon := \varepsilon_a/2 > 0$. Then condition (13) becomes

$$\lambda \left(\max\{fx \,|\, x \in P\} - \min\{fx \,|\, x \in G\} \right) \leq \varepsilon_a/2 \ . \tag{14}$$

It is not difficult to see that for all possible four cases of signs of $f_L - f_0$ and $\max\{fx \,|\, x \in P\} - \min\{fx \,|\, x \in G\}$, we can find a $\lambda > 0$ such that conditions (12) and (14) are satisfied. Hence, there exist λ and ε such that properties (i) and (ii) hold. This completes the proof of the theorem. □

In case that P_I has full dimension, Theorem 2.2 represents a natural analogy to the fact that every hyperplane that is associated with a facet-defining Gomory-Chvátal cut is spanned by n affinely independent integral points. However, while this property of the Gomory-Chvátal closure is a direct consequence of its definition (only inequalities with integral normal vectors are considered for the computation of the Gomory-Chvátal closure), the corresponding property for the M-closure does not immediately follow from its definition in (2). Observe, furthermore, that in the above proof we specifically used the fact that F can be described as the intersection of $M(P)$ with a hyperplane $(ax = a_0)$ that corresponds to an M-cut. Theorem 2.2 does not apply to arbitrary faces of $M(P)$. In particular, there are face-defining inequalities for which the associated hyperplanes do not contain any 0/1 points (see Figure 4).

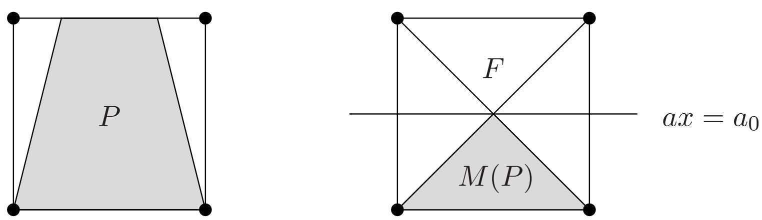

Figure 4: The inequality $ax \leq a_0$ is face-defining for $M(P)$, but the hyperplane $(ax = a_0)$ does not contain any 0/1 points.

3. Comparison of the M-Closure with Other Elementary Closures Numerous families of cutting planes and their associated elementary closures have been introduced and studied in the literature. We refer to Cornuéjols and Li (2000) for definitions and a detailed comparison of the associated elementary closures. Figure 5 provides a summary of their results. In this section, we discuss how the M-closure fits into this picture. Specifically, we compare the M-closure of a polytope with each of the closures depicted in Figure 5.

As demonstrated in Theorem 2.2, the hyperplanes in $\mathbb{R}^n$ that are spanned by n affinely independent 0/1-points are of fundamental importance for the M-closure of a polytope. Therefore, we will introduce a special notation for the M-cuts that define hyperplanes with this property. We will denote by $\mathcal{F}^n$ the set of pairs $(a, a_0) \in \mathbb{Z}^{n+1}$ with $\gcd(a) = 1$ for which the hyperplane $(ax = a_0)$ is spanned by n affinely independent points in $\{0,1\}^n$. Thus, for any facet-defining inequality $ax \leq a_0$ of an arbitrary full-dimenensional 0/1-polytope, the pair (a, a_0) is contained in $\mathcal{F}^n$. Conversely, every pair $(a, a_0) \in \mathcal{F}^n$ defines a facet of some full-dimensional 0/1-polytope. When we say in the sequel that an inequality $ax \leq a_0$ is an inequality in $\mathcal{F}^n$, we refer to the fact that $(a, a_0) \in \mathcal{F}^n$.

3.1 Comparison of the M-Closure with the Gomory-Chvátal Closure The close relationship between the M-closure and the classic Gomory-Chvátal cutting plane procedure is self-evident. We introduced the M-closure as a natural strengthening of the Gomory-Chvátal closure for

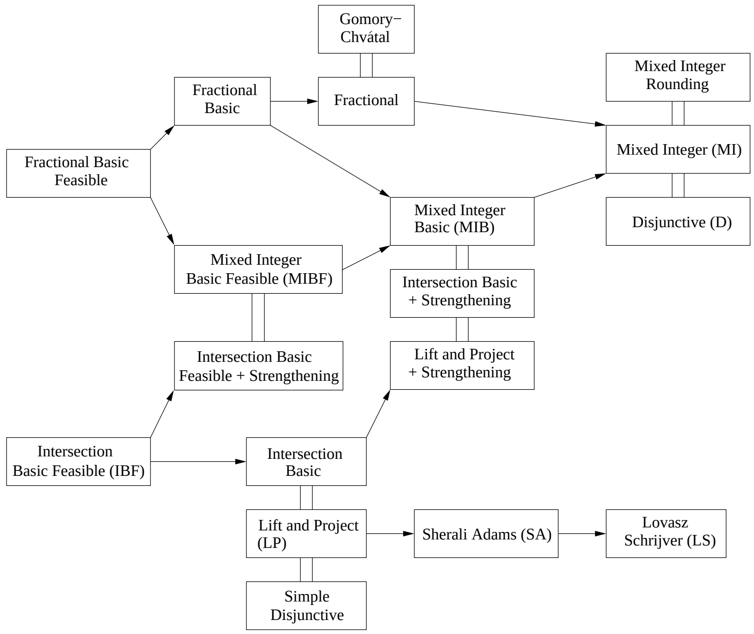

Figure 5: Comparison of elementary closures of 0/1-integer programs (taken from Cornuéjols and Li (2000)). An arrow from one family of cuts F_1 to another family of cuts F_2 indicates that the closure P_{F_2} is contained in the closure P_{F_1}, and there exist instances for which the inclusion is strict. If two families F_1 and F_2 are unrelated, there exist instances such that $P_{F_1} \not\subseteq P_{F_2}$ and instances such that $P_{F_2} \not\subseteq P_{F_1}$.

