

# The Split Common Null Point Problem

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April 15, 2012

## Abstract

We introduce and study the Split Common Null Point Problem (SCNPP) for set-valued maximal monotone mappings in Hilbert spaces. This problem generalizes our Split Variational Inequality Problem (SVIP) [Y. Censor, A. Gibali and S. Reich, Algorithms for the split variational inequality problem, Numerical Algorithms 59 (2012), 301–323]. The SCNPP with only two set-valued mappings entails finding a zero of a maximal monotone mapping in one space, the image of which

under a given bounded linear transformation is a zero of another maximal monotone mapping. We present four iterative algorithms that solve such problems in Hilbert spaces, and establish weak convergence for one and strong convergence for the other three.

## 1 Introduction

In this paper we introduce and study the *Split Common Null Point Problem* for set-valued mappings in Hilbert spaces. Let  $\mathcal{H}_1$  and  $\mathcal{H}_2$  be two real Hilbert spaces. Given set-valued mappings  $B_i : \mathcal{H}_1 \rightarrow 2^{\mathcal{H}_1}$ ,  $1 \leq i \leq p$ , and  $F_j : \mathcal{H}_2 \rightarrow 2^{\mathcal{H}_2}$ ,  $1 \leq j \leq r$ , respectively, and bounded linear operators  $A_j : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ ,  $1 \leq j \leq r$ , the problem is formulated as follows:

$$\text{find a point } x^* \in \mathcal{H}_1 \text{ such that } 0 \in \bigcap_{i=1}^p B_i(x^*) \quad (1.1)$$

and such that the points

$$y_j^* = A_j(x^*) \in \mathcal{H}_2 \text{ solve } 0 \in \bigcap_{j=1}^r F_j(y_j^*). \quad (1.2)$$

We denote this problem by  $\text{SCNPP}(p, r)$  to emphasize the multiplicity of mappings. To motivate this new problem and to understand its relationship with other problems, we first look at the prototypical *Split Inverse Problem* formulated in [22, Section 2]. It concerns a model in which there are given two vector spaces  $X$  and  $Y$  and a linear operator  $A : X \rightarrow Y$ . In addition, two inverse problems are involved. The first one, denoted by  $\text{IP}_1$ , is formulated in the space  $X$  and the second one, denoted by  $\text{IP}_2$ , is formulated in the space  $Y$ . Given these data, the *Split Inverse Problem* (SIP) is formulated as follows:

$$\text{find a point } x^* \in X \text{ that solves } \text{IP}_1 \quad (1.3)$$

and such that

$$\text{the point } y^* = A(x^*) \in Y \text{ solves } \text{IP}_2. \quad (1.4)$$

Real-world inverse problems can be cast into this framework by making different choices of the spaces  $X$  and  $Y$  (including the case  $X = Y$ ), and by choosing appropriate inverse problems for  $\text{IP}_1$  and  $\text{IP}_2$ . The *Split Convex Feasibility Problem* (SCFP) [20] is the first instance of an SIP. The two problems  $\text{IP}_1$  and  $\text{IP}_2$  there are of the *Convex Feasibility Problem* (CFP) type. This formulation was used for solving an inverse problem in radiation therapy treatment planning [21, 17]. The SCFP has been well studied during

the last two decades both theoretically and practically; see, e.g., [12, 21] and the references therein. Two leading candidates for  $IP_1$  and  $IP_2$  are the mathematical models of the CFP and problems of constrained optimization. In particular, the CFP formalism is in itself at the core of the modeling of many inverse problems in various areas of mathematics and the physical sciences; see, e.g., [16] and references therein for an early example. Over the past four decades, the CFP has been used to model significant real-world inverse problems in sensor networks, radiation therapy treatment planning, resolution enhancement and in many other areas; see [18] for exact references to all of the above. More work on the CFP can be found in [1, 11, 13, 19].

It is therefore natural to ask whether other inverse problems can be used for  $IP_1$  and  $IP_2$ , besides the CFP, and be embedded in the SIP methodology. For example, can  $IP_1 = \text{CFP}$  in the space  $X$  and can a constrained optimization problem be  $IP_2$  in the space  $Y$ ? In our recent paper [22] we have made a step in this direction by formulating an SIP with a *Variational Inequality Problem* (VIP) in each of the two spaces of the SIP, reaching a *Split Variational Inequality Problem* (SVIP). In the present paper we study an SIP with a *Null Point Problem* in each of the two spaces. As we explain below, this formulation includes the earlier formulation with VIPs and all its special cases such as the CFP and constrained optimization problems.

## 1.1 Relations with previous work and the contribution of the present paper

To further motivate our study, let us look at the various problem formulations from the point of view of their structure only, without reference to the various assumptions made in order to prove results regarding these problems. We put the  $\text{SCNPP}(p, r)$  in the context of other SIPs and related works. First recall the *Split Variational Inequality Problem* (SVIP), which is an SIP with a VIP in each one of the two spaces [22]. Let  $\mathcal{H}_1$  and  $\mathcal{H}_2$  be two real Hilbert spaces, and assume that there are given two operators  $f : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  and  $g : \mathcal{H}_2 \rightarrow \mathcal{H}_2$ , a bounded linear operator  $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ , and nonempty, closed and convex subsets  $C \subset \mathcal{H}_1$  and  $Q \subset \mathcal{H}_2$ . The SVIP is then formulated as

follows:

$$\text{find a point } x^* \in C \text{ such that } \langle f(x^*), x - x^* \rangle \geq 0 \text{ for all } x \in C \quad (1.5)$$

and such that

$$\text{the point } y^* = A(x^*) \in Q \text{ and solves } \langle g(y^*), y - y^* \rangle \geq 0 \text{ for all } y \in Q. \quad (1.6)$$

This can be structurally considered a special case of SCNPP(1, 1). Denoting by  $\text{SOL}(f, C)$  and  $\text{SOL}(g, Q)$  the solution sets of the VIPs in (1.5) and (1.6), respectively, we can also write the SVIP in the following way:

$$\text{find a point } x^* \in \text{SOL}(f, C) \text{ such that } A(x^*) \in \text{SOL}(g, Q). \quad (1.7)$$

Taking in (1.5)–(1.6)  $C = \mathcal{H}_1$ ,  $Q = \mathcal{H}_2$ , and choosing  $x := x^* - f(x^*) \in \mathcal{H}_1$  in (1.5) and  $y = A(x^*) - g(A(x^*)) \in \mathcal{H}_2$  in (1.6), we obtain the *Split Zeros Problem* (SZP) for two operators  $f : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  and  $g : \mathcal{H}_2 \rightarrow \mathcal{H}_2$ , which we introduced in [22, Subsection 7.3]. It is formulated as follows:

$$\text{find a point } x^* \in \mathcal{H}_1 \text{ such that } f(x^*) = 0 \text{ and } g(A(x^*)) = 0. \quad (1.8)$$

An important observation that should be made at this point is that if we denote by  $N_C(v)$  the *normal cone* of some nonempty, closed and convex set  $C$  at a point  $v \in C$ , i.e.,

$$N_C(v) := \{d \in \mathcal{H} \mid \langle d, y - v \rangle \leq 0 \text{ for all } y \in C\}, \quad (1.9)$$

and define the set-valued mapping  $B$  by

$$B(v) := \begin{cases} f(v) + N_C(v), & v \in C, \\ \emptyset, & \text{otherwise,} \end{cases} \quad (1.10)$$

where  $f$  is some given operator, then, under a certain continuity assumption on  $f$ , Rockafellar in [46, Theorem 3] showed that  $B$  is a maximal monotone mapping and  $B^{-1}(0) = \text{SOL}(f, C)$ .

