The optimal harvesting problem under risk aversion*

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April 3, 2012

Abstract

We study the exploitation of a one species forest plantation when timber price is uncertain. The work focuses on providing optimality conditions for the optimal harvesting policy in terms of the parameters of the price process and the discount factor. We use risk averse stochastic dynamic programming and use the Conditional Value-at-Risk (CVaR) as our main risk measure. We consider two important cases: when prices follow a geometric Brownian motion we completely characterize the optimal policy for all possible choices of drift and discount factor. When prices are governed by a mean-reverting (Ornstein-Uhlenbeck) process we provide sufficient conditions, based on explicit expressions for a reservation price period above which harvesting everything available is optimal. In both cases we solve the problem for *every* initial condition and the best policy is obtained endogenously, that is, without imposing any ad hoc restrictions such as maximum sustained yield or convergence to a predefined final state. We compare our results with the risk neutral framework and highlight the differences between the two cases. We generalize our results to any coherent risk measure that is affine on the current price and calculate the coefficients for risk measures other than the CVaR.

Keywords Forest management, risk aversion, coherent risk measures, stochastic dynamic programming.

1 Introduction

The presence of uncertainty in decision making can be seen in virtually all areas. For example in finance the returns of the assets are uncertain and the investor usually wants to maximize her returns (or minimize her risk) while having risk under control (or returns above some given threshold), see for example Bonami and Lejeune [2009], Mansini, Ogryczak, and Speranza [2007] and Pagnoncelli, Ahmed, and Shapiro [2009]. In hydrothermal scheduling, rain inflow is usually unknown and system operators want to avoid the higher cost of thermal fuel and penalties coming from energy shortages Guigues and Sagastizabal [2010], Philpott and de Matos [2012]. In those settings the presence of uncertainty comes along with the problem of risk management: in finance, a decision maker fears losing a significant part of his earnings and in the context of hydrothermal scheduling she wants to avoid disruptions in the system that could generate heavy penalty costs. In both cases, the decision maker seeks a less risky policy that avoids extreme risks at the expense of an increase in the expected cost. Maximizing the expected value or minimizing the expected costs are certainly valid approaches to deal with uncertainty, but there is a growing tendency of trying to reach a

*This research was partially supported by Project Anillo ACT-88 and Programa Basal PFB 03, CMM. Adriana Piazza also acknowledges the financial support of Fondecyt under Project 11090254.

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compromise between optimizing the objective on average and at the same time reducing the risk associated with a decision. Risk averse optimization has received an increasing amount of attention in the recent years, see for example Guigues and Romisch [2010], Kovacevic and Pflug [2009], Miller and Ruszczynski [2011], Philpott and de Matos [2012], Ruszczyński [2010], Shapiro [2009].

In this paper we are interested in modeling risk averseness in the context of natural resources planning, focusing on harvest scheduling. We survey some relevant results from the literature of natural resources. In Alvarez and Koskela [2006] the authors study the effect of risk aversion in the expected length of the forest rotation period. Some of the results obtained include showing that higher risk aversion shortens the expected rotation period and that increased forest value volatility decreases the optimal harvesting threshold, which is not true under risk neutrality. In Mosquera, Henig, and Weintraub [2011] the authors study a tactical planning problem subject to uncertainty in prices. Risk aversion is measured by a piecewise linear utility function. The paper Gong and Lofgren [2003] investigates the effect of risk aversion on the optimal harvesting behavior and show that it depends on the sign of a marginal variance function.

The concern to include risk aversion in the optimization of natural resources also appears in fisheries. In Ewald and Wang [2010] the authors consider the maximum expected sustainable yield problem in which the objective value is the expected value minus a positive constant times the variance, where the constant represent the level of risk aversion. Similarly, Yamazaki, Kompas, and Grafton [2009] compares the management outcomes of Total Allowable Catch (TAC) and Total Allowable Effort (TAE) policies in a scenario of uncertainty. The authors conclude that neither policy is always preferred and the optimal choice depends on the level of risk aversion of the decision maker. In water management, the paper Petersen and Schilizzi [2010] models risk by means of an expected utility function and determines the impact of price and yield risk on a farmer's expected level and variance of net revenue using a Monte-Carlo methodology.

While the works Brazee and Mendelsohn [1988], Clarke William and Harry [1989], Thomson [1992] assume normality and adopt a Geometric Brownian Motion (GBM) to represent the evolution of timber prices, others, like Alvarez and Koskela [2005], Gjolberg and Guttormsen [2002], argue that a mean reverting process, or Ornstein-Uhlenbeck (O-U) is a better description of the timber price path due to empirical data that has been collected for several species. It is not our intention to go any further into this discussion, we refer the reader to Insley and Rollins [2005] and references therein. As the issue seems far from being settled, we decided to consider both stochastic processes most commonly used to describe natural resource prices: GBM and O-U.

We propose a stochastic dynamic model for harvest scheduling in which the decision maker wishes to minimize the overall risk of her decisions. We build upon our previous work [Pagnoncelli and Piazza, 2011] where we considered a simple model that neglects natural mortality and assumes that only mature trees can be harvested. In Pagnoncelli and Piazza [2011] we completely characterized the optimal policy in the case where prices followed GBM and, for the O-U case, we found a sufficient condition for the optimality of harvesting every mature tree that can be translated into a reservation price. However, the results are valid only for the risk-neutral case. The two goals of this paper are to characterize the optimal policy under risk-averseness and to establish a connection between the risk-neutral and risk averse worlds, that is, to understand exactly how the inclusion of risk affects the optimal harvest policy. We will consider several risk measures but our main focus will be on the Conditional Value-at-Risk (CVaR).

The rest of the paper is as follows: In Section 2 we give a complete description of our model. In Section 3 we write dynamic programming equations for the model and discuss the introduction of risk measures in a dynamic framework. With the assumption that prices follow a Geometric Brownian Motion, we completely characterize the optimal policy in Section 4 and compare the results obtained with the risk neutral case. In Section 5 we assume prices follow an Ornstein-Uhlenbeck process and obtain sufficient conditions depending on the parameters of the process and on the discount factor for the optimality of harvesting every mature tree. In sections 4 and 5, we consider CVaR as our risk measure, while in Section 6 we generalize some of our results for risk measures other than the CVaR and construct an efficient frontier based on different levels of risk aversion. Section 7 concludes the paper and discuss possible extensions.

2 Model formulation

Let us consider a forest of total area S occupied by one species forest with maturity age of n years. In contrast with the case of wild forests, the state of a forest plantation may be described by specifying the areas occupied by trees of different ages, making the assumption that tress are planted within a pre-specified and constant distance of each other.

For each period $t \in \mathbb{N}$ we denote $x_a(t) \geq 0$ the area of trees of age $a = 1, \dots, n$ in year t, and $\bar{x}(t) \geq 0$ the area occupied by trees beyond maturity (older than n). Using a single state variable to represent the over-mature trees conveys the underlying assumption that the growth of trees is negligible beyond maturity. Each period we must decide how much land $c(t) \geq 0$ to harvest. Assuming that only mature trees can be harvested we must have

$$0 \leq c(t) \leq \bar{x}(t) + x_n(t), \tag{1}$$

and then the area not harvested in that period will comprise the over-mature trees at the next step, namely

$$\bar{x}(t+1) = \bar{x}(t) + x_n(t) - c(t). \tag{2}$$

We neglect natural mortality at every age, again an assumption valid in managed forest plantations but not in wild forests. Hence, the transition between age classes is given by

$$x_{a+1}(t+1) = x_a(t) \ \forall \ a = 1, \dots, n-1. \tag{3}$$

The harvested area is immediately allocated to new seedlings that will comprise the 1 year old tress in the following year:

$$x_1(t+1) = c(t). \tag{4}$$

We represent the *state* of the tree population by the vector state

$$\mathbb{X} = (\bar{x}, x_n, \dots, x_1)$$

and the dynamics described by equations (2), (3) and (4) can be represented as follows:

$$\mathbb{X}(t+1) = A\mathbb{X}(t) + Bc(t), \tag{5}$$

where

$$A = \begin{pmatrix} 1 & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ 0 & \dots & \dots & \dots & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

The expression of constraint (1) in terms of the defined matrices is

$$0 \leq c(t) \leq CA\mathbb{X}(t) \quad \text{where} \quad C = \begin{pmatrix} 1 & 0 & \dots\dots & 0 \end{pmatrix}. \tag{6}$$

The order of events is the following: at every period t the decision maker observes the state of the forest and the current price and decides how much she will harvest, $c(t)$, obtaining a benefit $p(t)c(t)$. Her decision changes the current state of the system from $\mathbb{X}(t)$ to $\mathbb{X}(t+1)$ according to (5). Then timber price $p(t+1)$ is observed, leaving her in position of deciding the following harvest $c(t+1)$. The process repeats itself until the last period T. We show a schematic view of the sequence of events in our model:

$$(p(1), \mathbb{X}(1)) \rightsquigarrow c(1) \rightsquigarrow \mathbb{X}(2) \rightsquigarrow p(2) \rightsquigarrow c(2) \rightsquigarrow \mathbb{X}(3) \rightsquigarrow \dots \rightsquigarrow c(T-1) \rightsquigarrow \mathbb{X}(T) \rightsquigarrow p(T) \rightsquigarrow c(T).$$

3 Risk averse formulation

The main goal of this paper is to depart from the classical risk-neutral framework by including risk measures into the objective function. A *risk measure* is a real-valued function $\rho(\cdot)$ defined on a space of measurable functions. The expected value $\mathbb{E}[\cdot]$ of a random variable is an example of a risk measure. Another well-known example is the so-called Value-at-Risk:

$$\text{VaR}_\alpha[X] = \inf\{x : \mathbb{P}(X \le x) \ge 1-\alpha\}, \quad \alpha \in (0,1).$$

In other words the Value-at-Risk is the left side $(1-\alpha)$ quantile of the distribution of X. A risk measure that gained increasing popularity since the publication of Rockafellar and Uryasev [2000] is the Conditional Value-at-Risk (CVaR). The CVaR is defined as the average of losses above the VaR level:

$$\text{CVaR}_\alpha[X] = \mathbb{E}[X | X > \text{VaR}_\alpha[X]].$$

We are going to use the CVaR extensively throughout our paper since it will be our prototype example of risk measure.

