Single-Row Equidistant Facility Layout as a Special Case of Single-Row Facility Layout

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Abstract. In this paper we discuss two particular layout problems, namely the Single-Row Equidistant Facility Layout Problem (SREFLP) and the Single-Row Facility Layout Problem (SRFLP). Our aim is to consolidate the two respective branches in the layout literature. We show that the (SREFLP) is not only a special case of the Quadratic Assignment Problem but also a special case of the (SRFLP). This new connection is relevant as the strongest exact methods for the (SRFLP) outperform the best approaches specialized to the (SREFLP). We describe and compare the exact approaches for the (SRFLP), the (SREFLP) and Linear Arrangement that is again a special case of the (SREFLP). In a computational study we showcase that the strongest exact approach for the (SRFLP) clearly outperforms the strongest exact approach tailored to the (SREFLP) on medium and large benchmark instances from the literature.

1 Introduction

Facility layout is concerned with the optimal location of departments inside a plant according to a given objective function. This is a well-known operations research problem that arises in different areas of applications. For example, in manufacturing systems, the placement of machines that form a production line inside a plant so that performance is optimized is a layout problem. Another example arises in the design of Very Large Scale Integration (VLSI) circuits in electrical engineering where one aims to arrange a set of rectangular modules on a rectangular chip area. In general, the objective function may reflect transportation costs and construction costs.

The variety of applications means that facility layout encompasses a broad class of optimization problems. The survey paper [56] divides facility layout research into three categories. The first categorry is concerned with models and algorithms for tackling different versions of the basic layout problem that asks for the optimal arrangement of a given number of departments with unequal area requirements within a facility so as to minimize the total material handling costs. The second category deals with extensions of the basic problem that take into account additional issues that arise in real-world applications, such as designing dynamic layouts by taking time-dependency issues into account, designing layouts under uncertainty conditions, and computing layouts that optimize two or more objectives simultaneously. The third category is concerned with specially structured instances of the problem, such as the layout of machines or departments along one row (e.g. a production line) or cellular layout design. In the following we will discuss two particular problems from this third area and aim to consolidate the two respective branches in the layout literature.

This Single-Row Facility Layout Problem (SRFLP), sometimes called the one-dimensional space allocation problem [64], consists of finding the optimal arrangement of rectangular machines or departments next to each other along one row so as to minimize the total weighted sum of the center-to-center distances between all pairs of departments. Hence an instance of the (SRFLP) consists of n one-dimensional departments, with given positive lengths $l_1, \dots, l_n$, and pairwise connectivities c_{ij}. The optimization problem can be written down as

$$\min_{\pi \in \Pi_n} \sum_{i<j \in n} c_{ij} z_{ij}^{\pi}, \tag{1}$$

where Π_n is the set of permutations of the indices $[n] := \{1, 2, \dots, n\}$ and z_{ij}^{π} is the center-to-center distance between departments i and j with respect to a particular permutation $\pi \in \Pi_n$.

Several practical applications of the (SRFLP) have been identified in the literature. It arises for example as the problem of ordering stations on a production line where the material flow is handled by an automated guided vehicle (AGV) travelling in both directions on a straight-line path [40]. Further applications are the arrangement of rooms on a corridor in hospitals, supermarkets, or offices [76], the assignment of airplanes to gates in an airport terminal [78] and the arrangement of books on a shelf and the assignment of disk cylinders to files [64].

The (SRFLP) has interesting connections to other combinatorial optimization problems. It is a special case of the Weighted Betweenness Problem which is again a special case of the Quadratic Ordering Problem .

Another extensively discussed combinatorial optimization problem in the layout literature is the Single-Row Equidistant Facility Layout Problem (SREFLP), sometimes called the one-dimensional machine location problem [71] or the linear machine-cell location problem [85]. The (SREFLP) has been investigated quite thoroughly in last three decades, as it models one of the major challenges encountered in designing a large manufacturing system. The (SREFLP) is normally formulated as follows. Given n machines and flows $f_{ij},\ i \neq j \in [n]$ between machines i and j, the aim is to find a one-to-one assignment of the machines to n locations equally spaced along a straight line so as to minimize the sum of the products of distances between the machines with the respective flows.

The (SREFLP) arises in many applications in manufacturing and logistics management, including sheet-metal fabrication [6], printed circuit board and disk drive assembly [22] and the optimal design of a flowline in a manufacturing system [85]. Furthermore Bhasker and Sahni [11] applied the (SREFLP) to minimize the total wire length needed when arranging circuit components on a straight line.

