

# A distribution-free risk-reward newsvendor model: Extending Scarf's min-max order formula

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Scarf's *min-max* order formula for the *distribution-free risk-neutral* newsvendor problem is a classical result in the field of inventory management. The min-max order formula provides, in closed-form, the order quantity that maximizes the worst-case expected profit associated with the demand of a single product when only the mean and variance of the product's demand distribution, rather than the full distribution itself, is assumed to be known. It has been a long-standing question whether a similar closed-form order formula exists for the *distribution-free risk-reward* newsvendor problem; that is, for the order quantity that maximizes the worst-case risk-reward associated with the demand of a single product when only the mean and variance of the product's demand distribution is assumed to be known. The main contribution of this work is to extend Scarf's closed-form order formula to one of the most important risk-reward criteria, namely when the reward is defined by the expected profit, and the risk by the profit's standard deviation. Furthermore, we provide managerial insights associated with this result.

*Key words*: Distribution-free, newsvendor, risk-averse, mean-standard deviation, closed-form, min-max rule

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## 1. Introduction

The classical newsvendor problem is one of the elementary and fundamental models in inventory management with vast applications in such areas as revenue management, supply chain management, and even accounting management (refer to Khouja (1999), Porteus (1990), and Cachon (2003) for further details on this problem along with variants). Numerous extensions and variations of this basic model have been proposed in the literature to suit the specific applications, such as airline seat reservation, supply chain contract, etc.

Formally the (risk-neutral) newsvendor problem can be defined as follows. A newsvendor, facing uncertain product demand  $S$  with cumulative distribution  $\pi$ , unit selling price  $p$  (normalized to be one without loss of generality for our purpose, i.e.,  $p = 1$ ) and unit purchasing price  $c < p = 1$ , wants to decide the optimal order quantity  $x$  so as to maximize the expected profit  $\max_{x \geq 0} \mathbb{E}_\pi [\min(x, S) - cx]$ . The solution to the optimal stocking quantity of the newsvendor is well-known:  $x^* = \pi^{-1}(1 - c)$ ; that is, the  $(1 - c)$ -percentile of the distribution  $\pi$ .

### 1.1. Scarf's min-max order formula

In the 50th anniversary issue of the journal *Operations Research*, Scarf, one of the earliest pioneers in inventory theory, provided a personalized reminiscence on his contributions to the area of inventory management, and the first one on his list is the *min-max* order formula (a.k.a., *distribution-free*, or *semi-parametric*, or *robust* order formula) for the newsvendor problem (Scarf (2002)).

One of the main criticisms of the basic newsvendor order formula discussed in the introduction, is the assumption of knowing the product's demand distribution  $\pi$ , an unrealistic or too strict assumption in practical applications. To mitigate this drawback, Scarf (1958, 2002) offered the aforementioned min-max order formula. Scarf's main idea is to assume that we know only the mean  $\mathbb{E}_\pi[S] = \mu$  and the variance  $\mathbb{V}_\pi[S] = (\gamma - 1)\mu^2$  ( $\gamma \geq 1$ ) instead of the whole distribution  $\pi$  of the demand  $S$ , and find the order quantity that maximizes the worst-case expected profit.

Let us denote  $\sigma = (\sigma_0, \sigma_1, \sigma_2) = (1, \mu, \gamma\mu^2)$ , and  $x^+ = \max\{x, 0\}$  for the rest of the paper. Also let  $f_\pi(x) = \mathbb{E}_\pi(\min\{x, S\} - cx)$  be the expected profit for any given order quantity  $x \geq 0$ . More precisely, Scarf's min-max order formula is obtained by maximizing the minimum expected profit among all demand distributions with the given mean and variance. This problem can be formulated as finding the solution of the following optimization problem:

$$\begin{aligned} f^s = \max_{x \geq 0} \min_{\pi} f_\pi(x) \\ \text{s.t.} \quad \mathbb{E}_\pi(S^i) = \sigma_i, \quad \text{for } i = 0, 1, 2, \\ \pi \quad \text{a distribution in } \mathbb{R}_+. \end{aligned} \quad (1)$$

The following theorem summarizes the classical result of Scarf.

**THEOREM 1.** (Scarf (1958, 2002)) The optimal solution  $x^s$  of the Problem (1) above is

$$x^s = \begin{cases} \mu \left( 1 + (1 - 2c) \sqrt{\frac{\gamma - 1}{4c(1 - c)}} \right), & \text{if } \gamma c \leq 1; \\ 0, & \text{if } \gamma c > 1, \end{cases}$$

along with the optimal expected profit  $f^s = \mu \left( (1 - c) - \sqrt{(\gamma - 1)c(1 - c)} \right)^+ . \quad \square$

As observed by Scarf (1958), the optimal expected profit  $f^s$  under the Scarf strategy consists of two terms, the first  $\mu(1 - c)$  is the profit earned for an mean demand, and the second  $\mu\sqrt{(\gamma - 1)c(1 - c)}$  is a loss term due to incomplete information about the demand. Nothing will be ordered whenever the loss is larger than the profit (or equivalently, when  $\gamma c > 1$ ).

Note that Scarf's min-max order formula is *robust* in the sense that it immunizes the expected profit against possible mis-specification of the actual product's demand distribution.

Although the ultimate goal of Scarf's analysis is to provide an alternative method to solve the newsvendor problem, an important ingredient of theoretically independent interest in his analysis amounts to the bounding of the expectation of the special random variable  $(S - 1)^+$  given the *moment* constraints  $\sigma_i$ ,  $i = 0, 1, 2$ ; that is:

$$\begin{aligned} \bar{p}(\sigma) = \max_{\pi} \mathbb{E}_\pi[(S - 1)^+] \\ \text{s.t.} \quad \mathbb{E}_\pi(S^i) = \sigma_i \quad \text{for } i = 0, 1, 2, \\ \pi \quad \text{a distribution in } \mathbb{R}_+. \end{aligned}$$

Lemma 1 in Appendix A1 recalls the solution to the above problem. It turns out that bounding expectation of this special random variable  $(S - 1)^+$  under moment constraints (a.k.a. moment problem) has vast applications in option pricing, portfolio management, robust optimization and stochastic programming etc. (see, e.g., Bertsimas et al (2000) and the references therein for further details along this line of research). In turn, the extensions to Scarf's min-max order formula that we will introduce here are related to Lemma 1's extensions, which similarly can also be applied to provide extensions to related results in other areas. In particular, Theorems 5 and 6 for Problems (7) and (8) respectively in Appendix A2 are of independent interests. We, however, focus our discussion on the newsvendor problem setting.

One of the nice characteristics of Scarf's min-max order formula is that it is provided by a simple closed-form. This property makes it simple to derive not only qualitative, but also quantitative managerial insights.

## 1.2. Distribution-free risk-reward newsvendor order formula

In the last 50+ years since the appearance of Scarf's formula, there have been numerous results in the literature related to extensions of Scarf's classical min-max order formula to different settings and applications. But one intriguing question still remains:

*Is there a similar closed-form formula for the distribution-free risk-award (risk-averse or risk-seeking) newsvendor problem? In particular, when the reward is defined by the expected profit, and the risk by the profit's standard deviation.*

The main contribution of this work is to answer this long-standing question in an affirmative way. Let  $g_\pi(x) = \sqrt{\text{Var}_\pi(\min\{x, S\} - cx)}$  be the standard deviation of the profit. Then we are interested in the following problem: for any given constant  $\lambda$ , which reflects the decision-maker's risk-attitude (namely risk-averse if  $\lambda > 0$ , risk-seeking if  $\lambda < 0$ , and risk-neutral if  $\lambda = 0$ ),

$$h^* = f^* - \lambda g^* = \max_{x \geq 0} \min_{\pi} f_\pi(x) - \lambda g_\pi(x) \quad (2)$$

s.t.  $\mathbb{E}_\pi(S^i) = \sigma_i, \quad \text{for } i = 0, 1, 2,$   
 $\pi \quad \text{a distribution in } \mathbb{R}_+.$

The main result of this work is summarized below.

**THEOREM 2.** The optimal order quantity  $x^*$  for Problem (2) above is given by:

$$x^* = \begin{cases} \mu \left( 1 + (1 - 2c) \sqrt{\frac{\gamma - 1}{\lambda^2 + 4c(1 - c)}} \right), & \text{if } \gamma c \leq 1 - \lambda \sqrt{\gamma - 1}; \\ 0, & \text{otherwise,} \end{cases}$$

along with the optimal objective value  $h^* = \mu \left( (1 - c) - 2^{-1} \sqrt{\gamma - 1} \left( \lambda + \sqrt{\lambda^2 + 4c(1 - c)} \right) \right)^+$ , and the corresponding expected profit  $f^*$  and the standard deviation of the profit  $g^*$

$$\begin{cases} f^* = \mu \left( (1 - c) - 2c(1 - c) \sqrt{\frac{\gamma - 1}{\lambda^2 + 4c(1 - c)}} \right), \\ g^* = \frac{\mu \sqrt{\gamma - 1}}{2} \left( 1 + \frac{\lambda}{\sqrt{\lambda^2 + 4c(1 - c)}} \right), \end{cases}$$

if  $\gamma c \leq 1 - \lambda \sqrt{\gamma - 1}$ , and 0 otherwise.  $\square$

We make a few observations on this result. First, one can immediately see that the above result includes Scarf's result in Theorem 1 as a special case when  $\lambda = 0$ . Second, as will be seen later on, the proof of this theorem is technically involved. It is therefore quite surprising to obtain these succinct closed-form formulas, which maintain the nice structural properties enjoyed by the original Scarf formula, yet provide extra managerial insights brought out by the inclusion of the risk parameter  $\lambda$ . Detailed observations and managerial insights will be provided in Section 3 along with some numerical results.

## 1.3. Related work

There are two lines of research that are related to this work. The first is on how to apply the min-max approach to variants of the newsvendor problem (e.g., Gallego (1992), Gallego and Moon (1993), Gallego (1998), Gallego (2001), Godfrey, and Powell (2001), Yue et al. (2006), Gallego (2007), Perakis and Roel (2008) etc.). The second is on how to incorporate risk into the newsvendor problem. Two extant approaches have been proposed in addressing risks in the newsvendor setting. One is the von Neumann and Morgenstern's *expected utility* approach widely used in decision theory (e.g., Lau (1980), Eeckhoudt et al. (1995)), and the other is the *risk-reward* approach first popularized by Nobel Laureate Markovitz in portfolio theory (e.g., Lau and Lau 1999, Chen and

Federgruen (2000), Choi et al., (2001), Chen et al (2007), Choi and Ruszczyński (2008), Chen et al (2009)). There is still an ongoing debate on which method is more sensible. One particular criticism on the expected utility approach is that it is highly nontrivial to select a proper utility in specific applications, while the mean-risk approach usually enjoys a much more intuitive explanation.

