Moneyless strategy-proof mechanism on single-sinked policy domain: characterization and applications

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We completely characterize deterministic strategy-proof and group strategy-proof mechanisms on single-sinked public policy domain. The single-sinked domain can be used to model any allocation problem where a single output must be chosen in an interval with the assumption that agents' preferences have a single most loathful point (the sink) in the interval, and the preferences are increasing as one moves away from that sink. Practical domains like this appear in political voting system where each voter has his most-hated candidate and alternative candidates are evaluated by their proximity to this candidate or in obnoxious location problem, where each agent prefers to have the obnoxious location to be distant from his own location, among others. Moreover, as applications of our characterization, we extend existing models and results and resolve several open questions from the literature.

Key words: Mechanism design, strategy-proof, group strategy-proof, efficiency, log-concave, Pareto-optimal, unanimous, anonymous, onto

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1. Introduction Formally, there is a set $N = \{1, \ldots, n\}$ of n agents, and the policy space is the unit interval $I = [0, 1]$. Each agent i has a strict preference order $\prec_i$ over the policy domain. The preference order $\prec_i$ is *single-sinked* if there exists a sink point $s_i \in I$ such that $\forall x \in I \backslash \{s_i\}$ and $\forall \alpha \in [0, 1)$, $\alpha x + (1 - \alpha)s_i \prec_i x$. The collection $\mathbf{s} = (s_1, \ldots, s_n)$ is a called a sink profile. An outcome of the model is a single point $y \in I$, specified by a social choice function f (or simply the mechanism f) that maps the announced sink profile $\mathbf{s} = (s_1, \ldots, s_n)$ to a point $y = f(\mathbf{s})$ on the policy domain I. Later on we also use the alternate notation $\mathbf{s} = (\mathbf{s}_S, \mathbf{s}_{-S})$ to represent any sink profile $\mathbf{s}$ for some given nonempty subset $S \subset N$.

Equivalently, for each single-sinked preference $\prec_i$, we can associate a quasiconvex (or unimodal) utility function $u_i(f - s_i)$, where $f \in I$ is the social choice and $s_i \in I$ is the sink of agent i, namely, $u_i(\cdot)$ is decreasing within $[-1, 0]$ and increasing within $[0, 1]$ (see Fig. 1 for an illustration). Without loss of generality, we assume that $u_i(0) = 0$, that is, the utility for each agent at their respective sink is zero. In the rest of this paper, we will use increasing or decreasing in its strict sense.

A social choice function (or mechanism) f is *strategy-proof* (a.k.a., incentive compatible or truthful) if it is a dominant strategy for each agent to truthfully report his preferences, or mathematically, given an agent i, a sink profile $\mathbf{s} = (s_1, \ldots, s_n)$, and a misreported sink profile $\mathbf{s}' = (s'_i, s_{-i})$:

$$u_i(f(s_i, \mathbf{s}_{-\mathbf{i}}) - s_i) \geq u_i(f(s'_i, \mathbf{s}_{-\mathbf{i}}) - s_i).$$

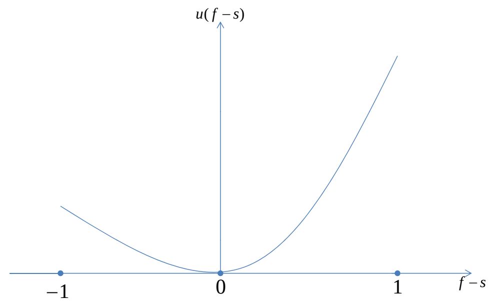


Figure 1: Utility function for any agent

A social choice function (or mechanism) f is *group strategy-proof* if for any group of agents, at least one of them cannot benefit if they misreport simultaneously, or mathematically, given an nonempty subset $A \subseteq N$, sink profile $\mathbf{s} = (s_1, \ldots, s_n)$, and misreported sinks $(\mathbf{s}'_A, \mathbf{s}_{-A})$, there exists an agent $i \in A$ such that

$$u_i(f(\mathbf{s}_A, \mathbf{s}_{-A}) - s_i) \geq u_i(f(\mathbf{s}'_A, \mathbf{s}_{-A}) - s_i).$$

Retrospectively, our model clearly is related to the famous single-peaked policy domain first investigated by Moulin [7], along with the popular *generalized median voter rule*. Extensive work on this special single-peaked domain have appeared in the literature after Moulin's pioneering paper (cited 400+ times according to Google Scholar).

Related work. Our work falls under the umbrella of mechanism design without money, a topic extensively investigated in economic, game theory, and public choice. More recently, in algorithmeric game theory, Procaccia and Tennenholtz [8] advocated the agenda of approximate mechanism without money to study moneyless strategy-proof approximation mechanisms for optimization problems. This agenda was actively followed in [1, 3, 6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 19].

These extant work focus mostly on single-peaked or sing-sinked discrete or continuous policy domain. Therefore the most related work is Cheng et al. [6] who studied the obnoxious facility location game, a special single-sinked domain. While the characterization of strategy-proofness on single-peaked domain is readily available from the work of Moulin [7], there is no analgous results on single-sinked domain. To the authors' best knowledge, our work here seems to be the first one to provide an analogous characterization for the single-sinked domain, which also have wide applications. However, some intricate differences exist between these two domains, as will be explained in Section 4 later on.