the special case that all integer points of a polyhedron are 0/1-points. By definition, every Gomory-Chvátal cut of a polytope P defines a valid inequality for its M-closure and, hence, $M(P) \subseteq P'$. On the other hand, an M-cut can strictly dominate the corresponding Gomory-Chvátal cutting plane. If $ax \leq a_0$ is a Gomory-Chvátal cut for P and facet-defining for P' and if the hyperplane $(ax = a_0)$ does not contain a 0/1-point, the right-hand side associated with the M-cut is strictly smaller than a_0, implying that $M(P) \subset P'$. Conversely, one can ask under which circumstances a facet-defining M-cut is also a Gomory-Chvátal cut. For this, consider polytopes P with full-dimensional integer hull. By Theorem 2.2, every facet F of $M(P)$ is contained in an affine space that is spanned by 0/1-points. Put differently, $F \subseteq (ax = a_0)$ for some pair $(a, a_0) \in \mathcal{F}^n$. If for every inequality $ax \leq a_0$ in $\mathcal{F}^n$ there was a 0/1-point in the hyperplane $(ax = a_0+1)$, any such inequality $ax \leq a_0$ would be valid for $M(P)$ if and only if it was also a Gomory-Chvátal cut. As a consequence, it would hold that $M(P) = P'$ for every polytope P with full-dimensional integer hull. On the other hand, if we were to find an inequality $ax \leq a_0$ in $\mathcal{F}^n$ for which $(ax = a_0 + 1) \cap \{0,1\}^n = \emptyset$, this would immediately enable us to construct a polytope P for which the M-closure $M(P)$ is strictly contained in its Gomory-Chvátal closure P' (see Figure 6 for an illustration).

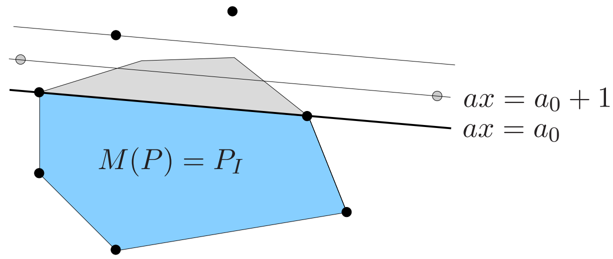

Figure 6: The hyperplane $(ax = a_0)$ with $\gcd(a) = 1$ is spanned by two 0/1-points and, therefore, facet-defining for some 0/1-polytope. If the hyperplane $(ax = a_0 + 1)$ does not contain a 0/1-point, we can construct a polytope P such that $(ax \leq a_0)$ is facet-defining for P_I and such that $M(P) \subset P'$. (Note that points in $\{0,1\}^n$ are illustrated as black points and general integer points are drawn in grey.)

The following lemma summarizes these observations.

LEMMA 3.1 *$M(P) = P'$ for every polytope $P \subseteq [0,1]^n$ with full-dimensional integer hull P_I if and only if $(ax = a_0 + 1) \cap \{0,1\}^n \neq \emptyset$ for every inequality $ax \leq a_0$ in $\mathcal{F}^n$.*

The characterization in Lemma 3.1 allows us to check whether the Gomory-Chvátal closures and the M-closures of all polytopes in the unit cube with full-dimensional integer hull coincide in small dimension. We simply enumerate all inequalities $ax \leq a_0$ in $\mathcal{F}^n$ and check whether $(ax = a_0 + 1) \cap \{0,1\}^n \neq \emptyset$. It turns out that this is computational feasible for $n \leq 7$. Interestingly, for every pair $(a, a_0) \in \mathcal{F}^n$ and for all $n \leq 7$, the hyperplane $(ax = a_0 + 1)$ contains a 0/1-point.

COROLLARY 3.1 *If $n \leq 7$, $M(P) = P'$ for every polytope $P \subseteq [0,1]^n$ with full-dimensional integer hull.*

We did not succeed in enumerating the inequalities in $\mathcal{F}^n$ for $n > 7$. However, the following example confirms the general intuition that there should exist polytopes of some dimension for which the M-closure is strictly contained in the Gomory-Chvátal closure.

EXAMPLE 3.1 *The integral vector $a \in \mathbb{R}^{31}$ with*

$$\begin{aligned} a \;:=\; (&3041115360, & 3216174509, && 3081631465, && 2598051222, && 2888963817, && -129947214, \\ &25001283, & -76282297, && -384543308, && 24697928, && -102658009, && -900040667, \\ &157047656, & -207693535, && -883266807, && -740003674, && -352003226, && 458140458, \\ &-261010564, & 566248994, && -360665679, && -185629660, && -305123253, && -182278292, \\ &-272079117, & -518683, && 311565566, && -569860449, && -691996681, && -303576202, \\ &1885519442) \end{aligned}$$

has relatively prime components. The hyperplane $(ax = 0)$ is spanned by affinely independent points in $\{0,1\}^{31}$, that is, $(a, 0) \in \mathcal{F}^{31}$. Furthermore, $(ax = 1) \cap \{0,1\}^n = \emptyset$. (The smallest right-hand side $a_0 > 0$ such that $ax = a_0$ contains a 0/1-point is $a_0 = 8$.)

The example is not arbitrary. Joswig published a list of 0/1-polytopes that have facet-defining inequalities with very large integer coefficients. These polytopes were generated based on a construction of 0/1-matrices with large determinants by Alon and Vu (1997). Geometrically, the hyperplanes described by 0/1-matrices with large determinants can be thought of as the most skewed hyperplanes spanned by 0/1-points. The inequality $ax \leq 0$ from Example 3.1 is facet-defining for one of these 0/1-polytopes (named "MJ:32-33"). Interestingly, we were unable to generate an example with the same property randomly. For every hyperplane H that we generated by randomly picking n affinely independent 0/1-points (for dimensions up to 35), the parallel hyperplanes obtained by shifting H to the next level of integer points above and below the hyperplane also contained a 0/1-point. This might suggest that the number of hyperplanes $(ax = a_0)$ that are spanned by 0/1-points and that satisfy $(ax = a_0 + 1) \cap \{0,1\}^n = \emptyset$ is small compared to the size of $\mathcal{F}^n$.

Example 3.1 allows us to construct polytopes in arbitrary dimension for which the M-closure is strictly contained in the Gomory-Chvátal closure.

