Analytical formulas for calculating the extremal ranks of the matrix-valued function $A + BXC$ when the rank of X is fixed

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Abstract. Analytical formulas are established for calculating the maximal and minimal ranks of the matrix-valued function $A + BXC$ when the rank of X is fixed. Some consequences are also given.

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1. Introduction

Consider the following two-sided linear matrix-valued function

$$\phi(X) = A + BXC, \tag{1.1}$$

where the triple matrices $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times p}$ and $C \in \mathbb{C}^{q\times n}$ are given, and $X \in \mathbb{C}^{p\times q}$ is a variable matrix. The variable matrix X may not be totally free, but subject to certain restrictions. One of the most popular restrictions is requiring the rank of X is fixed. In this paper, we take the rank of (1.1) as an inter-valued objective function, and solve the following optimization problem:

Problem 1.1. For the function in (1.1) and $0 \leqslant s \leqslant t \leqslant \min\{ p,\ q \}$, establish explicit formulas for calculating the following extremal ranks

$$\begin{aligned}
&\text{maximize } r(A + BXC) && \text{subject to } \ X \in \mathbb{C}^{p\times q} \ \text{ and } \ r(X) = t, && (1.2)\\
&\text{minimize } r(A + BXC) && \text{subject to } \ X \in \mathbb{C}^{p\times q} \ \text{ and } \ r(X) = t, && (1.3)\\
&\text{maximize } r(A + BXC) && \text{subject to } \ X \in \mathbb{C}^{p\times q} \ \text{ and } \ s \leqslant r(X) \leqslant t, && (1.4)\\
&\text{minimize } r(A + BXC) && \text{subject to } \ X \in \mathbb{C}^{p\times q} \ \text{ and } \ s \leqslant r(X) \leqslant t. && (1.5)
\end{aligned}$$

Matrix rank optimization problems are a class of discontinuous optimization problems, in which the decision variables are matrices running over certain matrix sets, while the ranks of the variable matrices are taken as integer-valued objective functions. This kind of optimization problems cannot be solved analytically by any approximation method. Motivations for finding the extremal ranks of (1.1) arise from both theoretical and applied points of view, and two well-known formulas in closed-form for calculating the global maximal and minimal ranks of (1.1) are given by

$$\max_{X\in\mathbb{C}^{p\times q}} r(A + BXC) = \min\left\{ r[\,A,\ B\,],\ r\begin{bmatrix} A \\ C \end{bmatrix} \right\}, \tag{1.6}$$

$$\min_{X\in\mathbb{C}^{p\times q}} r(A + BXC) = r[\,A,\ B\,] + r\begin{bmatrix} A \\ C \end{bmatrix} - r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}. \tag{1.7}$$

A very beginner who knows the concept of matrix rank in linear algebra can enjoy the simplicity and beauty of (1.6) and (1.7). Some people made essential contributions for the establishments of (1.6) and (1.7) through generalized inverses of matrices and simultaneous matrix decompositions; see, e.g., [1, 2, 5, 7]. The general expressions of the variable matrix X satisfying these two equalities were given in [2, 7] through generalized inverses of matrices and simultaneous matrix decompositions. When the two integers s and t are small in comparison with the dimensions of X, (1.2)–(1.5) can be regarded as some standard examples of lower-rank approximation problems.

Throughout this paper, $\mathbb{C}^{m\times n}$ stands for the set of all $m \times n$ complex matrices; A^*, $r(A)$ and $\mathscr{R}(A)$ stand for the conjugate transpose, rank and range (column space) of a matrix $A \in \mathbb{C}^{m\times n}$, respectively; I_m denotes the identity matrix of order m; $[\,A,\ B\,]$ denotes a row block matrix consisting of A and B.

The following simultaneous decomposition was due to Zha [8, 9].