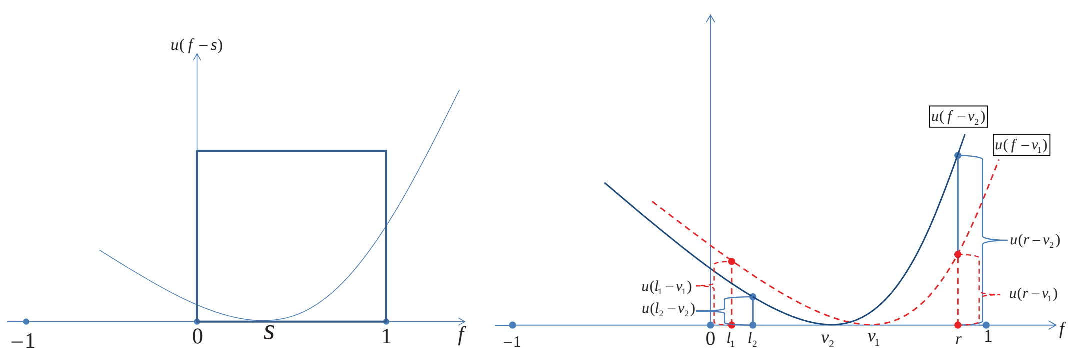
It should be noted that moneyless mechanism design is different from another body of extensive research on mechanism design with money transfer, high-lightened by the famous VCG mechanisms (See [18] for a review on this topic from algorithmic point of view).

While we are submitting our paper, we became aware of an independent work by Ibara and Nagamochi [5], who obtained some overlapping results with this work. But there are essential difference between the model assumptions and the derived results between these two bodies of work. They consider symmetric distance utility functions d in any metric space (Ω, d), where $d : \Omega \times \Omega \mapsto \mathbb{R}^+$ such that $d(x, y) = d(y, x)$ for any $x, y \in \Omega$, and $d(x, y) + d(y, z) \geq d(x, z)$ for any $x, y, z \in \Omega$. Note that our quasi-convex utility function $u_i(f - s_i)$ can be viewed as a function of the form $d_i(f, s_i)$. But d_i is not a distance function since symmetry and triangle inequality can be violated. Therefore, the utility functions from these two work are two overlapping classes and one is not a strict subclass of the other, and results obtained from these two work coincide when restricted to the common part. Moreover, the methods and analysis in these two work are completely different. An even more salient feature of this work, different from previous literature (including [5]), is the removal of identical utility function assumption for all agents. Therefore different agent can assume different utility function forms, which should have significant practical relevance.

Our results and the organization of the paper. This work makes contributions on several fronts. We first completely characterize the (deterministic) strategy-proof mechanisms for the single-sinked policy domain in Sections 2, and as an immediate corollary we show that the same characterization is also valid for group strategy-proofness in Section 2.4. These characterization complements the well-know results on single-peaked policy domain. We then offer a detailed comparison between the single-sinked and single-peaked domains, concerning some other desirable properties in mechanism design, such as onto, unanimous, Pareto optimal, and anonymous in Section 4. Finally, this characterization allows us to extend some existing algorithmic mechanism design models, in particular the obnoxious facility location game [6], and resolve several open problems in the literature, as presented in Section 5. Concluding remarks are given in Section 6.

2. The characterization of strategy-proof mechanism on the line We characterize the strategy-proof mechanisms on the line by first investigating the single-agent and two-agent cases in Sections 2.1 and 2.2, respectively. Then the general n-agent case will be followed in Section 2.3.

2.1 Single agent First of all, we state the following simple property of any quasi-convex utility function.



(a) (b)

Figure 2: The utility function

Proposition 1 *For any fixed s, if the utility function $u(\cdot)$ is quasi-convex, then* *(i) for any fixed s, $u(f-s)$ is decreasing within $f \in [0,s]$ and increasing within $f \in (s,1]$ (see Fig. 2(a) for an illustration);*

(ii) for any fixed ℓ and r satisfying $0 \leq \ell < r \leq 1$, the following equation in terms of v has a unique root within $[\ell, r]$:

$$u(\ell - v) = u(r - v); \tag{1}$$

(iii) moreover, the unique root v is component-wise increasing with respect to ℓ and r, respectively.

Proof. (i) being obvious, we now prove (ii) and (iii). For (ii), let $g(x) = u(r-x) - u(\ell - x) : [\ell, r] \to R$. Then $g(\ell) > 0, g(r) < 0$, and $g(x)$ is decreasing within $[\ell, r]$, implying the desired claim. For (iii), we only prove the case for ℓ and the case for r is analogous. For any fixed r, consider $0 \leq \ell_1 < \ell_2 < r \leq 1$. Let v_1 and v_2 be the roots with respect to ℓ_1 and ℓ_2. Suppose on the contrary, $v_1 \geq v_2$, implying that $\ell_1 - v_1 < \ell_2 - v_2$ and $r - v_1 \leq r - v_2$. However it follows from the quasi-convexity of $u(\cdot)$ and the definition of v_1 that $u(r-v_2) \geq u(r-v_1) = u(\ell_1 - v_1) > u(\ell_2 - v_2)$, a contraction to the definition of v_2 that $u(r-v_2) = u(\ell_2 - v_2)$ (see Fig. 2(b) for an illustration). □

Lemma 1 *Assume there is only one agent and the utility function $u(\cdot)$ is quasi-convex. A deterministic mechanism f on the line is strategy-proof if and only if it assumes one of the following two forms: for any location $i \in [0,1]$, reported by the agent:*

(F-1): there exists a $c \in [0,1]$ such that $f(i) = c$ (this will be called the trivial form);

(F-2): there exist ℓ and r satisfying $0 \leq \ell < r \leq 1$, such that

$$f(i) = \begin{cases} r & \text{if } i < v, \\ \ell & \text{if } i > v, \\ r \text{ or } \ell & \text{if } i = v, \end{cases}$$

where v is the unique root specified in (1) from Proposition 1. (See Fig. 3 for an illustration).