Following this idea, Moudafi [43] introduced the *Split Monotone Variational Inclusion* (SMVI) which generalized the SVIP of [22]. Given two operators  $f : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  and  $g : \mathcal{H}_2 \rightarrow \mathcal{H}_2$ , a bounded linear operator  $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ , and two set-valued mappings  $B_1 : \mathcal{H}_1 \rightarrow 2^{\mathcal{H}_1}$  and  $B_2 : \mathcal{H}_2 \rightarrow 2^{\mathcal{H}_2}$ , the SMVI is formulated as follows:

$$\text{find a point } x^* \in \mathcal{H}_1 \text{ such that } 0 \in f(x^*) + B_1(x^*) \quad (1.11)$$

and such that the point

$$y^* = A(x^*) \in \mathcal{H}_2 \text{ solves } 0 \in g(y^*) + B_2(y^*). \quad (1.12)$$

With the aid of simple substitutions it is clear that, structurally, SMVI is identical with SCNPP(1, 1) (use only two set-valued mappings, i.e.,  $p = r = 1$ , and put in (1.11)–(1.12) above,  $f = g = 0$ ). The applications presented in [43] only deal with this situation.

Masad and Reich [41] studied the *Constrained Multiple-Set Split Convex Feasibility Problem* (CMSSCFP). Let  $r$  and  $p$  be two natural numbers. Let  $C_i$ ,  $1 \leq i \leq p$ , and  $Q_j$ ,  $1 \leq j \leq r$ , be closed and convex subsets of  $\mathcal{H}_1$  and  $\mathcal{H}_2$ , respectively; further, for each  $1 \leq j \leq r$ , let  $A_j : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  be a bounded linear operator. Finally, let  $\Omega$  be another closed and convex subset of  $\mathcal{H}_1$ . The CMSSCFP is formulated as follows:

$$\text{find a point } x^* \in \Omega \tag{1.13}$$

such that

$$x^* \in \bigcap_{i=1}^p C_i \text{ and } A_j(x^*) \in Q_j \text{ for each } j = 1, 2, \dots, r. \tag{1.14}$$

This is also structurally a special case of SCNPP( $p, r$ ). Another related split problem is the *Split Common Fixed Point Problem* (SCFPP), first introduced in Euclidean spaces in [25] and later studied by Moudafi [42] in Hilbert spaces. Given operators  $U_i : \mathcal{H}_1 \rightarrow \mathcal{H}_1$ ,  $i = 1, 2, \dots, p$ , and  $T_j : \mathcal{H}_2 \rightarrow \mathcal{H}_2$ ,  $j = 1, 2, \dots, r$ , with nonempty fixed points sets  $C_i$ ,  $i = 1, 2, \dots, p$ , and  $Q_j$ ,  $j = 1, 2, \dots, r$ , respectively, and a bounded linear operator  $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ , the SCFPP is formulated as follows:

$$\text{find a point } x^* \in C := \bigcap_{i=1}^p C_i \text{ such that } A(x^*) \in Q := \bigcap_{j=1}^r Q_j. \tag{1.15}$$

This is also structurally a special case of SCNPP( $p, r$ ).

Motivated by the CMSSCFP of [41], see (1.13)–(1.14) above, the purpose of the present paper is to introduce the SCNPP( $p, r$ ) and present algorithms for solving it. Following [41], [34] and [35], we are able to establish strong convergence of three of the algorithms that we propose. These strongly convergent algorithms can be easily adapted to the SMVI and to other special cases of the SCNPP( $p, r$ ).

Our paper is organized as follows. In Section 2 we list several known facts regarding operators and set-valued mappings that are needed in the sequel. In Section 3 we present an algorithm for solving the SCNPP( $p, r$ ) and obtain its weak convergence. In Section 4 we propose three additional algorithms for solving the SCNPP( $p, r$ ) and present strong convergence theorems for them. Some further comments are presented in Section 5.

## 2 Preliminaries

Let  $\mathcal{H}$  be a real Hilbert space with inner product  $\langle \cdot, \cdot \rangle$  and induced norm  $\|\cdot\|$ , and let  $D \subset \mathcal{H}$  be a nonempty, closed and convex subset of it. We write either  $x^k \rightharpoonup x$  or  $x^k \rightarrow x$  to indicate that the sequence  $\{x^k\}_{k=0}^{\infty}$  converges either weakly or strongly, respectively, to  $x$ . Next we present several properties of operators and set-valued mappings which will be useful later on. For more details on many of the notions and results quoted here see, e.g., the recent books [4, 10].

**Definition 2.1** *Let  $\mathcal{H}$  be a real Hilbert space. Let  $D \subset \mathcal{H}$  be a subset of  $\mathcal{H}$  and  $h : D \rightarrow \mathcal{H}$  be an operator from  $D$  to  $\mathcal{H}$ .*

1.  *$h$  is called  **$\nu$ -inverse strongly monotone** ( $\nu$ -ism) on  $D$  if there exists a number  $\nu > 0$  such that*

$$\langle h(x) - h(y), x - y \rangle \geq \nu \|h(x) - h(y)\|^2 \text{ for all } x, y \in D. \quad (2.1)$$

2.  *$h$  is called **firmly nonexpansive** on  $D$  if*

$$\langle h(x) - h(y), x - y \rangle \geq \|h(x) - h(y)\|^2 \text{ for all } x, y \in D, \quad (2.2)$$

*i.e., if it is 1-ism.*

3.  *$h$  is called **Lipschitz continuous** with constant  $\kappa > 0$  on  $D$  if*

$$\|h(x) - h(y)\| \leq \kappa \|x - y\| \text{ for all } x, y \in D. \quad (2.3)$$

4.  *$h$  is called **nonexpansive** on  $D$  if*

$$\|h(x) - h(y)\| \leq \|x - y\| \text{ for all } x, y \in D, \quad (2.4)$$

*i.e., if it is 1-Lipschitz.*

5.  *$h$  is called a **strict contraction** if it is Lipschitz continuous with constant  $\kappa < 1$ .*
6.  *$h$  is called **hemicontinuous** if it is continuous along each line segment in  $D$ .*

7.  $h$  is called **asymptotically regular** at  $x \in D$  [8] if

$$\lim_{k \rightarrow \infty} (h^k(x) - h^{k+1}(x)) = 0 \text{ for all } x \in \mathcal{H}, \quad (2.5)$$

where  $h^k$  denotes the  $k$ -th iterate of  $h$ .