The (SREFLP) is a special case of the quadratic assignment problem (QAP) formulated by Koopmans and Beckmann [49] and hence any algorithm for the latter can also be applied to the former. However, studying special cases of a more general problem may lead to the discovery of more efficient algorithms tailored to solve these special cases. For the (QAP) the algorithms of Christofides and Benavent [23] for the Tree (QAP) and Drezner [26] for the grey pattern (QAP) are such examples.

But the (SREFLP) is also a special case of the (SRFLP) where all facilities have the same length and the pairwise connectivities $c_{ij},\ i < j \in [n]$ are given as the sum of the flows f_{ij} and f_{ji}. While exact methods and heuristics especially designed for the (SREFLP) clearly outperform general methods for the (QAP), this is not the case for approaches to the (SRFLP) (for details see the computational study in Section 4).

Furthermore minimum Linear Arrangement (LA), which is NP-hard [31] (even if the underlying graph is bipartite [30]), is a special case of the (SREFLP) where all connectivities are equal. Hence the (SREFLP) and the (SRFLP) are also NP-hard and weighted (LA) and the (SREFLP) are equivalent problems.

(LA) belongs to the graph layout problems that ask for a permutation of the nodes of the underlying graph that optimizes some function of pairwise node distances. (LA) was originally proposed by Harper [35, 36] to develop error-correcting codes with minimal average absolute errors and was since then applied to VLSI design [80], single machine job scheduling [1, 66] and computational biology [47, 57]. It is also used for the layout of entity relationship models [18] and data flow diagrams [29]. There exist approximation algorithms for (LA) with performance guarantee $O(\log n)$ [12,65] and $O(\sqrt{\log n}\log\log n)$ [17,27]. For further details on graph layout problems we refer to the survey paper of Díaz et al. [25].

In the following we will discuss the relations between (LA), the (SREFLP) and the (SRFLP) that have been disregarded in the literature so far in more detail. The main contributions of this paper are:

1. We describe and compare the most successful modelling and algorithmic approaches to (LA), the (SREFLP) and the (SRFLP).
2. We apply the strongest exact approach for the (SRFLP) to all (SREFLP) benchmark instances and favorably compare it to the leading exact algorithm for the (SREFLP).
3. We relate the heuristics for the (SREFLP) and the (SRFLP).

The paper is structured as follows. In Section 2 we put the most competitive exact approaches and heuristics for (LA), the (SREFLP) and the (SRFLP) into perspective and compare them from a theoretical point of view. In Section 3 we present a semidefinite optimization approach for Quadratic Ordering Problems and apply it to all (SREFLP) benchmark instances in Section 4. Finally, conclusions and future research directions are given in Section 5.

2 Survey and Comparison of Exact Approaches and Heuristics for Single-Row Facility Layout

Due to their strong combinatorial nature and the involved NP-hardness, numerous heuristic and metaheuristic approaches have been proposed for (LA) [46,55,62,63,68], the (SREFLP) [58,71,73,74,81,84,85] and the (SRFLP) [24,33,34,39,41,52,69,70].

For most facility layout problems there exist few methods that provide global optimal solutions, or at least a measure of nearness to global optimality, for large instances. The main exceptions are the problems in consideration.

Exact methods for (LA), the (SREFLP) and the (SRFLP) can be divided with respect to the variable and relaxation type that they are using. The problems are formulated either in $\binom{n}{2}$ integer distance variables modelling the distances between all pairs of departments or $\binom{n}{2}$ binary position variables modelling the positions of the departments. It is also possible that $\binom{n}{3}$ betweenness variables or $\binom{n}{2}$ ordering variables modeling the relative order between all triples or pairs of departments are used. Furthermore the enumerative scheme (in general a Branch & Bound approach) is based either on cheap linear relaxations or on more expensive but also stronger semidefinite relaxations.