Corollary 3.2 *For every polytope $P \subseteq [0,1]^n$, $M(P) \subseteq P'$. Furthermore, there exists a number $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ there exists a polytope $P \subseteq [0,1]^n$ with full-dimensional integer hull such that $M(P) \subset P'$.*

Proof. Since every M-cut associated with a valid inequality for P dominates the corresponding Gomory-Chvátal cut, we have $M(P) \subseteq P'$. For the second part of the corollary, we construct an explicit example: Let $n := 31$ and $P := [0,1]^n \cap (ax \leq 7)$, where a is the normal vector defined in Example 3.1. Since $(a, 0) \in \mathcal{F}^n$ and $(ax > 0) \cap (ax \leq 7) \cap \{0,1\}^n = \emptyset$, the inequality $ax \leq 0$ must be facet-defining for P_I. In particular, it cannot be implied by other inequalities that are valid for P_I. Clearly, $ax \leq 0$ is valid for $M(P)$. However, the inequality is not a Gomory-Chvátal cut and, since it is facet-defining for P_I, it cannot be implied by other Gomory-Chvátal cuts, that is, $M(P) \subset P'$. Furthermore, P_I is full-dimensional, since the 0/1-points that span the hyperplane $(ax = 0)$ together with the unit vector $e_6 \in \{0,1\}^{31}$ are $n+1$ affinely independent points (note that $a\, e_6 = a_6 < 0$). For every $n > n_0 := 31$, we can extend a to an n-dimensional vector by adding $n - n_0$ zero components and the same argument applies. □

3.2 Comparison of the M-Closure with the Knapsack Closure Another elementary closure that is very closely related to the M-closure is the knapsack closure, which was formally introduced by Fischetti and Lodi (2010) for polytopes in the unit cube. The definition of the knapsack closure is inspired by an observation of Crowder, Johnson, and Padberg (1983): if $P \subseteq [0,1]^n$ is defined by inequalities $a_i x \leq b_i$, for $i = 1, \ldots, m$, then

$$P_I \subseteq \bigcap_{i=1}^{m} P_I^i \ ,$$

where $P_I^i = \operatorname{conv}\{x \in \{0,1\}^n \mid a_i x \leq b_i\}$. That is, P_I is contained in the intersection of all knapsack polytopes defined by the constraints of P. As the example in Figure 7 illustrates, it is possible that the intersection of these knapsack polytopes results in P itself. However, if one intersects the

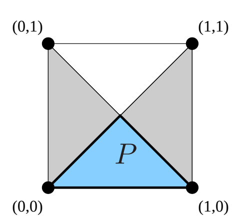

Figure 7: The polytope $P \subseteq [0,1]^2$ is defined by the inequalities $-x_1 + x_2 \leq 0$, $x_1 + x_2 \leq 1$, and $x_2 \geq 0$, and $P_I = \operatorname{conv}\{(0,0);(1,0)\}$. Since each constraint defines an integral half-space, $P_I^1 \cap P_I^2 \cap P_I^3 = P$.

knapsack polytopes associated with *all* valid inequalites for P, the resulting set certainly represents a better approximation of the integer hull P_I. The knapsack closure of a polytope $P \subseteq [0,1]^n$ is defined as

$$K(P) := \bigcap_{a \in \mathbb{Z}^n} \operatorname{conv}\{x \in \{0,1\}^n \mid ax \leq a_P\} \ . \tag{15}$$

As the knapsack closure clearly dominates the Gomory-Chvátal closure, its iterative application to a polytope will generate its integer hull in a finite number of steps. Furthermore, it is easy to see that the knapsack closure of a polytope is a polytope itself. (In fact, the knapsack closure of an arbitrary set in $[0,1]^n$ is a polytope.)

Lemma 3.2 *For any polytope $P \subseteq [0,1]^n$, the knapsack closure $K(P)$ is a polytope.*

Proof. The knapsack closure is an infinite intersection of knapsack polytopes. However, only a finite number of different knapsack polytopes in $\mathbb{R}^n$ exist, since every such polytope is uniquely defined by the set of 0/1 points that violate the knapsack inequality. □

We can characterize the facets of the knapsack closure of a polytope in the following way:

Lemma 3.3 *Let $P \subseteq [0,1]^n$ be a polytope with full-dimensional integer hull. If $ax \leq a_0$ is a facet-defining inequality for $K(P)$ with $(a, a_0) \in \mathbb{Z}^{n+1}$ and $\gcd(a) = 1$, then the following properties hold:*

(i) *$ax \leq a_0$ is in $\mathcal{F}^n$.*

(ii) *There exists a valid inequality $cx \leq c_0$ for P such that*

$$(cx \leq c_0) \cap \{0,1\}^n \subseteq (ax \leq a_0) \cap \{0,1\}^n \ .$$

Proof. Property (i) follows from the definition of the knapsack closure as the intersection of knapsack polytopes: Since P_I is full-dimensional, every knapsack polytope containing P is full-dimensional. Therefore, every facet of $K(P)$ is a facet of some full-dimensional 0/1-polytope, that is, it contains n affinely independent 0/1-points. Now consider property (ii): If $ax \leq a_0$ defines a facet of $K(P)$, there must exist a valid inequality $cx \leq c_0$ for P such that $ax \leq a_0$ is a facet of $\operatorname{conv}\{x \in \{0,1\}^n \mid cx \leq c_0\}$. Since $\operatorname{conv}\{x \in \{0,1\}^n \mid cx \leq c_0\} \subseteq (ax \leq a_0)$, we have $(cx \leq c_0) \cap \{0,1\}^n \subseteq (ax \leq a_0)$ and (ii) follows. □

As shown in Theorem 2.2, property (i) of Lemma 3.3 also applies to the M-closure of P. Property (ii) provides that for every facet-defining inequality $ax \leq a_0$ of $K(P)$ there must be a valid inequality $cx \leq c_0$ for P that is violated by at least the 0/1-points in $(ax > a_0)$. This property is also true for the M-closure, since we can choose $cx \leq c_0$ as $ax \leq \min\{ax \mid x \in \{0,1\}, ax > a_0\} - \varepsilon$, for some $\varepsilon > 0$. However, there is a significant difference between the M-closure and the knapsack closure:

Corollary 3.3 *Let $P \subseteq [0,1]^n$ be a polytope and $S \subseteq \{0,1\}^n$. If there exists a valid inequality $cx \leq c_0$ for P with $(cx > c_0) \cap \{0,1\}^n = S$, then any inequality $ax \leq a_0$ with $(ax > a_0) \cap \{0,1\}^n \subseteq S$ is valid for $K(P)$.*

Proof. The inequality $ax \leq a_0$ is valid for $\operatorname{conv}\{x \in \{0,1\}^n \mid cx \leq c_0\}$. □

It is not immediately obvious whether this property also holds for the M-closure. If it did, every cut that is valid for $K(P)$ would also be valid for $M(P)$ and, hence, the two closures would be identical. Equivalently, the property would imply that the M-closure of any polytope defined by a single constraint (in addition to the cube constraints) would give its integer hull. That is, for any polytope $P = \{x \in [0,1]^n \,|\, cx \le c_0\}$, it would hold that $M(P) = P_I$. The following example shows that this is, in general, not true.