3.1 Dynamic programming formulation

In this paper we deal with risk averse stochastic dynamic problems and in order to write dynamic programming equations we need to extend the concept of conditional expectation to accommodate arbitrary risk measures. If X and Y are random variables, we can consider the value of ρ at the conditional distribution of X given $Y = y$, which we denote as $\rho_{|Y}[X]$ and refer to $\rho_{|Y}[\cdot]$ as a *conditional risk measure* Ruszczynski and Shapiro [2005], Shapiro [2012]. Following Shapiro [2009], we consider a nested risk averse formulation based on conditional risk mappings:

$$\left\{ \begin{array}{rcl} V_0(p(0), \mathbb{X}(0)) & = & \text{Min}_{c(0)} \Big\{ -p(0)c(0) + \delta\rho_{|p(0)}\Big[\text{Min}_{c(1)} -p(1)c(1) + \dots \\ & & \dots + \delta\rho_{|p(T-1)}\Big[\text{Min}_{c(T)} -p(T)c(T)\Big]\Big]\Big\} \\ \text{s.t.} & & (5) \text{ and } (6), \end{array} \right. \tag{7}$$

where $\rho_{|p(t)}[\cdot]$ is a conditional risk mapping for $t = 0, \dots, T-1$ and $\delta \in (0,1)$ is the discount factor. We refer the reader to Ruszczynski and Shapiro [2005] for details. [1]

In Ruszczynski and Shapiro [2005], it is shown that the corresponding dynamic programming equations are,

$$V_t(p(t), \mathbb{X}(t)) = \underset{c(t)\in[0, CA\mathbb{X}(t)]}{\text{Min}} \Big\{ -p(t)c(t) + \delta\rho_{|p(t)}\left[(V_{t+1}(p(t+1), A\mathbb{X}(t) + Bc(t)) \right] \Big\}, \tag{8}$$

for all $t = 1, \dots, T-1$ and

$$V_T(p(T), \mathbb{X}(T)) = \underset{c(T)\in[0, CA\mathbb{X}(T)]}{\text{Min}} \Big\{ -p(T)c(T)\Big\}.$$

For the last period $t = T$ it is easy to solve the underlying optimization problem: just consume the maximum possible amount, subject to the dynamic constraints. Hence the optimal solution of the problem is $c^*(T) = CA\mathbb{X}(T)$ and $V_T(p(T), \mathbb{X}(T)) = -p(T)c^*(T)$. If the conditional risk mappings is positive homogeneous (see Shapiro [2009]) we can write the dynamic programming equation for period $t = T-1$,

$$\begin{aligned} V_{T-1}(p(T-1), \mathbb{X}(T-1)) &= \underset{c(T-1)}{\text{Min}} \{-p(T-1)c(T-1) + \delta\rho_{|p(T-1)}[V_T(p(T), \mathbb{X}(T))]\} \\ &= \underset{c(T-1)}{\text{Min}} \{-p(T-1)c(T-1) + \delta CA\mathbb{X}(T)\rho_{|p(T-1)}[-p(T)]\} \\ &= \delta(\bar{x}(T-1) + x_n(T-1) + x_{n-1}(T-1))\rho_{|p(T-1)}[-p(T)] \\ &\quad + \underset{c(T-1)}{\text{Min}} \{c(T-1)(-p(T-1) - \delta\rho_{|p(T-1)}[-p(T)])\}. \end{aligned}$$

[1] In the cited references, the conditional risk mapping is conditional on the whole history of the process. As we will consider only Markovian price processes, we condition only on the present price.

We have that if

$$-p(T-1) - \delta\rho_{|p(T-1)}[-p(T)] \le 0, \tag{9}$$

the optimal solution at $T-1$ is $c^*(T-1) = CA\mathbb{X}(T-1)$, with corresponding optimal value

$$V_{T-1}(p(T-1), \mathbb{X}(T-1)) = -p(T-1)(\bar{x}(T-1) + x_n(T-1)) + \delta\rho_{|p(T-1)}[-p(T)]x_{n-1}(T-1).$$

Remark 3.1 *In the calculus above, we are able to characterize the optimal policy for $t = T-1$ according to a simple condition* (9)*, because the future only comprises $t = T$ when the optimal policy is known. In the general case, the determination of the optimal policy for an arbitrary t is much more complicated, as we do not know future policies. Solving this difficulty is the main purpose of this article. We will see that even if the situation is much more complicated for $t < T-1$, conditions analogous to* (9)*, i.e.,*

$$-p(t) - \delta\rho_{|p(t)}[-p(t+1)] \le 0 \quad \textit{for all } t = 0, \ldots, T-1, \tag{10}$$

play a fundamental role in the characterization of the optimal policy in this article. Observe that these conditions do not consider the entire future, but only the price of timber one period ahead *of time.*

3.2 Conditional risk mappings

The expressions obtained so far are valid for any positive homogeneous conditional risk mapping and any price process $p(t)$. In this paper we will consider different risk measures but will focus on the Conditional Value-at-Risk (CVaR), which is a very popular risk measure studied in Rockafellar and Uryasev [2000]. In the seminal work of Artzner et al. [1999], the authors defined a set of axioms that a risk measure must satisfy in order to be called *coherent*: positive homogeneity, translation invariance, monotonicity and subadditivity. The CVaR satisfies all those properties and therefore is a coherent risk measure. The Value-at-Risk, however, is not coherent since it lacks subadditivity [see for example Cornuejols and Tütüncü, 2007].

For a random variable X in $L_1(\Omega, \mathcal{F}, \mathbb{P})$ and for a risk aversion parameter $\alpha \in (0, 1]$, we have

$$\mathrm{CVaR}_\alpha[X] := \mathbb{E}[X | X > \mathrm{VaR}_\alpha[X]] = \inf\left\{t \in \mathbb{R} : t + \alpha^{-1}\mathbb{E}[X-t]_+\right\}, \tag{11}$$

where the symbol $[a]_+$ denotes the maximum between a and 0 and the equality was proved in Rockafellar and Uryasev [2000]. The corresponding conditional risk mapping can be defined analogously using the second definition of CVaR given in (11):

$$\mathrm{CVaR}_{\alpha|Y}[X] = \inf\left\{t \in \mathbb{R} : t + \alpha^{-1}\mathbb{E}_{|Y}[X-t]_+\right\}, \quad \alpha \in (0, 1].$$

In the following two sections we derive explicit conditions for the optimality of the *greedy policy*, i.e., harvesting every mature tree, under GBM and O-U prices using the CVaR as our risk measure.

4 Geometric Brownian Motion

We first study problem (7) when the dynamics of prices follow a Geometric Brownian Motion (GBM). This dynamics has been extensively used to model asset prices in financial markets and therefore represents a natural choice for timber prices. It is a well known process, hence, we define it without further detail.

$$dp(t) = \mu p(t)dt + \sigma p(t)dW_t \qquad \text{(GBM)}, \tag{12}$$

where $\mu \in \mathbb{R}$ is the *drift* of the GBM, $\sigma > 0$ is the constant variance and W_t denotes the Wiener process.

For the GBM, at each time $t+1$ the price $p(t+1)$ conditional on price t follows a lognormal distribution, and it is possible to compute $\mathrm{CVaR}_{\alpha|p(t)}[-p(t+1)]$ explicitly:

$$\mathrm{CVaR}_{\alpha|p(t)}[-p(t+1)] = -p(t)\frac{e^\mu}{\alpha}\Phi(z_{1-\alpha} - \sigma),$$

where Φ is the cumulative distribution function of the normal random variable with mean 0 and variance 1 and $z_\alpha = \Phi^{-1}(1-\alpha)$. We present the details in the Appendix.

It is easy to see that (10) is equivalent to

$$p(t)(-1+\delta e^{\mu} C_{gbm}) \leq 0, \text{ with } 0 < C_{gbm} = \frac{1}{\alpha}\Phi(z_{1-\alpha} - \sigma) < 1.$$

Therefore, condition (10) is equivalent to

$$\delta e^{\mu} C_{gbm} \leq 1, \tag{13}$$

that depends only on the parameters of the problem (δ, α) those of the price process (μ, σ) and the α-level of the CVaR_α. This implies that when prices follow a GBM, condition (10) is satisfied for all $p(t)$ or for none. We are now in condition to state the main result of this section, the characterization of the *greedy policy*, i.e., $c^*(t) = CA\mathbb{X}(t)$ for all t, when prices follow a GBM.

Theorem 4.1 *Consider problem* (7) *and assume prices evolve according to* (12). *If condition* (13) *holds, the greedy policy is optimal.*

Proof is relegated to the appendix.

In Pagnoncelli and Piazza [2011], the authors showed that in the risk neutral setting under GBM, the greedy policy is optimal if and only if

$$\delta e^{\mu} \leq 1. \tag{14}$$

In the next subsection we compare the optimality conditions for the risk neutral and risk averse cases through numerical examples.

4.1 Numerical illustration and insights

We interpret constant C_{gbm} as a *multiplicative risk factor* that characterizes risk averse behavior. Since the constant C_{gbm} is less than one, we have that whenever the greedy policy is optimal for the risk neutral case it will be automatically optimal for the risk averse case for the same choice of parameters. This is rather intuitive: consider a risk neutral decision maker who maximizes expectation and is not particularly concerned about risk. If she decides to harvest everything available instead of waiting for prices to rise, a risk averse decision maker will certainly not be tempted to gamble and go for higher gains: she will follow the same policy. In Figure 1 we can see that for values of σ close to zero, that is, in the case where GBM is less volatile, the constant C_{gbm} is close to one and conditions (13) and (14) become essentially equivalent. For larger values of σ, that is, for higher volatility, we observe that C_{gbm} is closer to zero and, unless the drift μ is sufficiently high, the optimal policy for a risk averse decision maker will be greedy.