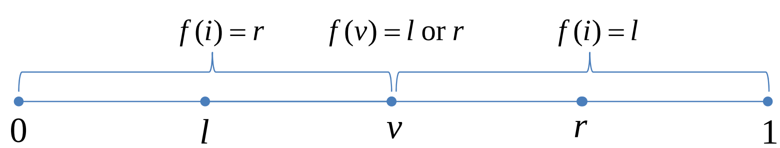


Figure 3: (F-2) in Lemma 1

Proof. The "if" part is easy to verify. We focus on the "only if" part. Let s, t be two profiles and $s < t$. Let $f(s)$ and $f(t)$ be the output by a deterministic strategy-proof mechanism for s and t, respectively. Assume that $f(s) \neq f(t)$. Firs note that $u(f-t)$ is a right-shift of $u(f-s)$ (See Fig 4), namely,

$$u(f-t) = u(f-(t-s)-s). \tag{2}$$

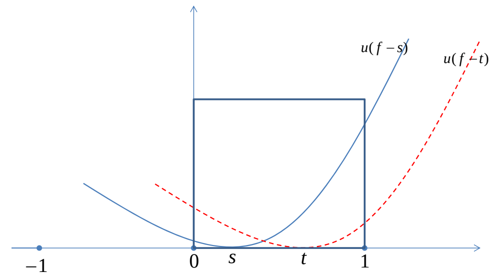


Figure 4: The utility function $u(f-s)$ and $u(f-t)$

Due to strategy-proof, we have that

$$u(f(s)-s) \geq u(f(t)-s),\ \forall t \tag{3}$$

and

$$u(f(t)-t) \geq u(f(s)-t),\ \forall s, \tag{4}$$

where u is quasi-convex with respect to f.

There are total nine cases dependent on the relative positions among $\{s, t, f(s), f(t)\}$, as shown in Table 1. For proof purpose, we classify all nine cases into four classes: Class I: (a) and (i);

	$0 \leq f(t) \leq s$	$s < f(t) \leq t$	$t < f(t) \leq 1$
$0 \leq f(s) \leq s$	(a) $\times$	(b) $\times$	(c) $\times$
$s < f(s) \leq t$	(d) $\surd$	(e) $\surd$	(f) $\times$
$t < f(s) \leq 1$	(g) $\surd$	(h) $\surd$	(i) $\times$

Table 1: Nine cases in the proof of Theorem 2.1

Class II: (b)(c); Class III: (f); and Class IV: (d), (e), (g) and (h).

Under the assumption that $f(s) \neq f(t)$, we first show that none of the classes I, II, and III is possible due to the strategy-proof requirement and the quasi-convexity of the utility function $u(\cdot)$.

Class I: We only prove the result for (a) and (i) is analogous. In Case (a), we have $f(s), f(t) \in [0, s]$. We have $f(s) \leq f(t)$ from Proposition 1(i) and Inequality (3), and $f(s) \geq f(t)$ from Proposition 1(i) and Inequality (4), implying that $f(s) = f(t)$, a contradiction to our assumption that $f(s) \neq f(t)$.

Class II: we only prove (b) and (f) is analogous. In Case (b), we have $f(s) \in [0, s]$ and $f(t) \in (s, t]$. It follows from Proposition 1(i) and Inequality (4) that $f(t) \leq f(s)$ since $f(s), f(t) \leq t$, a contradiction to the case assumption that $f(s) \leq s < f(t)$.

Class III: In case (c), we have $f(s) \in [0, s]$ and $f(t) \in (t, 1]$. Note that $u(f(s)-t) > u(f(s)-s) \geq u(f(t)-s) > u(f(t)-t) \geq u(f(s)-t)$, a contradiction, where the first and third inequalities follow from Proposition 1(i) and Equality (2), while the second and fourth inequalities follow from Inequalities (3) and (4), respectively .

Class IV: We now prove the last class IV. First of all, we have that $f(s) > f(t)$ due to assumption $f(s) \neq f(t)$ and since $u(f(s)-s) \geq u(f(t)-s)$ from Inequality (3) and Proposition 1(i) for case (e). We also have that $f(s) > f(t)$ for other cases due to the case assumptions. Note that the equation $u(f(s)-v) = u(f(t)-v)$ in terms of v has a unique root v within $[f(t), f(s)]$. Moreover, $u(f(s)-s) \geq u(f(t)-s)$ from Inequality (3) implies that $s \leq v$ due to Proposition 1(ii). We only prove (e), the rest is analogous. In case (e), we have $f(s) \in (s,t]$ and $f(t) \in (s,t]$. Let $f(s')$ be the facility location output by the mechanism for s'.

(i) $f(t) < s' < f(s)$. Using the same analysis as that in Case (a), we have that $f(s') = f(s)$ if $f(s') \geq s'$, $f(s') = f(t)$ otherwise. And further more, from

$$u(f(s')-s') \geq \max\{u(f(s)-s'), u(f(t)-s')\},$$

we have that $f(s') = f(s)$ for $f(t) < s' < v$, $f(s') = f(t)$ for $v < s' < f(s)$, and $f(v) = f(s)$ or $f(t)$.

(ii) $s' \leq f(t)$. If $f(s') \geq s'$, we have that $f(s') \geq f(s)$ from $u(f(s')-s') \geq u(f(s)-s')$. Then $f(s') = f(s)$ by using the same analysis as that in Case (a). If $f(s') < s'$, we have that $f(s') \geq f(t)$ from $u(f(t)-t) \geq u(f(s')-t)$, implying that $f(s') \geq f(t) \geq s'$, which is a contradiction. So $f(s') < s'$ can not occur. Therefore, we have that $f(s') = f(s)$ for any s' with $0 \leq s' < f(t)$.

(iii) $s' \geq f(s)$. Similarly to (ii), we have that $f(s') = f(t)$ for any s' with $f(s) < s' \leq 1$.

Finally, set $\ell = f(t), r = f(s)$, the "only if" is concluded. □

The following property is obvious and will be needed in later in the proof of Lemma 2.

Corollary 1 *The mechanism $f(i)$ in Lemma 1 is an non-increasing function of i.*

2.2 Two agents For the rest of argument of this section, we use i and j to represent the two agents or their locations whenever there is no confusion from the context.