Example 3.2 *For $n = 7$, let $cx \le c_0$ denote the inequality*

$$-3x_1 - 6x_2 + 7x_3 + 3x_4 + x_5 + x_6 + 2x_7 \le 5 \;,$$

and let $P = \{x \in [0,1]^n \,|\, cx \le c_0\}$. One can verify that $ax \le a_0$ given by

$$-3x_1 - 2x_2 + 3x_3 + 3x_4 + x_5 + x_6 + 2x_7 \le 5$$

is facet-defining for P_I. Furthermore, it holds that

$$(cx \le c_0) \cap \{0,1\}^n \subseteq (ax \le a_0) \cap \{0,1\}^n \;.$$

However, $\max\{ax \,|\, x \in P\} > 6$, and $(ax = 6) \cap \{0,1\}^n \ne \emptyset$. Therefore, $ax \le a_0$ is not an M-cut for P, implying $M(P) \supset K(P)$.

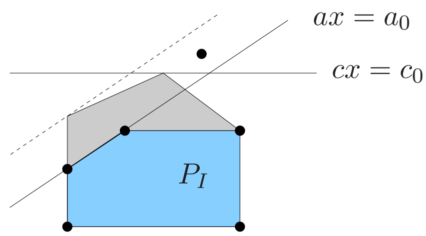

Figure 8: Schematic illustration of the difference between the M-closure and the knapsack closure: The inequality $ax \le a_0$ is valid for $K(P)$, but not for $M(P)$. The inequality $cx \le c_0$ is valid for P and cuts off the same set of 0/1-points as $ax \le a_0$. (Note that only integer points in $\{0,1\}^n$ are illustrated.)

Example 3.2 shows that Corollary 3.3 does not apply to the M-closure. It also highlights a key difference between the M-closure and the knapsack closure of a polytope P: In order for an inequality $ax \le a_0$ in $\mathcal{F}^n$ to be valid for $K(P)$, it is sufficient that there exists some valid inequality $cx \le c_0$ for P that cuts off the same set S of 0/1-points as $ax \le a_0$, or a larger set. In contrast, for $ax \le a_0$ to be a valid for the M-closure, there must exist a valid inequality for P that separates the points in S in a specific way. Intuitively, the angle between the normal vectors a and c and, therefore, the angle between the associated hyperplanes, must not be arbitrarily large (see Figure 8).

Corollary 3.4 *For every polytope $P \subseteq [0,1]^n$, $K(P) \subseteq M(P)$. Furthermore, there exists a number $n_0 \in N$ such that for every $n \ge n_0$ there is a polytope $P \subseteq [0,1]^n$ with full-dimensional integer hull such that $K(P) \subset M(P)$.*

3.3 Comparison of the M-Closure with Other Elementary Closures Cornuéjols and Li showed that there is no universal relationship between the Gomory-Chvátal closure and most of the elementary closures depicted in Figure 5. More precisely, they gave examples of polytopes for which the Gomory-Chvátal closure is not contained in the other elementary closures and vice versa.

Example 3.3 (Cornuéjols and Li 2000) *Consider the 2-dimensional polytope*

$$P = \{x \in \mathbb{R}^2 \,|\, -2x_1 + x_2 \le 0;\; 2x_1 + x_2 \le 2;\; x_2 \ge 0\} \tag{16}$$

with $P_I = \operatorname{conv}\{(0,0);(1,0)\} = \{x \in \mathbb{R}^2 \,|\, 0 \le x_1 \le 1;\; -x_2 \le 0;\; x_2 \le 0\}$ *(see Figure 9 for an illustration). The inequality* $x_2 \le 0$ *is an intersection cut derived from the basic feasible solution* $(1/2, 1)$*. Therefore,* $P_{IBF} = P_I$*, that is, the elementary closure with respect to intersection cuts derived from basic feasible solutions is equal to the integer hull of* P*. However, every Gomory-Chvátal cut is satisfied by the point* $(1/2, 1/2)$*, implying* $P' \neq P_I$*. Thus,* $P' \not\subseteq P_{IBF}$*.*

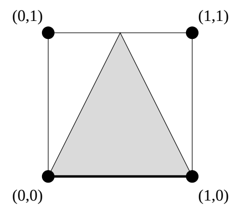

Figure 9: Example of a polytope $P \subseteq [0,1]^n$ for which the Gomory-Chvátal closure is not contained in the elementary closure defined by intersection cuts derived from all basic feasible solutions; that is, $P' \not\subseteq P_{IBF}$.

As can be seen in Figure 5, the elementary closure P_{IBF} of P associated with the family of intersection cuts derived from basic feasible solutions is the weakest relaxation of the integer hull P_I among $P_{IBF}, P_{MIBF}, P_{LP}, P_{SA}, P_{LS}, P_{MIB}$, and P_{MI}. Consequently, Example 3.3 establishes for all these closures the existence of instances of polytopes such that the Gomory-Chvátal closure is not contained in them. Moreover, since $K(P) = M(P) = P'$ for the polytope defined by (16), Example 3.3 establishes the same result for the M-closure $M(P)$ and the knapsack closure $K(P)$.

Corollary 3.5 *For every* $n \ge 2$*, there exist instances of full-dimensional polytopes* $P \subseteq [0,1]^n$*, such that* $M(P) = K(P) \not\subseteq P_{IBF}, P_{MIBF}, P_{LP}, P_{SA}, P_{LS}, P_{MIB}, P_{MI}$*.*

Cornuéjols and Li (2000) also provided examples of polytopes for which $P_{MIB} \not\subseteq P'$ and polytopes such that $P_{LS} \not\subseteq P_{FBF}$. Since $K(P) \subseteq M(P) \subseteq P' \subseteq P_{FBF}$ and because of the relationships between the closures depicted in Figure 5, the following observation is obtained.

Corollary 3.6 *For every* $n \ge 2$*, there exist instances of polytopes* $P \subseteq [0,1]^n$ *such that* P_{IBF}*,* P_{MIBF}*,* P_{LP}*,* P_{SA}*,* P_{LS}*,* $P_{MIB} \not\subseteq M(P) \supseteq K(P)$*.*

So far, we have established the incomparability between the M-closure and the knapsack closure and $P_{IBF}, P_{MIBF}, P_{LP}, P_{SA}, P_{LS}, P_{MIB}$ (and all closures equivalent to these). For mixed integer cuts or, equivalently, disjunctive cuts, we only know from Corollary 3.5 that there are instances such that $K(P) = M(P) \not\subseteq P_{MI} = P_D$, but no inferences for the other direction can be drawn. However, in the next lemma, we construct an example for the reverse inclusion.