8.  $h$  is called **demiclosed** at  $y \in \mathcal{H}$  if for any sequence  $\{x^k\}_{k=0}^{\infty} \subset D$  such that  $x^k \rightharpoonup \bar{x} \in D$  and  $h(x^k) \rightarrow y$ , we have  $h(\bar{x}) = y$ .

9.  $h$  is called **averaged** [2] if there exists a nonexpansive operator  $N : D \rightarrow \mathcal{H}$  and a number  $c \in (0, 1)$  such that

$$h = (1 - c)I + cN, \quad (2.6)$$

where  $I$  is the identity operator. In this case we also say that  $h$  is  $c$ -av [13].

10.  $h$  is called **odd** if  $D$  is symmetric, i.e.,  $D = -D$ , and if

$$h(-x) = -h(x) \text{ for all } x \in D. \quad (2.7)$$

**Remark 2.2** (i) It can be verified that if  $h$  is  $\nu$ -ism, then it is Lipschitz continuous with constant  $\kappa = 1/\nu$ .

(ii) It is known that an operator  $h$  is averaged if and only if its complement  $I - h$  is  $\nu$ -ism for some  $\nu > 1/2$ ; see, e.g., [13, Lemma 2.1].

(iii) The operator  $h$  is firmly nonexpansive if and only if its complement  $I - h$  is firmly nonexpansive. The operator  $h$  is firmly nonexpansive if and only if  $h$  is  $(1/2)$ -av (see [33, Proposition 11.2] and [13, Lemma 2.3]).

(iv) If  $h_1$  and  $h_2$  are  $c_1$ -av and  $c_2$ -av, respectively, then their composition  $S = h_1 h_2$  is  $(c_1 + c_2 - c_1 c_2)$ -av. See [13, Lemma 2.2].

**Definition 2.3** Let  $\mathcal{H}$  be a real Hilbert space. Let  $B : \mathcal{H} \rightarrow 2^{\mathcal{H}}$  and  $\lambda > 0$ .

(i)  $B$  is called **odd** if

$$B(-x) = -B(x) \text{ for all } x \in \mathcal{H}. \quad (2.8)$$

(ii)  $B$  is called a **maximal monotone mapping** if  $B$  is monotone, i.e.,

$$\langle u - v, x - y \rangle \geq 0 \text{ for all } u \in B(x) \text{ and } v \in B(y), \quad (2.9)$$

and the **graph**  $G(B)$  of  $B$ ,

$$G(B) := \{(x, u) \in \mathcal{H} \times \mathcal{H} \mid u \in B(x)\}, \quad (2.10)$$

is not properly contained in the graph of any other monotone mapping.

(iii) The **domain** of  $B$  is

$$\text{dom}(B) := \{x \in \mathcal{H} \mid B(x) \neq \emptyset\}. \quad (2.11)$$

(iv) The **resolvent** of  $B$  with parameter  $\lambda$  is denoted and defined by  $J_\lambda^B := (I + \lambda B)^{-1}$ , where  $I$  is the identity operator.

**Remark 2.4** It is well known that for  $\lambda > 0$ ,

(i)  $B$  is monotone if and only if the resolvent  $J_\lambda^B$  of  $B$  is single-valued and firmly nonexpansive.

(ii)  $B$  is maximal monotone if and only if  $J_\lambda^B$  is single-valued, firmly nonexpansive and  $\text{dom}(J_\lambda^B) = \mathcal{H}$ .

(iii) The following equivalence holds:

$$0 \in B(x^*) \Leftrightarrow x^* \in \text{Fix}(J_\lambda^B). \quad (2.12)$$

It follows from (2.12) that the SCNPP( $p, r$ ) with two set-valued maximal monotone mappings ( $p = r = 1$ ) can be seen as an SCFPP with respect to their resolvents. In addition, Moudafi's SMVI can also be considered an SCFPP with respect to  $J_\lambda^{B_1}(I - \lambda f)$  and  $J_\lambda^{B_2}(I - \lambda g)$  [43, Fact 1]. Now we present another known result; see, e.g., [43, Fact 2].

**Remark 2.5** Let  $\mathcal{H}$  be a real Hilbert space, and let a maximal monotone mapping  $B : \mathcal{H} \rightarrow 2^\mathcal{H}$  and an  $\alpha$ -ism operator  $h : \mathcal{H} \rightarrow \mathcal{H}$  be given. Then the operator  $J_\lambda^B(I - \lambda h)$  is averaged for each  $\lambda \in (0, 2\alpha)$ .

Next we present an important class of operators, the  $\mathfrak{T}$ -class operators. This class was introduced and investigated by Bauschke and Combettes in [3, Definition 2.2] and by Combettes in [27]. Operators in this class were named *directed operators* by Zaknoon [56] and further employed under this name by Segal [47], and by Censor and Segal [24, 25]. Cegielski [14, Def. 2.1] studied these operators under the name *separating operators*. Since both *directed* and *separating* are keywords of other, widely-used, mathematical entities, Cegielski and Censor have recently introduced the term *cutter operators* [15]. This class coincides with the class  $\mathcal{F}^\nu$  for  $\nu = 1$  [31] and with the class  $\text{DC}_p$

for  $\mathbf{p} = -1$  [40]. The term *firmly quasi-nonexpansive* (FQNE) for  $\mathfrak{T}$ -class operators was used by Yamada and Ogura [55, 54, Section B] because every *firmly nonexpansive* (FNE) mapping [33, page 42] is obviously FQNE.