In this work we will adopt the second approach in addressing risk by using the *risk-reward* approach. The most related previous research is therefore Lau (1980) who proposed the mean-standard deviation newsvendor model along with a numerical approach to solve the resultant optimization problem, assuming known demand distribution. Our model in this paper can be viewed as the *robust* version of the Lau's model with the benefit of a closed-form solution.

The rest of this work is organized as follows. In Section 2 we make a high-level presentation of the main idea of the proof of Theorem 2, leaving the details of the proof for the Appendices. We then present illustrative numerical results along with managerial insights in Section 3.

## 2. Proof of Theorem 2: high-level

The proof of Theorem 2 for Problem (2) can be divided into three steps, and each step involves two separated cases depending on the sign of  $\lambda$ , namely,  $\lambda \geq 0$  or  $\lambda < 0$ .

*Step 1.* First of all, we consider the problem (3) of *maximizing* (for the case of  $\lambda \geq 0$ ) the variance of the profit subject to an extra constraint  $\mathbb{E}_\pi((\frac{S}{x} - 1)^+) = \hat{\mu} \in \Omega_x$ ,

$$\begin{aligned} \bar{v}(x; \hat{\mu}) \equiv \max \quad & g_\pi^2(x) \\ \text{s.t.} \quad & \mathbb{E}_\pi(S^i) = \sigma_i, \quad \text{for } i = 0, 1, 2, \\ & \mathbb{E}_\pi((\frac{S}{x} - 1)^+) = \hat{\mu} \in \Omega_x, \\ & \pi \text{ a distribution on } \mathbb{R}_+, \end{aligned} \quad (3)$$

and the problem (4) of *minimizing* (for the case of  $\lambda < 0$ ) the variance of the profit subject to the same constraints

$$\begin{aligned} \underline{v}(x; \hat{\mu}) \equiv \min \quad & g_\pi^2(x) \\ \text{s.t.} \quad & \mathbb{E}_\pi(S^i) = \sigma_i, \quad \text{for } i = 0, 1, 2, \\ & \mathbb{E}_\pi((\frac{S}{x} - 1)^+) = \hat{\mu} \in \Omega_x, \\ & \pi \text{ a distribution on } \mathbb{R}_+. \end{aligned} \quad (4)$$

In the above, for any  $x > 0$ ,

$$\Omega_x = \left\{ \hat{\mu} : \left(\frac{\mu}{x} - 1\right)^+ \leq \hat{\mu} \leq \begin{cases} \frac{\mu}{x} - \frac{1}{\gamma}, & \text{if } \frac{\mu}{x} \geq \frac{2}{\gamma}, \\ \frac{1}{2} \left( \left(\frac{\mu}{x} - 1\right) + \sqrt{(\gamma - 1) \left(\frac{\mu}{x}\right)^2 + \left(\frac{\mu}{x} - 1\right)^2} \right), & \text{if } \frac{\mu}{x} \leq \frac{2}{\gamma}. \end{cases} \right\} \quad (5)$$

Here  $\Omega_x$  is the feasible region of  $\hat{\mu}$  (See Appendix A1 for details).

**THEOREM 3.** For any given  $\hat{\mu} \in \Omega_x$ , the optimal objective values of the above problems are given by

$$\bar{v}(x; \hat{\mu}) = \begin{cases} -\mu^2 + \mu x(1 + 2\hat{\mu}) - x^2\hat{\mu}(1 + \hat{\mu}), & \text{if } 0 < x \leq \gamma\mu \text{ and} \\ & \left(\frac{\mu}{x} - 1\right)^+ \leq \hat{\mu} \leq \frac{\mu}{x} - \frac{1}{\gamma}, \\ \left( \sqrt{\frac{(\gamma-1)\mu^2}{4} + x\hat{\mu}(\mu - x)} - x^2\hat{\mu}^2 + \frac{\mu}{2}\sqrt{\gamma-1} \right)^2, & \text{if } x \geq \frac{\gamma\mu}{2} \text{ and} \\ & \frac{\mu}{x} - \frac{1}{\gamma} \leq \hat{\mu} \leq \frac{(\frac{\mu}{x}-1) + \sqrt{(\gamma-1)(\frac{\mu}{x})^2 + (\frac{\mu}{x}-1)^2}}{2}. \end{cases}$$

and  $\underline{v}(x; \hat{\mu}) = \left( \sqrt{\frac{(\gamma-1)\mu^2}{4} + x\hat{\mu}(\mu - x)} - x^2\hat{\mu}^2 - \frac{\mu}{2}\sqrt{\gamma-1} \right)^2$ , respectively.

*Step 2.* Second of all, note that

$$f_\pi(x) - \lambda g_\pi(x) \geq \begin{cases} \mu - x\hat{\mu} - cx - \lambda\sqrt{\bar{v}(x; \hat{\mu})}, & \text{if } \lambda \geq 0, \\ \mu - x\hat{\mu} - cx - \lambda\sqrt{\underline{v}(x; \hat{\mu})}, & \text{otherwise.} \end{cases}$$

So we have that

$$h(x) := \min_{\pi} f_\pi(x) - \lambda g_\pi(x) = \begin{cases} \min_{\hat{\mu} \in \Omega_x} \mu - x\hat{\mu} - cx - \lambda\sqrt{\bar{v}(x; \hat{\mu})}, & \text{if } \lambda \geq 0, \\ \min_{\hat{\mu} \in \Omega_x} \mu - x\hat{\mu} - cx - \lambda\sqrt{\underline{v}(x; \hat{\mu})}, & \text{otherwise.} \end{cases} \quad (6)$$

Denote the following quantities:

$$\begin{aligned} \hat{\mu}_1(x) &= \frac{\mu}{x} - \frac{1}{\gamma}, \\ h_1(x) &= \frac{x}{\gamma} \left( (1 - \lambda\sqrt{\gamma-1}) - \gamma c \right), \\ \hat{\mu}_2(x) &= \frac{\mu}{x} - \frac{1}{2} + \frac{1}{2\sqrt{\lambda^2+1}}, \\ h_2(x) &= -\left( \frac{1}{2}(\sqrt{\lambda^2+1} - 1) + c \right)x, \\ \hat{\mu}_3(x) &= \frac{1}{2} \left( \frac{\mu}{x} - 1 + \sqrt{\frac{(\gamma-1)(\frac{\mu}{x})^2 + (1 - \frac{\mu}{x})^2}{\lambda^2+1}} \right), \\ h_3(x) &= \frac{1}{2} \left( x - 2cx - \sqrt{\lambda^2+1} \sqrt{(\mu-x)^2 + (\gamma-1)\mu^2} + (1 - \lambda\sqrt{\gamma-1})\mu \right). \end{aligned}$$

**THEOREM 4.** For a given  $x$ , if  $\hat{\mu}_1(x), \hat{\mu}_2(x), \hat{\mu}_3(x) \in \Omega_x$ , then the optimal solution  $\hat{\mu}^*$  and the corresponding value of the objective function  $h(x)$  for Problem (6) are given as follows.

(I)  $\lambda \geq 0$ :

(i) If  $\gamma \leq 2$ , then

$$(\hat{\mu}^*, h(x)) = \begin{cases} (\hat{\mu}_1(x), h_1(x)), & \text{if } \lambda > \frac{(2x-\gamma\mu)\sqrt{\gamma-1}}{|\gamma\mu+\gamma x-2x|} \text{ for } x \geq \frac{\gamma\mu}{2}; \\ (\hat{\mu}_3(x), h_3(x)), & \text{otherwise.} \end{cases}$$

(ii) Otherwise,

$$(\hat{\mu}^*, h(x)) = \begin{cases} (\hat{\mu}_3(x), h_3(x)) & \text{if } \lambda \leq \frac{(2x-\gamma\mu)\sqrt{\gamma-1}}{\gamma\mu+\gamma x-2x} \text{ for } x \geq \frac{\gamma\mu}{2}; \\ (\hat{\mu}_1(x), h_1(x)), & \text{if } \frac{(2x-\gamma\mu)\sqrt{\gamma-1}}{\gamma\mu+\gamma x-2x} < \lambda < \frac{2\sqrt{\gamma-1}}{\gamma-2}; \\ (\hat{\mu}_2(x), h_2(x)), & \text{otherwise.} \end{cases}$$

(II)  $\lambda < 0$ :  $\hat{\mu}^* = \hat{\mu}_3(x)$  and  $h(x) = h_3(x)$ .

*Step 3.* Finally, Problem (2) is reduced to  $\max_{x \geq 0} h(x)$ , whose resolution leads to Theorem 2.

### 3. Numerical examples and managerial insights

With the closed-form formula given in Theorem 2, we are able to derive deeper managerial insights. It is worth noting that extant literature without closed-form formulas either use simulation or fix a specific distribution or rely upon comparative analysis in order to derive managerial insights, resulting in limited or restricted applications or only qualitative insights.

Note that  $r = p/c = 1/c$  is the relative profit margin (due to normalized  $p = 1$ ), and  $\sqrt{\gamma-1}$  is nothing but the coefficient of variation (CV). Therefore the cut-off point  $\gamma c = 1 - \lambda\sqrt{\gamma-1}$  in the order quantity formula can be rewritten as:  $1 + \text{CV}^2 = r(1 - \lambda\text{CV})$ .

### 3.1. Some basic observations

First of all, we can make the following enriched observations compared to those by Scarf (1958) due to the inclusion of the risk-attitude parameter  $\lambda$ .