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Lemma 1.2. *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times p}$ *and* $C \in \mathbb{C}^{q\times n}$. *Then there exist two nonsingular matrices* $P \in \mathbb{C}^{m\times m}$, $Q \in \mathbb{C}^{n\times n}$ *and two unitary matrices* $U \in \mathbb{C}^{p\times p}$, $V \in \mathbb{C}^{q\times q}$ *such that*

$$A = P\Sigma_A Q, \quad B = P\Sigma_B U, \;\; C = V\Sigma_C Q, \tag{1.8}$$

in which

$$\Sigma_A = \begin{array}{c} \left[\begin{array}{cccccc} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & S_A & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} j \\ k \\ l \\ r \\ s_2 \\ t_2 \end{array} \\ \begin{array}{cccccc} j & k & l & r & s_1 & t_1 \end{array} \end{array}, \tag{1.9}$$

$$\Sigma_B = \begin{array}{c} \left[\begin{array}{cccc} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & S_B & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} j \\ k+l \\ r \\ s_2 \\ t_2 \end{array} \\ \begin{array}{cccc} j & p-j-r-s_2 & r & s_2 \end{array} \end{array}, \tag{1.10}$$

$$\Sigma_C = \begin{array}{c} \left[\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & S_C & 0 & 0 \\ 0 & 0 & 0 & I & 0 \end{array}\right] \begin{array}{l} q-l-r-s_1 \\ l \\ r \\ s_1 \end{array} \\ \begin{array}{ccccc} j+k & l & r & s_1 & t_1 \end{array} \end{array}, \tag{1.11}$$

where S_A, S_B *and* S_C *are diagonal matrices with positive diagonal entries, and*

$$\begin{aligned} j &= r\begin{bmatrix} A \\ C \end{bmatrix} + r(B) - r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}, \\ k &= r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} - r(B) - r(C), \\ l &= r[\,A,\, B\,] + r(C) - r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}, \\ r &= r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} + r(A) - r\begin{bmatrix} A \\ C \end{bmatrix} - r[\,A,\, B\,], \\ s_1 &= r\begin{bmatrix} A \\ C \end{bmatrix} - r(A), \\ s_2 &= r[\,A,\, B\,] - r(A), \\ t_1 &= n - r\begin{bmatrix} A \\ C \end{bmatrix}, \\ t_2 &= m - r[\,A,\, B\,]. \end{aligned}$$

Lemma 1.3. *Let* $X \in \mathbb{C}^{m\times n}$, $Y \in \mathbb{C}^{m\times p}$ *and* $Z \in \mathbb{C}^{q\times n}$ *be three variable matrices, and let*

$$\phi(X,\, Y,\, Z\,) = \begin{bmatrix} X & Y \\ Z & 0 \end{bmatrix}. \tag{1.12}$$

Then,

$$\max_{X\in\mathbb{C}^{m\times n},\, Y\in\mathbb{C}^{m\times p},\, Z\in\mathbb{C}^{q\times n}} r[\phi(X,\, Y,\, Z\,)] = \min\{\, m+q, \quad n+p, \quad m+n\,\}, \tag{1.13}$$

$$\min_{X\in\mathbb{C}^{m\times n},\, Y\in\mathbb{C}^{m\times p},\, Z\in\mathbb{C}^{q\times n}} r[\phi(X,\, Y,\, Z\,)] = 0. \tag{1.14}$$

Further, for any integer t *with* $0 \leqslant t \leqslant \min\{\, m+q,\ n+p,\ m+n\,\}$, *there exist* $X \in \mathbb{C}^{m\times n}$, $Y \in \mathbb{C}^{m\times p}$ *and* $Z \in \mathbb{C}^{q\times n}$ *such that*

$$r\begin{bmatrix} X & Y \\ Z & 0 \end{bmatrix} = t. \tag{1.15}$$

Proof. It is obvious that the right-hand side of (1.13) is an upper bound of $r[\phi(X, Y, Z)]$.

(I) Under $m+q \leqslant \min\{n+p, \quad m+n\}$ and $m \leqslant p$, setting

$$X = 0, \quad Y = [I_m, \ 0], \quad Z = [I_q, \ 0]$$

leads to $r[\phi(X, Y, Z)] = r(Y) + r(Z) = m + q$; under $m+q \leqslant \min\{n+p, \quad m+n\}$ and $m > p$, setting

$$[X, Y] = [0, \ I_m], \quad Z = [I_q, \ 0]$$

leads to $r[\phi(X, Y, Z)] = m + q$;