Let us perform numerical computations for some values of the parameters of the model. If we define the discount parameter as $\delta = 0.95$, the drift $\mu = 0.3$ and the variance $\sigma = 0.2$, the left-hand side of (14) is equal to 1.28 and the optimality condition for the risk neutral case is not satisfied, which tells us that the greedy policy is not optimal in this case. On the other hand, we have that (13) is satisfied for any $0 < \alpha \leq 0.30$, resulting in the optimality of the greedy policy in the risk averse case. Our interpretation is that the volatility σ in this case was too high for the risk averse decision maker. Even though the drift is positive and the discount factor is high the risk averse decision maker is afraid of the price oscillation and prefers to cash the gains as soon as a mature tree is available for harvesting.

Let us now fix the value of σ and turn our attention to the role of the risk aversion parameter α. With $\delta = 0.95$, $\mu = 0.3$ and $\sigma = 0.2$, we have for a risk aversion level of $\alpha = 0.10$

$$\delta e^{\mu} = 1.28 > 1,$$
$$\delta e^{\mu} C_{gbm} = 0.98 < 1.$$

For these parameter choices, according to Theorem 1 in Pagnoncelli and Piazza [2011] the greedy policy is *not* optimal for the risk neutral case, but it is under risk averseness. In this case the risk neutral decision maker considers future price expectations attractive enough to postpone harvesting. On the other hand for the risk averse decision maker the possibility of higher future benefits is less attractive compared to the risk of eventual losses, and therefore the optimal solution is to harvest everything available.

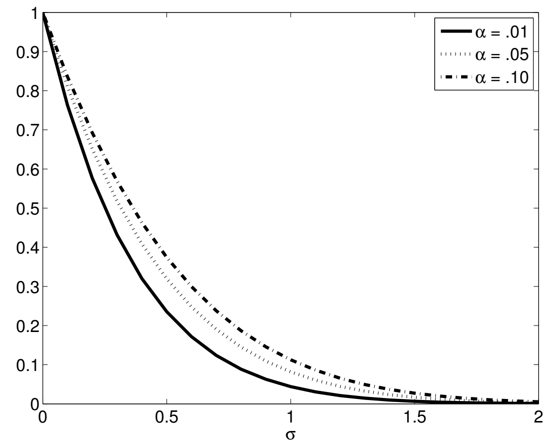

Figure 1: C_{gbm} as a function of σ for $\alpha = 0.10, 0.05$ and 0.01.

4.2 Another optimal policy

When condition (10) does not hold then it is natural to think that the decision maker should postpone the harvest as much as possible. Hence, before the final time T, harvesting should be stopped altogether in order to have the maximum surface available at time T. However, observe that every land plot harvested and planted n or more time steps before T will contain mature tress available for harvesting at time T. Hence, it is convenient to harvest every mature tree at time $T-n$, since there is enough time for seedlings to mature before reaching T. Repeating this reasoning we can conjecture that the only time steps when harvesting is allowed are $T-kn$ for $k=0,1,\ldots,\lfloor T/n \rfloor$ and that everything available then should be harvested. This is to say,

$$c(t) = \begin{cases} CA\mathbb{X}(t) & \text{if } t = T-kn \\ 0 & \text{else.} \end{cases}$$

We call this harvesting policy the *accumulating policy.*

This intuitive claim is formalized in Theorem 4.2, whose proof is in the appendix. Before stating the formal results, we will present a toy example: let $n=3, S=6, T=8$ and suppose the initial state is $\mathbb{X}(1) = (0,1,2,3)$. If we apply the policy detailed above, then it is only possible to harvest at times 2, 5 and 8, and the evolution of the state is

$$\begin{pmatrix}0\\1\\2\\3\end{pmatrix} \to \begin{pmatrix}1\\2\\3\\0\end{pmatrix} \to \begin{pmatrix}0\\3\\0\\3\end{pmatrix} \to \begin{pmatrix}3\\0\\3\\0\end{pmatrix} \to \begin{pmatrix}3\\3\\0\\0\end{pmatrix} \to \begin{pmatrix}0\\0\\0\\6\end{pmatrix} \to \begin{pmatrix}0\\0\\6\\0\end{pmatrix} \to \begin{pmatrix}0\\6\\0\\0\end{pmatrix},$$

with corresponding harvests

$$c(t) = \begin{cases} 3 & t=2 \\ 6 & t=5 \\ 6 & t=8 \\ 0 & \text{else.} \end{cases}$$

Theorem 4.2 *Consider problem* (7) *and assume prices evolve according to* (12)*. If condition* (13) *does not hold, the accumulating policy is optimal.*

Proof is also relegated to the appendix.

5 Ornstein-Uhlenbeck process

We now study problem (7) when prices follow a mean-reverting Ornstein-Uhlenbeck process (O-U). It is well-know that GBM does not capture some behaviors of price movement such as mean-reversion, which is best emulated by the O-U process. Since, O-U is a well-known process, we define it without further detail.

$$dp(t) = \eta(\bar{p} - p(t))dt + \sigma dW_t \qquad \text{(O-U)}, \tag{15}$$

where $\eta > 0$ is the rate of mean-reversion to an equilibrium $\bar{p}$, $\sigma > 0$ is the constant variance and W_t denotes the Wiener process. A closed expression for the value of the price $p(t+1)$ given the price $p(t)$ can be written as follows:

$$p(t+1) = e^{-\eta}p(t) + (1 - e^{-\eta})\bar{p} + \int_t^{t+1} \sigma e^{\eta(s-(t+1))} dW_s$$

In an O-U process, it can be shown (see for example Maller, Mller, and Szimayer [2009]) that at every time period t the random variable $p(t)$ follows a normal distribution[2] with conditional mean

$$\mathbb{E}[p(t)|p(s)] = \bar{p} + (p(s) - \bar{p})e^{-\eta(t-s)}, \qquad \text{for } s \leq t \tag{16}$$

and conditional variance

$$\text{Variance}[p(t)|p(s)] = \frac{\sigma^2}{2\eta}(1 - e^{-2\eta(t-s)}), \qquad \text{for } s \leq t. \tag{17}$$

5.1 Explicit expression for the CVaR of the O-U process

If X follows a normal distribution with mean μ and variance σ, the $\text{CVaR}_\alpha[X]$ can be explicitly computed:

$$\text{CVaR}_\alpha[X] = \mu + \frac{\sigma}{\alpha\sqrt{2\pi}} e^{-z_\alpha/2}, \tag{18}$$

where $z_\alpha = \Phi^{-1}(1 - \alpha)$ and Φ is the cumulative distribution function of the normal random variable with mean 0 and variance 1. See the appendix for the details of the computation.

Therefore, using (18) we can calculate the conditional CVaR at each time period as follows:

$$\text{CVaR}_{\alpha|p(t)}[-p(t+1)] = -p(t)e^{-\eta} - \bar{p}(1 - e^{-\eta}) + \frac{\sigma}{\sqrt{2\pi}} \sqrt{\frac{(1 - e^{-2\eta})}{2\eta}} \frac{e^{-z_\alpha^2/2}}{\alpha}. \tag{19}$$

To simplify notation we name

$$a = e^{-\eta} \quad \text{and} \quad b = \bar{p}(1 - e^{-\eta}) - \tfrac{\sigma}{\sqrt{2\pi}} \sqrt{\tfrac{(1-e^{-2\eta})}{2\eta}} \tfrac{e^{-z_\alpha^2/2}}{\alpha}, \tag{20}$$

and hence $\text{CVaR}_{\alpha|p(t)}[-p(t+1)] = -ap(t) - b$. Using that $a \in (0, 1)$, condition (10) can be expressed as

$$-p(t)(1 - \delta a) + \delta b \leq 0 \iff p(t) \geq \frac{\delta b}{1 - \delta a}, \tag{21}$$

with $\delta a < 1$. In the previous section, when prices follow a GBM, we have condition (13) that does not depend on $p(t)$ and assures the satisfaction of (10) for all t. We do not have anything similar when prices follow an O-U process, as we see that condition (21) depends on $p(t)$. Hence, when solving Bellman equation

[2] Even though the arithmetic O-U can lead to negative values, the process is frequently used to model the evolution of prices [see Alvarez and Koskela, 2005, Gjolberg and Guttormsen, 2002]. Following Schwartz [1997], one could consider that the arithmetic O-U represents the log of the actual timber prices. There are other versions of the O-U process that generate only positive values, but the technical difficulties would lead to a mathematically intractable model. The discussion of which process best represents timber prices is far from being settled. We refer the reader to Dixit and Pindyck [1994], Insley and Rollins [2005] and the references therein for more information.

(8) for t, we have no information of what may happen at $t+1$ or after and we have to consider every possible situation.

However, we will show in Theorem 5.1 that condition (21) is sufficient to assure that the greedy policy is optimal at t, i.e., $c(t) = CA\mathbb{X}(t)$. We must stress here, that (21) is *not* necessary and that we do not have any information about what the optimal policy is when it does not hold.

The idea of the proof consists in showing that the coefficient of c in (8) is composed of terms of the form

$$\Delta_{j_i}^{m_i}(p(t)) = \delta^{m_i}\left\{-p(t)a^{m_i}(1-\delta^{j_i}a^{j_i}) - b\Big[\sum_{l=0}^{m-1} a^l - \delta^{j_i}\sum_{l=0}^{m_i+j_i-1} a^l\Big]\right\} \quad (22)$$

$$= \delta^{m_i}\left\{-p(t)a^{m_i}(1-\delta^{j_i}a^{j_i}) - \frac{b}{1-a}\Big[1-\delta^{j_i} - a^{m_i}(1-\delta^{j_i}a^{j_i})\Big]\right\}, \quad (23)$$

for some values of $m_i \le T-t$ and $j_i \in \{0,\ldots,n-1\}$ plus possibly one negative term.

Instead of checking the sign of $\Delta_j^m(p(t))$ for every value of m and j, we use the equivalence

$$\Delta_j^m(p(t)) \le 0 \iff p(t) \ge \frac{b}{1-a}\Big[1 - \frac{1-\delta^j}{a^m(1-\delta^j a^j)}\Big]. \quad (24)$$

to prove in the next lemma that $\Delta_j^m(p(t))$ is non-positive when condition (21) holds.

Lemma 5.1 *If $a \in (0,1)$ and $b \ge 0$, condition* (21) *implies that*

$$\Delta_j^m(p(t)) \le 0, \quad \textit{for all } m \textit{ and } j \in I\!N.$$

When price follows an O-U process, we have that $a < 1$, but for some values of the parameters we could have $b < 0$. Hence, the second hypothesis of Lemma 5.1 has to be explicitly required in the statement of the main result of this section, whose proof has been relegated to the appendix.