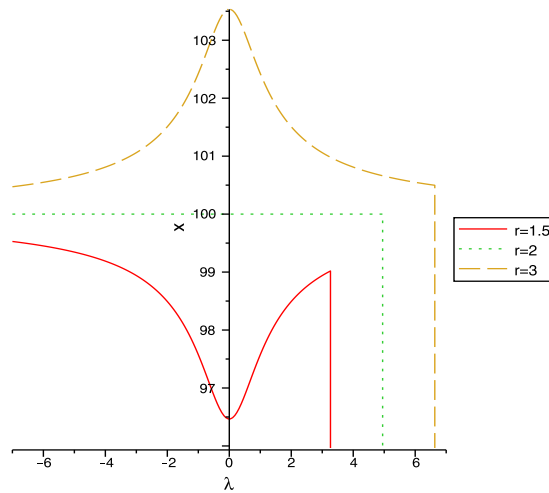
1. The optimal objective value  $h^*$  also consists of two terms, the first  $\mu(1-c)$  is still the profit for an mean demand, and the second  $2^{-1}\mu\sqrt{\gamma-1}\left(\lambda+\sqrt{\lambda^2+4c(1-c)}\right)$  is a *different* loss term due to incomplete information about the demand *and* the risk averseness  $\lambda$  of the decision-maker. Nothing will be ordered whenever the loss is larger than the profit (i.e.,  $\gamma c > 1 - \lambda\sqrt{\gamma-1}$ ).
2. The optimal order quantity can be interpreted as follows (see Fig. 1 for illustration):
  - (a) if  $r > 2$ , order more than the mean demand  $\mu$ , and for fixed  $\mu$ , the higher the CV, the higher the order quantity, unless  $1 + \text{CV}^2 > r(1 - \lambda\text{CV})$ .
  - (b) if  $r < 2$ , order less than the mean demand  $\mu$ , and for fixed  $\mu$ , the higher the CV, the lower the order quantity, unless  $1 + \text{CV}^2 > r(1 - \lambda\text{CV})$ .
  - (c) if  $r = 2$ , order the mean demand  $\mu$ , independently of the CV, unless  $1 + \text{CV}^2 > 2(1 - \lambda\text{CV})$ , or equivalently  $\text{CV}^2 + 2\lambda\text{CV} - 1 > 0$ , or  $\text{CV} > -\lambda + \sqrt{1 + \lambda^2}$ , in which case order zero.
3. We compare the optimal order quantities for risk-neutral, risk-averse, and risk-seeking. Note that when  $\lambda > 0$ , a risk-aversion person stocks a positive quantity whenever  $\gamma c \leq 1 - \lambda\sqrt{\gamma-1} < 1$  under a stricter condition compared to a risk-neutral person ( $\gamma c \leq 1$ ). On the other hand, when  $\lambda < 0$ , a risk-seeking person stocks a positive quantity whenever  $\gamma c \leq 1 - \lambda\sqrt{\gamma-1} > 1$  under a more relaxed condition compared to a risk-neutral person.

### 3.2. Impact of the risk $\lambda$

For the following discussion, with fixed expected demand  $\mu$ , uncertainty of the demand  $\gamma$  and relative profit margin  $r$ , we consider the effects on the order quantity  $x^*$  and various objective values  $h^*$ ,  $f^*$  and  $g^*$  in terms of the decision-maker's risk attitude  $\lambda$ .

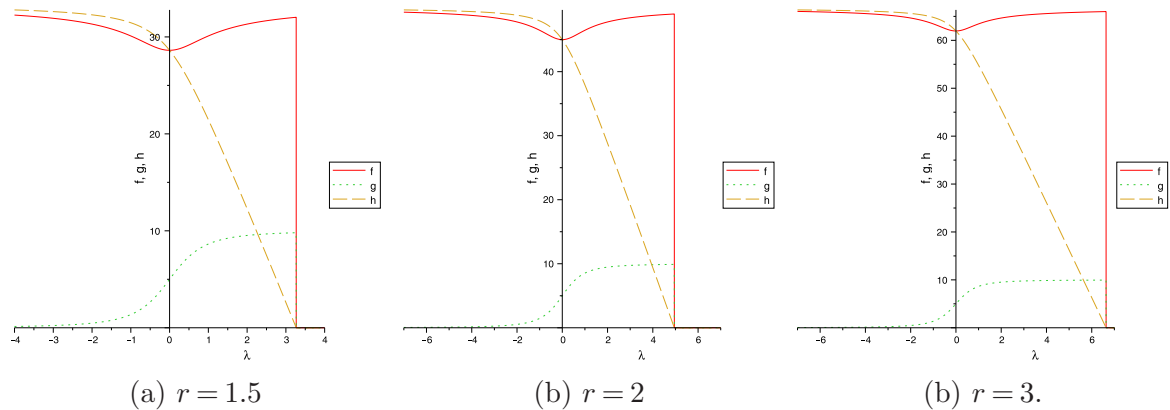
1. **Impacts of risk measure  $\lambda$  on the order quantity  $x^*$ .** The order quantity is almost symmetric for risk-averse and risk seeking decision-maker. Everything else being equal, the order quantity  $x^*$  decreases with higher risk-averseness (more positive) or higher risk-seekingness (more negative) when the relative profit margin  $r > 2$ , and increases when the relative profit margin  $r < 2$ . Note that the Scarf order corresponds to  $\lambda = 0$ . Therefore, this monotonicity also tells us the difference between the Scarf strategy and the mean-standard deviation strategy, namely, with higher risk-averseness or risk-seekingness, the former orders more than the latter when the relative profit margin  $r$  is large ( $r > 2$ ) and order less when it is small ( $r < 2$ ).  
 Moreover, there is a threshold beyond which the risk-averseness is valued so high by the decision-maker that order quantity is forced to be zero, though the threshold depends on both the relative profit margin  $r$  and the demand uncertainty  $\gamma$ . Fig. 1 shows the effect for three choices of  $r$ , namely  $r = 1.5, 2, 3$ . The order quantity is zero when  $\lambda > 3.267$  for  $r = 1.5$ ,  $\lambda > 4.95$  for  $r = 2$ , and  $\lambda > 6.633$  for  $r = 3$ .
2. **Impacts of risk measure  $\lambda$  on the objective values ( $h^*, f^*, g^*$ ).** The expected profit  $f^*$  is still symmetric for risk-averse and risk seeking decision-maker, but the standard deviation of the profit  $g^*$  and the overall man-standard deviation  $h^*$  shows an asymmetric behavior. Everything else being equal, the expected profit  $f^*$  behaves similarly to the order quantity, that is, increasing with higher risk-averseness and higher risk-seekingness; the standard deviation of the profit  $g^*$  is increasing with higher risk-averseness, but decreasing with higher risk-seekingness; and finally the mean-standard deviation objective  $h^*$  is decreasing with higher risk-averseness, but increasing with higher risk-seeking.

All these quantities  $f^*$ ,  $g^*$  and  $h^*$  drop to zero once the risk-averseness is beyond certain positive threshold. Below the threshold, the mean-standard deviation strategy, compared to



**Figure 1** Order quantity  $x^*$  vs risk measure  $\lambda$  for  $\mu = 100$ , and  $\sigma = 10$

Scarf's strategy, always generate more profit  $f^*(\lambda) > f^*(0) = f^s$  along with higher profit risk  $g^*(\lambda) > g^*(0) = g(x^s)$  for risk-averse decision-maker  $\lambda > 0$  and lower profit risk  $g^*(\lambda) < g^*(0) = g(x^s)$  for risk-seeking decision-maker  $\lambda < 0$ . But the counter-intuitive fact is that this can be achieved with less order quantity compared to Scarf's strategy when the profit margin is large (i.e.,  $r > 2$ ) because  $x^s = x^*(0) \geq x^*(\lambda)$  (See Fig. 1). Fig. 2 illustrates these observations for three choices of  $r$ , namely  $r = 1.5, 2, 3$ .



**Figure 2** Various objective values ( $f^*$ ,  $g^*$ ,  $h^*$ ) vs risk measure  $\lambda$  for  $\mu = 100$ , and  $\sigma = 10$

### 3.3. Risk-averse, risk neutral or risk-seeking

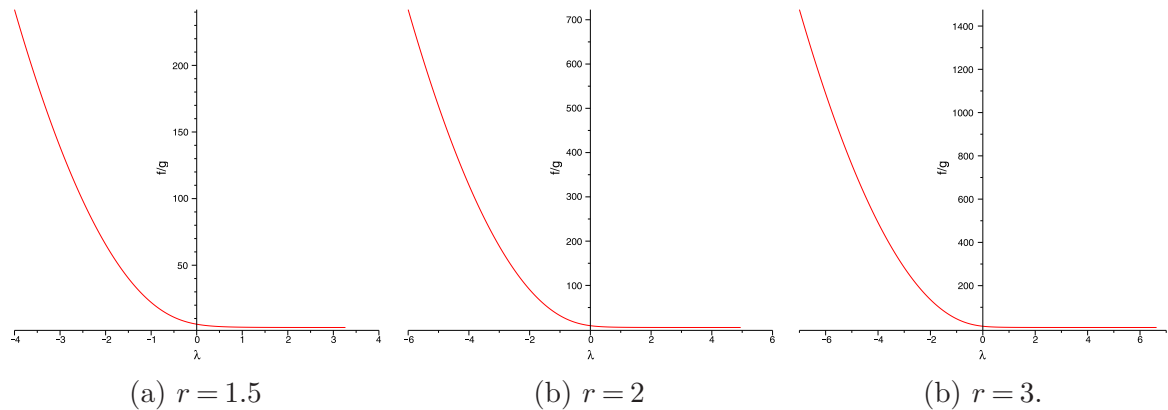
Based on the discussion above, the formulas we derived in this work (namely Theorem 2) provide some guidance for decision-makers with different risk attitude along the risk spectra.

- On one extreme of the risk spectra, there exists certain threshold  $\lambda_0 > 0$  such that no quantity is stocked for any  $\lambda > \lambda_0$ . Therefore, an extremely risk-averse decision-maker will order zero as shown in both Fig. 1 and Fig. 2.

- On the other extreme of the risk spectra, namely when  $\lambda \rightarrow -\infty$ , an extremely risk-seeking decision-maker will stock the mean demand  $\mu$  with expected profit approaching  $\mu(1 - c)$  along with diminishing profit uncertainty.

- For any decision-maker with risk attitude in between, the situation becomes more interesting. We use the Sharpe ratio  $f^*/g^*$  to examine how well the profit return compensates the decision-maker for the risk taken. As illustrated clearly in Fig 3, below the threshold, the higher the Sharpe ratio, namely the higher risk-seeking of the decision-making, the more profit for the same risk.

- Note that the min-max rule is a worst-case analysis method. Therefore, one should exercise caution and care when applying the observations based on worst-case analysis to any specific demand distribution.