(II) under $n+p \leqslant \min\{m+q, \quad m+n\}$ and $n \leqslant q$, setting

$$X = 0, \quad Y = [I_n, \ 0]^T, \quad Z = [I_p, \ 0]^T$$

leads to $r[\phi(X, Y, Z)] = r(Y) + r(Z) = n + p$; under $n+p \leqslant \min\{m+q, \quad m+n\}$ and $n > q$, setting

$$\begin{bmatrix} X \\ Z \end{bmatrix} = \begin{bmatrix} 0 \\ I_n \end{bmatrix}, \quad Y = \begin{bmatrix} I_p \\ 0 \end{bmatrix}$$

leads to $r[\phi(X, Y, Z)] = r(Y) + r(Z) = n + p$;

(III) under $m+n \leqslant \min\{m+q, \quad n+p\}$, setting

$$X = 0, \quad Y = [I_m, \ 0], \quad Z = [I_n, \ 0]^T$$

leads to $r[\phi(X, Y, Z)] = r(Y) + r(Z) = m + n$; establishing (1.13).

Setting $X = 0$ and $Y = 0$ leads to (1.14).

(a) for any integer $0 \leqslant t \leqslant \min\{m+q, \quad n+p, \quad m+n\}$ with $m+q \leqslant \min\{n+p, \quad m+n\}$ and $m \leqslant p$, setting

$$X = 0, \quad Y = [Y_1, \ 0], \quad Z = [Z_1, \ 0], \quad r(Y_1) + r(Z_1) = t$$

leads to $r[\phi(X, Y, Z)] = r(Y_1) + r(Z_1) = t$; with $m+q \leqslant \min\{n+p, \quad m+n\}$ and $m > p$, setting

$$[X, Y] = [0, \ Y_1], \quad Z = [Z_1, \ 0], \quad r(Y_1) + r(Z_1) = t$$

leads to $r[\phi(X, Y, Z)] = r(Y_1) + r(Z_1) = t$;

(b) for any integer $0 \leqslant t \leqslant \min\{m+q, \quad n+p, \quad m+n\}$ with $n+p \leqslant \min\{m+q, \quad m+n\}$ and $n \leqslant q$, setting

$$X = 0, \quad Y = [Y_1, \ 0]^T, \quad Z = [Z_1, \ 0]^T, \quad r(Y_1) + r(Z_1) = t$$

leads to $r[\phi(X, Y, Z)] = r(Y_1) + r(Z_1) = t$; with $n+p \leqslant \min\{m+q, \quad m+n\}$ and $n > q$, setting

$$\begin{bmatrix} X \\ Z \end{bmatrix} = \begin{bmatrix} 0 \\ Z_1 \end{bmatrix}, \quad Y = \begin{bmatrix} Y_1 \\ 0 \end{bmatrix}$$

leads to $r[\phi(X, Y, Z)] = r(Y_1) + r(Z_1) = t$;

(c) for any integer $0 \leqslant t \leqslant \min\{m+q, \quad n+p, \quad m+n\}$ with $m+n \leqslant \min\{m+q, \quad n+p\}$, setting

$$X = 0, \quad Y = [Y_1, \ 0], \quad Z = [Z_1, \ 0]^T, \quad r(Y_1) + r(Z_1) = t$$

leads to $r[\phi(X, Y, Z)] = r(Y_1) + r(Z_1) = t$, establishing (1.15). $\square$

Lemma 1.4. *Let $A \in \mathbb{C}^{m\times n}$ be given, $Y \in \mathbb{C}^{m\times p}$, $Z \in \mathbb{C}^{q\times n}$ and $U \in \mathbb{C}^{q\times p}$ be three variable matrices, and define*

$$\phi(Y, Z, U) = \begin{bmatrix} A & Y \\ Z & U \end{bmatrix}. \tag{1.16}$$

Then,

$$\max_{Y\in\mathbb{C}^{m\times p}, \ Z\in\mathbb{C}^{q\times n}, U\in\mathbb{C}^{q\times p}} r[\phi(X, Y, U)] = \min\{m+p, \quad n+q, \quad p+q-r(A)\}, \tag{1.17}$$

$$\min_{Y\in\mathbb{C}^{m\times p}, \ Z\in\mathbb{C}^{q\times n}, U\in\mathbb{C}^{q\times p}} r[\phi(X, Y, U)] = r(A). \tag{1.18}$$