**Definition 2.6** *Let  $\mathcal{H}$  be a real Hilbert space. An operator  $h : \mathcal{H} \rightarrow \mathcal{H}$  is called a **cutter operator** if  $\text{dom}(h) = \mathcal{H}$  and*

$$\langle h(x) - x, h(x) - q \rangle \leq 0 \text{ for all } (x, q) \in \mathcal{H} \times \text{Fix}(h), \quad (2.13)$$

where the fixed point set  $\text{Fix}(h)$  of  $h$  is defined by

$$\text{Fix}(h) := \{x \in \mathcal{H} \mid h(x) = x\}. \quad (2.14)$$

It can be seen that this class of operators coincides with the class of *firmly quasi-nonexpansive operators* (FQNE), which satisfy the inequality

$$\|h(x) - q\|^2 \leq \|x - q\|^2 - \|x - h(x)\|^2 \text{ for all } (x, q) \in \mathcal{H} \times \text{Fix}(h). \quad (2.15)$$

Note that the  $\mathfrak{T}$ -class operators include, among others, orthogonal projections, subgradient projectors, resolvents of maximal monotone mappings, and firmly nonexpansive operators. This last class was first introduced by Browder [7, Definition 6] under the name *firmly contractive operators*. Every  $\mathfrak{T}$ -class operator belongs to the class  $\mathcal{F}^0$  of operators, defined by Crombez [31, p. 161]:

$$\mathcal{F}^0 := \{h : \mathcal{H} \rightarrow \mathcal{H} \mid \|h(x) - q\| \leq \|x - q\| \text{ for all } (x, q) \in \mathcal{H} \times \text{Fix}(h)\}. \quad (2.16)$$

The elements of  $\mathcal{F}^0$  are called *quasi-nonexpansive* or *paracontracting operators*. A more general class of operators is the class of *demiccontractive operators* (see, e.g., [40]).

**Definition 2.7** *Let  $\mathcal{H}$  be a real Hilbert space and let  $h : \mathcal{H} \rightarrow \mathcal{H}$  be an operator.*

(i)  *$h$  is called a **demiccontractive operator** if there exists a number  $\beta \in [0, 1)$  such that*

$$\|h(x) - q\|^2 \leq \|x - q\|^2 + \beta \|x - h(x)\|^2 \text{ for all } (x, q) \in \mathcal{H} \times \text{Fix}(h). \quad (2.17)$$

*This is equivalent to*

$$\langle x - h(x), x - q \rangle \geq \frac{1 - \beta}{2} \|x - h(x)\|^2 \text{ for all } (x, q) \in \mathcal{H} \times \text{Fix}(h). \quad (2.18)$$

Another useful observation, already hinted to above, is that if  $h : \mathcal{H} \rightarrow \mathcal{H}$  is monotone and hemicontinuous on a nonempty, closed and convex subset  $D$ , then the set-valued mapping

$$M(v) = \begin{cases} h(v) + N_D(v), & v \in D, \\ \emptyset, & \text{otherwise,} \end{cases} \quad (2.19)$$

is, by [46, Theorem 3], maximal monotone and  $M^{-1}(0) = \text{SOL}(h, D)$ . Therefore, as mentioned in [43], if we choose  $B_1 = N_C$  and  $B_2 = N_Q$  in (1.11) and (1.12), respectively, then we get the SVIP of (1.5)–(1.6). Of course, this assertion also holds for our SCNPP( $p, r$ ) with two set-valued maximal monotone mappings ( $p = r = 1$ ) when we take  $B_1$  and  $F_1$  to be similar to  $M$  in (2.19). This enables us to solve the SVIP for monotone and hemicontinuous operators (which constitute a larger class than the class of inverse strongly monotone operators) by using our convergence theorem for the SVIP [22, Theorem 6.3]. In [22, Theorem 6.3] we also assumed [22, Equation (5.9)] that for all  $x^* \in \text{SOL}(f, C)$ ,

$$\langle f(x), P_C(I - \lambda f)(x) - x^* \rangle \geq 0 \text{ for all } x \in \mathcal{H}_1, \quad (2.20)$$

an assumption which is not needed for the convergence theorems we establish in the present paper.

The next lemma is the well-known *Demiclosedness Principle* [6].

**Lemma 2.8** *Let  $\mathcal{H}$  be a Hilbert space,  $D$  a closed and convex subset of  $\mathcal{H}$ , and let  $h : D \rightarrow \mathcal{H}$  be a nonexpansive operator. Then  $I - h$  is demiclosed at any  $y \in \mathcal{H}$ .*

The next definition is due to Clarkson [26].

**Definition 2.9** *A Banach space  $\mathcal{B}$  is said to be **uniformly convex** if to each  $\varepsilon \in (0, 2]$ , there corresponds a positive number  $\delta(\varepsilon)$  such that the conditions  $\|x\| = \|y\| = 1$  and  $\|x - y\| \geq \varepsilon$  imply that  $\|(x + y)/2\| \leq 1 - \delta(\varepsilon)$ .*

It follows from the Parallelogram Identity that every Hilbert space is uniformly convex. Next we present two known theorems, the Krasnosel'skiĭ-Mann-Opial theorem [37, 39, 44] and the Halpern-Suzuki theorem [34, 48].

**Theorem 2.10** [37, 39, 44] *Let  $\mathcal{H}$  be a real Hilbert space and  $D \subset \mathcal{H}$  be a nonempty, closed and convex subset of  $\mathcal{H}$ . Given an averaged operator  $h : D \rightarrow D$  with  $\text{Fix}(h) \neq \emptyset$  and an arbitrary  $x^0 \in D$ , the sequence generated by the recursion  $x^{k+1} = h(x^k)$ ,  $k \geq 0$ , converges weakly to a point  $z \in \text{Fix}(h)$ .*

**Theorem 2.11** [34, 48] *Let  $\mathcal{H}$  be a real Hilbert space and  $D \subset \mathcal{H}$  be a closed and convex subset of  $\mathcal{H}$ . Given an averaged operator  $h : D \rightarrow D$ , and a sequence  $\{\alpha_k\}_{k=0}^{\infty} \subset [0, 1]$  satisfying  $\lim_{k \rightarrow \infty} \alpha_k = 0$  and  $\sum_{k=0}^{\infty} \alpha_k = \infty$ , the sequence  $\{x^k\}_{k=0}^{\infty}$  generated by  $x^0 \in D$  and  $x^{k+1} = \alpha_k x^0 + (1 - \alpha_k)h(x^k)$ ,  $k \geq 0$ , converges strongly to a point  $z \in \text{Fix}(h)$ .*

### 3 Weak convergence

In this section we first present an algorithm for solving the SCNPP( $p, r$ ) for two set-valued maximal monotone mappings. Then, for the general case of more than two such set-valued mappings, we employ a product space formulation in order to transform it into an SCNPP(1, 1) for two set-valued maximal monotone mappings, in a similar fashion to what has been done in [25, Section 4] and [22, Subsection 6.1].

#### 3.1 The SCNPP(1, 1) for set-valued maximal monotone mappings

Consider the SCNPP( $p, r$ ) (1.1)–(1.2) with  $p = r = 1$ . That is, given two set-valued mappings  $B_1 : \mathcal{H}_1 \rightarrow 2^{\mathcal{H}_1}$  and  $F_1 : \mathcal{H}_2 \rightarrow 2^{\mathcal{H}_2}$ , and a bounded linear operator  $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ , we want to

$$\text{find a point } x^* \in \mathcal{H}_1 \text{ such that } 0 \in B_1(x^*) \text{ and } 0 \in F_1(A(x^*)). \quad (3.1)$$

Here is our proposed algorithm for solving (3.1).