Semidefinite Programming (SDP) is the extension of Linear Programming (LP) from the cone of non-negative real vectors to the cone of symmetric positive semidefinite matrices. (SDP) includes (LP) as a special case, namely when all the matrices involved are diagonal. A (primal) SDP can be expressed as the following optimization problem

$$\begin{aligned} &\inf_X \{ \langle C, X \rangle \ : \ X \in \mathcal{P} \}, \\ &\mathcal{P} := \{ X \mid \langle A_i, X \rangle = b_i, \ i \in \{1, \ldots, m\}, \ X \succcurlyeq 0 \}, \end{aligned} \qquad \text{(SDP)}$$

where the data matrices $A_i, \ i \in \{1, \ldots, m\}$ and C are symmetric. We refer the reader to the handbooks [8, 82] for a thorough coverage of the theory, algorithms and software in this field, as well as a discussion of many application areas where (SDP) has had a major impact.

The theoretically fastest known exact algorithm for (LA) is based on dynamic programming [50]. Back in 2009 this approach that is restricted to instances of size $n \leq 30$ was the strongest exact method for (LA). Hence finding global optimal solutions or at least tight global bounds for large (LA) instances is already very challenging. Recently one (SDP)-based method and two (LP)-based approaches were developed that are applicable to large instances with $n \geq 40$. The algorithm of Caprara et al. [15] that is using betweenness variables and was realized by Schwarz [75], is the most competitive exact method for small graphs and large, sparse graphs with $n \leq 200$. For even larger graphs the algorithm proposed by Caprara et al. [16] that is using position variables is the method of choice as it can provide reasonable bounds for sparse graphs with up to $n \approx 1000$. The semidefinite approach of Hungerländer and Rendl [45] yields competitive results for most sparse instances with $n \leq 100$ and is the method of choice for dense instances (edge density $\geq 30\%$) with $n \geq 30$. For a detailed survey and comparison of exact methods for (LA) see [42,45].

In contrast to (LA), the benchmark instances for the (SREFLP) and also the (SRFLP) are in general very dense. Hence it is difficult to efficiently generalize methods based on exploiting sparsity from (LA) to these problems. For the (SREFLP) Kouvelis et al. [51] used a dynamic programming algorithm to solve instances with $n \leq 20$ and Palubeckis [61] proposed an (LP)-based Branch & Bound approach based on distance variables in conjunction with a tabu search heuristic to obtain optimal solutions for instances with $n \leq 35$ and global bounds for instances with $n \leq 60$.

For the (SRFLP) several exact approaches have been proposed. Simmons [76] first studied the (SRFLP) and suggested a Branch & Bound algorithm. Later on Simmons [77] pointed out the possibility of extending the dynamic programming algorithm of Karp and Held [48] to the (SRFLP). This was later on implemented by Picard and Queyranne [64]. A nonlinear model was presented by Heragu and Kusiak [41]. (LP)-based approaches using distance variables were proposed by Love and Wong [54] and Amaral [2]. Amaral [3] achieved a more efficient (LP)-based method by linearizing a quadratic model formulated in ordering variables. However all these models suffer from weak lower bounds and hence have high computation times and memory requirements. But just recently Amaral and Letchford [5] achieved significant progress in that direction through the first polyhedral study of the distance polytope for the (SRFLP) and showed that their approach is quite effective for instances with challenging size ($n \geq 30$). Amaral [4] suggested an (LP)-based cutting plane algorithm using betweenness variables that proved to be highly competitive and solved instances with up to 35 facilities to optimality.

Anjos et al. [7] proposed the first semidefinite relaxation for the (SRFLP) yielding bounds for instances with up to 80 facilities. Anjos and Vanelli [9] further tightened this relaxation using triangle inequalities as cutting planes and obtained optimal solutions for instances with up to 30 facilities that remained unsolved since 1988. Anjos and Yen [10] suggested an alternative semidefinite relaxation and achieved optimality gaps no greater than 5 % for large instances with up to 100 departments. Recently Hungerländer and Rendl [45] proposed a general (SDP)-based approach for quadratic ordering problems, where they further improved on the tightness of the existing relaxations. They used a suitable combination of optimization methods to deal with their stronger but more expensive relaxations and applied their method among others to all known benchmark instances for the (SRFLP). They provided global optimal solutions for instances with up to 42 departments, and obtained tighter bounds than the Anjos-Yen relaxation for instances with up to 100 departments. Hence their algorithm achieved the best practical performance to date among all exact approaches to the (SRFLP). For a detailed survey and comparison of exact methods for the (SRFLP) see [42, 44]. We will recap the (SDP)-based algorithm of Hungerländer and Rendl in the following section and apply it to all known benchmark instances for the (SREFLP) in Section 4.