In particular, for any integer t with $r(A) \leqslant t \leqslant \min\{m+p, \quad n+q, \quad r(A)+p+q\}$, there exist $Y \in \mathbb{C}^{m\times p}$, $Z \in \mathbb{C}^{q\times n}$ and $U \in \mathbb{C}^{q\times p}$ such that

$$r[\phi(X, Y, U)] = t. \tag{1.19}$$

Proof. Without lost generality, we assume that A is given by

$$A = \mathrm{diag}(\, I_d, \ 0\,). \tag{1.20}$$

Correspondingly,

$$\phi(Y,\, Z,\, U) = \begin{bmatrix} I_d & 0 & \widehat{Y}_1 \\ 0 & 0 & \widehat{Y}_2 \\ \widehat{Z}_1 & \widehat{Z}_2 & U \end{bmatrix}, \tag{1.21}$$

and

$$r[\phi(Y,\, Z,\, U)] = d + r\begin{bmatrix} 0 & \widehat{Y}_2 \\ \widehat{Z}_2 & U - \widehat{Z}_1\widehat{Y}_1 \end{bmatrix}. \tag{1.22}$$

Applying (1.13) and (1.14) to the block matrix in (1.22) leads to

$$\max_{Y\in\mathbb{C}^{m\times p},\, Z\in\mathbb{C}^{q\times n},\, U\in\mathbb{C}^{q\times p}} r\begin{bmatrix} 0 & \widehat{Y}_2 \\ \widehat{Z}_2 & U - \widehat{Z}_1\widehat{Y}_1 \end{bmatrix} = \min\{\, m+p-r(A), \ \ n+q-r(A), \ \ p+q-2r(A)\,\}, \tag{1.23}$$

$$\min_{Y\in\mathbb{C}^{m\times p}, Z\in\mathbb{C}^{q\times n},\, U\in\mathbb{C}^{q\times p}} r\begin{bmatrix} 0 & \widehat{Y}_2 \\ \widehat{Z}_2 & U - \widehat{Z}_1\widehat{Y}_1 \end{bmatrix} = 0. \tag{1.24}$$

Substituting (1.23) and (1.24) into (1.22) yields (1.17) and (1.18). Applying (1.15) to (1.23) and (1.24) leads to (1.19). □

2. The rank of $A + X$

One of the special cases in (1.1) is the ordinary sum $A + X$. In this section, we derive explicit formulas for calculating the extremum ranks of $A + X$ subject to X with a fixed rank. The formulas obtained will be used in Sections 3.

Theorem 2.1. *Let $A \in \mathbb{C}^{m\times n}$ be given, $X \in \mathbb{C}^{m\times n}$ be a variable matrix, and assume that s and t are two integers satisfying*

$$0 \leqslant s \leqslant t \leqslant \min\{\, m, \ n\,\}. \tag{2.1}$$

Then,

(a) *The following equalities hold*

$$\max_{X\in\mathbb{C}^{m\times n},\, r(X)=t} r(\, A + X\,) = \min\{\, m, \ \ n, \ \ r(A) + t\,\}, \tag{2.2}$$

$$\min_{X\in\mathbb{C}^{m\times n},\, r(X)=t} r(\, A + X\,) = |\, r(A) - t\,|. \tag{2.3}$$

(b) *The following equalities hold*

$$\max_{X\in\mathbb{C}^{m\times n},\, s\leqslant r(X)\leqslant t} r(\, A + X\,) = \min\{\, m, \ \ n, \ r(A) + t\,\}, \tag{2.4}$$

$$\min_{X\in\mathbb{C}^{m\times n},\, s\leqslant r(X)\leqslant t} r(\, A + X\,) = \max\{\, 0, \ s - r(A), \ r(A) - t\}. \tag{2.5}$$

(c) *The following equalities hold*

$$\max_{X\in\mathbb{C}^{m\times n},\, 0\leqslant r(X)\leqslant t} r(\, A + X\,) = \min\{\, m, \ n, \ r(A) + t\,\}, \tag{2.6}$$

$$\min_{X\in\mathbb{C}^{m\times n},\, 0\leqslant r(X)\leqslant q} r(\, A + X\,) = \max\{\, 0, \ \ r(A) - t\}. \tag{2.7}$$

(d) *The following equalities hold*

$$\max_{X\in\mathbb{C}^{m\times n},\, s\leqslant r(X)\leqslant \min\{m,\, n\}} r(\, A + X\,) = \min\{m, \ \ n\}, \tag{2.8}$$

$$\min_{X\in\mathbb{C}^{m\times n},\, s\leqslant r(X)\leqslant \min\{m,\, n\}} r(\, A + X\,) = \max\{\, 0, \ s - r(A)\}. \tag{2.9}$$

The matrices X satisfying these equalities can be formulated from the canonical form of A.