**Algorithm 3.1**

**Initialization:** Let  $\lambda > 0$  and select an arbitrary starting point  $x^0 \in \mathcal{H}_1$ .

**Iterative step:** Given the current iterate  $x^k$ , compute

$$x^{k+1} = J_{\lambda}^{B_1} (x^k - \gamma A^*(I - J_{\lambda}^{F_1})A(x^k)), \quad (3.2)$$

where  $A^*$  is the adjoint of  $A$ ,  $L = \|A^*A\|$  and  $\gamma \in (0, 2/L)$ .

The convergence theorem for this algorithm is presented next. We denote by  $\Gamma$  the solution set of (3.1).

**Theorem 3.2** *Let  $\mathcal{H}_1$  and  $\mathcal{H}_2$  be two real Hilbert spaces. Given two set-valued maximal monotone mappings  $B_1 : \mathcal{H}_1 \rightarrow 2^{\mathcal{H}_1}$  and  $F_1 : \mathcal{H}_2 \rightarrow 2^{\mathcal{H}_2}$ , and a bounded linear operator  $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ , any sequence  $\{x^k\}_{k=0}^\infty$  generated by Algorithm 3.1 converges weakly to a point  $x^* \in \Gamma$ , provided that  $\Gamma \neq \emptyset$  and  $\gamma \in (0, 2/L)$ , where  $L = \|A^*A\|$ .*

**Proof.** In view of the connection between our SCNPP( $p, r$ ) and Moudafi's SMVI, this theorem can be obtained as a corollary of [43, Theorem 3.1], the proof of which is based on the Krasnosel'skiĭ-Mann-Opial theorem [37, 39, 44].  
 ■

**Remark 3.3** *Observe that in Theorem 3.2 we assume that  $\gamma \in (0, 2/L)$ , while in [22, Theorem 6.3],  $\gamma$  is assumed to be in  $(0, 1/L)$ , which obviously was a more restrictive assumption.*

To describe the relationship of our work with splitting methods, let  $\mathcal{H}$  be a real Hilbert space, and let  $B : \mathcal{H} \rightarrow 2^{\mathcal{H}}$  and  $F : \mathcal{H} \rightarrow 2^{\mathcal{H}}$  be two maximal monotone mappings. Consider the following problem:

$$\text{find a point } x^* \in \mathcal{H} \text{ such that } 0 \in B(x^*) + F(x^*). \quad (3.3)$$

Many algorithms were developed for solving this problem. An important class of such algorithms is the class of splitting methods. References on splitting methods and their applications can be found in Eckstein's Ph.D. thesis [32], in Tseng's work [49, 50, 51] and more recently in Combettes et al. [28, 29, 30].

One splitting method of interest is the following forward-backward algorithm:

$$x^{k+1} = J^B(I - h)(x^k), \quad (3.4)$$

where  $F = h$  is single-valued. Combettes [28, Section 6] was interested in (3.4) under the assumption that  $B : \mathcal{H} \rightarrow 2^{\mathcal{H}}$  and  $h : \mathcal{H} \rightarrow \mathcal{H}$  are maximal monotone, and  $\beta h$  is firmly nonexpansive (i.e., 1/2-av) for some  $\beta \in (0, \infty)$ . He proposed the following algorithm:

$$x^{k+1} = x^k + \lambda_k (J_{\gamma_k}^B(x^k - \gamma_k(h(x^k) + b_k)) + a_k - x^k), \quad (3.5)$$

where the sequence  $\{\gamma_k\}_{k=0}^\infty$  is bounded and the sequences  $\{a_k\}_{k=0}^\infty$  and  $\{b_k\}_{k=0}^\infty$  are absolutely summable errors in the computation of the resolvents. It can be seen that the iterative step (3.2) is a special case of (3.4) with

$h = \gamma A^*(I - J_\lambda^{F_1})A$ . In the setting of Theorem 3.2 here,  $h$  is  $1/(\gamma L)$ -ism and therefore for  $\beta = (\gamma L)^{-1}$ , the operator  $\beta\gamma A^*(I - J_\lambda^{F_1})A$  is 1-ism, that is, firmly nonexpansive. Now by [4, Example 20.27], this operator is maximal monotone. Therefore Algorithm 3.1 is a special case of (3.5) without relaxation and we also need to calculate the exact resolvent. It may be somewhat surprising that our SCNPP is formulated in two different spaces, while (3.3) is only defined in one space and still we arrive at the same algorithm. Further related results on proximal feasibility problems appear in Combettes and Wajs [29, Subsection 4.3].

### 3.2 The general SCNPP( $p, r$ )

In view of Remark 2.4, we can show, by applying similar arguments to those used in [25], that our SCNPP( $p, r$ ) can be transformed into a split common fixed point problem (SCFPP) (see (1.15)) with two operators  $T$  and  $U$  in a product space. Next, we show how the general SCNPP( $p, r$ ) can be transformed into an SCNPP(1, 1) for two set-valued maximal monotone mappings.

Consider the space  $\mathbf{H} = \mathcal{H}_1^p \times \mathcal{H}_2^r$ , and the set-valued maximal monotone mappings  $D : \mathcal{H}_1 \rightarrow 2^{\mathcal{H}_1}$  and  $\mathbf{F} : \mathbf{H} \rightarrow 2^{\mathbf{H}}$  defined by  $D(x) = \{0\}$  for all  $x \in \mathcal{H}_1$  and  $\mathbf{F}((x^1, \dots, x^p, y^1, \dots, y^r)) = B_1(x^1) \times \dots \times B_p(x^p) \times F_1(y^1) \times \dots \times F_r(y^r)$  for each  $(x^1, \dots, x^p, y^1, \dots, y^r) \in \mathbf{H}$ . In addition, let the bounded linear operator  $\mathbf{A} : \mathcal{H}_1 \rightarrow \mathbf{H}$  be defined by  $\mathbf{A}(x) = (x, \dots, x, A_1(x), \dots, A_r(x))$  for all  $x \in \mathcal{H}_1$ . Then the general SCNPP( $p, r$ ) (1.1)–(1.2) is equivalent to

$$\text{find a point } x \in \mathcal{H}_1 \text{ such that } 0 \in D(x) \text{ and } 0 \in \mathbf{F}(\mathbf{A}(x)). \quad (3.6)$$

When Algorithm 3.1 is applied to this two-set problem in the product space  $\mathbf{H}$  and then translated back to the original spaces, it takes the following form.