Theorem 5.1 *Consider problem* (7) *and assume prices evolve according to* (15)*. If $b \ge 0$, and condition* (21) *holds, then $c(t) = CA\mathbb{X}(t)$ is optimal at time t.*

The right-hand side of (21) can be used as a reservation price, as we know that if $p(t)$ is above that value it is optimal to harvest every mature tree. The condition is sufficient but not necessary: if (21) does not hold, we are not able to discard the greedy policy.

5.2 Policy insights

In this section we will take a closer look to expression (19) in order to obtain a deeper understanding of how each parameter affects condition (21). In the expression of b in (20), the term $\sqrt{\frac{1-e^{-2\eta}}{2\eta}}$ is strictly decreasing with η and it lies between zero and one. When the speed of the mean reversion η vanishes, the whole expression converges to 1. Since the first term in the expression of b goes to zero as η approaches zero, we have in this case that $b < 0$. It is interesting to note that in this case Theorem 5.1 does not apply and we will deal with this particular situation in the next section. When η goes to infinity, b goes to $\bar{p}$ and a goes to zero. The reservation price (the right hand side of (21)) simplifies to $\delta\bar{p}$. In this case one can expect that the optimal policy will be greedy for any t since the speed of the mean reversion is so high that the price is essentially equal to $\bar{p}$ for all time periods and therefore greater than the reservation price $\delta\bar{p}$ because $\delta \in (0,1)$. This is consistent with the result for constant deterministic prices presented in Rapaport, Sraidi, and Terreaux [2003] that establishes that the greedy policy is optimal for all time periods.

If we choose $\alpha = 1$ in the calculation of the CVaR we obtain the usual expected value operator. Observe that the term $e^{-z_\alpha^2/2}/\alpha$ in (19) goes to zero when α approaches 1, implying that $b = \bar{p}(1-e^{-\eta})$. The resulting reservation price would be

$$p_r = \bar{p}\frac{\delta(1-e^{-\eta})}{1-\delta e^{-\eta}}, \quad (25)$$

which coincides with the results presented in the paper Pagnoncelli and Piazza [2011], in which the authors study the harvesting problem in a risk neutral setting.

It is straightforward to see that if the volatility σ goes to zero we have that the reservation price also coincides with the results obtained in Pagnoncelli and Piazza [2011] for the risk neutral case. This result is rather intuitive and in some sense it mimics what we obtained for the GBM case: if the volatility is small the sufficient conditions for the optimality of the greedy policy are essentially the same under risk neutrality and risk averseness.

However, when the volatility σ increases the two cases are significantly different. Note that the reservation price in the risk neutral case (25) is greater or equal to the right-hand side of (21):

$$\frac{\delta b}{1-\delta a} = p_r - \frac{\delta}{1-\delta e^{-\eta}} \frac{\sigma}{\sqrt{2\pi}} \sqrt{\frac{(1-e^{-2\eta})}{2\eta}} \frac{e^{-z_\alpha^2/2}}{\alpha}.$$

Therefore, for every value of the parameters η, α and σ, if condition (21) is satisfied for the risk neutral case it is also satisfied for the risk averse case and the resulting policy is greedy.

6 Extension to affine coherent risk measures

It is worth investigating whether the results of the previous section can be generalized for other values of a and b. This would eliminate the hypothesis $b \geq 0$ of Theorem 5.1, and, more importantly, it would allow the extension of the results to other price processes and to risk measures other than CVaR_α, whenever the conditional risk measure is affine on $p(t)$. We will see by the end of this section that there are several examples of affine conditional risk measures if prices follow GBM and O-U process.

The proof of Theorem 5.1 relies heavily in the fact that

$$\rho_{|p(t)}[-p(t+1)] = -ap(t) - b$$

and is valid for $(a, b) \in (0,1) \times \mathbb{R}_+$. In the following we study the extension of this theorem to $(a, b) \in \mathbb{R}_+ \times \mathbb{R}$.

We divide the semi-plane of parameters in six regions as shown in Table 1 and study the variation of the right-hand side of (24) in each of these regions. The following theorem summarizes the sufficient conditions for the optimality of the greedy policy, that can be found with this method.

	$0 < a \leq 1$	$1 < a < 1/\delta$	$a > 1/\delta$
$b \geq 0$	(i)	(ii)	(iii)
$b < 0$	(iv)	(v)	(vi)

Table 1: Parameters regions.

Theorem 6.1 *Consider problem* (7) *and assume that* $\rho_{|p(t)}[-p(t+1)] = -ap(t) - b$.

1. *In region (i),* $p(t) \geq b\delta/(1-\delta a)$ *is sufficient to assure* $c^*(t) = CA\mathbb{X}(t)$.
2. *In region (ii),* $p(t) \geq b\delta/(1-\delta a)$ *is sufficient to assure* $c^*(t) = CA\mathbb{X}(t)$.
3. *In region (iv),* $p(t) \geq \frac{b}{1-a}[1 - \frac{1-\delta}{a^{T-t}(1-\delta a)}]$ *is sufficient to assure* $c^*(t) = CA\mathbb{X}(t)$.
4. *In region (v),* $p(t) \geq b/(1-a)$ *is sufficient to assure* $c^*(t) = CA\mathbb{X}(t)$.
5. *In region (vi),* $p(t) \leq b/(1-a)$ *is sufficient to assure* $c^*(t) = CA\mathbb{X}(t)$.

Remark 6.1 *It is curious that the theorem does not provide information about the region (iii). For this region, the method of proof used delivers a condition that would imply $p(t) < 0$. Although a price following a O-U process could take negative values (see footnote 2), we do not withdraw conclusions from this condition, in the understanding that it will not hold in any real application. It is also worth commenting that, although regions (i) and (ii) deliver the same sufficient condition, we decided to keep them separated as the former is exactly Theorem 5.1.*

Theorem 1 is a corollary of Theorem 6.1. Indeed, when prices follow a GBM and we use CVaR as the objective function, we have $a = e^{\mu}$ and $b = 0$. When $b = 0$, by simple inspection we see that conditions (i) and (ii) reduce to $p(t) \geq 0$, that is always satisfied. Hence,

Corollary 6.1 *If $b = 0$ and $0 < a < 1/\delta$ and prices follow a GBM, the greedy policy is always optimal.*

Furthermore, if $e^{\mu} \geq 1/\delta$, condition (vi) reduces to $p(t) \leq 0$ that is never satisfied. In this region, we cannot withdraw any conclusions from the theorem above.

We now prove that some conditional risk measures are affine on $p(t)$ when prices follow a GBM or O-U process and give the values of a and b in either cases. In some cases, we are able to provide closed expressions of a and b as functions of the parameters of the model.

6.1 Examples

A new coherent risk measure can be constructed as a weighted combination of the expected value and the CVaR:

$$\text{WCVaR}_{\alpha}[X] = \lambda \mathbb{E}[X] + (1-\lambda)\text{CVaR}_{\alpha}[X], \ \lambda \in [0,1]. \tag{26}$$

When λ is zero, WCVaR reduces to the CVaR and when λ equals to one we have the risk neutral case, studied in Pagnoncelli and Piazza [2011]. For intermediate values we have a combination of both measures and λ is a risk averse parameter: values close to zero emphasize the risk averse behavior by putting more weight on the CVaR while values close to one emphasize the expected value and therefore are closer to the behavior of a risk neutral decision maker. It is straightforward to check from definition (26) that the WCVaR is a coherent risk measure. If prices follow GBM, we can show that it is affine in $p(t)$ as follows.

$$\text{WCVaR}_{\alpha|p(t)}[p(t+1)] = -e^{\mu}\left[\lambda + \frac{1}{\alpha}(1-\lambda)\Phi(z_{1-\alpha}-\sigma)\right]p(t),$$

where we used that the expected value of $p(t+1)$ given $p(t)$ is equal to $p(t)e^{\mu}$. Using the notation of (6) we see that in this case $a = e^{\mu}[\lambda + \alpha^{-1}(1-\lambda)\Phi(z_{1-\alpha}-\sigma)]$ and $b = 0$. If prices follow an O-U process we have

$$\text{WCVaR}_{\alpha|p(t)}[p(t+1)] = -p(t)e^{\mu} - \bar{p}(1-e^{-\eta}) + (1-\lambda)\frac{\sigma}{\sqrt{2\pi}\alpha}\sqrt{\frac{1-e^{-2\eta}}{2\eta}}e^{-z_{\alpha}^2/2},$$

which shows that the WCVaR is also affine when prices follow an O-U process, with $a = e^{-\eta}$ and $b = p(1-e^{-\eta}) - (1-\lambda)\frac{\sigma}{\sqrt{2\pi}\alpha}\sqrt{\frac{1-e^{-2\eta}}{2\eta}}e^{-z_{\alpha}^2/2}$.

In the spirit of Markowitz [1952], we construct an efficient frontier of optimal policies for the GBM case. For $\lambda \in [0,1]$, we calculate the total value at time $t = 0$ under the greedy and accumulating policies. From Corollary 6.1 we know that the greedy policy will be optimal for δa less or equal to one. Regarding the accumulating policy, it is straightforward to prove a version of Theorem 4.2 applicable to the WCVaR establishing that it is optimal when $\delta a > 1$.

For a value of λ in the y axis, Figure 2 shows the negative of the total value at time $t = 0$ under the greedy policy (continuous line) and the accumulated policy (dashed line) in the x-axis. The figure is generated with parameters $\mu = 0.2$, $\sigma = 0.4$, $p(0) = 1$, $\delta = 0.95$, $\alpha = 0.1$ and with $T = 20$ periods for the initial state $\mathbb{X}(0) = (0,0,0,1)$. With these values, condition $\delta a < 1$ gives a value of $\lambda = 0.742$. Observe that the two lines intersect and the change of optimal policies occur for λ between 0.74 and 0.75, as we predicted.