Lemma 2 *Assume there are two agents with quasi-convex utility functions $u_1(\cdot)$ and $u_2(\cdot)$, respectively. A deterministic mechanism f on the line is strategy-proof if and only if it assumes one of the following three forms:*

(F-1): there exists a constant $c \in [0,1]$ such that $f(i,j) = c$, $\forall i,j \in [0,1]$ (this will be called the trivial form);

(F-2): there exist ℓ and r satisfying $0 \leq \ell < r \leq 1$, such that for any $0 \leq i \leq 1$,

$$f(i,j) = \begin{cases} r, & \text{if } 0 \leq j < v, \\ \ell, & \text{if } v < j \leq 1, \\ \ell \text{ or } r, & \text{if } j = v \end{cases}$$

(See Fig 5(a)), or for $0 \leq j \leq 1$,

$$f(i,j) = \begin{cases} r, & \text{if } 0 \leq i < \bar{v}, \\ \ell, & \text{if } \bar{v} < i \leq 1, \\ \ell \text{ or } r, & \text{if } i = \bar{v} \end{cases}$$

(See Fig 5(b)). In the above v and $\bar{v}$ are the unique roots of the equations $u_2(\ell - v) = u_2(r - v)$, and $u_1(\ell - \bar{v}) = u_1(r - \bar{v})$, respectively.

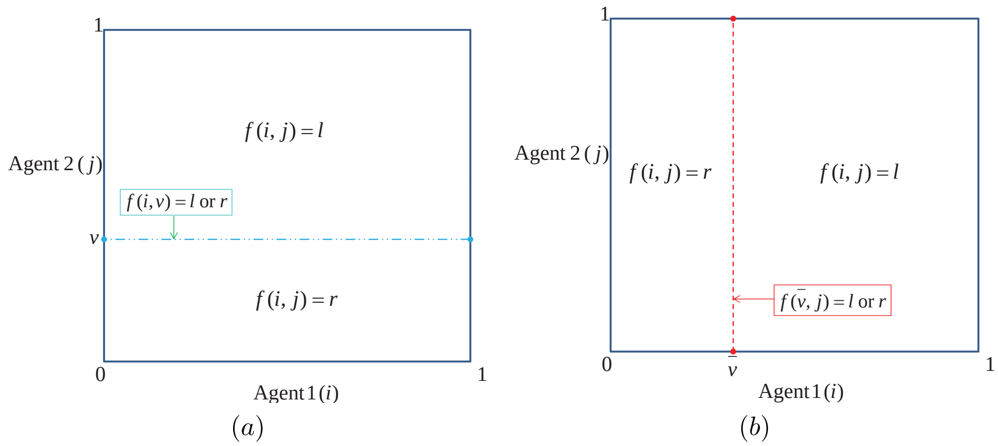


Figure 5: (F-2) in Lemma 2

(F-3): there exist ℓ and r satisfying $0 \le \ell < r \le 1$, such that

$$f(i,j) = \begin{cases} r, & \text{if} \quad 0 \le i < \bar{v},\ 0 \le j < v, \\ \ell, & \text{if} \quad \bar{v} < i \le 1,\ 0 \le j < v, \ \text{or} \quad 0 \le i \le 1, v < j \le 1, \\ \ell \ \text{or} \ r, & \text{if} \quad i = \bar{v},\ 0 \le j < v, \ \text{or} \quad 0 \le i < \bar{v},\ j = v, \end{cases}$$

(See Fig 6(a)) or

$$f(i,j) = \begin{cases} r, & \text{if} \quad 0 \le i < \bar{v}, 0 \le j \le 1, \ \text{or} \quad \bar{v} \le i \le 1, 0 \le j < v, \\ \ell, & \text{if} \quad \bar{v} < i \le 1, v < j \le 1 \\ \ell \ \text{or} \ r, & \text{if} \quad \bar{v} \le i \le 1, j = v, \ \text{or} \quad i = \bar{v}, v < j \le 1. \end{cases}$$

(See Fig 6(b)). In the above v and $\bar{v}$ are the unique roots of the equations $u_2(\ell - v) = u_2(r - v)$, and $u_1(\ell - \bar{v}) = u_1(r - \bar{v})$, respectively.

Proof. The "if" part is trivial. We only consider the "only if" part. For the rest of the proof, we use s and t to represent any two locations of the same agent.

Assume the mechanism is not (F-1), then there exists either $\{i, s, t\}$ with $f(i,s) \neq f(i,t)$, or $\{j, s, t\}$ with $f(s,j) \neq f(t,j)$. We consider the case of $f(i,s) \neq f(i,t)$ only and the other case for $f(s,j) \neq f(t,j)$ is analogous.

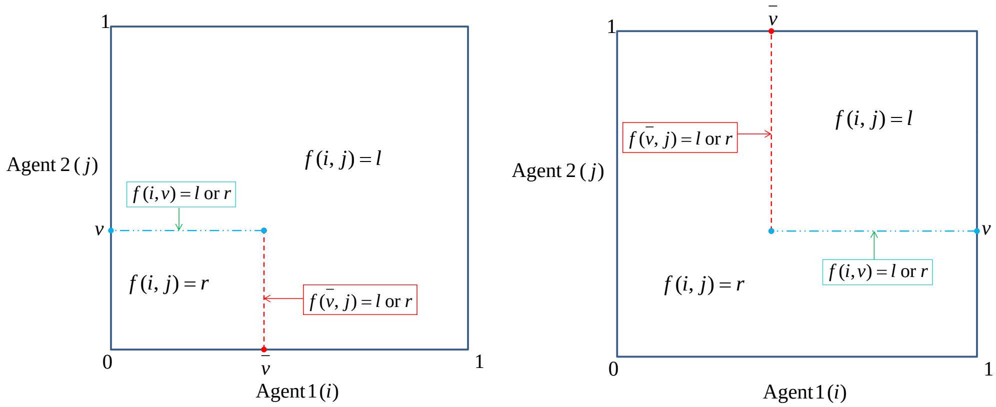
From Lemma 1, for the second agent, there exist $\{\ell, r\}$ with $0 \le \ell < v \le r \le 1$ such that

$$f(i,x) = \begin{cases} r & \text{if } x < v, \\ \ell & \text{if } x > v, \\ r \text{ or } \ell & \text{if } x = v, \end{cases}$$

where v comes from Proposition 1(ii) with respect to the second agent's utility, namely v is the unique root of equation

$$u_2(r - v) = u_2(\ell - v). \tag{5}$$

Moreover, for any fixed $i' \neq i$, applying Lemma 1 with respect to the second agent results in that the mechanism f assumes one of the following two forms:



(a) (b)

Figure 6: (F-3) in Lemma 2

(F$'$-1): $f(i', x) = c$ for any x, where c is a constant within $[0, 1]$;

(F$'$-2): there exists $\{\ell', r'\}$ with $0 \leq \ell' < v' < r' \leq 1$ such that

$$f(i, x) = \begin{cases} r' & \text{if } x < v', \\ \ell' & \text{if } x > v', \\ r' \text{ or } \ell' & \text{if } x = v', \end{cases}$$

where v' comes from Proposition 1(ii) with respect to the second agent's utility, namely v' is the unique root of equation $u_2(r' - v') = u_2(\ell' - v')$.

For the case of (F$'$-1). We claim that $c = \ell$ or r, implying the desired form (F-3). Suppose that $c \neq \ell$ and $c \neq r$.

We only prove the case when $i' > i$, and the other case $i' < i$ is analogous. Note that $i' > i$ and $c \neq \ell, r$ imply that $c < \ell < r$ from Corollary 1. Furthermore, applying Lemma 1 with respect to the first agent results in that,

$$f(i, j) = \begin{cases} r, & \text{if } (i, j) \in \Omega_1 = \{0 \leq i < v_1, 0 \leq j < v\}, \\ c, & \text{if } (i, j) \in \Omega_2 = \{v_2 < i \leq 1, v < j \leq 1\}, \end{cases}$$

where v_1 and v_2 come from Proposition 1(ii) using the first agent's utility, namely v_1 and v_2 are the unique roots of the two equations $u_1(r - v_1) = u_1(c - v_1)$, and $u_1(\ell - v_2) = u_1(c - v_2)$, respectively.

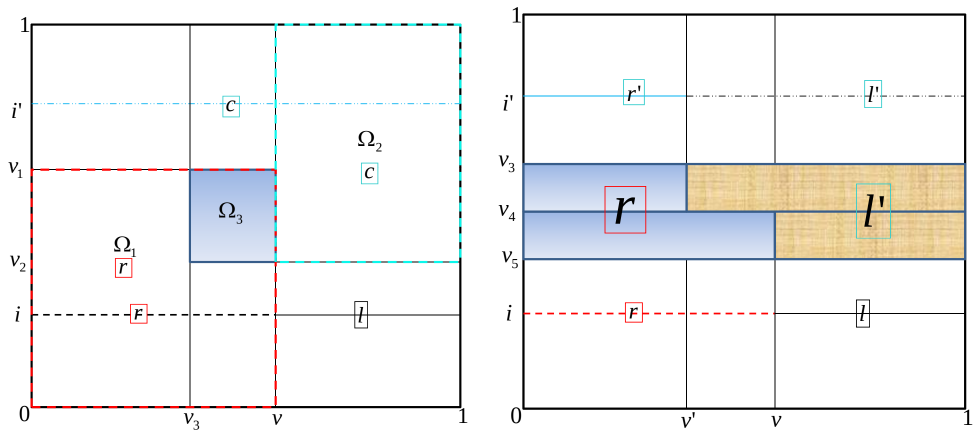
Moreover, it follows from Proposition 1(iii) that $v_1 > v_2$ since $\ell < r$. Let v_3 be the unique root of $u_2(r - v_3) = u_2(c - v_3)$ as in the Proposition 1(ii), which together with (5) imply that $v_3 < v$. Define $\Omega_3 = \{(i, j) : v_2 \leq i \leq v_1, v_3 \leq j \leq v\} \subseteq \Omega_1$. Now, for fixed $i'' \in [v_2, v_1]$, applying (F-2) in Lemma 1 with respect to the second agent, we obtain that $f(i'', j) = c$ for any $j \in [v_3, v]$, a contradiction to $f(i'', j) = r$ since $(i'', j) \in \Omega_3 \subseteq \Omega_1$, implying the desired result that $c = r$ or ℓ. Moreover, $i' > i$ implies that $c \leq \ell < r$ due to Corollary 1, and hence $c = \ell$ when $i' > i$; and analogously $c = r$ when $i' < i$ (See Fig 7(a) for illustration).

For the case of (F$'$-2). We claim that $v' = v$ and $r' = r$ for all $i' \neq i$, implying the desired form (F-2). If $v' \neq v$, we assume that $i' > i$ (the case $i' < i$ is analogous). Then we have that $\ell' < \ell$ and

$r' < r$ from Corollary 1. It follows that $v' < v$ from Proposition 1(iii). Let v_3, v_4 and $_5$ be the roots of the three equations: $u_1(r - v_3) = u_1(r' - v_3)$, $u_1(r - v_4) = u_1(\ell' - v_4)$, and $u_1(\ell - v_5) = u_1(\ell' - v_5)$, respectively. It follows that $v_5 < v_4 < v_3$ from Proposition 1(iii). Applying Theorem 1 with respect to the first agent, we have that $f(i, j) = r$ for (i, j) satisfying $v_5 < i < v_3, 0 \le j < v'$ or $v_5 \le i < v_4, 0 \le j < v$; and $f(i, j) = \ell'$ for any (i, j) satisfying $v_4 < i < v_3, v' < j \le 1$ or $v_5 < i < v_4, v < j < 1$. Now, for any fixed $i'' \in [r_5, r_3]$, applying Theorem 1 with respect to the second agent results in that v and v' both satisfy equation $u_2(\ell' - x) = u_2(r - x)$, implying that $v = v'$ from Proposition 1(ii), a contradiction to $v' \ne v$. Therefore we have shown that $v' = v$ (See Fig 7(b) for illustration).

Now we show that $r = r'$. Suppose on the contract that $r \ne r'$. Assume that $i' > i$ (the case $i' < i$ is analogous). Then we have that $\ell' \le \ell$ and $r' < r$ from Corollary 1, implying that $v' < v$ from Proposition 1(iii), a contradiction to $v = v'$.