Proof. It is obvious that the right-hand sides of (2.2) and (2.3) are upper and lower bounds. Without loss of generality, we assume that A is of the form

$$A = \mathrm{diag}(\, I_d, \ 0\,). \tag{2.10}$$

Let $X = \begin{bmatrix} 0 & 0 \\ 0 & I_t \end{bmatrix}$. If $m \leqslant \min\{\, n, \ \ r(A)+t\,\}$, then $r(\, A+X\,) = m$; if $n \leqslant \min\{\, m, \ \ r(A)+t\,\}$, then $r(\, A+X\,) = n$; if $r(A)+t \leqslant \min\{\, m, \ \ n\,\}$, then $r(\, A+X\,) = r(A)+t = r(A)+t$, so that (2.2) holds.

If $r(A) \leqslant t$, then setting $X = \begin{bmatrix} -I_d & 0 & 0 \\ 0 & I_{t-d} & 0 \\ 0 & 0 & 0 \end{bmatrix}$ gives $r(\, A+X\,) = t-d$; if $r(A) > t$, then setting $X = \begin{bmatrix} -I_t & 0 \\ 0 & 0 \end{bmatrix}$ gives $r(\, A+X\,) = d-t$, so that (2.3) holds.

Note that

$$\{X \in \mathbb{C}^{m\times n} \,|\, s \leqslant r(X) \leqslant t\} = \{X \in \mathbb{C}^{m\times n} \,|\, r(X) = s\} \cup \{X \in \mathbb{C}^{m\times n} \,|\, r(X) = s+1\} \cup \ldots \cup \{X \in \mathbb{C}^{m\times n} \,|\, r(X) = t\}.$$

So that

$$\begin{aligned} &\max_{X\in\mathbb{C}^{m\times n},\, s\leqslant r(X)\leqslant t} r(\, A+X\,) \\ &= \max\left\{ \max_{X\in\mathbb{C}^{m\times n},\, r(X)=s} r(\, A+X\,), \ \max_{X\in\mathbb{C}^{m\times n},\, r(X)=s+1} r(\, A+X\,), \ldots, \max_{X\in\mathbb{C}^{m\times n},\, r(X)=t} r(\, A+X\,) \right\}, \end{aligned} \tag{2.11}$$

$$\begin{aligned} &\min_{X\in\mathbb{C}^{m\times n},\, s\leqslant r(X)\leqslant t} r(\, A+X\,) \\ &= \min\left\{ \min_{X\in\mathbb{C}^{m\times n},\, r(X)=s} r(\, A+X\,), \ \min_{X\in\mathbb{C}^{m\times n},\, r(X)=s+1} r(\, A+X\,), \ldots, \min_{X\in\mathbb{C}^{m\times n},\, r(X)=t} r(\, A+X\,) \right\}. \end{aligned} \tag{2.12}$$

Substituting (2.2) and (2.3) for $r(X) = s, \, s+1, \ldots, \, t$ into (2.11) and (2.12) and making the max-min comparison, we obtain

$$\begin{aligned} \max_{X\in\mathbb{C}^{m\times n},\, s\leqslant r(X)\leqslant t} r(\, A+X\,) &= \max\{\min\{\, m, \ n, \ r(A)+s\,\}, \ \min\{\, m, \ n, \ r(A)+s+1\,\}, \ \ldots, \ \min\{\, m, \ n, \ r(A)+t\,\}\} \\ &= \min\{\, m, \ n, \ r(A)+t\,\}, \qquad (2.13) \\ \min_{X\in\mathbb{C}^{m\times n},\, s\leqslant r(X)\leqslant t} r(\, A+X\,) &= \min\{|\, r(A)-s\,|, \ |\, r(A)-s-1\,|, \ \ldots, \ |\, r(A)-t\,|\} \\ &= \max\{\, 0, \ s-r(A), \ r(A)-t\}, \qquad (2.14) \end{aligned}$$

establishing (2.4) and (2.5), as well as (2.6)–(2.9). □

It is easy to see that (2.4)–(2.9) also hold for any $s < 0$ and $t > \min\{m, \ n\}$. So that we can use (2.4)–(2.9) under the condition

$$s \leqslant \min\{m, \ n\}, \quad 0 \leqslant t, \quad s \leqslant t. \tag{2.15}$$