3 A Semidefinite Optimization Approach for Quadratic Ordering Problems

In this section we will elaborate on the (SDP)-based approach by Hungerländer and Rendl [45] that has not only been successfully applied to (LA) and the (SRFLP) but also to several further Quadratic Ordering Problems [19–21, 42]. First we will deduce a quadratic programming formulation in ordering variables of the (SRFLP). We will rewrite this intractable formulation in matrix notation and use standard techniques for relaxing it. Hence we will obtain a basic semidefinite relaxation that can be tightened by adding several classes of valid constraints. Finally we will describe a suitable combination of optimization methods that can be used to obtain strong lower bounds and feasible layouts from the semidefinite relaxation.

3.1 A Matrix-Based Formulation for Single-Row Facility Layout

The center-to-center distances between departments can be encoded using betweenness variables that again can be expressed as products of ordering variables $y_{ij} (i, j \in \mathcal{N}, i < j)$

$$y_{ij} = \begin{cases} 1, & \text{if department } i \text{ lies before department } j, \\ -1, & \text{otherwise.} \end{cases} \tag{2}$$

Any feasible ordering of the departments has to fulfill the 3-cycle inequalities

$$-1 \leq y_{ij} + y_{jk} - y_{ik} \leq 1, \quad i, j, k \in [n],\ i < j < k. \tag{3}$$

It is well-known that the 3-cycle inequalities together with integrality conditions on the ordering variables suffice to describe feasible orderings, see e.g. [79, 83]. Now we can rewrite the objective function (1) in terms of ordering variables

$$K - \sum_{\substack{i,j \in \mathcal{N} \\ i<j}} \frac{c_{ij}}{2} \left(\sum_{\substack{k \in \mathcal{N} \\ k<i}} l_k y_{ki} y_{kj} - \sum_{\substack{k \in \mathcal{N} \\ i<k<j}} l_k y_{ik} y_{kj} + \sum_{\substack{k \in \mathcal{N} \\ k>j}} l_k y_{ik} y_{jk} \right), \tag{4}$$

where

$$K := \left(\sum_{\substack{i,j \in \mathcal{N} \\ i<j}} \frac{c_{ij}}{2} \right) \left(\sum_{k \in \mathcal{N}} l_k \right). \tag{5}$$

We collect the ordering variables in a vector y and reformulate the (SRFLP) as a quadratic program in ordering variables.

Theorem 1 *Minimizing* (4) *over* $y \in \{-1, 1\}^{\binom{n}{2}}$ *and* (3) *solves the* (SRFLP).

Finally we rewrite (4) in matrix notation as follows:

$$\min \left\{ \langle C_Y, Y \rangle + K : \ y \in \{-1, 1\}^{\binom{n}{2}} \text{ satisfies (3)} \right\}, \tag{SRFLP}$$

where $Y := yy^\top$ and the cost matrix C_Y is deduced from (4).

In the following subsection we use matrix-based relaxations to get tight lower bounds for the (SRFLP). Similar relaxations have already been successfully applied to combinatorial optimization problems arising in the area of graph drawing [14, 19–21, 42].

3.2 Semidefinite Relaxations

We apply standard techniques to construct `(SDP)` relaxations over the linear-quadratic ordering polytope

$$\mathcal{P}_{LQO} := \text{conv}\left\{ \begin{pmatrix}1\\y\end{pmatrix}\begin{pmatrix}1\\y\end{pmatrix}^\top \;:\; y \in \{-1,1\}^{\binom{n}{2}},\; y \text{ satisfies (3)} \right\}.$$

First we relax the nonconvex equation $Y - yy^\top = 0$ to the positive semidefinite constraint

$$Y - yy^\top \succcurlyeq 0.$$

Moreover, the main diagonal entries of Y correspond to squared $\{-1,1\}$ variables, hence $\text{diag}(Y) = e$, the vector of all ones. To simplify notation let us introduce

$$Z = Z(y, Y) := \begin{pmatrix}1 & y^\top\\ y & Y\end{pmatrix}, \tag{6}$$

where $\dim(Z) = \binom{n}{2}+1 =: \Delta$. The Schur complement lemma [13, Appendix A.5.5] implies $Y - yy^\top \succcurlyeq 0 \Leftrightarrow Z \succcurlyeq 0$. We therefore conclude that $\mathcal{P}_{LQO}$ is contained in the elliptope

$$\mathcal{E} := \{\, Z :\; \text{diag}(Z) = e,\; Z \succcurlyeq 0 \,\}.$$

In order to express constraints on y in terms of Y, they have to be reformulated as quadratic conditions in y. A natural way to do this for the 3-cycle inequalities $|y_{ij} + y_{jk} - y_{ik}| = 1$ consists in squaring both sides. Additionally using $y_{ij}^2 = 1$, we obtain

$$y_{ij,jk} - y_{ij,ik} - y_{ik,jk} = -1, \quad i,j,k \in [n],\; i < j < k. \tag{7}$$