#### Algorithm 3.4

**Initialization:** Select an arbitrary starting point  $x^0 \in \mathcal{H}_1$ .

**Iterative step:** Given the current iterate  $x^k$ , compute

$$x^{k+1} = x^k + \gamma \left( \sum_{i=1}^p (J_\lambda^{B_i}(x^k) - x^k) + \sum_{j=1}^r A_j^*(J_\lambda^{F_j} - I)A_j(x^k) \right), \quad (3.7)$$

where  $\gamma \in (0, 2/L)$ , with  $L = p + \sum_{j=1}^r \|A_j\|^2$ .

The convergence of this algorithm follows from Theorem 3.2. We may also introduce relaxation parameters into the above algorithm as has been done in the relaxed version of [42, equation 2.10].

## 4 Strong convergence

We focus on the SCNPP( $p, r$ ) for two set-valued maximal monotone mappings, keeping in mind that for the general case we can always apply the above product space formulation and then translate back the algorithms to the original spaces. In this section we first present a strong convergence theorem for Algorithm 3.1 under an additional assumption. This result relies on the work of Browder and Petryshyn [8, Theorem 5], and on that of Baillon, Bruck and Reich [2, Theorem 1.1] (see also [41, Lemma 7]). Then we study a second algorithm which is a modification of Algorithm 3.1 that results in a Halpern-type algorithm. The third algorithm in this section is inspired by Haugazeau's method [35]; see also [3].

### 4.1 Strong convergence of Algorithm 3.1

The next two theorems are needed for our proof of Theorem 4.3. We present their full proofs for the reader's convenience.

**Theorem 4.1** [8, Theorem 5], [36] *Let  $\mathcal{B}$  be a uniformly convex Banach space. If the operator  $S : \mathcal{B} \rightarrow \mathcal{B}$  is nonexpansive with a nonempty fixed point set  $\text{Fix}(S) \neq \emptyset$ , then for any given constant  $c \in (0, 1)$ , the operator  $S_c := cI + (1 - c)S$  is asymptotically regular and has the same fixed points as  $S$ .*

**Proof.** It is obvious that  $\text{Fix}(S) = \text{Fix}(S_c)$  and that  $S_c$  is also a non-expansive self-mapping of  $\mathcal{B}$ . Let  $u \in \text{Fix}(S_c)$  and for a given  $x \in \mathcal{B}$ , let  $x^k = S_c^k(x)$ . Since  $S_c$  is nonexpansive and  $u \in \text{Fix}(S_c)$ , it follows that

$$\|x^{k+1} - u\| \leq \|x^k - u\| \text{ for all } k \geq 0. \quad (4.1)$$

Therefore there exists  $\lim_{k \rightarrow \infty} \|x^k - u\| = \ell \geq 0$ . Assume that  $\ell > 0$ . Then

$$\begin{aligned} x^{k+1} - u &= S_c^{k+1}(x) - u = S_c(x^k) - u \\ &= (cI + (1 - c)S)(x^k) - u \\ &= c(x^k - u) + (1 - c)(S(x^k) - u). \end{aligned} \quad (4.2)$$

Since

$$\lim_{k \rightarrow \infty} \|x^k - u\| = \lim_{k \rightarrow \infty} \|x^{k+1} - u\| = \ell \quad (4.3)$$

and

$$\|x^{k+1} - u\| = \|S(x^k) - u\| \leq \|x^k - u\|, \quad (4.4)$$

the uniform convexity of  $\mathcal{B}$  implies that

$$\lim_{k \rightarrow \infty} \|(x^k - u) - (S(x^k) - u)\| = 0, \quad (4.5)$$

i.e.,  $x^k - S(x^k) \rightarrow 0$ . Hence  $x^{k+1} - x^k \rightarrow 0$ , which means that  $S_c$  is asymptotically regular, as claimed. ■

**Theorem 4.2** [2, Theorem 1.1] *Let  $\mathcal{B}$  be a uniformly convex Banach space. If the operator  $S : \mathcal{B} \rightarrow \mathcal{B}$  is nonexpansive, odd and asymptotically regular at  $x \in \mathcal{B}$ , then the sequence  $\{S^k(x)\}_{k=0}^{\infty}$  converges strongly to a fixed point of  $S$ .*

**Proof.** Since  $S$  is odd,  $S(0) = -S(0)$  and  $S(0) = 0$ . Since  $S$  is nonexpansive, we have by the triangle inequality,

$$\begin{aligned} \|S^k(x)\| &= \|S^k(x)\| - \|S^k(0)\| \leq \|S^k(x) - S^k(0)\| \\ &\leq \|S^{k-1}(x) - S^{k-1}(0)\| = \|S^{k-1}(x)\| \leq \dots \leq \|x - 0\| = \|x\|, \end{aligned} \quad (4.6)$$

which means that the sequence  $\{\|S^k(x)\|\}_{k=0}^{\infty}$  is decreasing and bounded. Therefore the limit  $\lim_{k \rightarrow \infty} \|S^k(x)\|$  exists and, for a fixed  $i$ , the sequence  $\{\|S^{k+i}(x) + S^k(x)\|\}_{k=0}^{\infty}$  is decreasing. Let  $\lim_{k \rightarrow \infty} \|S^k(x)\| = d$ . Then by the triangle inequality,

$$\begin{aligned} 2d &\leq \|2S^k(x)\| = \|S^k(x) - S^{k+i}(x) + S^{k+i}(x) + S^k(x)\| \\ &\leq \|S^k(x) - S^{k+i}(x)\| + \|S^k(x) + S^{k+i}(x)\|. \end{aligned} \quad (4.7)$$

Since  $S$  is asymptotically regular at  $x$ ,  $\lim_{k \rightarrow \infty} \|S^k(x) - S^{k+i}(x)\| = 0$ . Thus  $\lim_{k \rightarrow \infty} \|S^k(x) + S^{k+i}(x)\| \geq 2d$ . But the sequence  $\{\|S^{k+i}(x) + S^k(x)\|\}_{k=0}^{\infty}$  is decreasing, so that  $\|S^k(x) + S^{k+i}(x)\| \geq 2d$  for all  $k$  and  $i$ . We now have  $\lim_{k \rightarrow \infty} \|S^k(x)\| = d$  and  $\lim_{m,n \rightarrow \infty} \|S^m(x) + S^n(x)\| = 2d$ . The uniform convexity of  $\mathcal{B}$  implies that  $\lim_{m,n \rightarrow \infty} \|S^m(x) - S^n(x)\| = 0$ , whence  $\{S^k(x)\}_{k=0}^{\infty}$  converges strongly to a fixed point of  $S$ . ■

In Theorem 4.3 we need the resolvent  $J_\lambda^B$  to be odd, which means that

$$((I + \lambda B)^{-1})(-x) = -((I + \lambda B)^{-1})(x) \text{ for all } x \in \mathcal{H}. \quad (4.8)$$

Denote

$$((I + \lambda B)^{-1})(-x) = y \text{ and } ((I + \lambda B)^{-1})(x) = z. \quad (4.9)$$

Then

$$-x \in y + \lambda B(y) \text{ and } x \in z + \lambda B(z). \quad (4.10)$$

If  $B$  is odd, then

$$x \in -y + \lambda B(-y). \quad (4.11)$$

Hence  $-y = z$ , which is (4.8). Therefore we assume in the following theorem that both  $B_1$  and  $F_1$  are odd.