3. Main results

According to [8], substituting (1.8) into (1.1) yields

$$\phi(X) = P\Sigma_A Q + P\Sigma_B UXV\Sigma_C Q = P(\,\Sigma_A + \Sigma_B UXV\Sigma_C\,)Q. \tag{3.1}$$

So that

$$r(\, A+BXC\,) = r(\,\Sigma_A + \Sigma_B Y \Sigma_C\,), \quad r(X) = r(Y), \tag{3.2}$$

where $Y = UXV$. Partition it as

$$Y = \begin{array}{c} \left[\begin{array}{cccc} Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ Y_{21} & Y_{22} & Y_{23} & Y_{24} \\ Y_{31} & Y_{32} & Y_{33} & Y_{34} \\ Y_{41} & Y_{42} & Y_{43} & Y_{44} \end{array}\right] \begin{array}{l} j \\ p-j-r-s_2 \\ r \\ s_2 \end{array} \\ \begin{array}{cccc} q-l-r-s_1 & l & r & s_1 \end{array} \end{array}. \tag{3.3}$$

Then we have

$$\Sigma_A + \Sigma_B Y \Sigma_C = \begin{bmatrix} I_j & 0 & Y_{12} & Y_{13}S_C & Y_{14} & 0 \\ 0 & I_k & 0 & 0 & 0 & 0 \\ 0 & 0 & I_l & 0 & 0 & 0 \\ 0 & 0 & S_BY_{32} & S_A + S_BY_{33}S_C & S_BY_{34} & 0 \\ 0 & 0 & Y_{42} & Y_{43}S_C & Y_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{3.4}$$

and

$$\begin{aligned} r(\,\Sigma_A + \Sigma_B Y \Sigma_C\,) &= j + k + l + r\begin{bmatrix} S_A + S_BY_{33}S_C & S_BY_{34} \\ Y_{43}S_C & Y_{44} \end{bmatrix} \\ &= r[\,A,\, B\,] + r\begin{bmatrix} A \\ C \end{bmatrix} - r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} + r\begin{bmatrix} S_B^{-1}S_AS_C^{-1} + Y_{33} & Y_{34} \\ Y_{43} & Y_{44} \end{bmatrix}. \end{aligned} \tag{3.5}$$

So that

$$\max_{Y \in \mathbb{C}^{p\times q},\, r(Y)=t} r(\,\Sigma_A + \Sigma_B Y \Sigma_C\,) = r[\,A,\, B\,] + r\begin{bmatrix} A \\ C \end{bmatrix} - r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} + \max_{Y \in \mathbb{C}^{p\times q}, r(Y)=t} r(\,S + \widehat{Y}\,), \tag{3.6}$$

$$\min_{Y \in \mathbb{C}^{p\times q},\, r(Y)=t} r(\,\Sigma_A + \Sigma_B Y \Sigma_C\,) = r[\,A,\, B\,] + r\begin{bmatrix} A \\ C \end{bmatrix} - r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} + \min_{Y \in \mathbb{C}^{p\times q}, r(Y)=t} r(\,S + \widehat{Y}\,), \tag{3.7}$$

where

$$S = \begin{bmatrix} S_B^{-1}S_AS_C^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \quad \widehat{Y} = \begin{bmatrix} Y_{33} & Y_{34} \\ Y_{43} & Y_{44} \end{bmatrix}.$$

Applying Lemma 1.4 to (3.5) yields the main result in the paper.