**Figure 3** Sharpe ratio  $f^*/g^*$  vs risk measure  $\lambda$  for  $\mu = 100$ , and  $\sigma = 10$

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## References

- [1] D. Bertsimas, I. Popescu and J. Sethuraman, Moment problems and semidefinite programming, In: H. Wolkovitz, (Eds.), Handbook on Semidefinite Programming: Theory, Algorithms, and Applications, (2000) 469-509.
- [2] G.P. Cachon, Supply chain coordination with contracts. In: A.G. deKok, S.C. Graves, S.C. (Eds.), Handbook in Operations Research and Management Science, Volume on Supply Chain Management: Design, Coordination and Operation. North Holland, Amsterdam, (2003) 229-339.
- [3] F. Chen and A. Federgruen, Mean-variance analysis of basic inventory models. Working paper, Columbia University, (2000).
- [4] X. Chen, M. Sim, D. Simchi-Levi, and P. Sun, Risk aversion in inventory management, Operations Research, 55(5) (2007) 828-842.



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- [5] Y. Chen, M. Xu, Z.G. Zhang, A risk-averse newsvendor model under the CVaR criterion, *Operations Research*, 57(4) (2009) 1040-1044.
- [6] T-M. Choi, D. Li and H. Yan, Newsvendor problem with mean variance objectives. *The Proceedings of 5th International Conference on Optimization: Techniques and Applications*, (2001).
- [7] S. Choi, and A. Ruszczyński, A risk-averse newsvendor with law invariant coherent measures of risk, *Operations Research Letters*, 36 (2008) 77–82.
- [8] L. Eeckhoudt, C. Gollier and H. Schlesinger, The Risk-averse (and prudent) newsboy, *Management Science*, 41(50) (1995) 786-794.
- [9] G. Gallego, (1992). A minimax distribution-free for the  $(Q, R)$  inventory model. *Operation Research Letters*, 11 (1992) 55-60.
- [10] G. Gallego and I. Moon, The distribution free newsboy problem: review and extensions, *Journal of Operational Research Society*., 44 (1993) 825-834.
- [11] G. Gallego, New bounds and heuristics for  $(Q, r)$  policies, *Management Science* 44(2) (1998) 219-233.
- [12] G. Gallego, K. Katircioglu, and B. Ramachandran, Inventory management under highly uncertain demand, *Operation Research Letters*, 35(3) (2007) 281-289.
- [13] G. Gallego, J.K. Ryan, and D. Simchi-Levi, Minimax analysis for finite-horizon inventory models. *IIE Transactions*, 33 (2001) 861-874.
- [14] G. Godfrey, and W.B. Powell,, An adaptive, distribution-free algorithm for the newsvendor problem with censored demands, with application to inventory and distribution problems. *Management Sciences*, 47 (2001) 1101-1112.
- [15] M. Khouja, The single-period (news-vendor) problem: literature review and suggestions for future research, *Omega*, 27 (1999) 537-553.
- [16] H.S. Lau, The newsboy problem under alternative optimization objectives. *Journal of the Operational Research Society*, 31 (1980) 525-535.
- [17] H.S. Lau, A.H.L. Lau, Manufacturer's pricing strategy and returns policy for a single-period commodity, *European Journal of Operational Research*, 116 (1999) 291-304.
- [18] A. Lo, Semiparametric upper bounds for option prices and expected payoffs, *Journal of Financial Economics*, 19, pp. 373–388, (1987).
- [19] Markowitz, H.M., 1959. *Portfolio selection: efficient diversification of investment*, John Wiley & Sons, New York.
- [20] A. Parmar and S. Ahmed, Mean-risk models for the newsboy problem, Working Paper, School of Industrial and Systems Engineering, Georgia Institute of Technology, Atlanta, April 2004.
- [21] G. Perakis and G. Roels, Regret in the newsvendor model with partial information, *Operations Research*, 56(1) (2008) 188-203.
- [22] E.L. Porteus, Stochastic inventory model, In: S.C. Graves, A.H.G.R. Kan, P.H. Zipkin (Eds.), *Handbook in Operations Research and Management Science*, Volume on Stochastic Models, North Holland, Amsterdam, (1990) 605-652.
- [23] H. Scarf, A min-max solution of an inventory problem, In: K. J. Arrow, S. Karlin, and H. Scarf, (Eds.), *Studies in the Mathematical Theory of Inventory and Production*, Stanford University Press, (1958) 201-209.
- [24] H. Scarf, Inventory theory, *Operations Research*, 50(1) (2002) 186-191.
- [25] M.E. Schweitzer and Gerard P. Cachon, Decision bias in the newsvendor problem with known demand distribution: experimental evidence, *Management Science*, 46 (2000) 404-420.
- [26] J. Yue, B. Chen, M-C. Wang, Expected value of distribution information for the newsvendor problem, *Operations Research*, 54(6) (2006) 1128-1136.
- [27] L. F. Zuluaga, J. Pena and D. Du, Extensions of Lo's semiparametric bound for European call options, *European Journal of Operational Research*, 198(2) (2009) 557-570.

## Appendices

We first show Theorems 3 and 4 in Appendices A and B, leading to the final proof of Theorem 2 in Appendix C.

### Appendix A: Proof of Theorem 3

First, simple algebra shows that the objective function of Problems (3) and (4) could be written as

$$g_{\pi}^2(x) = \mathbb{V}\text{ar}_{\pi}(p \min\{x, S\} - cx) = (\gamma - 1)\mu^2 - x^2 \mathbb{V}\text{ar}_{\pi}\left(\left(\frac{S}{x} - 1\right)^+\right) - 2x^2 \widehat{\mu}^2 - 2x(x - \mu)\widehat{\mu}.$$

Therefore, Problem (3) is reduced to

$$\begin{aligned} \min \quad & \mathbb{V}\text{ar}_{\pi}\left(\left(\frac{S}{x} - 1\right)^+\right) \\ \text{s.t.} \quad & \mathbb{E}_{\pi}\left(\left(\frac{S}{x}\right)^i\right) = \frac{\sigma_i}{x^i}, \quad \text{for } i = 0, 1, 2, \\ & \mathbb{E}_{\pi}\left(\left(\frac{S}{x} - 1\right)^+\right) = \widehat{\mu} \\ & \pi \text{ a distribution on } \mathbb{R}_+; \end{aligned}$$

and Problem (4) is reduced to

$$\begin{aligned} \max \quad & \mathbb{V}\text{ar}_{\pi}\left(\left(\frac{S}{x} - 1\right)^+\right) \\ \text{s.t.} \quad & \mathbb{E}_{\pi}\left(\left(\frac{S}{x}\right)^i\right) = \frac{\sigma_i}{x^i}, \quad \text{for } i = 0, 1, 2, \\ & \mathbb{E}_{\pi}\left(\left(\frac{S}{x} - 1\right)^+\right) = \widehat{\mu} \\ & \pi \text{ a distribution on } \mathbb{R}_+. \end{aligned}$$

By abusing the notations ( $S := S/x$ ,  $\sigma_i := \sigma_i/x^i$ ), the above two problems are simplified respectively as

$$\begin{aligned} \underline{q}(\mu, \gamma, \widehat{\mu}) = \min \quad & \mathbb{V}\text{ar}_{\pi}\left((S - 1)^+\right) \\ \text{s.t.} \quad & \mathbb{E}_{\pi}(S^i) = \sigma_i, \quad \text{for } i = 0, 1, 2, \\ & \mathbb{E}_{\pi}(S - 1)^+ = \widehat{\mu}, \\ & \pi \text{ a distribution on } \mathbb{R}_+, \end{aligned} \tag{7}$$

and

$$\begin{aligned} \bar{q}(\mu, \gamma, \widehat{\mu}) = \max \quad & \mathbb{V}\text{ar}_{\pi}\left((S - 1)^+\right) \\ \text{s.t.} \quad & \mathbb{E}_{\pi}(S^i) = \sigma_i, \quad \text{for } i = 0, 1, 2, \\ & \mathbb{E}_{\pi}(S - 1)^+ = \widehat{\mu}, \\ & \pi \text{ a distribution on } \mathbb{R}_+. \end{aligned} \tag{8}$$

Consequently, the rest of this section will focus on solving Problems (7) and (8). We first address the feasibility issue in Appendix A1, and then present closed-form solutions to Problems (7) and (8) in Appendix A2, leading to the proof of Theorem 3.

#### Appendix A1: Feasibility

We recall two existing results for bounding the expectation of the any random variable  $(S - 1)^+$  subject to moments constraints:

$$\begin{aligned} \bar{p}(\sigma) = \max \quad & \mathbb{E}_{\pi}((S - 1)^+) \\ \text{s.t.} \quad & \mathbb{E}_{\pi}(S^i) = \sigma_i, \quad \text{for } i = 0, 1, 2, \\ & \pi \text{ a distribution on } \mathbb{R}_+, \end{aligned}$$

and

$$\begin{aligned} \underline{p}(\sigma) = \min \quad & \mathbb{E}_\pi((S-1)^+) \\ \text{s.t.} \quad & \mathbb{E}_\pi(S^i) = \sigma_i, \quad \text{for } i = 0, 1, 2, \\ & \pi \text{ a distribution on } \mathbb{R}_+. \end{aligned}$$

LEMMA 1. (Scarf (1958)) *If  $\mu \geq 0$  and  $\gamma \geq 1$ , then*

$$\bar{p}(\sigma) = \begin{cases} \mu - \frac{1}{\gamma}, & \text{if } \mu > \frac{2}{\gamma}, \\ \frac{1}{2} \left( (\mu - 1) + \sqrt{(\gamma - 1)\mu^2 + (\mu - 1)^2} \right), & \text{if } \mu \leq \frac{2}{\gamma}, \end{cases}$$

and  $\underline{p}(\sigma) = (\mu - 1)^+$ , respectively.  $\square$

Therefore, Problem (7) or (8) is feasible if and only if  $\mu \geq 0, \gamma \geq 1$ , and  $\hat{\mu} \in \Omega_1$  (see the definition of  $\Omega_x$  in (5) in Section 2).

### Appendix A2: Closed-form solution

In Appendices A2-1 and A2-2 respectively, we solve Problems (7) and (8) by demonstrating a pair of primal and dual feasible solutions with equal objective values.