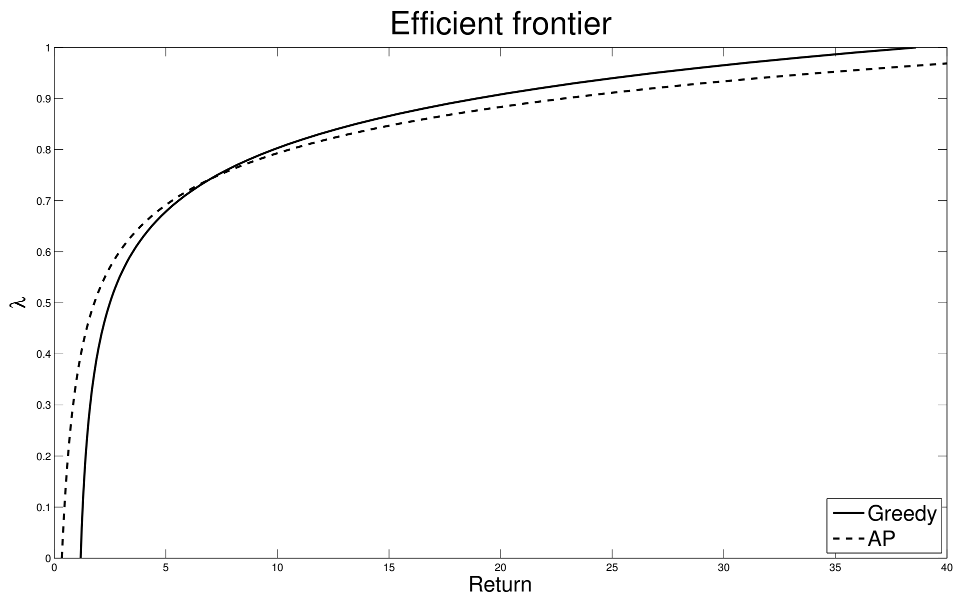

Figure 2: Efficient frontier for $\lambda \in [0,1]$.

Another important coherent risk measure is the so-called Mean-Deviation Risk (MDR) measure of order p [see Shapiro, Dentcheva, and Ruszczyński, 2009, for a complete description]:

$$\mathrm{MDR}[X] := \mathbb{E}[X] + c(\mathbb{E}\left[|X - \mathbb{E}[X]|^p\right])^{1/p},$$

where $c > 0$ and $p \in [1, +\infty)$. For this risk measure we also have that $\rho_{|p(t)}[p(t+1)]$ is affine on $p(t)$ when $p(t)$ follows a GBM or a O-U. Indeed we have:

$$\mathrm{MDR}_{|p(t)}(-p(t+1)) = -p(t)e^{\mu}\left(1 + c\left(\mathbb{E}\left[\left|\exp(-\tfrac{\sigma^2}{2} + \sigma W) - 1\right|^p\right]\right)^{1/p}\right), \tag{GBM}$$

$$\mathrm{MDR}_{|p(t)}(-p(t+1)) = -p(t)e^{-\eta} - \bar{p}(1 - e^{-\theta}) + c\left(\mathbb{E}\left[\left|\int_t^{t+1} \sigma e^{\eta(s-(t+1))} dW(s)\right|^p\right]\right)^{1/p}. \tag{O-U}$$

For the particular case of $p = 2$ and a O-U process, it is possible to explicitly compute the conditional MDR as follows:

$$\mathrm{MDR}(-p(t+1)|p(t)) = -p(t)e^{-\eta} - \bar{p}(1 - e^{-\theta}) + c\left(\mathbb{E}\left[\left|\int_t^{t+1} \sigma e^{\eta(s-(t+1))} dW(s)\right|^2\right]\right)^{1/2}$$

In order to calculate the stochastic integral we apply Itô's isometry:

$$\mathbb{E}\left(\left|\int_0^T G(t, W(t))dW(t)\right|^2\right) = \mathbb{E}\left(\int_0^T |G(t, W(t))|^2\, dt\right), \tag{27}$$

for a stochastic process $G(t, W(t)) \in \mathbb{L}^2(0, T)$. Using (27) and noting that in our case the process G is deterministic, we have

$$\left(\mathbb{E}\left[\left|\int_t^{t+1} \sigma e^{\eta(s-(t+1))} dW(s)\right|^2\right]\right)^{1/2} = \left(\mathbb{E}\left[\int_t^{t+1} (\sigma e^{\eta(s-(t+1))})^2 ds\right]\right)^{1/2}$$
$$= \sigma\left(\mathbb{E}\left[\int_t^{t+1} e^{2\eta(s-(t+1))} ds\right]\right)^{1/2} = \sigma\left(\frac{1}{2\eta} - \frac{e^{-\eta}}{2\eta}\right)^{1/2} > 0.$$

We have that the conditional MDR of order 2 is affine in $p(t)$ for the O-U case and the corresponding coefficients are $a = e^{-\eta}$ and $b = \bar{p}(1 - e^{-\theta}) - c\sigma\left(\frac{1-e^{-\eta}}{2\eta}\right)^{1/2}$.

6.2 Summary of results

We showed that several important coherent risk measures are affine in the price $p(t)$. We present a summary of all parameters values a and b for the risk measures considered in this paper and for both processes used to represent timber price. The computations for the expected value can be found in Pagnoncelli and Piazza [2011].

ρ	a	b
$\mathbb{E}$	e^{μ}	0
CVaR_α	$e^{\mu}\Phi(z_{1-\alpha} - \sigma)/\alpha$	0
WCVaR_α	$e^{\mu}[\lambda + (1-\lambda)\Phi(z_{1-\alpha} - \sigma)/\alpha]$	0
MDR of order 2	$e^{\mu}(1 + c\mathbb{E}^{1/2}[\lvert\exp(-\frac{\sigma^2}{2} + \sigma W_1) - 1\rvert^2])$	0

Table 2: Values of a and b when prices follow a GBM.

ρ	a	b
$\mathbb{E}$	$e^{-\eta}$	$\bar{p}(1 - e^{-\eta})$
CVaR_α	$e^{-\eta}$	$\bar{p}(1 - e^{-\eta}) - \frac{\sigma}{\sqrt{2\pi}\alpha}\sqrt{\frac{1-e^{-2\eta}}{2\eta}}e^{-z_\alpha^2/2}$
WCVaR_α	$e^{-\eta}$	$\bar{p}(1 - e^{-\eta}) - (1-\lambda)\frac{\sigma}{\sqrt{2\pi}\alpha}\sqrt{\frac{1-e^{-2\eta}}{2\eta}}e^{-z_\alpha^2/2}$
MDR of order 2	$e^{-\eta}$	$\bar{p}(1 - e^{-\eta}) - c\sigma\left(\frac{1-e^{-\eta}}{2\eta}\right)^{1/2}$

Table 3: Values of a and b when prices follow an O-U.

7 Conclusions

We study a harvest scheduling problem under price uncertainty. We depart from the usual risk neutral framework and incorporate risk measures in the objective function. We focus on the Conditional Value at Risk and we obtain conditions on the parameters of the model that characterize the optimal policy.

When prices follow a Geometric Brownian Motion, we obtain an optimality condition for the greedy policy that depends only on the drift of the process and on the discount factor. Furthermore, when the condition is not satisfied we also characterize the optimal policy and prove it is not greedy. We relate those results with the conditions obtained for the risk neutral case in Pagnoncelli and Piazza [2011] and construct an efficient frontier based on different levels of risk aversion.

When prices follow an Ornstein-Uhlenbeck process we find a sufficient condition that guarantees optimality of the greedy policy. Such condition can be interpreted as a reservation price: if the current price is above the threshold then it is optimal to harvest everything available. We perform asymptotic analysis on the expression of the reservation price and discuss the implications for the optimal policy when the parameters approach zero or infinity.

We extend the results obtained for the Conditional Value at Risk to any coherent risk measure that is affine in the price at time t. We show two explicit examples of affine measures: the weighted Conditional Value at Risk and the Mean-Deviation-Risk.

The publication of Artzner et al. [1999] stimulated a great amount of theoretical research on coherent risk measures, but the incorporation of those measures in applied problems has not followed the same pace. The majority of applied papers that incorporate uncertainty usually consider the expected value as a risk measure. In forestry, most papers present numerical results that provide intuition about the problems under study but do not offer complete descriptions of the optimal policy. Our theoretical results give a complete understanding of the GBM case and characterize the optimal policy for the O-U case by providing a closed expression for a reservation price.

There are very few benchmark problems in stochastic harvest scheduling and we hope our results will help future applied research in the field. We believe this is the first paper to incorporate coherent risk measures into forestry and we believe it is important to extend the uncertainty treatment in order to incorporate risk averse behavior.

Proof. CVaR of a lognormal random variable

The first step is to compute the Value-at-Risk (VaR_α) of a lognormal random variable $Y = e^X$, where $X \sim \mathrm{N}(\mu, \sigma^2)$ is a normal random variable. We remind the reader that the VaR_α of a random variable W is equal to

$$\mathrm{VaR}_\alpha[W] := \inf_t \{\mathbb{P}(W \leq t) \geq (1-\alpha)\}. \tag{28}$$

For the lognormal random variable Y, we have

$$\begin{aligned} \mathbb{P}(Y \leq t) \leq 1-\alpha &\Leftrightarrow \mathbb{P}(e^X \leq t) \leq 1-\alpha, \\ \mathbb{P}(X \leq \ln t) \leq 1-\alpha &\Leftrightarrow \mathbb{P}(Z \leq (\ln t - \mu)/\sigma) \leq 1-\alpha, \\ \frac{\ln t - \mu}{\sigma} \geq \Phi^{-1}(1-\alpha) &\Leftrightarrow t \geq e^{\mu + \sigma z_\alpha}, \end{aligned}$$

where $Z \sim \mathrm{N}(0,1)$, $\Phi(\cdot)$ is the cumulative distribution function of Z and $z_\alpha = \Phi^{-1}(-\alpha)$. From the definition (28) we have

$$\mathrm{VaR}_\alpha[Y] = e^\mu + \sigma z_\alpha.$$

Denoting by $f(\cdot)$ the density function of the random variable Y and letting $t^* = \mathrm{VaR}_\alpha[Y]$, we can compute the $\mathrm{CVaR}_\alpha[Y]$ as follows:

$$\begin{aligned} \mathrm{CVaR}_\alpha[Y] &= \frac{1}{\alpha}\int_{t^*}^{\infty} y f(y) dy = \frac{1}{\alpha}\int_{t^*}^{\infty} y \frac{1}{\sqrt{2\pi\sigma^2} y} e^{-\frac{(\ln y - \mu)^2}{2\sigma^2}} dy \\ &= \frac{1}{\alpha\sqrt{2\pi\sigma^2}} \int_{t^*}^{\infty} e^{-\frac{(\ln y - \mu)^2}{2\sigma^2}} dy. \end{aligned}$$