Lemma 3.4 *For every* $n \ge 31$*, there exist instances of polytopes* $P \subseteq [0,1]^n$ *such that* $P_D = P_{MI} \not\subseteq M(P)$*, and thus also* $P_D = P_{MI} \not\subseteq K(P)$*.*

Proof. Let $n := 31$ and $P := [0,1]^n \cap (ax \le 7)$, where a is the normal vector defined in Example 3.1. As seen above, $ax \le 0$ is an M-cut for P and facet-defining for P_I. We will show that $ax \le 0$ is not valid for P_D. Suppose there exists a pair $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$ such that the corresponding disjunction implies $ax \le 0$. That is, $P_1 := P \cap (\pi x \le \pi_0) \subseteq (ax \le 0)$ and $P_2 := P \cap (\pi x \ge \pi_0 + 1) \subseteq (ax \le 0)$. If $P_1 \ne \emptyset$ and $P_2 \ne \emptyset$, then

$$\begin{array}{lcl} \max\left\{ax \,\middle|\, x \in [0,1]^n,\ ax \le 7,\ \pi x \le \pi_0\right\} & \le & 0\ , \\ \max\left\{ax \,\middle|\, x \in [0,1]^n,\ ax \le 7,\ \pi x \ge \pi_0 + 1\right\} & \le & 0 \end{array}$$

implies

$$\begin{array}{lcl} z_1 := \max\left\{ax \,\middle|\, x \in [0,1]^n,\ \pi x \le \pi_0\right\} & \le & 0\ , \\ z_2 := \max\left\{ax \,\middle|\, x \in [0,1]^n,\ \pi x \ge \pi_0 + 1\right\} & \le & 0\ . \end{array}$$

Hence, both $\pi x \le \pi_0$ and $\pi x \ge \pi_0 + 1$ have to dominate $ax \le 0$ over the unit cube, which is not possible. (To see this, observe the following: We can assume w.l.o.g. that $0 \in [0,1]^n \cap (\pi x \le \pi_0)$. Then $\pi_0 \ge 0$ and $\max\{ax \,|\, x \in [0,1]^n,\ \pi x \le \pi_0\} = 0$. This implies that for every index i such that $a_i > 0$, we have $\pi_i > 0$, which on the other hand implies $\pi_0 = 0$. As $a_1 > 0$, we get $\pi_1 \ge 1$ and, therefore, e_1 satisfies $\pi x \ge \pi_0 + 1 = 1$. But then $z_2 > 0$, which is a contradiction.) Therefore, let us assume w.l.o.g. that $P_2 = \emptyset$. Then $ax \le 7$ implies $\pi x < \pi_0 + 1$ for any point $x \in [0,1]^n$ and we get that every $x \in P \cap \{0,1\}^n$ satisfies $\pi x \le \pi_0$. Therefore, $\pi x \le \pi_0$ is valid for P_I and $P_1 \ne \emptyset$. In particular, every 0/1-point in $(ax = 0)$ satisfies $\pi x \le \pi_0$. Since $P_1 \subseteq (ax \le 0)$ by assumption, $\pi x \le \pi_0$ has to dominate $ax \le 0$ over the unit cube which is, given the other observations made above, only possible if $\pi x = \pi_0$ contains the same 0/1-points as $(ax = 0)$. This is only possible if $\pi x \le \pi_0$ and $ax \le 0$ define the same half-space. But then $P_2 = P \cap (\pi x \ge \pi_0 + 1) \supseteq P \cap (ax \ge 1) \ne \emptyset$, which is a contradiction. □

Figure 10 summarizes the results of this subsection.

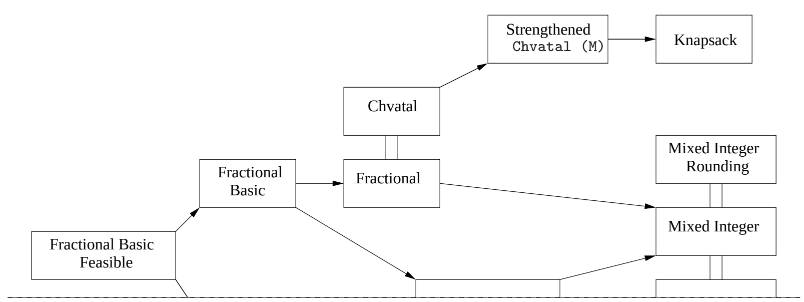

Figure 10: Placement of the M-closure and the knapsack closure in Figure 5.

4. Bounds on the M-Rank The M-closure of a polytope $P \subseteq [0,1]^n$ defines a tighter approximation of its integer hull P_I than P itself, assuming $P \ne P_I$. More precisely, it is at least as tight as the Gomory-Chvátal closure of the polytope, and in certain cases it strictly dominates P'. Therefore, it is interesting to compare the sequences of successively tighter relaxations that arise from an iterative application of the M-closure operation with the sequences obtained during the Gomory-Chvátal procedure.

Clearly, as $M(P) \subseteq P'$, the M-rank cannot exceed the Chvátal rank of a polytope. Hence, by Theorem 3.3 of Eisenbrand and Schulz (2003), we can conclude that for any polytope in the unit cube the M-rank is polynomially bounded in the dimension.

COROLLARY 4.1 *If $P \subseteq [0,1]^n$ is a polytope, then the M-rank of P is bounded by* $\mathrm{O}(n^2 \log n)$.

Since we know that there are polytopes for which the M-closure is strictly contained in the Gomory-Chvátal closure, this raises the question of whether a better general upper bound for the M-rank can be obtained. This question is also of interest in light of the fact that the upper bound of Eisenbrand and Schulz for the Chvátal rank is not known to be tight; in fact, there is a significant gap between this bound and the rank of the best worst-case example that has been exhibited so far. However, we will show in the next subsection that, in the special case of a polytope with empty integer hull, the worst-case bound that holds for the Chvátal rank also applies to the M-rank.

4.1 Upper Bounds for Polytopes in the Unit Cube without Integral Points It is known that the Chvátal rank of a polytope $P \subseteq [0,1]^n$ without integral points does not exceed n, and this bound is tight. More precisely, Bockmayr et al. (1999) proved the following result.