Theorem 3.1. *Let* $\phi(X)$ *be as given in* (1.1), *and assume that* t *is an integer satisfying* $0 \leqslant t \leqslant \min\{\, p, \quad q\,\}$. *Also define*

$$G = [\,A,\, B\,], \quad H = \begin{bmatrix} A \\ C \end{bmatrix}, \quad M = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}. \tag{3.8}$$

Then,

$$\max_{X \in \mathbb{C}^{p\times q},\, r(X)=t} r(\,A + BXC\,) = \min\{r(G), \quad r(H), \quad r(A) + t\}\,, \tag{3.9}$$

$$\min_{X \in \mathbb{C}^{p\times q},\, r(X)=t} r(\,A + BXC\,) = \max\{r(G) + r(H) - r(M), \quad r(G) + r(H) - r(A) + t - p - q, \;\; r(A) - t\}\,. \tag{3.10}$$

In consequences,

(i) *Under* $m = n$, *there exists an* $X \in \mathbb{C}^{p\times q}$ *with* $r(X) = t$ *such that* $A + BXC$ *is nonsingular if and only if*

$$r(G) = r(H) = m \;\; \text{and} \;\; r(A) \geqslant m - t. \tag{3.11}$$

(ii) *There exists an* $X \in \mathbb{C}^{p\times q}$ *with* $r(X) = t$ *such that* $A + BXC = 0$ *if and only if*

$$\mathscr{R}(A) \subseteq \mathscr{R}(B), \quad \mathscr{R}(A^*) \subseteq \mathscr{R}(C^*), \quad r(G) + r(H) \leqslant r(A) - t + p + q, \quad r(A) \leqslant t. \tag{3.12}$$

(iii) *Under* $\mathscr{R}(A) \subseteq \mathscr{R}(B)$ *and* $\mathscr{R}(A^*) \subseteq \mathscr{R}(C^*)$,

$$\max_{X \in \mathbb{C}^{p\times q},\, r(X)=t} r(\,A + BXC\,) = \min\{r(B), \quad r(C), \quad r(A) + t\}\,, \tag{3.13}$$

$$\min_{X \in \mathbb{C}^{p\times q},\, r(X)=t} r(\,A + BXC\,) = \max\{0, \quad r(B) + r(C) - r(A) + t - p - q, \;\; r(A) - t\}\,. \tag{3.14}$$

Proof. Let $z = p + q - 2r(M) + r(G) + r(H)$. Then we find by (1.19), (2.4) and (2.5) that

$$\begin{aligned} \max_{Y \in \mathbb{C}^{p\times q},\, r(Y)=t} r(\,S + \widehat{Y}\,) &= \max_{t - z \leqslant r(\widehat{Y}) \leqslant t} r(\,S + \widehat{Y}\,) = \min\{\,r + s_1, \quad r + s_2, \quad r(S) + t\,\} \\ &= \min\{\,r(M) - r(G), \quad r(M) - r(H), \quad r(M) + r(A) - r(G) - r(H) + t\}\,, \end{aligned} \tag{3.15}$$

$$\begin{aligned} \min_{Y \in \mathbb{C}^{p\times q},\, r(Y)=t} r(\,S + \widehat{Y}\,) &= \min_{t - z \leqslant r(\widehat{Y}) \leqslant t} r(\,S + \widehat{Y}\,) = \max\{\,0, \quad t - z - r(S), \quad r(S) - t\,\} \\ &= \max\{\,0, \quad t - p - q + r(M) - r(A), \quad r(M) + r(A) - r(G) - r(H) - t\}\,. \end{aligned} \tag{3.16}$$

Substituting (3.15) and (3.16) into (3.6) and (3.7) yields (3.9) and (3.10). Setting (3.9) equal to m yields (3.11); setting (3.10) equal to 0 yields (3.12). $\square$

Corollary 3.2. *Let* $\phi(X)$ *be as given in* (1.1), G, H *and* M *be the matrices in* (3.8), *and assume that* s *and* t *are two integers satisfying*

$$0 \leqslant s \leqslant t \leqslant \min\{\, p, \quad q\,\}. \tag{3.17}$$

Then,

$$\max_{X\in\mathbb{C}^{p\times q},\, s\leqslant r(X)\leqslant t} r(\, A + BXC\,) = \min\{r(G), \quad r(H), \quad r(A)+t\}\,, \tag{3.18}$$

$$\min_{X\in\mathbb{C}^{p\times q},\, s\leqslant r(X)\leqslant t} r(\, A + BXC\,) = \min\{u_s, \; u_{s+1}, \ldots, u_t\}, \tag{3.19}$$

where

$$u_l = \max\{r(G) + r(H) - r(M), \quad r(G) + r(H) - r(A) + l - p - q, \quad r(A) - l\}\,, \quad l = s, \; s+1, \ldots, t.$$