In [14] it is shown that these 3-cycle equations formulated in the $\{0,1\}$ model[1] describe the smallest linear subspace that contains $\mathcal{P}_{LQO}$. The 3-cycle inequalities are implicitly ensured by the 3-cycle equations together with $Z \succcurlyeq 0$ [42, Proposition 4.2].

Next we can formulate the `(SRFLP)` as a semidefinite optimization problem in binary variables.

Theorem 2 (see [42,45] for a proof) *The problem*

$$\min\; \left\{ K + \langle C_Z, Z\rangle : Z \textit{ satisfies } (7),\; Z \in \mathcal{E},\; y \in \{-1,1\}^{\binom{n}{2}} \right\}$$

where Z is given by (6)*, K is defined in* (5) *and the cost matrix C_Z is given by*

$$C_Z := \begin{pmatrix}0 & 0\\ 0 & C_Y\end{pmatrix},$$

is equivalent to the `(SRFLP)`.

Dropping the integrality condition on the first row and column of Z yields the basic semidefinite relaxation of the `(SRFLP)`:

$$\min\; \{K + \langle C_Z, Z\rangle : Z \text{ satisfies (7)},\; Z \in \mathcal{E}\,\}. \tag{$\texttt{SDP}_{\texttt{basic}}$}$$

There are several ways to tighten the above relaxation. We will concentrate on two of them that have been successfully applied for the `(SRFLP)`.

First we notice that Z is generated as the outer product of the vector $\begin{pmatrix}1 & y\end{pmatrix}^\top$ that has merely $\{-1,1\}$ entries in the non-relaxed SDP formulation. Hence any feasible solution the `(SRFLP)` also belongs to the metric polytope $\mathcal{M}$ that is defined through $4\binom{\Delta}{3} \approx \frac{1}{12}n^6$ facets.

$$\mathcal{M} = \left\{ Z : \begin{pmatrix}-1 & -1 & -1\\ -1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1\end{pmatrix}\begin{pmatrix}z_{ij}\\ z_{jk}\\ z_{ik}\end{pmatrix} \le e,\;\; 1 \le i < j < k \le \Delta \right\}. \tag{8}$$

[1] In [37] it is shown that one can easily switch between the $\{0,1\}$ and $\{-1,1\}$ formulations of bivalent problems so that the resulting bounds remain the same and structural properties are preserved.

A second class of strengthening constraints for Quadratic Ordering Problems was proposed by Lovász and Schrijver [53]. They suggest to multiply the 3-cycle inequalities

$$1 - y_{ij} - y_{jk} + y_{ik} \geq 0, \qquad 1 + y_{ij} + y_{jk} - y_{ik} \geq 0. \tag{9}$$

by the nonnegative expressions

$$1 - y_{lo} \geq 0, \qquad 1 + y_{lo} \geq 0, \qquad l, o \in [n],\ l < o. \tag{10}$$

This results in the following $4\binom{n}{3}\binom{n}{2} \approx \frac{1}{3}n^5$ inequalities:

$$\begin{aligned} -1 - y_{lo} &\leq y_{ij} + y_{jk} - y_{ik} + y_{ij,lo} + y_{jk,lo} - y_{ik,lo} \leq 1 + y_{lo}, \\ -1 + y_{lo} &\leq y_{ij} + y_{jk} - y_{ik} - y_{ij,lo} - y_{jk,lo} + y_{ik,lo} \leq 1 - y_{lo}, \end{aligned} \tag{11}$$

for $i, j, k, l, o \in [n],\ i < j < k,\ l < o$. We define the corresponding polytope $\mathcal{LS}$:

$$\mathcal{LS} := \{\ Z : Z \text{ satisfies } (11)\ \}. \tag{12}$$

In summary we get the following tractable semidefinite relaxation of $\mathcal{P}_{LQO}$:

$$\min \{K + \langle C_Z, Z\rangle : Z \text{ satisfies (7)},\ Z \in (\mathcal{E} \cap \mathcal{M} \cap \mathcal{LS})\ \}. \tag{SDP$_{\texttt{full}}$}$$