#### Appendix A2-1: Closed-form solution for Problem (7)

The dual problem corresponding to Problem (7) can be written as:

$$\begin{aligned} (\underline{Q}^d) \quad \underline{q}^d = \max \quad & y_o + y_1\mu + y_2\gamma\mu^2 + \hat{y}_1\hat{\mu} - \hat{\mu}^2 \\ \text{s.t.} \quad & y_o + y_1s + y_2s^2 + \hat{y}_1(s-1)^+ \leq ((s-1)^+)^2, \quad \forall s \in \mathbb{R}_+ \end{aligned}$$

Clearly, weak duality holds between Problem (7) and  $(\underline{Q}^d)$ , that is,  $\underline{q}(\mu, \gamma, \hat{\mu}) \geq \underline{q}^d$ .

LEMMA 2. *If  $(\mu, \gamma, \hat{\mu})$  is feasible for Problem (7), and  $\hat{\mu} < \mu - \gamma^{-1}$ , then*

$$\pi(s) = \begin{cases} p_1 = \frac{\hat{\mu}^2}{\mu(\gamma\mu-1)-\hat{\mu}}, & \text{if } s = s_1 = \frac{\mu(\gamma\mu-1)}{\hat{\mu}}, \\ p_2 = \frac{\gamma\mu^2(\mu-\hat{\mu}-\frac{1}{\gamma})}{\mu(\gamma\mu-1)-\hat{\mu}}, & \text{if } s = 1, \\ p_3 = 1 - \mu + \hat{\mu}, & \text{if } s = 0. \end{cases} \quad (9)$$

is feasible for Problem (7).

*Proof of Lemma 2* First, the lemma assumption implies that  $\mu \geq 0, \gamma \geq 1$  and  $\hat{\mu} \in \Omega_1$ , namely,

$$(\mu - 1)^+ \leq \hat{\mu} < \mu - \frac{1}{\gamma}. \quad (10)$$

The first inequality in (10) implies that  $p_3 = 1 - \mu + \hat{\mu} \geq 1 - \mu + (\mu - 1)^+ \geq 0$ . The second inequality in (10) implies that  $s_1 > 0$  since  $\gamma\mu > 1 + \gamma\hat{\mu} \geq 1$ , and  $p_1 \geq 0, p_2 \geq 0$  since

$$\mu(\gamma\mu - 1) - \hat{\mu} = \gamma\mu \left( \mu - \frac{1}{\gamma} \right) - \hat{\mu} > 1 \left( \mu - \frac{1}{\gamma} \right) - \hat{\mu} > 0. \quad (11)$$

So  $\pi(s)$  is a probability distribution. Moreover,  $s_1 > 1$  from (11), which, along with simple algebra, shows that  $\mathbb{E}_\pi(1) = 1, \mathbb{E}_\pi(S) = \mu, \mathbb{E}_\pi(S^2) = \gamma\mu^2$ , and  $\mathbb{E}_\pi((S-1)^+) = \hat{\mu}$ .  $\square$

LEMMA 3. If  $(\mu, \gamma, \hat{\mu})$  is feasible for Problem (7), and  $\hat{\mu} \geq \mu - \gamma^{-1}$ , then

$$\pi(s) = \begin{cases} p_1 = \frac{\mu(2\hat{\mu} + (\gamma-1)\mu + r(\mu, \gamma, \hat{\mu})) - 2\hat{\mu}}{2(1-2\mu + \gamma\mu^2)}, & \text{if } s = s_1 = \mu \left(1 + \frac{(\gamma-1)\mu - r(\mu, \gamma, \hat{\mu})}{2\hat{\mu}}\right), \\ 1 - p_1, & \text{if } s = s_2 = \mu \left(1 - \frac{(\gamma-1)\mu + r(\mu, \gamma, \hat{\mu})}{2(1-\mu + \hat{\mu})}\right). \end{cases} \quad (12)$$

is feasible for Problem (7), where

$$r(\mu, \gamma, \hat{\mu}) = \sqrt{(\gamma-1)((\gamma-1)\mu^2 + 4\hat{\mu}(\mu-1) - 4\hat{\mu}^2)}. \quad (13)$$

*Proof of Lemma 3* We first show that  $0 \leq s_2 \leq 1$ . The lemma assumption implies that  $\hat{\mu} \geq \mu - \gamma^{-1}$  and  $\mu \leq 2\gamma^{-1}$ . Therefore  $2(1 - \mu + \hat{\mu}) - (\gamma - 1)\mu = 2(1 + \hat{\mu}) - (\gamma + 1)\mu \geq 2(1 + \mu - \gamma^{-1}) - (\gamma - 1)\mu = (\gamma - 1)(2\gamma^{-1} - \mu) \geq 0$ . Note that  $s_2 \geq 0$  if and only if  $(2(1 - \mu + \hat{\mu}) - (\gamma - 1)\mu)^2 \geq r^2(\mu, \gamma, \hat{\mu}) = (\gamma - 1)((\gamma - 1)\mu^2 + 4\hat{\mu}(\mu - 1) - 4\hat{\mu}^2)$  or equivalently  $(\hat{\mu} - \mu + 1)(\gamma(\hat{\mu} - \mu) + 1) \geq 0$  after simple algebra, which is evidently true. On the other hand, we will show that  $s_2 < 1$ , which is trivially true if  $\mu \leq 1$ . In the case of  $\mu > 1$ ,  $\hat{\mu} \leq 2^{-1} \left( (\mu - 1) + \sqrt{(\gamma - 1)\mu^2 + (\mu - 1)^2} \right)$  implies that  $2(1 - \mu^{-1})(1 - \mu + \hat{\mu}) - (\gamma - 1)\mu < 0$ , implying further that  $2(1 - \mu^{-1})(1 - \mu + \hat{\mu}) - (\gamma - 1)\mu < r(\mu, \gamma, \hat{\mu})$ , or equivalently,  $s_2 < 1$ .

Then we will show that  $s_1 > 1$ , which is clearly true in case of  $\mu > 1$ , since  $(\gamma - 1)\mu - r(\mu, \gamma, \hat{\mu}) \geq 0$  from  $\hat{\mu} \geq \mu - 1$ . Consider the case of  $\mu \leq 1$ . It follows from  $\hat{\mu} \geq \mu - \gamma^{-1}$  that  $(\gamma - 1)\mu - 2(\mu^{-1} - 1)\hat{\mu} \geq \mu^{-1}((\gamma + 1)(\mu - \gamma^{-1})^2 + (\gamma - 1)\gamma - 2) \geq 0$ . Then simple algebra shows that  $(\gamma - 1)\mu - 2(\mu^{-1} - 1)\hat{\mu} > r(\mu, \gamma, \hat{\mu})$ , or equivalently  $s_1 > 1$ .

We may rewrite  $p_1$  as  $p_1 = \frac{(\mu - s_2)^2}{(\mu - s_2)^2 + \sigma^2}$ , implying that  $0 \leq p_1 \leq 1$ . So  $\pi(s)$  is a probability distribution. Finally,  $s_1 > 1$  and  $s_2 < 1$ , along with simple algebra, show that  $\mathbb{E}_\pi(1) = 1$ ,  $\mathbb{E}_\pi(S) = \mu$ ,  $\mathbb{E}_\pi(S^2) = \gamma\mu^2$ ,  $\mathbb{E}_\pi((S - 1)^+) = \hat{\mu}$ .  $\square$

We now present the closed-form solution for Problem (7).

THEOREM 5. If  $(\mu, \gamma, \hat{\mu})$  is feasible for Problem (7), then

$$\underline{q}(\mu, \gamma, \hat{\mu}) = \begin{cases} \mu(\gamma\mu - 1) - \hat{\mu} - \hat{\mu}^2, & \text{if } \hat{\mu} < \mu - \frac{1}{\gamma}, \\ \frac{1}{2} \left( 2\hat{\mu}(\mu - 1) + \mu((\gamma - 1)\mu - r(\mu, \gamma, \hat{\mu})) \right) - \hat{\mu}^2, & \text{if } \hat{\mu} \geq \mu - \frac{1}{\gamma}, \end{cases}$$

where  $r(\mu, \gamma, \hat{\mu})$  is defined in (13).

*Proof of Theorem 5* We consider two cases.

CASE 1.  $\hat{\mu} < \mu - \gamma^{-1}$ . If  $(\mu, \gamma, \hat{\mu})$  is feasible for Problem (7), then  $\pi(s)$  in (9) is feasible for Problem (7) from Lemma 2 and its corresponding objective value is equal to  $\mu(\gamma\mu - 1) - \hat{\mu} - \hat{\mu}^2$ . Thus  $\underline{q} \leq \mu(\gamma\mu - 1) - \hat{\mu} - \hat{\mu}^2$  if  $\mu \geq 2\gamma^{-1}$ . Consider the dual solution for  $(\underline{Q}^d)$  given by  $y_o = 0$ ,  $y_1 = -1$ ,  $y_2 = 1$ , and  $\hat{y}_1 = -1$ . This solution is feasible for  $(\underline{Q}^d)$ , and its corresponding objective value is equal to  $\mu(\gamma\mu - 1) - \hat{\mu} - \hat{\mu}^2 \leq \underline{q}^d$ . The weak duality implies that  $\underline{q}(\mu, \gamma, \hat{\mu}) = \mu(\gamma\mu - 1) - \hat{\mu} - \hat{\mu}^2$  when  $\hat{\mu} < \mu - \gamma^{-1}$ .