Corollary 3.3. *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times p}$ *and* $C \in \mathbb{C}^{p\times n}$ *be given. Then,*

$$\max_{X\in\mathbb{C}^{p\times p},\, r(X)=p} r(\, A + BXC\,) = \min\left\{r[\, A, \, B\,], \quad r\begin{bmatrix} A \\ C \end{bmatrix}\right\}, \tag{3.20}$$

$$\min_{X\in\mathbb{C}^{p\times p},\, r(X)=p} r(\, A + BXC\,) = \max\left\{r[\, A, \, B\,] + r\begin{bmatrix} A \\ C \end{bmatrix} - r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}, \quad r[\, A, \, B\,] + r\begin{bmatrix} A \\ C \end{bmatrix} - r(A) - p\right\}. \tag{3.21}$$

Corollary 3.4. *Let* $0 \neq B \in \mathbb{C}^{m\times p}$ *and* $0 \neq C \in \mathbb{C}^{q\times n}$ *be given, and assume that* t *is an integer satisfying* $1 \leqslant t \leqslant \min\{\, p, \quad q\,\}$. *Then,*

$$\max_{X\in\mathbb{C}^{p\times q},\, r(X)=t} r(\, BXC\,) = \min\{r(B), \quad r(C), \quad t\}\,, \tag{3.22}$$

$$\min_{X\in\mathbb{C}^{p\times q},\, r(X)=t} r(\, BXC\,) = \max\{0, \quad r(B) + r(C) + t - p - q\}\,. \tag{3.23}$$

In consequences,

(i) *Under* $m = n$, *there exists an* $X \in \mathbb{C}^{p\times q}$ *with* $r(X) = t$ *such that* BXC *is nonsingular if and only if*

$$r(B) = r(C) = m \quad \textit{and} \quad t \geqslant m. \tag{3.24}$$

(ii) *There exists an* $X \in \mathbb{C}^{p\times q}$ *with* $r(X) = t$ *such that* $BXC = 0$ *if and only if*

$$r(B) + r(C) \leqslant p + q - t. \tag{3.25}$$

(iii) *The rank of* BXC *is invariant for all* $X \in \mathbb{C}^{p\times q}$ *with* $r(X) = t$ *if and only if*

$$r(B) = p + q - t, \quad \textit{or} \quad r(C) = p + q - t, \quad \textit{or} \quad r(B) = p \textit{ and } r(C) = q. \tag{3.26}$$

A special case of (1.1) with symmetric pattern is $A + BXB^*$, where $A = \pm A^*$. Explicit formulas for calculating the extremal ranks and inertias of $A + BXB^*$ subject to $X = \pm X^*$ and $r(X) = t$ can be established by a similar approach.

References

[1] B. De Moor and G.H. Golub, The restricted singular value decomposition: properties and applications. SIAM J. Matrix Anal. Appl. 12(1991), 401–425.

[2] Y. Liu and Y. Tian, How to use RSVD to solve the matrix equation $A = BXC$, Linear Multilinear Algebra 58(2010), 537–543.

[3] B.De Moor and H. Zha, A tree of generalizations of the ordinary singular value decomposition, Linear Algebra Appl. 147(1991), 469–500.

[4] S.K. Mitra, Fixed rank solutions of linear matrix equations, Sankhya Ser. A 35(1972), 387–392.

[5] Y. Tian, The maximal and minimal ranks of some expressions of generalized inverses of matrices, Southeast Asian Bull. Math. 25(2002), 745–755.

[6] Y. Tian, Ranks of solutions of the matrix equation $AXB = C$, Linear and Multilinear Algebra 51(2003), 111–125.

[7] Y. Tian and S. Cheng, The maximal and minimal ranks of $A - BXC$ with applications, New York J. Math. 9(2003), 345–362.

[8] H. Zha, The restricted singular value decomposition of matrix triplets, SIAM. J. Matrix. Anal. Appl. 12(1991), 172–194.

[9] H. Zha, a Numerical algorithm for computing the restricted singular value decomposition of matrix tripletss, Linear Algebra Appl. 168(1992), 1–25.