Defining $\ln y = x$, we have $y = e^x$ and $dx/dy = 1/y$. The integral becomes

$$\begin{aligned}
&\frac{1}{\alpha\sqrt{2\pi\sigma^2}} \int_{\ln t^*}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} e^x dx \\
&= \frac{1}{\alpha} \int_{\ln t^*}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-(\mu+\sigma^2))^2}{2\sigma^2}} e^{\mu+\sigma^2/2} dx \\
&= \frac{e^{\mu+\sigma^2/2}}{\alpha} \left[1 - \Phi\left(\frac{\ln t^* - (\mu+\sigma^2)}{\sigma}\right)\right] \\
&= \frac{e^{\mu+\sigma^2/2}}{\alpha} \Phi\left(\frac{(\mu+\sigma^2) - \ln(e^{\mu+\sigma z_\alpha})}{\sigma}\right) \\
&= \frac{e^{\mu+\sigma^2/2}}{\alpha} \Phi\left(\sigma - z_\alpha\right).
\end{aligned}$$

Taking into account that (i) $p(t+1)|p(t)$ follows a lognormal with parameters $(\mu - \sigma^2/2, \sigma)$ in the GBM process and (ii) we actually need the $\text{CVaR}_{\alpha|p(t)}[-p(t+1)]$, a simple change of variables gives us

$$\text{CVaR}_{\alpha|p(t)}[-p(t+1)] = -p(t)\frac{e^{\mu}}{\alpha}\Phi(z_{1-\alpha} - \sigma).$$

CVaR of a normal random variable: Let $Y \sim \text{N}(\mu, \sigma^2)$ and density function $f(\cdot)$. Denoting by $\Phi(\cdot)$ the cumulative distribution function of $Z \sim \text{N}(0,1)$ we can easily compute the $\text{VaR}_\alpha[Y]$ as follows:

$$\begin{aligned}
\mathbb{P}\left(Y \le t\right) &\ge 1 - \alpha \Leftrightarrow \\
\mathbb{P}\left(\frac{Y-\mu}{\sigma} \le \frac{t-\mu}{\sigma}\right) &\ge 1 - \alpha \Leftrightarrow \\
\Phi\left(\frac{t-\mu}{\sigma}\right) &\ge 1 - \alpha \Leftrightarrow \\
\frac{t-\mu}{\sigma} &\ge \Phi^{-1}(1-\alpha) \Leftrightarrow \\
t &\ge \mu + \sigma\Phi^{-1}(1-\alpha).
\end{aligned}$$

From the definition of the $\text{VaR}_\alpha[Y]$, we have that we have that

$$\text{VaR}_\alpha[Y] = \mu + \sigma z_\alpha \tag{29}$$

The computation of $\text{CVaR}_\alpha[Y]$ is similar to the lognormal case and uses (29). Defining $t^* = \text{VaR}_\alpha[Y]$, we have

$$\begin{aligned}
\text{CVaR}_\alpha[Y] = \mathbb{E}\left[Y | Y > \text{VaR}_\alpha[Y]\right] &= \frac{1}{\alpha}\int_{t^*}^{\infty} y f(y) dy \\
&= \frac{1}{\alpha}\int_{t^*}^{\infty} (y-\mu) f(y) dy + \frac{\mu}{\alpha}\int_{t^*}^{\infty} f(y) dy \\
&= \mu + \frac{1}{\alpha}\int_{t^*}^{\infty} (y-\mu)\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy.
\end{aligned}$$

Letting $x = y - \mu$, we have

$$\begin{aligned}
\text{CVaR}_\alpha[Y] &= \mu + \frac{1}{\alpha\sqrt{2\pi}\sigma}\int_{\sigma z_\alpha}^{\infty} x e^{-x^2/(2\sigma^2)} dy \\
&= \mu + \frac{1}{\alpha\sqrt{2\pi}\sigma}\left[-\sigma^2 e^{-x^2/2\sigma^2}\Big|_{\sigma z_\alpha}^{\infty}\right] \\
&= \mu + \frac{\sigma}{\alpha\sqrt{2\pi}} e^{-z_\alpha^2/2}.
\end{aligned}$$

Taking into account that (i) $p(t+1)|p(t)$ follows a normal random variable for the O-U process and that (ii) the negative of a normal random variable is also a normal random variable we have

$$\text{CVaR}_{\alpha|p(t)}[-p(t+1)] = -p(t)e^{-\eta} - \bar{p}(1-e^{-\eta}) + \frac{\sigma}{\sqrt{4\eta\pi}\alpha}\sqrt{1-e^{-2\eta}}e^{-z_\alpha^2/2}.$$

Proof. Proof of Theorem 4.1: Knowing the particular expression of the $\text{CVaR}_\alpha(-p(t+1)|p(t))$ allows us to write (7) in the following equivalent form

$$V_t(p(t),\mathbb{X}(t)) = \begin{cases} \text{Min}_{c(t),\ldots,c(T)} & -p(t)\sum_{k=t}^{T}(\delta e^{\mu}C_{gbm})^{k-t}c(k) \\ \text{s.t.} & (5) \text{ and } (6) \end{cases} \tag{30}$$

For the last period $t = T$ the proof follows trivially.

We now proceed by backwards induction on t. Let us assume that the greedy policy is optimal for every $s > t$, $(t \leq T)$. We claim that it is also optimal for t.

Let us compute explicitly $\mathbb{X}(t+1)$ in terms of $\mathbb{X}(t)$ and c,

$$\mathbb{X}(t) = \begin{pmatrix} \bar{x}(t) \\ x_n(t) \\ x_{n-1}(t) \\ \vdots \\ \vdots \\ x_1(t) \end{pmatrix} \longrightarrow \mathbb{X}(t+1) = A\mathbb{X}(t) + Bc = \begin{pmatrix} \bar{x}(t) + x_n(t) - c \\ x_{n-1}(t) \\ x_{n-2}(t) \\ \vdots \\ \vdots \\ c \end{pmatrix}$$

Knowing that for $s > t$ the greedy policy is optimal, we can calculate the coefficient of c in the expression of $V_{t+1}(p(t+1),\mathbb{X}(t+1))$. We know that the fraction of surface $\bar{x}(t) + x_n(t) - c + x_{n-1}(t)$ will be harvested at $t+1$ and every n steps afterwards, until reaching T. This together with (30) allows to conclude that the coefficient affecting the non-negative term $(\bar{x}(t) + x_n(t) - c + x_{n-1}(t))$ in the expression of $V_{t+1}(\cdot,\cdot)$ is:

$$-p(t+1)\big(1 + (\delta e^{\mu}C_{gbm})^{n} + \cdots + (\delta e^{\mu}C_{gbm})^{kn}\big)$$

where $k = \lfloor \frac{T-(t+1)}{n} \rfloor$. In addition to this, the area of trees that are one year old at $t+1$ is c. This fraction surface will be harvested at $t+n$ and every n afterwards, until reaching T. Thus, the coefficient affecting the term c in the expression of $V_{t+1}(\cdot,\cdot)$ at (30) will be:

$$-p(t+1)\big((\delta e^{\mu}C_{gbm})^{n-1} + (\delta e^{\mu}C_{gbm})^{2n-1} + \cdots + (\delta e^{\mu}C_{gbm})^{k'n-1}\big)$$

where $k' = \lfloor \frac{T-(t+1)}{n} \rfloor$. The benefits at all the other time steps do not depend on c and will be represented by a generic function $g(p(t+1),\mathbb{X}(t))$.

Observe that either $k' = k$ or $k' = k+1$. In the case that $k' = k$, the coefficient of c at (8) is

$$\begin{aligned} & -p(t) + \delta\rho_{|p(t)}\Big(-p(t+1)\big((\delta e^{\mu}C_{gbm})^{n-1} + (\delta e^{\mu}C_{gbm})^{2n-1} + \cdots + (\delta e^{\mu}C_{gbm})^{k'n-1}\big) \\ + \ & p(t+1)\big(1 + (\delta e^{\mu}C_{gbm})^{n} + \cdots + (\delta e^{\mu}C_{gbm})^{kn}\big)\Big) \\ = \ & -p(t)\big[(1-\delta e^{\mu}C_{gbm}) + (\delta e^{\mu}C_{gbm})^{n}(1-\delta e^{\mu}C_{gbm}) + \cdots + (\delta e^{\mu}C_{gbm})^{kn}(1-\delta e^{\mu}C_{gbm})\big] \\ = \ & -p(t)(1-\delta e^{\mu}C_{gbm})\sum_{l=0}^{k}(\delta e^{\mu}C_{gbm})^{l} \end{aligned} \tag{31}$$

Hence, (8) can be written as

$$\begin{aligned} V_t(p(t),\mathbb{X}(t)) = \min_{c\in[0,CA\mathbb{X}(t)]} \ & -cp(t)(1-\delta e^{\mu}C_{gbm})\textstyle\sum_{l=0}^{k}(\delta e^{\mu}C_{gbm})^{l} \\ & +\delta\rho_{|p(t)}\big[g(p(t+1),\mathbb{X}(t))\big] \end{aligned}$$

It is evident that the coefficient affecting c will be negative whenever (13) holds with strict inequality, and the minimum will be attained at $c = CA\mathbb{X}(t)$. If condition (13) holds with equality, there is no influence of c in the value of $V_t(p(t), \mathbb{X}(t))$ and we can freely chose the value of c provided that it is feasible. We impose $c(t) = CA\mathbb{X}(t)$.