CASE 2.  $\hat{\mu} \geq \mu - \gamma^{-1}$ . If  $(\mu, \gamma, \hat{\mu})$  is feasible for Problem (7), then  $\pi(s)$  in (12) is feasible for Problem (7) from Lemma 3, and its corresponding objective value is equal to  $2^{-1} (2\hat{\mu}(\mu - 1) + \mu((\gamma - 1)\mu - r(\mu, \gamma, \hat{\mu}))) - \hat{\mu}^2$ . Thus  $\underline{q} \leq 2^{-1} (2\hat{\mu}(\mu - 1) + \mu((\gamma - 1)\mu - r(\mu, \gamma, \hat{\mu}))) - \hat{\mu}^2$ . Consider the dual solution for  $(\underline{Q}^d)$  given by

$$\begin{cases} y_o = -\frac{1}{2} \left( (2\hat{\mu} - \mu)\mu + \frac{\mu}{r(\mu, \gamma, \hat{\mu})} (2(\gamma - 2)\hat{\mu}^2 + (\gamma - 1)\mu^2 - 2\hat{\mu}(1 + (\gamma - 2)\mu)) \right) \\ y_1 = \hat{\mu} - \mu - \frac{1}{r(\mu, \gamma, \hat{\mu})} (2\hat{\mu}^2 - (\gamma - 1)\mu^2 + \hat{\mu}(2 + (\gamma - 3)\mu)) \\ y_2 = -\frac{1}{2} \left( \frac{1}{\mu r(\mu, \gamma, \hat{\mu})} ((\gamma - 1)\mu^2 + 2\hat{\mu}(\mu - 1) - 2\hat{\mu}^2) - 1 \right) \\ \hat{y}_1 = (\mu - 1) - \frac{1}{r(\mu, \gamma, \hat{\mu})} ((\gamma - 1)\mu(\mu - 1 - 2\hat{\mu})). \end{cases} \quad (14)$$

Note that

$$\begin{aligned} y_2 &= -\frac{1}{2} \left( \frac{1}{\mu r(\mu, \gamma, \hat{\mu})} ((\gamma - 1)\mu^2 + 2\hat{\mu}(\mu - 1) - 2\hat{\mu}^2) - 1 \right) \\ &= -\frac{1}{4} \left( \frac{r(\mu, \gamma, \hat{\mu})}{(\gamma - 1)\mu} + \frac{(\gamma - 1)\mu}{r(\mu, \gamma, \hat{\mu})} \right) + \frac{1}{2} < 0, \end{aligned}$$

and that

$$\begin{aligned} y_0 + y_1 s + y_2 s^2 &= y_2 (s - s_2)^2 \leq 0, \\ y_0 + y_1 s + y_2 s^2 + \hat{y}_1 (s - 1) - (s - 1)^2 &= (y_2 - 1)(s - s_1)^2 \leq 0, \end{aligned}$$

where  $s_1$  and  $s_2$  are defined in (12). That is, the solution in (14) is feasible for  $(Q^d)$ , and its corresponding objective value is equal to  $2^{-1} (2\hat{\mu}(\mu - 1) + \mu ((\gamma - 1)\mu - r(\mu, \gamma, \hat{\mu}))) - \hat{\mu}^2 \leq \bar{q}^d$ . The weak duality implies that  $\underline{q}(\mu, \gamma, \hat{\mu}) = 2^{-1} (2\hat{\mu}(\mu - 1) + \mu ((\gamma - 1)\mu - r(\mu, \gamma, \hat{\mu}))) - \hat{\mu}^2$  when  $\hat{\mu} \geq \mu - \gamma^{-1}$ .  $\square$

## Appendix A2-2: Closed-form solution for Problem (8)

The dual problem corresponding to Problem (8) can be written as:

$$\begin{aligned} \bar{q}^d &= \min y_0 + y_1 \mu + y_2 \gamma \mu^2 + \hat{y}_1 \hat{\mu} - \hat{\mu}^2 \\ (\bar{Q}^d) \quad \text{s.t.} \quad &y_0 + y_1 s + y_2 s^2 + \hat{y}_1 (s - 1)^+ \geq ((s - 1)^+)^2 \\ &\forall s \in \mathbb{R}_+ \end{aligned}$$

Clearly, weak duality holds between Problem (8) and  $(\bar{Q}^d)$ , that is,  $\bar{q}(\mu, \gamma, \hat{\mu}) \leq \bar{q}^d$ .

LEMMA 4. *If  $(\mu, \gamma, \hat{\mu})$  is feasible for Problem (8), and  $\hat{\mu} \geq \mu - \gamma^{-1}$ , then*

$$\pi(s) = \begin{cases} p_1 = \frac{\mu(2\hat{\mu} + (\gamma - 1)\mu - r(\mu, \gamma, \hat{\mu})) - 2\hat{\mu}}{2(1 - 2\mu + \gamma\mu^2)}, & \text{if } s = s_1 = \mu \left( 1 + \frac{(\gamma - 1)\mu + r(\mu, \gamma, \hat{\mu})}{2\hat{\mu}} \right), \\ 1 - p_1, & \text{if } s = s_2 = \mu \left( 1 - \frac{2(\gamma - 1)\hat{\mu}}{(\gamma - 1)\mu + r(\mu, \gamma, \hat{\mu})} \right), \end{cases} \quad (15)$$

is feasible for Problem (8), where  $r(\mu, \gamma, \hat{\mu})$  is defined in (13).

*Proof of Lemma 4* If  $\hat{\mu} \geq \mu - \gamma^{-1}$ , it follow from  $\mu \leq 2\gamma^{-1}$  that  $2(1 + \hat{\mu}) - (\gamma + 1)\mu \geq 0$ , implying that  $s_2 \geq 0$ . Otherwise,  $(\mu - 1)^+ \leq \hat{\mu} < \mu - \gamma^{-1}$ . Then it follows that  $(\hat{\mu} - \mu + 1)(\gamma(\hat{\mu} - \mu) + 1) \leq 0$ , or equivalently  $s_2 \geq 0$  by simple algebra. On the other hand, we will show that  $s_2 < 1$ . In the case of  $\mu \leq 1$ , it is trivially true. In the case of  $\mu > 1$ ,  $\hat{\mu} \geq \mu - 1$  implies that  $(\gamma - 1)\mu + r(\mu, \gamma, \hat{\mu}) < 2\hat{\mu}(\gamma - 1)(1 - \mu^{-1})^{-1}$ , or equivalently,  $s_2 < 1$ .

Then we will show that  $s_1 > 1$  in the follows. In case of  $\mu > 1$ ,  $s_1 > 1$  is clearly true, since  $(\gamma - 1)\mu + r(\mu, \gamma, \hat{\mu}) > 0$ . Consider the case of  $\mu \leq 1$ . We will shows that  $(\gamma - 1)\mu > 2(\mu^{-1} - 1)\hat{\mu}$ , or equivalently  $s_1 > 1$ . If  $\hat{\mu} < \mu - \gamma^{-1}$ , then  $(\gamma - 1)\mu - 2(\mu^{-1} - 1)\hat{\mu} \geq (\gamma + 1)(\mu - \gamma^{-1})^2 + (\gamma - 1)\gamma^{-2} > 0$  by simple algebra. Otherwise,  $\hat{\mu} \leq \left( (\mu - 1) + \sqrt{(\gamma - 1)\mu^2 + (\mu - 1)^2} \right) / 2$  implies that  $(\gamma - 1)\mu - 2(\mu^{-1} - 1)\hat{\mu} \geq \sqrt{(\gamma - 1)\mu^2 + (\mu - 1)^2} (\sqrt{(\gamma - 1)\mu^2 + (\mu - 1)^2} - (1 - \mu)) > 0$ .

We may rewrite  $p_1$  as  $p_1 = \frac{(\mu - s_2)^2}{(\mu - s_2)^2 + \sigma^2}$ , implying that  $0 \leq p_1 \leq 1$ . So  $\pi(s)$  is a probability distribution. Finally,  $s_1 > 1$  and  $s_2 < 1$ , along with simple algebra, show that  $\mathbb{E}_\pi(1) = 1$ ,  $\mathbb{E}_\pi(S) = \mu$ ,  $\mathbb{E}_\pi(S^2) = \gamma\mu^2$ ,  $\mathbb{E}_\pi((S - 1)^+) = \hat{\mu}$ .  $\square$

Now, we present the closed-form solution for Problem (8).

THEOREM 6. *If  $(\mu, \gamma, \hat{\mu})$  is feasible for Problem (8), then*

$$\bar{q}(\mu, \gamma, \hat{\mu}) = \frac{2\hat{\mu}(\mu - 1) + \mu((\gamma - 1)\mu + r(\mu, \gamma, \hat{\mu}))}{2} - \hat{\mu}^2, \quad (16)$$

where  $r(\mu, \gamma, \hat{\mu})$  is defined in (13).

*Proof of Theorem 6* Consider the dual solution for  $(\bar{Q}^d)$  given by

$$\begin{cases} y_0 = -\frac{1}{2} \left( (2\hat{\mu} - \mu)\mu - \frac{\mu}{r(\mu, \gamma, \hat{\mu})} (2(\gamma - 2)\hat{\mu}^2 + (\gamma - 1)\mu^2 - 2\hat{\mu}(1 + (\gamma - 2)\mu)) \right), \\ y_1 = \hat{\mu} - \mu + \frac{1}{r(\mu, \gamma, \hat{\mu})} (2\hat{\mu}^2 - (\gamma - 1)\mu^2 + \hat{\mu}(2 + (\gamma - 3)\mu)), \\ y_2 = \frac{1}{2} \left( \frac{1}{\mu r(\mu, \gamma, \hat{\mu})} ((\gamma - 1)\mu^2 + 2\hat{\mu}(\mu - 1) - 2\hat{\mu}^2) + 1 \right), \\ \hat{y}_1 = (\mu - 1) + \frac{1}{r(\mu, \gamma, \hat{\mu})} (\gamma - 1)\mu(\mu - 1 - 2\hat{\mu}). \end{cases} \quad (17)$$

Note that

$$\begin{aligned} y_2 &= \frac{1}{2} \left( \frac{1}{\mu r(\mu, \gamma, \hat{\mu})} ((\gamma - 1)\mu^2 + 2\hat{\mu}(\mu - 1) - 2\hat{\mu}^2) + 1 \right) \\ &= \frac{1}{4} \left( \frac{r(\mu, \gamma, \hat{\mu})}{(\gamma - 1)\mu} + \frac{(\gamma - 1)\mu}{r(\mu, \gamma, \hat{\mu})} \right) + \frac{1}{2} > 1, \end{aligned}$$

and that

$$\begin{aligned} y_0 + y_1 s + y_2 s^2 &= y_2 (s - s_2)^2 \geq 0, \\ y_0 + y_1 s + y_2 s^2 + \hat{y}_1 (s - 1) - (s - 1)^2 &= (y_2 - 1)(s - s_1)^2 \geq 0, \end{aligned}$$

where  $s_1$  and  $s_2$  are defined in (15). That is, the solution in (17) is feasible for  $(\bar{Q}^d)$ , and its corresponding objective value is equal to  $2^{-1} (2\hat{\mu}(\mu - 1) + \mu((\gamma - 1)\mu + r(\mu, \gamma, \hat{\mu}))) - \hat{\mu}^2 \geq \bar{q}^d$ . The weak duality implies that  $2^{-1} (2\hat{\mu}(\mu - 1) + \mu((\gamma - 1)\mu + r(\mu, \gamma, \hat{\mu}))) - \hat{\mu}^2 \geq \bar{q}(\mu, \gamma, \hat{\mu})$ .