$\square$



(a) Case of (F'-1). (b)Case of (F'-2).

Figure 7: Proof of Theorem 2.2

We have the following corollary if the utility functions are symmetric, namely $u_i(x) = u_i(-x)$. Therefore, $v = \bar{v} = \frac{\ell + r}{2}$ in Lemma 2.

Corollary 2 *Assume there are two agents and their utility functions are quasi-convex. A deterministic mechanism f on the line is strategy-proof if and only if it assumes one of the following three forms:*

(F-1): there exists a constant $c \in [0, 1]$ such that $f(i, j) = c$, $\forall i, j \in [0, 1]$;

(F-2): there exist ℓ and r satisfying $0 \le \ell < r \le 1$, such that for any $0 \le i \le 1$,

$$f(i, j) = \begin{cases} r, & \text{if} \quad 0 \le j < v, \\ \ell, & \text{if} \quad v < j \le 1, \\ \ell \text{ or } r, & \text{if} \quad j = v, \end{cases}$$

or for $0 \leq j \leq 1$,

$$f(i,j) = \begin{cases} r, & \text{if} \quad 0 \leq i < v, \\ \ell, & \text{if} \quad v < i \leq 1, \\ \ell \text{ or } r, & \text{if} \quad i = v, \end{cases}$$

where $v = \frac{\ell+r}{2}$.

(F-3): there exist ℓ *and* r *satisfying* $0 \leq \ell < r \leq 1$, *such that*

$$f(i,j) = \begin{cases} r, & \text{if} \quad 0 \leq i < v, \ 0 \leq j < v, \\ \ell, & \text{if} \quad v < i \leq 1, \ 0 \leq j < v, \ \text{or} \quad 0 \leq i \leq 1, v < j \leq 1, \\ \ell \text{ or } r, & \text{if} \quad i = v, \ 0 \leq j < v, \ \text{or} \quad 0 \leq i < v, \ j = v, \end{cases}$$

or

$$f(i,j) = \begin{cases} r, & \text{if} \quad 0 \leq i < v, 0 \leq j \leq 1, \ \text{or} \quad v \leq i \leq 1, 0 \leq j < v, \\ \ell, & \text{if} \quad v < i \leq 1, v < j \leq 1 \\ \ell \text{ or } r, & \text{if} \quad v \leq i \leq 1, j = v, \ \text{or} \quad i = v, v < j \leq 1, \end{cases}$$

where $v = \frac{\ell+r}{2}$.

2.3 n agents We are now ready to present the characterization for the general n-agent case.

Theorem 1 *Assume there are* n *agents and all utility functions* $u_i(\cdot)$ $(i = 1, \ldots, n)$ *are quasi-convex. Let* $\mathbf{s} = (s_1, \ldots, s_n) \in I^n$ *be a sink profile. Then a deterministic mechanism* f *on the line is strategy-proof if and only if it assumes one of the following two forms*

(1) there exists a constant c *such that* $f(\mathbf{s}) = c$ *for any* $\mathbf{s}$ *(this will be called the trivial form);*

(2) there exist ℓ *and* r *satisfying* $0 \leq \ell < r \leq 1$ *such that that* $f(\mathbf{s}) = r$ *or* $f(\mathbf{s}) = \ell$ *for any* $\mathbf{s}$. *Let* $v_1, \ldots, v_n$ *be the roots of the* n *equations* $u_i(\ell - v_i) = u_i(r - v_i)$ $(i = 1, \ldots, n)$. *Let* $n_1(\mathbf{s}) = |\{i : s_i \in [0, v_i)\}|$ *(or* $n_1'(\mathbf{s}') = |\{i : s_i \in [0, v_i]\}|$*). There exists* $0 \leq k < n$ *such that*

$$f(\mathbf{s}) = \begin{cases} \ell, & \text{if } n_1(\mathbf{s}) \leq k \ (\text{or } n_1'(\mathbf{s}') \leq k), \\ r, & \text{if } n_1(\mathbf{s}) > k \ (\text{or } n_1'(\mathbf{s}') > k). \end{cases}$$

Proof. The "if" part is trivial. We only consider the "only if" part. If (1) does not occur, then applying Lemma 2 to any two agents, we obtain that there exist two constants ℓ and r satisfying $0 \leq \ell < r \leq 1$ such that that $f(\mathbf{s}) = r$ or $f(\mathbf{s}) = \ell$ for any $\mathbf{s}$. Suppose on the contrary that there exist two sink profiles $\mathbf{s}$ and $\mathbf{s}'$ such that $f(\mathbf{s}) = r$ and $f(\mathbf{s}') = \ell$ with $n_1(\mathbf{s}') = n_1(\mathbf{s})+1$. Among such pairs of $\mathbf{s}$ and $\mathbf{s}'$, consider one with smallest $n_1(\mathbf{s})$. There exists at least one agent i with $s_i > v_i$, who will benefit by misreporting its sink to $s_i' < v_i$ since the new utility $u_i(\ell - s_i)$ is larger than the original utility $u_i(r - s_i)$ due to quasi-convexity of u_i, a contradiction to the strategy-proofness.

□

For symmetric utility function, we have $v_i = \frac{\ell+r}{2}$, $i = 1, \ldots, n$.