Theorem 4.1 (Bockmayr et al. 1999) *Let P be a d-dimensional polytope in $[0,1]^n$ with $P_I = \emptyset$. If $d = 0$, then $P' = \emptyset$. If $d > 0$, then $P^{(d)} = \emptyset$.*

A family of polytopes for which exactly n iterations of the Gomory-Chvátal procedure are necessary in order to obtain its empty integer hull is given by

$$P_n := \left\{ x \in [0,1]^n \;\middle|\; \sum_{i \in I} x_i + \sum_{i \notin I} (1 - x_i) \geq \frac{1}{2},\ I \subseteq \{1, \ldots, n\} \right\} . \tag{17}$$

P_n is the convex hull of all midpoints of the edges of the unit cube (see Figure 11). To see that the

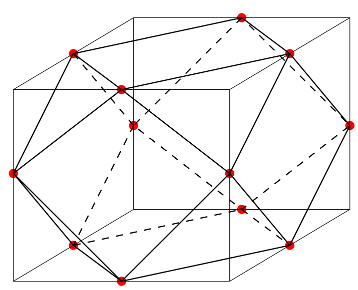

Figure 11: A family of polytopes in the n-dimensional unit cube with empty integer hull and M-rank n.

Chvátal rank of P_n is n, let F_j denote the set of all vectors in $\mathbb{R}^n$ such that j components are 1/2 and each of the remaining components is either 0 or 1. Chvátal, Cook, and Hartmann (1989) showed that, if a polytope P contains the set F_j, then F_{j+1} must be contained in its Gomory-Chvátal closure P'. Since it holds that $F_1 \subseteq P_n$, it follows that $F_n \subseteq P_n^{(n-1)}$. Hence, the Chvátal rank of P_n is at least n. With the upper bound from Theorem 4.1, the rank must be precisely n.

As we show next, the same family of polytopes also requires n iterations of the M-procedure to obtain the empty integer hull.

Lemma 4.1 *Let P be a d-dimensional polytope in $[0,1]^n$ with $P_I = \emptyset$. If $d = 0$, then $M(P) = \emptyset$. If $d > 0$, then $M^{(d)}(P) = \emptyset$. Furthermore, for every $n \geq 1$, there exists a polytope $P \subseteq [0,1]^n$ with $P_I = \emptyset$ and M-rank n.*

Proof. The first part of the lemma follows directly from Theorem 4.1 and the fact that $M(P) \subseteq P'$. In order to establish the tightness of this bound, we first show that if P contains F_j, then $M(P)$ contains F_{j+1}, for all $j = 1, \ldots, n-1$.

Let $(a, a_0) \in \mathbb{Z}^{n+1}$ such that $ax \leq a_0$ is a valid inequality for P. As the set F_j is symmetric with respect to permutations of the coordinates and flipping signs of coordinates, we can, w.l.o.g., assume that $a_1 \geq a_2 \geq \ldots \geq a_n \geq 0$. Suppose that F_j is contained in $(ax \leq a_0)$. We need to show that

$$F_{j+1} \subseteq \Big(ax \leq \max \big\{ ax \,\big|\, x \in \{0,1\}^n, ax \leq a_0 \big\} \Big) \ .$$

If j is even, then

$$\begin{aligned}
\max\{a\,y \,|\, y \in F_{j+1}\} &= a_1 + \ldots + a_{n-(j+1)} + 1/2\ (a_{n-j} + \ldots + a_n) \\
&\leq a_1 + \ldots + a_{n-(j+1)} + 1/2\ (2a_{n-j} + 2a_{n-j+2} + \ldots + 2a_{n-2} + a_n) \\
&\leq a_1 + \ldots + a_{n-(j+1)} + (a_{n-j} + a_{n-j+2} + \ldots + a_{n-2}) + a_n \\
&= a_1 + \ldots + a_{n-j} + 1/2\ (2a_{n-j+2} + \ldots + 2a_{n-2} + 2a_n) \\
&\leq a_1 + \ldots + a_{n-j} + 1/2\ (a_{n-j+1} + \ldots + a_n) \\
&= \max\{a\,y \,|\, y \in F_j\} \ .
\end{aligned}$$

If j is odd, then

$$\begin{aligned}
\max\{a\,y \,|\, y \in F_{j+1}\} &= a_1 + \ldots + a_{n-(j+1)} + 1/2\ (a_{n-j} + \ldots + a_n) \\
&\leq a_1 + \ldots + a_{n-(j+1)} + 1/2\ (2a_{n-j} + 2a_{n-j+2} + \ldots + 2a_{n-1}) \\
&= a_1 + \ldots + a_{n-(j+1)} + (a_{n-j} + a_{n-j+2} + \ldots + a_{n-1}) \\
&\leq a_1 + \ldots + a_{n-j} + 1/2\ (2a_{n-j+2} + \ldots + 2a_{n-1}) + 1/2\ a_n \\
&\leq a_1 + \ldots + a_{n-j} + 1/2\ (a_{n-j+1} + \ldots + a_n) \\
&= \max\{a\,y \,|\, y \in F_j\} \ .
\end{aligned}$$

In both cases, the sequence of inequalities contains a line in which all coefficients of the a_i are either 0 or 1. Therefore, there always exists some $\bar{x} \in \{0,1\}^n$ such that

$$\max\{a\,y \,|\, y \in F_{j+1}\} \leq a\bar{x} \leq \max\{a\,y \,|\, y \in F_j\} \ .$$

Since $\max\{a\,y \,|\, y \in F_j\} \leq a_0$, we get for all $y \in F_{j+1}$,

$$a\,y \leq \max\{ax \,|\, x \in \{0,1\}^n, ax \leq a_0\} \ .$$

As $F_1 \subseteq P_n$, the above observations imply that $F_n \subseteq M^{(n-1)}(P_n)$ and, hence, the M-rank of P_n is at least n. Since the M-rank cannot exceed n, it is exactly n. □

We want to mention here that the cutting plane procedure associated with the family of knapsack cuts (see Section 3.2) has the same worst-case bound, that is, not more than n successive applications of the knapsack closure operation are necessary for polytopes with empty integer hull to obtain P_I. Again, the family of polytopes defined by (17) serves as an example for which this bound is achieved.