On the other hand, if  $(\mu, \gamma, \hat{\mu})$  is feasible for Problem (8), then  $\pi(s)$  in (15) is feasible for Problem (8) from Lemma 4, and its corresponding objective value is equal to  $2^{-1} (2\hat{\mu}(\mu - 1) + \mu((\gamma - 1)\mu + r(\mu, \gamma, \hat{\mu}))) - \hat{\mu}^2 \leq \bar{q}(\mu, \gamma, \hat{\mu})$ .  $\square$

Theorems 5 and 6, after simple algebra, imply that the optimal solution for Problem (3) as given in Theorem 3.

## Appendix B: Proof of Theorem 4

*Proof of Theorem 4* We first consider the simpler case of  $\lambda < 0$ , where we need to solve the optimization problem (see (6) in Section 2):

$$\min_{\hat{\mu} \in \Omega_x} \theta(\hat{\mu}) = \mu - x\hat{\mu} - cx - \lambda \sqrt{\underline{v}(x; \hat{\mu})}.$$

From Theorem 3, we have that

$$\theta(\hat{\mu}) = \mu - x\hat{\mu} - cx - \lambda \sqrt{\left( \frac{\mu}{2} \sqrt{\gamma - 1} - y \right)^2} = \mu - x\hat{\mu} - cx - \lambda \left( \frac{\mu}{2} \sqrt{\gamma - 1} - y \right),$$

where  $y = \sqrt{\frac{(\gamma - 1)\mu^2}{4} + x\hat{\mu}(\mu - x) - x^2\hat{\mu}^2}$  and the second equality follows from that  $y \leq \frac{\mu}{2} \sqrt{\gamma - 1}$  by simple algebra. The derivative of  $\theta(\hat{\mu})$  is given by

$$\theta'(\hat{\mu}) = -\frac{x}{2} \left( 2 - \frac{\lambda(\mu - x - 2x\hat{\mu})}{y} \right).$$

Let  $\theta'(\hat{\mu}) = 0$ , that is,  $2y = \lambda(\mu - x - 2x\hat{\mu})$ . Squaring both sides, we have

$$4(\lambda^2 + 1)\hat{\mu}^2 - 4(\lambda^2 + 1)\left(\frac{\mu}{x} - 1\right)\hat{\mu} + \lambda^2\left(\frac{\mu}{x} - 1\right)^2 - (\gamma - 1)\left(\frac{\mu}{x}\right)^2 = 0.$$

Then we get the unique stationary point of  $\theta(\hat{\mu})$  as

$$\hat{\mu}_3(x) = \frac{1}{2} \left( \frac{\mu}{x} - 1 + \sqrt{\frac{(\gamma - 1)\left(\frac{\mu}{x}\right)^2 + \left(1 - \frac{\mu}{x}\right)^2}{\lambda^2 + 1}} \right).$$

And moreover,  $\theta'(\hat{\mu}) \leq 0$  when  $\hat{\mu} \leq \hat{\mu}_3(x)$  and  $\theta'(\hat{\mu}) \geq 0$  when  $\hat{\mu} \geq \hat{\mu}_3(x)$ . Hence,  $\hat{\mu}_3(x)$  is the minimizer of  $\theta(\hat{\mu})$  if  $\hat{\mu}_3(x) \in \Omega_x$ , or sufficiently if  $\hat{\mu}_3(x) \geq (\frac{\mu}{x} - 1)^+$ .

We will then proceed with the proof for  $\lambda \geq 0$ , where we need to solve the following optimization problem (see (6) in Section 2):

$$\min_{\hat{\mu} \in \Omega_x} \theta(\hat{\mu}) = \mu - x\hat{\mu} - cx - \lambda\sqrt{\bar{v}(x; \hat{\mu})}.$$

We consider two cases based on the range of  $\hat{\mu}$ .

CASE 1.  $(\frac{\mu}{x} - 1)^+ \leq \hat{\mu} \leq \frac{\mu}{x} - \frac{1}{\gamma}$ , where  $0 < x \leq \gamma\mu$ . From Theorem 3, we have that

$$\theta(\hat{\mu}) = \mu - x\hat{\mu} - cx - \lambda\sqrt{-\mu^2 + \mu x(1 + 2\hat{\mu}) - x^2\hat{\mu}(1 + \hat{\mu})},$$

and the derivative of  $\theta(\hat{\mu})$  is given by

$$\theta'(\hat{\mu}) = -\frac{x}{2} \left( 2 + \frac{\lambda(2\mu - x - 2x\hat{\mu})}{\sqrt{-\mu^2 + \mu x(1 + 2\hat{\mu}) - x^2\hat{\mu}(1 + \hat{\mu})}} \right).$$

For  $\gamma \leq 2$ , we have that  $\theta'(\hat{\mu}) \leq 0$  (hence  $\theta(\hat{\mu})$  is non-increasing) since  $\hat{\mu} \leq \frac{\mu}{x} - \frac{1}{\gamma} \leq \frac{\mu}{x} - \frac{1}{2}$ , implying that the minimizer of  $\theta(\hat{\mu})$  is achieved at the right boundary  $\hat{\mu}^* = \frac{\mu}{x} - \frac{1}{\gamma} = \hat{\mu}_1(x)$ . For  $\gamma > 2$ , the first-order condition  $\theta'(\hat{\mu}) = 0$  offers the unique stationary point  $\hat{\mu} = \frac{\mu}{x} - \frac{1}{2} + \frac{1}{2\sqrt{\lambda^2 + 1}} = \hat{\mu}_2(x)$ . The minimizer of  $\theta(\hat{\mu})$  can be therefore decided as follows:

(i) If  $\lambda < \frac{2\sqrt{\gamma-1}}{\gamma-2}$ , then we have that

$$\frac{\mu}{x} - \frac{1}{2} + \frac{1}{2\sqrt{\lambda^2 + 1}} > \frac{\mu}{x} - \frac{1}{\gamma};$$

that is,  $\theta'(\hat{\mu}) < 0$  (hence  $\theta(\hat{\mu})$  is decreasing) when  $(\frac{\mu}{x} - 1)^+ \leq \hat{\mu} \leq \frac{\mu}{x} - \frac{1}{\gamma}$ , implying that the minimizer of  $\theta(\hat{\mu})$  is achieved at the right boundary

$$\hat{\mu}^* = \frac{\mu}{x} - \frac{1}{\gamma} = \hat{\mu}_1(x).$$

(ii) Otherwise, if  $\lambda \geq \frac{2\sqrt{\gamma-1}}{\gamma-2}$ , then  $(\frac{\mu}{x} - 1)^+ \leq \frac{\mu}{x} - \frac{1}{2} + \frac{1}{2\sqrt{\lambda^2 + 1}} \leq \frac{\mu}{x} - \frac{1}{\gamma}$ , and hence the minimizer of  $\theta(\hat{\mu})$  is given by

$$\hat{\mu}^* = \frac{\mu}{x} - \frac{1}{2} + \frac{1}{2\sqrt{\lambda^2 + 1}} = \hat{\mu}_2(x).$$

CASE 2.  $\frac{\mu}{x} - \frac{1}{\gamma} \leq \hat{\mu} \leq \frac{(\frac{\mu}{x} - 1) + \sqrt{(\gamma-1)(\frac{\mu}{x})^2 + (\frac{\mu}{x} - 1)^2}}{2}$ , where  $x \geq \frac{\gamma\mu}{2}$ . From Theorem 3, we have that

$$\theta(\hat{\mu}) = \mu - x\hat{\mu} - cx - \lambda \left( y + \frac{\mu}{2} \sqrt{\gamma - 1} \right),$$

where  $y = \sqrt{\frac{(\gamma-1)\mu^2}{4} + x\hat{\mu}(\mu - x) - x^2\hat{\mu}^2}$ . The derivative of  $\theta(\hat{\mu})$  is given by

$$\theta'(\hat{\mu}) = -\frac{x}{2} \left( 2 + \frac{\lambda(\mu - x - 2x\hat{\mu})}{y} \right).$$

The first-order condition  $\theta'(\hat{\mu}) = 0$  offers the unique stationary point:

$$\hat{\mu}_3(x) = \frac{1}{2} \left( \frac{\mu}{x} - 1 + \sqrt{\frac{(\gamma-1)(\frac{\mu}{x})^2 + (1 - \frac{\mu}{x})^2}{\lambda^2 + 1}} \right).$$

The minimizer of  $\theta(\hat{\mu})$  can be therefore decided as follows:

- (i) If  $\lambda > \frac{(2x - \gamma\mu)\sqrt{\gamma-1}}{|\gamma\mu + \gamma x - 2x|}$ , then we have that  $\hat{\mu}_3(x) < \frac{\mu}{x} - \frac{1}{\gamma}$ ; that is,  $\theta'(\hat{\mu}) > 0$  (hence  $\theta(\hat{\mu})$  is increasing) when  $\frac{\mu}{x} - \frac{1}{\gamma} \leq \hat{\mu} \leq \frac{(\frac{\mu}{x} - 1) + \sqrt{(\gamma-1)(\frac{\mu}{x})^2 + (\frac{\mu}{x} - 1)^2}}{2}$ , implying that  $\hat{\mu}^* = \frac{\mu}{x} - \frac{1}{\gamma} = \hat{\mu}_1(x)$ .
- (ii) Otherwise, we have  $(\frac{\mu}{x} - 1)^+ \leq \hat{\mu}_3(x) \leq \frac{\mu}{x} - \frac{1}{\gamma}$ , implying that the optimal solution is achieved at the stationary point, namely  $\hat{\mu}^* = \hat{\mu}_3(x)$ .  $\square$

## Appendix C: Proof of Theorem 2

We need two lemmas in Appendix C1 before the final proof Theorem 2 in Appendix C2.

### Appendix C1: Two lemmas

LEMMA 5. If  $1 - \lambda\sqrt{\gamma-1} \geq \gamma c$ ,

$$1 + \frac{(1-2c)\sqrt{\gamma-1}}{\sqrt{\lambda^2 + 4c(1-c)}} \geq \frac{\gamma}{2}.$$

*Proof of Lemma 5* We consider three cases.