Now we are ready to present the strong convergence theorem for Algorithm 3.1. Its proof relies on Theorem 4.2.

**Theorem 4.3** *Let  $\mathcal{H}_1$  and  $\mathcal{H}_2$  be two real Hilbert spaces. Let two set-valued, odd and maximal monotone mappings  $B_1 : \mathcal{H}_1 \rightarrow 2^{\mathcal{H}_1}$  and  $F_1 : \mathcal{H}_2 \rightarrow 2^{\mathcal{H}_2}$ , and a bounded linear operator  $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  be given. If  $\gamma \in (0, 2/L)$ , then any sequence  $\{x^k\}_{k=0}^\infty$  generated by Algorithm 3.1 converges strongly to  $x^* \in \Gamma$ .*

**Proof.** The operator  $J_\lambda^{B_1}(I - \gamma A^*(I - J_\lambda^{F_1})A)$  is averaged by the proof of [43, Theorem 3.1]. Therefore, by [8, Theorem 5] and [36] (see Theorem 4.1), the operator  $J_\lambda^{B_1}(I - \gamma A^*(I - J_\lambda^{F_1})A)$  is also asymptotically regular. Since  $B_1$  and  $F_1$  are odd, so are their resolvents  $J_\lambda^{B_1}$  and  $J_\lambda^{F_1}$ , and therefore  $J_\lambda^{B_1}(I - \gamma A^*(I - J_\lambda^{F_1})A)$  is odd. Finally, the strong convergence of Algorithm 3.1 is now seen to follow from Theorem 4.2. ■

For the general SCNPP( $p, r$ ) we can again employ a product space formulation as in Subsection 3.2 and under the additional oddness assumption also get strong convergence.

## 4.2 A Halpern-type algorithm

Next, we consider a modification of Algorithm 3.1 inspired by Halpern's iterative method and prove its strong convergence. Let  $T : C \rightarrow C$  be a nonexpansive operator, where  $C$  is a nonempty, closed and convex subset of a Banach space  $\mathcal{B}$ . A classical way to study nonexpansive mappings is to use

strict contractions to approximate  $T$ , i.e., for  $t \in (0, 1)$ , we define the strict contraction  $T_t : C \rightarrow C$  by

$$T_t(x) = tu + (1 - t)T(x) \text{ for } x \in C, \quad (4.12)$$

where  $u \in C$  is fixed. Banach's Contraction Mapping Principle (see, e.g., [33]) guarantees that each  $T_t$  has a unique fixed point  $x_t \in C$ . In case  $\text{Fix}(T) \neq \emptyset$ , Browder [6] proved that if  $\mathcal{B}$  is a Hilbert space, then  $x_t$  converges strongly as  $t \rightarrow 0^+$  to the fixed point of  $T$  nearest to  $u$ . Motivated by Browder's results, Halpern [34] proposed an explicit iterative scheme and proved its strong convergence to a point  $z \in \text{Fix}(T)$ . In the last decades many authors modified Halpern's iterative scheme and found necessary and sufficient conditions, concerning the control sequence, that guarantee the strong convergence of Halpern-type schemes (see, e.g., [38, 45, 52, 53, 48]). Our algorithm for the SCNPP( $p, r$ ) with two set-valued maximal monotone mappings is presented next.

**Algorithm 4.4**

**Initialization:** Select some  $\lambda > 0$  and an arbitrary starting point  $x^0 \in \mathcal{H}_1$ .

**Iterative step:** Given the current iterate  $x^k$ , compute

$$x^{k+1} = \alpha_k x^0 + (1 - \alpha_k) J_\lambda^{B_1} (I - \gamma A^*(I - J_\lambda^{F_1})A) (x^k), \quad (4.13)$$

where  $\gamma \in (0, 2/L)$  with  $L = \|A^*A\|$  and the sequence  $\{\alpha_k\}_{k=0}^\infty \subset [0, 1]$  satisfies  $\lim_{k \rightarrow \infty} \alpha_k = 0$  and  $\sum_{k=0}^\infty \alpha_k = \infty$ .

Here is our strong convergence theorem for this algorithm.

**Theorem 4.5** Let  $\mathcal{H}_1$  and  $\mathcal{H}_2$  be two real Hilbert spaces. Let there be given two set-valued maximal monotone mappings  $B_1 : \mathcal{H}_1 \rightarrow 2^{\mathcal{H}_1}$  and  $F_1 : \mathcal{H}_2 \rightarrow 2^{\mathcal{H}_2}$ , and a bounded linear operator  $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ . If  $\Gamma \neq \emptyset$ ,  $\gamma \in (0, 2/L)$  and  $\{\alpha_k\}_{k=0}^\infty \subset [0, 1]$  satisfies  $\lim_{k \rightarrow \infty} \alpha_k = 0$  and  $\sum_{k=0}^\infty \alpha_k = \infty$ , then any sequence  $\{x^k\}_{k=0}^\infty$  generated by Algorithm 4.4 converges strongly to  $x^* \in \Gamma$ .

**Proof.** As we already know, the operator  $J_\lambda^{B_1} (I - \gamma A^*(I - J_\lambda^{F_1})A)$  is averaged. So, according to Theorem 2.11, any sequence  $\{x^k\}_{k=0}^\infty$  generated by Algorithm 4.4 converges strongly to a point in the fixed point set of the operator, i.e.,  $x^* \in \text{Fix} (J_\lambda^{B_1} (I - \gamma A^*(I - J_\lambda^{F_1})A))$  as long as this set is nonempty. As in the proof of [43, Theorem 3.1], we conclude that  $x^* \in \Gamma$ , as claimed. ■