Lemma 4.2 *Let P be a d-dimensional polytope in $[0,1]^n$ with $P_I = \emptyset$. If $d = 0$, then $K(P) = \emptyset$. If $d > 0$, then $K^{(d)}(P) = \emptyset$. Furthermore, for every $n \geq 1$, there exists a polytope $P \subseteq [0,1]^n$ with $P_I = \emptyset$ such that n applications of the knapsack-closure operation are necessary to obtain P_I.*

Proof. As in the proof of Lemma 4.1, we will show that, if a polytope P contains F_j, then $K(P)$ contains F_{j+1}, for all $j = 1, \ldots, n-1$. For this, it suffices to show that for any inequality $ax \leq a_0$ such that $F_j \subseteq (ax \leq a_0)$, it holds that

$$F_{j+1} \subseteq \text{conv}\{x \in \{0,1\}^n \,|\, ax \leq a_0\} \ .$$

Let $(a, a_0) \in \mathbb{Z}^{n+1}$, such that $F_j \subseteq (ax \le a_0)$. As F_j is invariant to coordinate flips, we can assume w.l.o.g. that $a \ge 0$. Consider an arbitrary $y \in F_{j+1}$. By renaming the indices, we can assume w.l.o.g. that $y_i = 1/2$ for $i = 1, \ldots, j+1$, $y_i \in \{0, 1\}$ for $i = j+2, \ldots, n$, and $a_1 \le a_2 \le \ldots \le a_{j+1}$. If j is even, then $F_j \subseteq (ax \le a_0)$ implies

$$\begin{aligned} a_0 &\ge \frac{1}{2}(a_1 + a_2 + \ldots + a_j) + a_{j+1} + y_{j+2}a_{j+2} + y_{j+3}a_{j+3} + \ldots + y_n a_n \\ &\ge a_1 + a_3 + \ldots + a_{j-1} + a_{j+1} + y_{j+2}a_{j+2} + y_{j+3}a_{j+3} + \ldots + y_n a_n \end{aligned}$$

and

$$\begin{aligned} a_0 &\ge \frac{1}{2}(a_1 + a_2 + \ldots + a_j) + a_{j+1} + y_{j+2}a_{j+2} + y_{j+3}a_{j+3} + \ldots + y_n a_n \\ &\ge \frac{1}{2}a_1 + a_2 + a_4 + \ldots + a_j + y_{j+2}a_{j+2} + y_{j+3}a_{j+3} + \ldots + y_n a_n \\ &\ge a_2 + a_4 + \ldots + a_j + y_{j+2}a_{j+2} + y_{j+3}a_{j+3} + \ldots + y_n a_n \ . \end{aligned}$$

From the first sequence, we get that $u \in \{0, 1\}^n$ with

$$u_i = \begin{cases} 1 & \text{if } i \le j+1 \text{ and } i \text{ odd} \\ 0 & \text{if } i \le j+1 \text{ and } i \text{ even} \\ y_i & \text{if } i \ge j+2 \end{cases}$$

is contained in $(ax \le a_0)$. The second sequence implies the same for $v \in \{0, 1\}^n$ defined by

$$v_i = \begin{cases} 0 & \text{if } i \le j+1 \text{ and } i \text{ odd} \\ 1 & \text{if } i \le j+1 \text{ and } i \text{ even} \\ y_i & \text{if } i \ge j+2 \end{cases}$$

Consequently, both u and v are contained in $\operatorname{conv}\{x \in \{0, 1\}^n \mid ax \le a_0\}$.

Similarly, if j is odd, then

$$\begin{aligned} a_0 &\ge \frac{1}{2}(a_1 + a_2 + \ldots + a_j) + a_{j+1} + y_{j+2}a_{j+2} + y_{j+3}a_{j+3} + \ldots + y_n a_n \\ &\ge a_1 + a_3 + \ldots + a_j + y_{j+2}a_{j+2} + y_{j+3}a_{j+3} + \ldots + y_n a_n \end{aligned}$$

and

$$\begin{aligned} a_0 &\ge \frac{1}{2}(a_1 + a_2 + \ldots + a_j) + a_{j+1} + y_{j+2}a_{j+2} + y_{j+3}a_{j+3} + \ldots + y_n a_n \\ &\ge a_2 + a_4 + \ldots + a_{j+1} + y_{j+2}a_{j+2} + y_{j+3}a_{j+3} + \ldots + y_n a_n \end{aligned}$$

imply again that u and v are points in $\operatorname{conv}\{x \in \{0, 1\}^n \mid ax \le a_0\}$. Therefore,

$$y = \frac{1}{2}(u + v) \in \operatorname{conv}\{x \in \{0, 1\}^n \mid ax \le a_0\} \ .$$

As a result, $y \in K(P)$. Now $F_1 \subseteq P$ implies that $F_n \subseteq K^{(n-1)}(P_n)$ and, hence, the knapsack-rank of P_n is at least n. Since the knapsack-rank cannot exceed n, it is exactly n. □

5. Complexity of the M-Closure From a computational perspective, one of the most important questions regarding elementary closures is whether one can efficiently optimize over a given closure. The following two problems are closely related (see Grötschel, Lovász, and Schrijver (1988) for details).

The *validity problem* for the elementary closure of a family $\mathcal{F}$ of cuts:

> Given a rational polyhedron $P \subseteq \mathbb{R}^n$ and a rational inequality $ax \leq a_0$, is $ax \leq a_0$ valid for the elementary closure of P defined by $\mathcal{F}$?

The *non-membership problem* for the elementary closure of a family $\mathcal{F}$ of cuts:

> Given a rational polyhedron $P \subseteq \mathbb{R}^n$ and a rational point $\bar{x} \in P$, does $\bar{x}$ not belong to the elementary closure of P defined by $\mathcal{F}$?

Both the non-membership problem and the validity problem are known to be in NP for the family of Gomory-Chvátal cuts. Eisenbrand (1999) observed that a proof by Caprara and Fischetti (1996) implies that the non-membership problem for the Gomory-Chvátal closure is NP-complete. In contrast, Bockmayr and Eisenbrand (2001) showed that for fixed dimension, the Gomory-Chvátal closure of a rational polyhedron P can be described by a polynomial number of inequalities. Moreover, in this case P' can be constructed in polynomial time. It turns out that the same is true for the M-closure of a rational polytope with full-dimensional integer hull. This fact is a direct consequence of Theorem 2.2.