CASE 1. If  $\gamma \leq 2$  and  $1 \geq 2c$ , then

$$1 + \frac{(1-2c)\sqrt{\gamma-1}}{\sqrt{\lambda^2 + 4c(1-c)}} \geq 1 \geq \frac{\gamma}{2}.$$

CASE 2. If  $\gamma \leq 2$  and  $1 < 2c$ , then it follow that

$$\gamma \leq \frac{1}{c} < \frac{1}{1-c},$$

implying that  $4((\gamma-1) - \gamma c) \cdot (1 - \gamma c) \leq 0 \leq (\gamma-2)^2 \lambda^2$ , which is equivalent to the desired inequality.

CASE 3. If  $\gamma > 2$ , and  $|\lambda| \leq \frac{1-\gamma c}{\sqrt{\gamma-1}}$ , then it follows that  $3\gamma - 4 > c(6\gamma - 8) \geq c(-\gamma^2 + 8\gamma - 8)$  from  $1 \geq \gamma c > 2c$ , implying that

$$4(\gamma - 1 - \gamma c) \geq \frac{(\gamma-2)^2}{\gamma-1}(1 - \gamma c),$$

which, together with the case assumption  $|\lambda| \leq \frac{1-\gamma c}{\sqrt{\gamma-1}}$ , implies that  $4((\gamma-1) - \gamma c) \cdot (1 - \gamma c) \geq (\gamma-2)^2 \lambda^2$ , which is equivalent to the desired inequality.

CASE 4. If  $\gamma > 2$ , and  $\lambda \leq -\frac{1-\gamma c}{\sqrt{\gamma-1}}$ , then

$$1 + \frac{(1-2c)\sqrt{\gamma-1}}{\sqrt{\lambda^2 + 4c(1-c)}},$$

is non-increasing with  $\lambda$  within the range specified in the case assumption, implying the desired result.  $\square$

LEMMA 6. Let

$$x^0 = \mu \left( 1 + \frac{(1-2c)\sqrt{\gamma-1}}{\sqrt{\lambda^2 + 4c(1-c)}} \right).$$

If  $1 - \lambda\sqrt{\gamma-1} \geq \gamma c$  and  $\lambda \geq 0$ , then

$$\lambda \leq \frac{(2x^0 - \gamma\mu)\sqrt{\gamma-1}}{|\gamma\mu + \gamma x^0 - 2x^0|}.$$



*Proof of Lemma 6* We consider two cases.

CASE 1.  $\gamma \geq 2$ . The case assumption implies that  $\gamma\mu + \gamma x - 2x > 0, \forall x \geq 0$ . Moreover,  $0 \leq \lambda \leq \frac{1-\gamma c}{\sqrt{\gamma-1}}$  implies that  $-(\gamma-1)\lambda^2 - (\gamma-2)\sqrt{\gamma-1}\lambda + (\gamma-1) - c(1-c)\gamma^2 \geq 0$ , or equivalently

$$(\lambda + \sqrt{\gamma-1})\gamma\mu \leq \left(2\sqrt{\gamma-1} - (\gamma-2)\lambda\right) \left(1 + \frac{(1-2c)\sqrt{\gamma-1}}{\sqrt{\lambda^2 + 4c(1-c)}}\right) \mu,$$

implying the desired claim after simple algebra.

CASE 2.  $\gamma < 2$ . We consider three subcases.

- (i) First of all, if  $x^0 < \frac{\gamma\mu}{2-\gamma}$ , then the proof follows from the same argument as in Case 1 since  $\gamma\mu + \gamma x^0 - 2x^0 > 0$ .
- (ii) Second of all, if  $x^0 \geq \frac{\gamma\mu}{2-\gamma}$ , and  $\lambda \leq \frac{2\sqrt{\gamma-1}}{2-\gamma}$ . Then  $x^0 \geq \frac{\gamma\mu}{2-\gamma}$  implies that  $\gamma\mu + \gamma x - 2x < 0$  for any  $x > x^0$ . Note that  $\frac{(2x-\gamma\mu)\sqrt{\gamma-1}}{|\gamma\mu + \gamma x - 2x|}$  is an increasing function of  $x$  in the range  $\left[\frac{\gamma\mu}{2}, \frac{\gamma\mu}{2-\gamma}\right)$  and decreasing in  $\left(\frac{\gamma\mu}{2-\gamma}, +\infty\right)$ . Moreover,

$$\lim_{x \rightarrow +\infty} \frac{(2x-\gamma\mu)\sqrt{\gamma-1}}{|\gamma\mu + \gamma x - 2x|} = \frac{2\sqrt{\gamma-1}}{2-\gamma}.$$

Therefore the desired result follows from the second case assumption:

$$\lambda \leq \frac{2\sqrt{\gamma-1}}{2-\gamma} < \frac{(2x^0 - \gamma\mu)\sqrt{\gamma-1}}{|\gamma\mu + \gamma x^0 - 2x^0|}.$$

- (iii) Now we consider the remaining case, where  $\gamma < 2$ ,  $x^0 > \frac{\gamma\mu}{2-\gamma}$  and  $\lambda > \frac{2\sqrt{\gamma-1}}{2-\gamma}$ . Note that  $\lambda > \frac{2\sqrt{\gamma-1}}{2-\gamma} > \frac{(\gamma-1)-\gamma c}{\sqrt{\gamma-1}}$  implies that  $(\gamma-1)\lambda^2 + (2-\gamma)\sqrt{\gamma-1}\lambda - (\gamma-1) + c(1-c)\gamma^2 \geq 0$ , or equivalently

$$\left(\lambda(2-\gamma) - 2\sqrt{\gamma-1}\right) \cdot \left(1 + \frac{(1-2c)\sqrt{\gamma-1}}{\sqrt{\lambda^2 + 4c(1-c)}}\right) \leq (\lambda - \sqrt{\gamma-1})\gamma,$$

implying the desired claim after simple algebra.

□

## Appendix C2: Proof of Theorem 2

Now, we have all the ingredients to prove Theorem 2.

*Proof of Theorem 2* We consider the two cases  $\lambda \geq 0$  and  $\lambda < 0$  separately.

(I) From Theorem 4, when  $\lambda \geq 0$ , we consider three cases as follows:

CASE 1.  $h(x) = h_1(x) = \frac{x}{\gamma}(1 - \lambda\sqrt{\gamma-1} - \gamma c)$  under the condition  $\lambda > \frac{(2x-\gamma\mu)\sqrt{\gamma-1}}{\gamma\mu + \gamma x - 2x}$ . If  $1 - \lambda\sqrt{\gamma-1} \leq \gamma c$  in  $h(x)$ , then  $h(x)$  being a linear function with a non-positive slope implies that the maximizer of  $h(x)$  is  $x^* = 0$ . Otherwise, if  $1 - \lambda\sqrt{\gamma-1} > \gamma c$ , namely,  $h(x)$  has a positive slope, then  $x^*$  should assume the largest possible value in its range. Note that the right-hand-side of the case condition  $\lambda > \frac{(2x-\gamma\mu)\sqrt{\gamma-1}}{\gamma\mu + \gamma x - 2x}$  as a function of  $x$  is increasing, implying that the largest feasible value of  $x$  satisfies that  $\lambda = \frac{(2x-\gamma\mu)\sqrt{\gamma-1}}{\gamma\mu + \gamma x - 2x}$ , which shares the same assumption as that for CASE 3, and hence the desired result is implied by the argument therein.

CASE 2.  $h(x) = h_2(x) = -(2^{-1}(\sqrt{\lambda^2 + 1} - 1) + c)x$ . Then  $h_2(x)$  being non-increasing implies that the maximizer of  $h(x)$  is  $x^* = 0$ .

CASE 3.  $h(x) = h_3(x)$ , where

$$h_3(x) = \frac{1}{2} \left( x - 2cx - \sqrt{\lambda^2 + 1} \cdot \sqrt{(\mu - x)^2 + (\gamma - 1)\mu^2} + (1 - \lambda\sqrt{\gamma-1})\mu \right).$$

The first-order condition offers the unique stationary point of  $h(x)$  as

$$x^* = \mu \left( 1 + \frac{(1-2c)\sqrt{\gamma-1}}{\sqrt{\lambda^2 + 4c(1-c)}} \right).$$

Lemmas 5 and 6 together imply that  $x^* \geq \frac{\gamma\mu}{2}$  and  $\lambda \leq \frac{(2x^* - \gamma\mu)\sqrt{\gamma-1}}{|\gamma\mu + \gamma x^* - 2x^*|}$ , and hence  $x^*$  is the maximizer of  $h(x)$ .

(II) When  $\lambda < 0$ ,  $h(x) = h_3(x)$ . Let  $h'(x) = 0$ , we obtain the unique stationary point of  $h(x)$ :

$$x^* = \mu \left( 1 + \frac{(1-2c)\sqrt{\gamma-1}}{\sqrt{\lambda^2 + 4c(1-c)}} \right).$$

Lemma 5 implies that  $x^* \geq \mu\gamma/2 \geq 0$ . Recall the definition of  $\hat{\mu}(x)$  from Section 2

$$\hat{\mu}_3(x^*) = \frac{1}{2} \left( \frac{\mu}{x^*} - 1 + \sqrt{\frac{(\gamma-1)\left(\frac{\mu}{x^*}\right)^2 + \left(1 - \frac{\mu}{x^*}\right)^2}{\lambda^2 + 1}} \right),$$

We now show that  $\hat{\mu}_3(x^*) \in \Omega_{x^*}$ , together with  $x^* \geq 0$ , imply that  $x^*$  is feasible for Problem (2). It is sufficient to show that  $\hat{\mu}_3(x^*) \geq \left(\frac{\mu}{x^*} - 1\right)^+$ , which is obviously true if  $0 \leq c \leq 0.5$ . For  $0.5 < c \leq 1$ , inequality  $(2c-1)^2 \leq 1$  is equivalent to  $\frac{x^*}{\mu} \geq 1 + \frac{\sqrt{\gamma-1}}{\lambda}$ , implying that  $\hat{\mu}_3(x^*) \geq \frac{\mu}{x^*} - 1$ .

Moreover, note that  $h'(x) \leq 0$  for  $x \leq x^*$  and  $h'(x) \geq 0$  for  $x \geq x^*$  by simple algebra. Hence,  $h(x)$  achieves its minimum at  $x^*$ .  $\square$