We are left with the case $k' = k + 1$. Here the coefficient of c will comprise the same expression we obtained in (31) plus the negative term $-p(t)(\delta e^{\mu} C_{gbm})^{(k+1)n}$. It is evident that if condition (13) holds, the coefficient of c is negative also in this case, and then the minimum will be attained at $c = CA\mathbb{X}(t)$. As a consequence, the greedy policy is optimal at t. ■

Proof. Proof of Theorem 4.2: To prove that the accumulating policy is optimal, we check that the benefit associated with it (Q^{AP}) satisfies the dynamic programming equation (8). After some computations we can prove that

$$\begin{aligned} Q_t^{AP}(\mathbb{X}(t), p(t)) &= \rho_{|p(t)}\Big[-\delta^j p(t+j)(\bar{x} + \sum_{l=0}^{j} x_{n-l}) - \sum_{i=1}^{k} \delta^{in+j} p(t+in+j) S\Big] \\ &= -p(t)\Big[(\delta e^{\mu} C_{gbm})^j (\bar{x} + \sum_{l=0}^{j} x_{n-l}) + \sum_{i=1}^{k} (\delta e^{\mu} C_{gbm})^{in+j} S\Big] \end{aligned} \tag{32}$$

where $k = \lfloor \frac{T-t}{n} \rfloor$, $j \in \{0, \ldots, n-1\}$ and S represents the total surface of the forest. We point out that for $k = 0$, we follow the convention $\sum_1^0 (\cdot) = 0$..

For the rest of the proof we divide the study into two cases depending on the value of j: (i) $j > 0$ and (ii) $j = 0$.

(i) Here we have $t + 1 = T - (k'n + j')$ where $k' = k$ and $j' = j - 1 \in \{0, \ldots, n-2\}$ and $Q_{t+1}^{AP}(A\mathbb{X}(t) + Bc, p(t+1))$ can be expressed as

$$\begin{aligned} Q_{t+1}^{AP}(A\mathbb{X}(t) + Bc, p(t+1)) &= -p(t+1)\Big[(\delta e^{\mu} C_{gbm})^{j-1}(\bar{x} + x_n - c + \sum_{l=0}^{j-1} x_{n-l-1}) \\ &\quad + \sum_{i=1}^{k} (\delta e^{\mu} C_{gbm})^{in+j-1} S\Big]. \end{aligned}$$

Inserting $V = Q^{AP}$ into the right-hand side of the dynamic programming equation (8), the argument of the min operator is

$$\begin{aligned} \Phi(c) &= -p(t)c + \delta \rho_{|p(t)}\big[-p(t+1)\big[(\delta e^{\mu} C_{gbm})^{j-1}(\bar{x} + x_n - c + \sum_{l=0}^{j-1} x_{n-l-1}) \\ &\quad + \sum_{i=1}^{k} (\delta e^{\mu} C_{gbm})^{in+j-1} S\big]\big] \\ &= -p(t)c - p(t)\big[(\delta e^{\mu} C_{gbm})^j (\bar{x} + x_n - c + \sum_{l=0}^{j-1} x_{n-l-1}) + \sum_{i=1}^{k} (\delta e^{\mu} C_{gbm})^{in+j} S\big]\big) \\ &= -p(t)c(1 - (\delta e^{\mu} C_{gbm})^j) - p(t)\big[(\delta e^{\mu} C_{gbm})^j (\bar{x} + x_n + \sum_{l=1}^{j} x_{n-l}) + \sum_{i=1}^{k} (\delta e^{\mu} C_{gbm})^{in+j} S\big] \\ &= -p(t)c(1 - (\delta e^{\mu} C_{gbm})^j) + Q_t^{AP}(\mathbb{X}(t), p(t)). \end{aligned}$$

As the coefficient of c is positive, the minimum is attained when $c = 0$ and $\Phi(0)$ is exactly $Q_t^{AP}(\mathbb{X}(t), p(t))$, showing that equation (8) holds.

(ii) Case $t = T - kn$. In this case, we have $t+1 = T - [(k-1)n + n - 1)]$ and Q^{AP}_{t+1} can be expressed as

$$\begin{aligned} Q^{AP}_{t+1}(A\mathbb{X}(t) + Bc, p(t+1)) &= -p(t+1)\Big[(\delta e^{\mu} C_{gbm})^{n-1}(\bar{x} + x_n - c + \sum_{l=0}^{n-2} x_{n-l-1} + c) \\ &+ \sum_{i=1}^{k-1} (\delta e^{\mu} C_{gbm})^{in+n-1} S\Big] \\ &= -p(t+1)\left[(\delta e^{\mu} C_{gbm})^{n-1} S + \sum_{i=2}^{k} (\delta e^{\mu} C_{gbm})^{in-1} S\right] \\ &= -p(t+1)\sum_{i=1}^{k} (\delta e^{\mu} C_{gbm})^{in-1} S. \end{aligned}$$

Inserting again $V = Q^{AP}$ into the right-hand side of the Bellman's equation (8), the argument of the min operator is

$$\Phi(c) = -p(t)c + \delta\rho_{|p(t)}\left[-p(t+1)\sum_{i=1}^{k} (\delta e^{\mu} C_{gbm})^{in-1} S\right].$$

The coefficient of c is negative, and thus, the minimum is attained when $c = \bar{x} + x_n$. So we have,

$$\Phi(\bar{x} + x_n) = -p(t)(\bar{x} + x_n) - p(t)\sum_{i=1}^{k} (\delta e^{\mu} C_{gbm})^{in} S.$$

The right-hand side is exactly (32) when $j = 0$, hence we have $\Phi(\bar{x} + x_n) = Q^{AP}_t(\cdot,\cdot)$ and equation (8) is satisfied.

In both cases, we have shown that $Q^{AP}_t(\cdot,\cdot)$ satisfies equation (8), hence it is the value function and the proposed policy is optimal.

■

Proof. Proof of Lemma 5.1: Due to (24), we only need to show that

$$\frac{\delta b}{1-\delta a} \geq \frac{b}{1-a}\Big[1 - \frac{1-\delta^j}{a^m(1-\delta^j a^j)}\Big]. \tag{33}$$

Using that

$$\frac{\delta b}{1-\delta a} = \frac{b}{1-a}\Big[1 - \frac{1-\delta}{1-\delta a}\Big],$$

we have that (33) is equivalent to

$$\begin{aligned} \frac{1}{1-a}\Big[\frac{1-\delta}{1-\delta a}\Big] &\leq \frac{1}{1-a}\Big[\frac{1-\delta^j}{a^m(1-\delta^j a^j)}\Big] \\ \iff \frac{1}{1-a}\Big[\frac{a^m(1-\delta^j a^j)}{1-\delta a}\Big] &\leq \frac{1}{1-a}\Big[\frac{1-\delta^j}{1-\delta}\Big] \\ \iff \frac{1}{1-a}\Big[a^m \sum_{l=0}^{j-1} (\delta a)^l\Big] &\leq \frac{1}{1-a}\Big[\sum_{l=0}^{j-1} \delta^l\Big]. \end{aligned}$$

Given that $a \in (0,1)$, the last inequality is always valid.

■

Proof. Proof of Theorem 5.1: For the period $t = T$ the proof follows trivially, so we only need to prove the result for $t < T$. As in the proof of Theorem 4.1, we state $\mathbb{X}(t+1)$ and equation (8) in terms of $\mathbb{X}(t)$ and c as follows.

$$\mathbb{X}(t) = \begin{pmatrix} \bar{x}(t) \\ x_n(t) \\ x_{n-1}(t) \\ \vdots \\ \vdots \\ x_1(t) \end{pmatrix} \longrightarrow \mathbb{X}(t+1) = A\mathbb{X}(t) + Bc = \begin{pmatrix} \bar{x}(t) + x_n(t) - c \\ x_{n-1}(t) \\ x_{n-2}(t) \\ \vdots \\ \vdots \\ c \end{pmatrix},$$

$$V_t(p(t), \mathbb{X}(t)) = \min_c \big\{ - p(t)c + \delta\rho_{|p(t)}(V_{t+1}(p(t+1), \mathbb{X}(t+1)))\big\}.$$

The main idea of the proof is to consider the role played by c in all the possible expressions of $V_{t+1}(\cdot,\cdot)$. This is not an easy task, but despite all the possible harvesting policies, the coefficient of c has a particular structure: it is the sum of a finite number of terms of the form $\Delta_j^m(p(t))$ (as defined in (22)), for some values of $m_i \leq T - t$ and $j_i \in \{0, \ldots, n-1\}$ plus possibly one negative term $-\delta^m[p(t)a^m + b\sum_{l=0}^{m-1} a^l]$. For a proof of this statement we refer the reader to Pagnoncelli and Piazza [2011] where the analogous result in the risk neutral case is presented. The construction of the coefficient of c follows the same lines, the reader only needs to substitute the operator $\mathbb{E}_{|p(t)}$ for $\rho_{|p(t)}$.

The proof is completed by showing that the coefficient of c is negative. But, Lemma 5.1 shows that $\Delta_j^m \leq 0$ when condition (21) holds, which finishes the proof.

■

Proof. Proof of Theorem 6.1: This theorem is a generalization of Theorem 5.1 from $(a, b) \in (0, 1) \times \mathbb{R}_+$ to $(a, b) \in \mathbb{R}_+ \times \mathbb{R}$. The proof of Theorem 5.1 consist in characterizing the coefficient of c in the Bellman equation (8). It is shown that this coefficient comprises a finite number of terms of the form $\Delta_j^m(p(t))$ plus possibly one negative term. This construction does not depend on the value of a and b but relies exclusively in the fact that the conditional risk measure is affine on $p(t)$. Hence, this part of the proof extends directly to the more general setting of this theorem.