### 4.3 An Haugazeau-type algorithm

Haugazeau [35] presented an algorithm for solving the *Best Approximation Problem* (BAP) of finding the projection of a point onto the intersection of  $m$  closed convex subsets  $\{C_i\}_{i=1}^m \subset \mathcal{H}$  of a real Hilbert space. Defining for any pair  $x, y \in \mathcal{H}$  the set

$$H(x, y) := \{u \in \mathcal{H} \mid \langle u - y, x - y \rangle \leq 0\}, \quad (4.14)$$

and denoting by  $T(x, y, z)$  the projection of  $x$  onto  $H(x, y) \cap H(y, z)$ , namely,  $T(x, y, z) = P_{H(x, y) \cap H(y, z)}(x)$ , Haugazeau showed that for an arbitrary starting point  $x^0 \in \mathcal{H}$ , any sequence  $\{x^k\}_{k=0}^\infty$ , generated by the iterative step

$$x^{k+1} = T(x^0, x^k, P_{k(\bmod m)+1}(x^k)) \quad (4.15)$$

converges strongly to the projection of  $x^0$  onto  $C = \bigcap_{i=1}^m C_i$ . The operator  $T$  requires projecting onto the intersection of two constructible half-spaces; this is not difficult to implement. In [35] Haugazeau introduced the operator  $T$  as an explicit description of the projector onto the intersection of the two half-spaces  $H(x, y)$  and  $H(y, z)$ . So, following, e.g., [5, Definition 3.1], and denoting  $\pi = \langle x - y, y - z \rangle$ ,  $\mu = \|x - y\|^2$ ,  $\nu = \|y - z\|^2$  and  $\rho = \mu\nu - \pi^2$ , we have

$$T(x, y, z) = \begin{cases} z, & \text{if } \rho = 0 \text{ and } \pi \geq 0, \\ x + \left(1 + \frac{\pi}{\nu}\right)(z - y), & \text{if } \rho > 0 \text{ and } \pi\nu \geq \rho, \\ y + \frac{\nu}{\rho}(\pi(x - y) + \mu(z - y)), & \text{if } \rho > 0 \text{ and } \pi\nu < \rho. \end{cases} \quad (4.16)$$

We already know that the operator  $S := J_\lambda^{B_1}(I - \gamma A^*(I - J_\lambda^{F_1})A)$  is averaged and therefore nonexpansive. Now consider the firmly nonexpansive operator  $S_{1/2} := (I + S)/2$ , which according to Theorem 4.1 has the same fixed points as  $S$ . Following the “weak-to-strong convergence principle” [3], strong convergence (without additional assumptions) can be obtained by replacing the updating rule (3.2) in Algorithm 3.1 with

$$\begin{aligned} x^{k+1} &= T(x^0, x^k, S_{1/2}(x^k)) \\ &= P_{H(x^0, x^k) \cap H(x^k, S_{1/2}(x^k))}(x^0). \end{aligned} \quad (4.17)$$

A similar technique can also be applied to the forward-backward splitting method in [28, Section 6].

## 5 Further comments

1. Since the  $\text{SCNPP}(p, r)$  generalizes the SVIP, it includes all the applications to which SVIP applies (see [22, Section 7]). In particular, it includes the Split Feasibility Problem (SFP) and the Convex Feasibility Problem (CFP). Since the Common Solutions to Variational Inequalities Problem (CSVIP) [23] with operators is a special case of the SVIP, the  $\text{SCNPP}(p, r)$  includes its applications as well. In addition, since all the applications of the SMVI presented in [43] are for  $f = g = 0$  in (1.11) and (1.12) above, it follows that these applications are also covered by our  $\text{SCNPP}(p, r)$ . They include the Split Minimization Problem (SMP), which has already been presented in [22, Subsection 7.3] with continuously differentiable convex functions, for which we can now drop this assumption, the Split Saddle-Point Problem (SSPP), the Split Minimax Problem (SMMP) and the Split Equilibrium Problem (SEP). Observe that if  $\mathcal{H}_1 = \mathcal{H}_2$  and  $A_j = I$  for for all  $j = 1, 2, \dots, r$ , then we can deal with all of the above applications with “Split” replaced by “Common”. We can even study mixtures of “split” and “common” applications.
2. According to Remark 2.5, the operator  $J_\lambda^B(I - \lambda f)$  is averaged, where  $B : \mathcal{H} \rightarrow 2^{\mathcal{H}}$  is maximal monotone, the operator  $f : \mathcal{H} \rightarrow \mathcal{H}$  is  $\alpha$ -ism and  $\lambda \in (0, 2\alpha)$ . Since our convergence theorems rely on the averagedness of the operators involved, we could modify our algorithms and obtain strong convergence for Moudafi’s SMVI ((1.11) and (1.12) above). In addition, our algorithms allow us to solve Moudafi’s SMVI with monotone and hemicontinuous operators  $f$  and  $g$  (which is a larger class than the class of inverse strongly monotone operators).
3. Assuming that the set-valued mappings  $B_1 : \mathcal{H}_1 \rightarrow 2^{\mathcal{H}_1}$  and  $B_2 : \mathcal{H}_2 \rightarrow 2^{\mathcal{H}_2}$  are maximal monotone, and  $f : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  and  $g : \mathcal{H}_2 \rightarrow \mathcal{H}_2$  are ism-operators, Moudafi presented an algorithm that converges weakly to a solution of the SMVI. By [46, Theorem 3], the sum of a maximal monotone mapping and an ism-operator is maximal monotone. Therefore, the SMVI reduces to our set-valued two-mapping  $\text{SCNPP}(p, r)$ . In addition, we can phrase the set-valued SVIP for maximal monotone mappings in the following way. Given two maximal monotone mappings  $B_1 : \mathcal{H}_1 \rightarrow 2^{\mathcal{H}_1}$  and  $B_2 : \mathcal{H}_2 \rightarrow 2^{\mathcal{H}_2}$ , a bounded linear operator  $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ , and nonempty, closed and convex subsets  $C \subset \mathcal{H}_1$  and

$Q \subset \mathcal{H}_2$ , the set-valued SVIP is formulated as follows:

$$\begin{aligned} &\text{find a point } x^* \in C \text{ and a point } u^* \in B_1(x^*) \\ &\quad \text{such that } \langle u^*, x - x^* \rangle \geq 0 \text{ for all } x \in C, \\ &\quad \text{and such that} \\ &\text{the points } y^* = A(x^*) \in Q \text{ and } v^* \in B_2(y^*) \\ &\quad \text{solve } \langle v^*, y - y^* \rangle \geq 0 \text{ for all } y \in Q. \end{aligned} \tag{5.1}$$

It is clear that if the zeros of the set-valued mappings  $B_1$  and  $B_2$  are in  $C$  and  $Q$ , respectively, then they are solutions of the set-valued SVIP, but in general not all solutions are zeros.

### Acknowledgments

This work was partially supported by United States-Israel Binational Science Foundation (BSF) Grant number 200912, US Department of Army Award number W81XWH-10-1-0170, Israel Science Foundation (ISF) Grant number 647/07, the Fund for the Promotion of Research at the Technion, and by the Technion VPR Fund.

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