All variables in Z with cost coefficient greater than zero appear in a 3-cycle equality (7) and thus are tightly constrained in the relaxation. This fact explains why the various linear and semidefinite relaxations for the (SRFLP) that are based on betweenness or ordering variables produce very tight bounds in the root node relaxation even for large instances. Notice that (SDP$_{\texttt{full}}$) can be easily applied to general Quadratic Ordering Problems by adapting its objective function. For the experiments in Section 4 we will apply (SDP$_{\texttt{full}}$) to the (SREFLP) to allow for a direct comparisons with the well-studied (SRFLP).

3.3 Computing Lower and Upper Bounds

The core of our semidefinite approach is to solve our semidefinite relaxation (SDP$_{\texttt{full}}$) by using the bundle method in conjunction with interior point methods. The resulting fractional solutions constitute lower bounds. By the use of a rounding strategy, we can exploit such fractional solutions to obtain upper bounds, i.e. integer solutions that describe a feasible layout of the departments. Hence, in the end we have some feasible solution, together with a proof how far this solution could possibly be from the true optimum. We will discuss these two steps in more detail in the following.

While theoretically tractable, it is clear that (SDP$_{\texttt{full}}$) has an impractically large number of constraints. Indeed, even including only $O(n^3)$ constraints is not realistic for instances of size $n \geq 20$. For this reason, we adopt an approach originally suggested in [28] and since then applied to the Max-Cut Problem [67] and several ordering problems [19,21,42,44,45]. Initially, we only explicitly ensure that Z lies in the elliptope $\mathcal{E}$. This can be achieved efficiently with standard interior-point methods, see e.g. [38]. All other constraints are handled through Lagrangian duality in the objective function f. By using the bundle method [28] that iteratively conducts function evaluations of f and makes improvement steps we obtain an approximate minimizer of f that is guaranteed to yield a lower bound to the optimal solution of (SDP$_{\texttt{full}}$). Since the bundle method has a rather weak local convergence behavior, we limit the number of function evaluations to control the overall computational effort.

To obtain feasible layouts, we apply the hyperplane rounding algorithm of Goemans-Williamson [32] to the approximate solution of (SDP$_{\texttt{full}}$). We take the resulting vector and flip the signs of some of its entries to make it feasible with respect to the 3-cycle inequalities (3). Computational experiments demonstrated that the repair strategy is not as critical as one might assume [21, 44, 45]. For example for the (SRFLP) the (SDP)-based rounding heuristic performs comparably to the strongest heuristics [24, 70]. Further notice that our heuristic clearly outperforms another (SDP)-based heuristic proposed by Anjos et al. [7].

4 Computational Comparison

We report the results for different computational experiments with our semidefinite relaxation (SDP$_{\texttt{full}}$). All benchmark instances used can be downloaded together with the best layouts found from `http://anjos.mgi.polymtl.`

ca/flplib. The (SDP) computations were conducted on an Intel Xeon E5160 (Dual-Core) with 2 GB RAM, running Debian 5.0 in 64-bit mode. The algorithm was implemented in Matlab 7.7. In Table 1 we compare our approach for Quadratic Ordering Problems described in Section 3 with the strongest approach specialized to the (SREFLP) designed by Palubeckis [61] on benchmark instances with up to 35 departments taken from [59,60,72,85]. Palubeckis coded his (LP)-based Branch & Bound approach in the C programming language and ran it on a Pentium M 1733 MHz notebook that is about four times slower than our machine.[2] Notice that we do not take into account the speed of the machines in Table 1, as it does not affect the conclusions drawn.

For small instances with up to 20 departments the specialized (LP)-based Branch & Bound algorithm is preferable to the (SDP)-based approach whereas the (SDP)-based approach clearly outperforms the (LP)-based algorithm on the larger instances with $n \geq 25$. The difference between the two approaches strongly grows with the problem size.