Corollary 3 *Assume there are* n *agents and all utility functions* $u_i(\cdot)$ $(i = 1, \ldots, n)$ *are quasi-convex. Let* $\mathbf{s} = (s_1, \ldots, s_n) \in I^n$ *be a sink profile. Then a deterministic mechanism* f *on the line is strategy-proof if and only if it assumes one of the following two forms*

(1) there exists a constant c such that $f(\mathbf{s}) = c$ for any $\mathbf{s}$;

(2) there exist ℓ and r satisfying $0 \le \ell < r \le 1$ such that that $f(\mathbf{s}) = r$ or $f(\mathbf{s}) = \ell$ for any $\mathbf{s}$. Let $v = (\ell + r)/2$. Let $n_1(\mathbf{s}) = |\{i : s_i \in [0, v)\}|$ (or $n_1'(\mathbf{s}') = |\{i : s_i \in [0, v]\}|$). There exists an integer k $(0 \le k < n)$ such that

$$f(\mathbf{s}) = \begin{cases} \ell, & \text{if } n_1(\mathbf{s}) \le k \ (\text{or } n_1'(\mathbf{s}') \le k), \\ r, & \text{if } n_1(\mathbf{s}) > k \ (\text{or } n_1'(\mathbf{s}') > k). \end{cases}$$

Note that when $n = 1, 2$, Theorem 1 reduces to Lemmas 1 and 2, respectively, where we can afford to list all the possible forms for small n.

2.4 Group strategy-proof

Corollary 4 *Any strategy-proof mechanism on the line is also group strategy-proof.*

Proof. Let $S \subseteq N$ be a coalition. We will prove that the agents in S can not all benefit by misreporting. For any given k $(0 \le k < n)$, we consider the corresponding strategy-proof mechanism f. Let $\mathbf{s}$ be a sink profile such that $n_1(\mathbf{s}) \le k$. Then the strategy-proof mechanism f outputs facility location ℓ. Assume that Agent i in S misreports his sink as s_i' $(i \in S)$. Let $\mathbf{s}' = (\mathbf{s}'_S, \mathbf{s}_{-S})$. If $n_1(\mathbf{s}') \le k$, then f still outputs location ℓ. If $n_1(\mathbf{s}') > k$, then f outputs location r. At least one agent $i \in S$ in $(v_i, 1]$ misreports his sink as $s_i' \in [0, v_i]$. If $r \le s_i$, then $u_i(\ell - s_i) > u_i(r - s_i)$; otherwise, we still have $u_i(\ell - s_i) > u_i(\ell - v_i) = u_i(r - v_i) > u_i(r - s_i)$, leading to the desired group strategy-proofness. □

3. Extensions We now extend the characterization on the line to the tree domain (Section 3.1) and the cycle domain (Section 3.2), respectively.

3.1 Characterization of strategy-proof on a tree For a given diameter in the tree, let one endpoint be 0 and another endpoint be 1. Then the vertices on the diameter have their (one-dimensional) coordinates. For the vertex P that is not on the diameter, there is a vertex W on the diameter such the distance between P and W is the smallest. Let the coordinate of P be the same as that of W.

Theorem 2 *Assume there are n agents and all the utility functions $u_i(\cdot)$ $(i = 1, \ldots, n)$ are quasi-convex. Let $\mathbf{s} = (s_1, \ldots, s_n) \in I^n$ be a sink profile. Then a deterministic mechanism f on the tree is strategy-proof if and only if it assumes one of the following two forms*

(1) there exists a constant c such that $f(\mathbf{s}) = c$ for any $\mathbf{s}$;

(2) there exist a diameter of the tree, and ℓ and r satisfying $\ell < r$ such that that $f(\mathbf{s}) = r$ or $f(\mathbf{s}) = \ell$ for any $\mathbf{s}$. Let $v_1, \ldots, v_n$ be the roots of the n equations $u_i(\ell - v_i) = u_i(r - v_i)$ $(i = 1, \ldots, n)$. Let $n_1(\mathbf{s}) = |\{i : s_i \in [0, v_i)\}|$ (or $n_1'(\mathbf{s}') = |\{i : s_i \in [0, v_i]\}|$). There exists $0 \le k < n$ such that

$$f(\mathbf{s}) = \begin{cases} \ell, & \text{if } n_1(\mathbf{s}) \le k \ (\text{or } n_1'(\mathbf{s}') \le k), \\ r, & \text{if } n_1(\mathbf{s}) > k \ (\text{or } n_1'(\mathbf{s}') > k). \end{cases}$$

3.2 Characterization of strategy-proof on a circle Consider the obnoxious facility game of n agents on a circle. Let O be a point on the circle and its antipodal point be O'. Let A be a point on the left half of the circle and B be a point on the right half part of the circle, which is symmetric with A about O (See Fig. 8). For a profile $\mathbf{s} = (s_1, \ldots, s_n)$, let n_1 be the number of agents located in the left half of the circle, namely the semi-circle OAO', and n_2 be the number of agents located in the right half of the circle, namely the semi-circle OBO'.

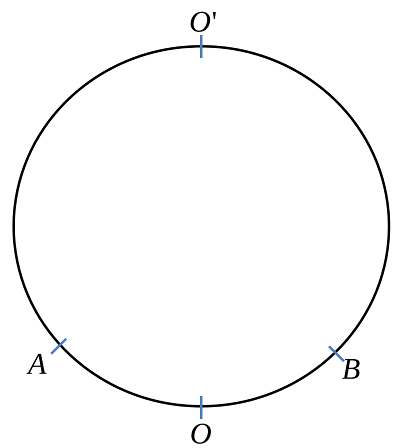


Figure 8: Facility location game on a circle

Theorem 3 *Assume that all the utility functions $u_i(\cdot)$ ($i = 1, \ldots, n$) are quasi-convex and symmetric. A mechanism f on the cycle is strategy-proof if and only if it assumes one of the following two forms:*

(1) $f(\mathbf{s}) = C$ for all profiles $\mathbf{s}$, where C is any fixed point on the circle.

(2) there exists A and O (hence O' and B), such that $f(\mathbf{s}) = A$ or B. Moreover, there exists $0 \le k < n$ such that $f(\mathbf{s}) = A$ if $n_1 \le k$; and $f(\mathbf{s}) = B$, otherwise.