THEOREM 5.1 *If $P \subseteq [0,1]^n$ is a polytope with full-dimensional integer hull, the M-closure of $M(P)$ can be computed in polynomial time, when the dimension n is fixed.*

PROOF. By Theorem 2.2, any facet of $M(P)$ is contained in $\mathcal{F}^n$. Obviously, $|\mathcal{F}^n| \leq \binom{2^n}{n}$. For each $ax \leq a_0$ in $\mathcal{F}^n$, we can check in polynomial time whether it is a valid inequality for $M(P)$. For this, we first compute $a_P = \max\{ax \,|\, x \in P\}$ by solving a linear programming problem. Then we determine $z_I = \max\{ax \,|\, x \in \{0,1\}^n, ax \leq a_P\}$ by enumerating all 2^n cube vertices and compare the value with a_0. The inequality $ax \leq a_0$ is valid for $M(P)$ if and only if $z_I \leq a_0$. The theorem follows. □

In general, however, things are more difficult.

THEOREM 5.2 *The validity problem problem for the family of M-cuts is (weakly) coNP-complete.*

PROOF. It is easy to see that the validity problem is in coNP. Just recall that an inequality $cx \leq c_0$ is valid for $M(P)$ if and only if $\max\left\{cx \,|\, x \in \{0,1\}^n, cx \leq c_P\right\} \leq c_0$. We prove completeness by describing a reduction from the SUBSET SUM problem, which is NP-complete: Given a set of numbers $a_1, \ldots, a_n \in \mathbb{Z}_+$ and a positive integer a_0, decide whether there is subset $I \subseteq \{1, \ldots, n\}$ such that $\sum_{i \in I} a_i = a_0$. Consider an arbitrary instance of SUBSET SUM, and let P be the polytope defined by $P := [0,1]^n \cap (ax \leq a_0)$. The inequality $ax \leq a_0 - 1$ is valid for $M(P)$ if and only if there is no point in $x \in P \cap \{0,1\}^n$ such that $ax = a_0$. In other words, $ax \leq a_0 - 1$ is not valid for $M(P)$ if and only if the subset sum instance is a YES-instance. □

6. Concluding Remarks Arguably, the most interesting open question in the theory of Gomory-Chvátal cuts for polytopes in $[0,1]^n$ and, by extension, for the M-closure is to determine the correct behavior of the worst-case rank as a function of n. As mentioned above, the best known upper bound for the Chvátal rank of a polytope in the unit cube is $O(n^2 \log n)$. The best known lower bound is $(1+\epsilon)n$, for some $\epsilon > 0$. Both bounds are due to Eisenbrand and Schulz (2003). In this paper, we have provided examples for which the M-closure is strictly contained in the Gomory-Chvátal closure, leading to the question as to whether there might be a significant difference in the worst-case behavior of the M-rank if compared to the Chvátal rank. We showed that there is no such difference for the class of polytopes in the unit cube with empty integer hull. For general polytopes in $[0,1]^n$, Theorem 2.2 might prove useful in establishing better bounds on the M-rank. The insights that we have gained so far let us believe that the following should definitely be true.

CONJECTURE: For every polytope $P \subseteq [0,1]^n$, the M-rank is bounded by $O(n^2)$.

Pokutta (2011) has given examples which show that the worst-case M-rank grows faster than n.

Acknowledgements The authors would like to thank Andrea Lodi for stimulating discussions.

References

Alon, N. and V. H. Vu (1997). Anti-Hadamard matrices, coin weighing, threshold gates, and indecomposable hypergraphs. *Journal of Combinatorial Theory A 79*, 133–160.

Bockmayr, A. and F. Eisenbrand (2001). Cutting planes and the elementary closure in fixed dimension. *Mathematics of Operations Research 26*, 304–312.

Bockmayr, A., F. Eisenbrand, M. Hartmann, and A. S. Schulz (1999). On the Chvátal rank of polytopes in the 0/1 cube. *Discrete Applied Mathematics 98*, 21–27.

Caprara, A. and M. Fischetti (1996). 0-1/2 chvátal-gomory cuts'. *Mathematical Programming 74*, 221–236.

Chvátal, V. (1973). Edmonds polytopes and a hierarchy of combinatorial problems. *Discrete Mathematics 4*, 305–337.

Chvátal, V., W. Cook, and M. Hartmann (1989). On cutting-plane proofs in combinatorial optimization. *Linear Algebra and its Applications 114/115*, 455–499.

Cornuéjols, G. (2008). Valid inequalities for mixed integer linear programs. *Mathematical Programming B 112*, 3–44.

Cornuéjols, G. and Y. Li (2000). Elementary closures for integer programs. *Operations Research Letters 28*, 1–8.

Crowder, H., E. Johnson, and M. Padberg (1983). Solving large-scale zero-one linear programming problems. *Operations Research 31*, 803–834.

Dunkel, J. (2011, June). *The Gomory-Chvátal Closure: Polyhedrality, Complexity, and Extensions*. Ph. D. thesis, Massachusetts Institute of Technology, Cambridge, MA.

Eisenbrand, F. (1999). On the membership problem for the elementary closure of a polyhedron. *Combinatorica 19*, 297–300.

Eisenbrand, F. and A. S. Schulz (2003). Bounds on the Chvátal rank of polytopes in the 0/1-cube. *Combinatorica 23*, 245–261.

Fischetti, M. and A. Lodi (2010). On the knapsack closure of 0-1 integer linear programs. In *Electronic Notes in Discrete Mathematics, ISCO 2010 - International Symposium on Combinatorial Optimization*, Volume 36, pp. 799–804.

Gomory, R. E. (1958). Outline of an algorithm for integer solutions to linear programs. *Bulletin of the American Mathematical Society 64*, 275–278.

Grötschel, M., L. Lovász, and A. Schrijver (1988). *Geometric Algorithms and Combinatorial Optimization*. Springer.

Joswig, M. `http://www.math.tu-berlin.de/polymake/examples/ZeroOne/`, G. M. Ziegler's collection of 0/1-polytopes.

Pokutta, S. (2011, September). Lower bounds for Chvátal-Gomory style operators. Optimization Online.

Pokutta, S. and A. S. Schulz (2010). On the rank of cutting-plane proof systems. In *Proceedings of the 14th Conference on Integer Programming and Combinatorial Optimization (IPCO 2010), Lecture Notes in Computer Science*, Volume 6080, pp. 450–463.

Schrijver, A. (1980). On cutting planes. *Annals of Discrete Mathematics 9*, 291–296.