The instances "N-25" and "N-30" from Table 1 were also implicitly considered in the (SRFLP) literature where they are denoted as "N25_01" and "N30_01" and each connectivity $c_{ij},\ i < j \in [n]$ is exactly the half of $f_{ij} + f_{ji}$ which results in an optimal objective value of half the size. The instances "N25_02" – "N25_5" and "N30_02" – "N30_5" were obtained by leaving the connectivities and assigning random integers $\leq$ 30 for the lengths of the departments [9]. In Table 2 we computationally compare the four most competitive approaches to the (SRFLP) for the instances discussed above. Again we do not take into account the speed of the machines, as it does not differ too much and thus does not affect the conclusions drawn. Our machine is the quickest and about 2.5 times faster than the one in [4], which is the slowest.[2]

Comparing the results in Tables 1 and 2 we find that the approach of Palubeckis [61] is also outperformed by the (LP)-based cutting plane algorithm of Amaral [4] on "N-25" and "N-30". Furthermore we can deduce that an instance does not become a lot harder for the approaches to the (SRFLP) if we allow arbitrary department lengths instead of isochronous ones.

Next we compare our approach with the one of Palubeckis on large benchmark instances with up to 60 departments taken from [85]. We restrict the bundle method to 500 function evaluations. This limitation of the number of function evaluations sacrifices some possible incremental improvement of the bounds. We summarize the results in Table 3.

The results show that the specialized (LP)-based approach by Palubeckis [61] allows for substantial improvement. While the tabu search heuristic generates very strong layouts, the lower bounds are quite weak and can be improved very quickly by our (SDP)-based approach.

The standard (SRFLP) benchmark set also contains some large instances with isochronous department lengths. For the sake of completeness we summarize the best known bounds for these instances lifted from [44] in Tables 4 and 5. It would be interesting to compare the tabu search heuristic of Palubeckis [61] on these instances with the strongest heuristics for the (SRFLP):

1. the tabu search based heuristic of Samarghandi and Eshghi [70],
2. the permutation-based genetic algorithm of Datta et al. [24],
3. and our (SDP)-based rounding heuristic.

Further notice that the gaps obtained are quite similar to the ones in Table 3 and also stay in the same order of magnitude if we allow arbitrary integer department lengths.

5 Conclusions And Future Research

This paper dealt with two particular layout problems, namely the (SREFLP) and the (SRFLP), and aimed to consolidate the two respective branches in the layout literature. We showed that the (SREFLP) is not only a special case of the (QAP) but also a special case of the (SRFLP). This new connection is relevant as the strongest exact methods for the (SRFLP) outperform the best approaches specialized to the (SREFLP). We described and compared the exact approaches for the (SRFLP), the (SREFLP) and (LA) that is again a special case of the (SREFLP) and showed that the (SDP)-based approach of Hungerländer and Rendl [45] for the (SRFLP) outperforms the strongest exact approach tailored to the (SREFLP) from Palubeckis [61] on medium and large (SREFLP) benchmark instances.

The strength of the discussed (SDP)-based approach is its general applicability. It can not only be successfully applied to the (SRFLP) and its special cases but to any Quadratic Ordering Problem, including for instances applications in the area of graph drawing

[2] For exact numbers of the speed differences see http://www.cpubenchmark.net/.

Instance	Source	n	Optimum	Specialized (LP)-based B & B approach [61]	General (SDP)-based approach [45]
O-5	[60]	5	150	1	1
O-6		6	292	1	1
O-7		7	472	1	1
O-8		8	784	1	1
O-9		9	1032	1	1
O-10		10	1402	1	1
O-15		15	5134	2	4
O-20		20	12924	37	37
S-12	[72]	12	4431	1	4
S-13		13	5897	1	4
S-14		14	7316	1	15
S-15		15	8942	2	19
S-16		16	11019	3	27
S-17		17	13172	5	50
S-18		18	15699	8	48
S-19		19	18700	22	1:43
S-20		20	21825	55	2:09
S-21		21	24891	1:41	3:20
S-22		22	28607	3:48	3:57
S-23		23	33046	8:12	3:15
S-24		24	37498	13:41	3:31
S-25		25	42349	36:21	7:07
Y-6	[85]	6	1372	1	1
Y-7		7	1801	1	1
Y-8		8	2302	1	1
Y-9		9	2808	1	1
Y-10		10	3508	1	2
Y-11		11	4022	1	4
Y-12		12	4793	1	10
Y-13		13	5471	1	9
Y-14		14	6445	1	34
Y-15		15	7359	2	26
Y-20		20	12185	23	1:14
Y-25		25	20357	22:38	5:14
Y-30		30	27673	16:17:07	17:46
Y-35		35	38194	459:08:51	25:50
N-12	[59]	12	1000	1	1
N-14		14	1866		4
N-15		15	2186	2	17
N-16a		16	3050		8
N-16b		16	2400		6
N-17		17	3388		13
N-18		18	3986		16
N-20		20	5642	41	1:11
N-21		21	5084		1:16
N-22		22	6184		1:20
N-24		24	8270		2:12
N-25		25	9236	24:03	2:48
N-30		30	16494	12:34:18	4:42