The characterization of the coefficient' sign relies on Lemma 5.1 presented in Section 5 that gives a sufficient condition assuring that

$$\Delta_j^m(p(t)) \leq 0 \text{ for all } m \leq T - t \text{ and for all } j \in \{1, \ldots, n\}. \tag{34}$$

Lemma 5.1 is valid for $(a, b) \in (0, 1) \times \mathbb{R}_+$. In the following we study the extension of this lemma to $(a, b) \in \mathbb{R}_+ \times \mathbb{R}$. We start by noticing that (23) is not valid for $a = 1$. We will use the following representation of $\Delta_j^m(p(t))$

$$\Delta_{j_i}^{m_i}(p(t)) = \begin{cases} \delta^{m_i}\left\{-p(t)a^{m_i}(1-\delta^{j_i}a^{j_i}) - \frac{b}{1-a}\left[1-\delta^{j_i} - a^{m_i}(1-\delta^{j_i}a^{j_i})\right]\right\} & \text{if } a \neq 1 \\ \delta^m\{-p(t)(1-\delta^j) - b[m - \delta^j(m+j)]\} & \text{if } a = 1, \end{cases}$$

We divide the semi-plane of parameters in six regions as shown in Figure 3 and Table 4. [3]

In the following, we look for conditions implying Condition (34), as this is sufficient to prove that the coefficient of $c^*(t)$ is negative and hence $c^*(t) = CA\mathbb{X}(t)$. We will see that:

Region (i): $p(t) \geq b\delta/(1 - a\delta)$ implies Condition (34) (this is Lemma 5.1).

Region (ii): $p(t) \geq b\delta/(1 - a\delta)$ implies Condition (34).

[3] In the particular case that $1 - \delta a = 0$ we observe that $\Delta_j^m(p(t))$ does not depend of $p(t)$ and that Condition (34) can be verified *a priori*. The study of this particular case is straightforward and we omit it.

	$0 < a \leq 1$	$1 < a < 1/\delta$	$a > 1/\delta$
$b \geq 0$	(i)	(ii)	(iii)
$b < 0$	(iv)	(v)	(vi)

Table 4: Parameters regions.

Region (iii): Condition (34) does not hold for $p(t) \geq 0$, hence, we have no information about $c^*(t)$.

Region (iv): There is no sufficient condition independent of t, assuring Condition (34). However, we do have that $p(t) \geq \frac{b}{1-a}[1 - \frac{1-\delta}{a^{T-t}(1-\delta a)}]$ is sufficient to assure Condition (34). For large values of $T - t$, this condition is likely to be very restrictive.

Region (v): $p(t) \geq b/(1-a)$ implies Condition (34).

Region (vi): $p(t) \leq b/(1-a)$ implies Condition (34).

Let us denote by $r_j^m(a, b)$ to the rhs of (24) when $a \neq 1$ and the corresponding expression for $a = 1$, i.e.,

$$r_j^m(a,b) = \begin{cases} \frac{b}{1-a}\left[1 - \frac{1-\delta^j}{a^m(1-\delta^j a^j)}\right] & \text{if } a \neq 1 \\ \frac{-b}{1-\delta^j}[m - \delta^j(m+j)] & \text{if } a = 1 \end{cases}$$

In the following, we prove the properties summarized in Figure 3. We observe in the first place that

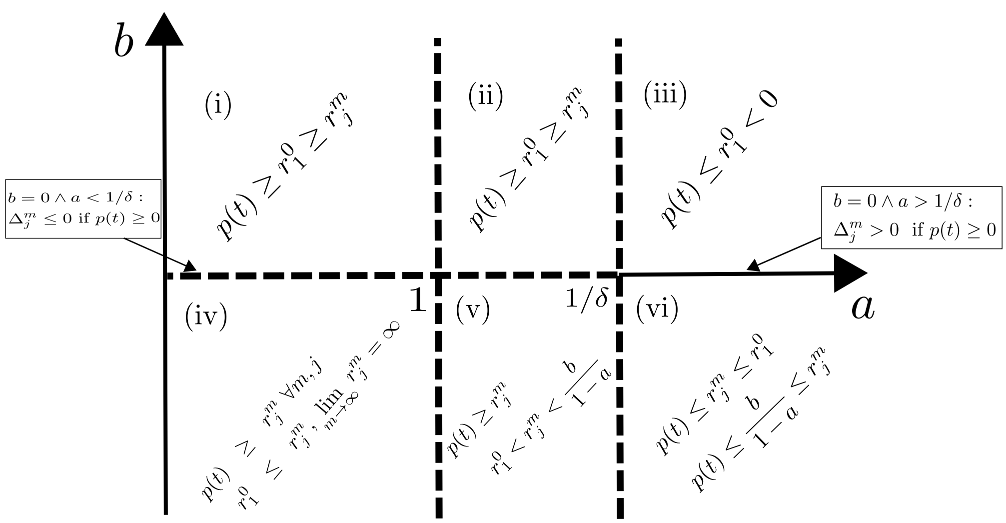

Figure 3: Semi-plane of parameters a and b.

$$\begin{aligned} \Delta_j^m(p(t)) \leq 0 &\iff p(t) \geq r_j^m(a,b) \qquad \text{if } 1 - \delta a > 0 \\ \Delta_j^m(p(t)) \leq 0 &\iff p(t) \leq r_j^m(a,b) \qquad \text{if } 1 - \delta a < 0 \end{aligned} \tag{35}$$

We start by determining whether $r_1^0(a,b) = \frac{b\delta}{1-\delta a}$ bounds $r_j^m(a,b)$ (below or above) for all m and $j = 1, \ldots, n$. We study the case $a \neq 1$, leaving the easier particular case $a = 1$ to the reader. We also make the observation that if $b = 0$ then $r_j^m = 0$ for all m and j.

$$
\begin{aligned}
r_1^0(a,b) &\lesseqqgtr r_j^m(a,b) \\
\iff \frac{b\delta}{1-\delta a} = \frac{b}{1-a}\Big[1-\frac{1-\delta}{(1-\delta a)}\Big] &\lesseqqgtr \frac{b}{1-a}\Big[1-\frac{1-\delta^j}{a^m(1-\delta^j a^j)}\Big] \\
\iff \frac{-b}{1-a}\Big[\frac{1-\delta}{(1-\delta a)}\Big] &\lesseqqgtr \frac{-b}{1-a}\Big[\frac{1-\delta^j}{a^m(1-\delta^j a^j)}\Big] \\
\iff \frac{b}{a-1}\Big[\frac{1-\delta}{1-\delta a}\Big] &\lesseqqgtr \frac{b}{a-1}\Big[\frac{1-\delta^j}{a^m(1-\delta^j a^j)}\Big] \qquad (36)
\end{aligned}
$$

If $sg(b)sg(a-1)sg(1-\delta a) > 0$, i.e., if we are in regions $(ii), (iv)$ and (vi), (36) is equivalent to

$$
\begin{aligned}
a^m \frac{1-\delta^j a^j}{1-\delta a} &\lesseqqgtr \frac{1-\delta^j}{1-\delta} \\
\iff a^m \sum_{l=0}^{j-1} (\delta a)^j &\lesseqqgtr \sum_{l=0}^{j-1} \delta^j
\end{aligned}
$$

- If $a > 1$ (regions (ii) and (vi)), the inequality above holds with "$\geq$". Hence, $r_1^0(a,b) \geq r_j^m(a,b)$.
- If $a < 1$ (region (iv)), it holds with "$\leq$". Hence, $r_1^0(a,b) \leq r_j^m(a,b)$.

On the other hand, if $sg(b)sg(a-1)sg(1-\delta a) < 0$, i.e., if we are in regions $(i), (iii)$ and (v), (36) is equivalent to

$$
\begin{aligned}
\frac{1-\delta^j}{1-\delta} &\lesseqqgtr a^m \frac{1-\delta^j a^j}{1-\delta a} \\
\sum_{l=0}^{j-1} \delta^j &\lesseqqgtr \sum_{l=0}^{j-1} \delta^j a^m \sum_{l=0}^{j-1} (\delta a)^j
\end{aligned}
$$

- If $a > 1$ (regions (iii) and (v)), the inequality above holds with "$\leq$". Hence, $r_1^0(a,b) \leq r_j^m(a,b)$.
- If $a < 1$ (region (i)), it holds with "$\geq$". Hence, $r_1^0(a,b) \geq r_j^m(a,b)$.

Table 5 summarizes these results.

	$0 < a \leq 1$	$1 < a < 1/\delta$	$a > 1/\delta$
$b \geq 0$	$r_1^0 \geq r_j^m$	$r_1^0 \geq r_j^m$	$r_1^0 \leq r_j^m$
$b < 0$	$r_1^0 \leq r_j^m$	$r_1^0 \leq r_j^m$	$r_1^0 \geq r_j^m$

Table 5: Bounds for $r_j^m(a,b)$.

Putting this information together with that of (35), we conclude that to assure Condition (34) it is sufficient to have the conditions indicated in Table 6

In regions (i) and (ii) we are ready to give a sufficient condition assuring Condition (34):

$$
p(t) \geq r_1^0(a,b) = \frac{\delta b}{1-\delta a}.
$$

In region (iii) we have $r_1^0 < 0$, hence the condition we deduced is never satisfied. Furthermore, we readily see that $\Delta_1^0(p(t)) < 0$ for all $p(t) > 0$.

	$0 < a \leq 1$	$1 < a < 1/\delta$	$a > 1/\delta$
$b \geq 0$	$p(t) \geq r_1^0$	$p(t) \geq r_1^0$	$p(t) \leq r_1^0$
$b < 0$	$p(t) \geq r_j^m \; \forall m, j$	$p(t) \geq r_j^m \; \forall m, j$	$p(t) \leq r_j^m \; \forall m, j$

Table 6: Bounds for $p(t)$.

To reach some conclusion in regions $(iv) - (vi)$ we need some extra information of $r_j^m(a,b)$.

In region (v) it is very easy to check that $r_j^m(a,b) \leq b/(1-a)$. Hence $p(t) \geq b/(1-a)$ is sufficient to assure Condition (34).

Analogously, in region (vi) it is very easy to check that $r_j^m(a,b) \geq b/(1-a)$, and we conclude that $p(t) \leq b/(1-a)$ is sufficient to assure Condition (34).

In region (iv) is a bit different. We have that $\lim_{m\to\infty} r_j^m(a,b) = +\infty$, hence $p(t)$ cannot be greater than r_j^m for all m. However, Condition (34) only requires having $p(t) \geq r_j^m$ for $m \leq T-t$. Furthermore, some calculation shows that $r_j^{m+1} > r_j^m$ and $r_j^m > r_{j+1}^m$. Hence, we can propose a condition depending on the value of $T-t$:

$$p(t) \geq r_1^{T-t} = \frac{b}{1-a}\Big[1 - \frac{1-\delta}{a^{T-t}(1-\delta a)}\Big].$$

■

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