Table 1. Results for (SREFLP) instances with up to 35 departments. “B & B” is a shortcut for “Branch & Bound”. The running times are given in sec, in min:sec or in h:min:sec respectively. A missing entry indicates that the instance was not considered by the respective approach.

Instance	n	Optimum	Anjos/Vanelli [9]	Amaral/Letchford [5]	Amaral [4]	Hungerländer/Rendl [45]
N25_01	25	4618	3:44:38	7:19:44	3:46	2:48
N25_02	25	37116.5	4:50:27	38:35	9:59	5:46
N25_03	25	24301	5:48:21	1:25:41	4:49	4:11
N25_04	25	48291.5	4:04:51	39:34	10:19	5:33
N25_05	25	15623	8:22:22	1:18:10	3:47	3:31
N30_01	30	8247	7:41:06	34:00:51	25:41	4:42
N30_02	30	21582.5	10:41:53	3:56:53	22:43	6:08
N30_03	30	45449	19:32:01	13:08:12	23:14	10:12
N30_04	30	56873.5	31:03:11	58:20	2:19:22	11:44
N30_05	30	115268	19:54:07	13:03:51	1:05:36	18:30

Table 2. Results for (SRFLP) instances with up to 35 facilities. The running times are given in sec, in min:sec or in h:min:sec respectively.

Instance	n	Specialized (LP)-based B & B approach [61]			General (SDP)-based approach [45]				
		Best lower bound	Best layout	Gap (%)	Best lower bound	Best layout	Gap (%)	Time	Improve B & B lower bound
Y-40	40	43891	47561	8.36	47561	47561	0	2:14:47	1:27
Y-45	45	58551	62890	7.41	62849	62904	0.09	3:44:43	4:09
Y-50	50	76520	83127	8.63	83086	83127	0.05	6:03:27	5:32
Y-60	60	102828	112055	8.97	111884	112126	0.22	19:57:35	15:06

Table 3. Results for well-known (SREFLP) instances with 40–60 departments. “B & B” is a shortcut for “Branch & Bound”. “Improve B & B lower bound” denotes the running times that the (SDP)-based approach needs to improve on the lower bound of the (LP)-based approach. The bundle method is restricted to 500 function evaluations and the running times are given in min:sec or in h:min:sec respectively.

Instance	n	General (SDP)-based approach [45]			
		Best lower bound	Best layout	Gap (%)	Time
ste36-1	36	10287	10287	0	14:50
sko42-1	42	25521	25525	0.02	2:23:09
sko49-1	49	40895	41012	0.29	4:36:21
sko56-1	56	63971	64027	0.09	12:36:33

Table 4. Results for well-known (SREFLP) instances with 36–56 departments that have isochronous lengths. The bundle method is restricted to 500 function evaluations and the running times are given in min:sec or in h:min:sec respectively.

Instance	n	General (SDP)-based approach [45]			
		Best lower bound	Best layout	Gap (%)	Time
sko64-1	64	96569	97194	0.65	13:08:05
sko72-1	72	138885	139231	0.25	29:33:19
sko81-1	81	203424	207063	1.79	52:44:10
sko100-1	100	375999	380562	1.21	191:47:21

Table 5. Results for well-known (SREFLP) instances with 64–100 departments that have isochronous lengths. The bundle method is restricted to 250 function evaluations and the running times are given in h:min:sec.

Therefore it seems to be worthwhile to think about ways to further improve the (SDP)-based approach. There are three (combinable) ways to do so. First we could include additional constraint classes to further tighten the underlying semidefinite relaxation. Secondly we could incorporate the bounds obtained in a Branch & Bound framework and thirdly we could try speed up the computations over the elliptope by using first-order instead of interior-point methods.

Furthermore one could try to identify other layout problems with quadratic ordering structure. A first attempt in this direction has been made by Hungerländer and Anjos [43] for the Multi-row Facility Layout Problem. Another promising application seems to be the Equidistant Unidirectional Cyclic Layout Problem.

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