PROOF. Note that the utility function for each agent i on the circle is defined as $u_i(f - s_i)$, where $f - s_i$ is the signed shorter arc length between f and s_i and it is positive whenever the shorter arc $\widehat{fs_i}$ follows clockwise direction and negative otherwise. This utility function is the same as that on the line except for the domain being $[-0.5, 0.5]$. So Theorems 1 and 2 are still valid under the new definition of utility $u_i(f - s_i)$, implying the desired result . □

4. Single-sink vs single-peak For single-sinked domain $I = [0, 1]$, a mechanism f is *onto* if for all $y \in I$, there exists a profile $\mathbf{s}$ such that $f(\mathbf{s}) = y$. It is *unanimous* whenever all agents report the same sinks $s_1 = \ldots = s_n = s$, then $f(\mathbf{s}) \neq s$. It is *Pareto optimal* if for any location profile $\mathbf{s}$ and $f(\mathbf{s}) = y$, there exists no $x \in I$ such that $u_i(x - s_i) \ge u_i(y - s_i)$ for all $i = 1, \ldots, n$, and $u_i(x - s_i) > u_i(y - s_i)$ for at least one i. It is *anonymous* if $f(s_1, \ldots, s_i, \ldots, s_j, \ldots, s_n) = f(s_1, \ldots, s_j, \ldots, s_i, \ldots, s_n)$ for any $i \neq j$.

The above concepts can be defined analogously for the single-peaked domain by replacing sinks with peaks expect for unanimous, where it is *unanimous* whenever all agents report the same peaks $p_1 = \ldots = p_n = p$, then $f(p, \ldots, p) = p$.

It is well-know that some of these concepts are closely related under the single-peaked domain, as stated below:

Theorem 4 ([20]) *Suppose f is strategy-proof under the single-peaked domain. Then f is onto if and only if it is unanimous if and only if it is Pareto optimal.*

However, these concepts behave completely different under the single-sinked domain. Actually, Theorem 1 implying the following result.

Theorem 5 *Assume f is strategy-proof under the single-sinked domain I. Then*

(i) f cannot be onto.

(ii) f is unanimous when f is trivial.

(iii) f is Pareto optimal.

(iv) f is anonymous if $v_1 = \ldots = v_n$ in Theorem 1.

PROOF. (i) follows since any strategy-proof mechanism f can assume at most two different values. For (ii), $f(r, \ldots, r) = \ell$ and $f(\ell, \ldots, \ell) = r$ since f is trivial. For other sink profiles $(s, \ldots, s)$, we have $f(s, \ldots, s) = \ell$ or r, and hence $f(s, \ldots, s) \neq s$. For (iii), Pareto optimality is obvious if f is trivial. If f is not Pareto optimal, then there exists a sink profile $\mathbf{s}$ such that either $f(\mathbf{s}) = \ell$ or $f(\mathbf{s}) = r$. We only consider the former case (the latter is analogous). When $f(\mathbf{s}) = \ell$, we have $u_i(r - s_i) \geq u_i(\ell - s_i)$ for all i due to the non-Pareto-optimality assumption. Since f is not trivial, it follows from Theorem 1 that there exists a sink profile $\mathbf{s}'$ with $f(\mathbf{s}') = r$. We now construct another sink profile $s'' = (s'_A, s_A)$, where $A = \{i \in N : s_i < v_i < s'_i \text{ or } s'_i < v_i < s_i\}$. Note that $f(s'') = f(s') = r$ since $n_1(s') = n_1(s'')$. If $A = \emptyset$, then $f(s'') = f(s) = \ell$, a contradiction to $f(s'') = r$, implying that $A \neq \emptyset$. Moreover for this new sink profile s'' we must have $u_i(r - s_i) > u_i(\ell - s_i)$ for every agent $i \in A$, a contradiction to the group-strategy-proofness. For (iv), $v_1 = \ldots = v_n$ implies that ℓ, r, v are independent of the agents. □

5. Efficiency We discuss efficiency issue by first presenting a general approximate efficiency result and then apply this result to the obnoxious facility location game investigated by Cheng et al. [6].

We focus on the single-sinked domain on the line I. The main interest is to design strategy-proof mechanisms that also fare well with respect to maximizing social welfare, i.e., *efficiency*, namely

$$\max_{f(\mathbf{s})\text{: strategy-proof}} \left[\sum_{i=1}^{n} u_i(f(\mathbf{s}) - s_i) \right].$$

It turns out that these two requirements, strategy-proof and efficiency, are in conflict with each other in general. Therefore we instead look for a strategy-proof mechanism f that is γ-efficient (a.k.a., γ-competitive, or γ-approximation), namely, for $\gamma \geq 1$:

$$\text{OPT}(\mathbf{s}) \leq \gamma \text{SW}(f(\mathbf{s})), \forall \mathbf{s} \in I^n.$$

where $\textsc{opt}(\mathbf{s})$ and $\textsc{sw}(f(\mathbf{s}))$ are the optimal social welfare without strategy-proof and the social welfare of the mechanism, respectively.

Note that the social welfare is quasi-convex, implying that the optimal social welfare (without strategy-proof requirement) is achieved at the two boundaries of I, namely $f(\mathbf{s}) = 0$ or 1, and hence the optimal social welfare is given by

$$\textsc{opt}(\mathbf{s}) = \max\left\{\sum_{i=1}^{n} u_i(-s_i), \sum_{i=1}^{n} u_i(1-s_i)\right\}. \tag{6}$$

However, as pointed out in [6], this optimal mechanism is not strategy-proof.

5.1 A general result We obtain a generic approximation ratio for the Mechanism 1 in Cheng et al. [6] for any quasi-convex identical symmetric utility function, extending the result by Cheng et al. [6]. Moreover, we show that this ratio is best possible for log-concave utility functions by applying our characterization in Lemma 2, answering an open question left by Cheng et al. [6].

Let $u(x) : \mathbb{R} \mapsto \mathbb{R}^+$ be a univariate function. Suppose $\{x : u(x) > 0\} = (a, b)$. Then $u(x)$ is log-concave if $\log u(x)$ is concave [4], or equivalently [2], for any $\beta \geq 0$, the following function is decreasing with $x \in (a, b)$:

$$\frac{u(x+\beta)}{u(x)}. \tag{7}$$

Mechanism 1. (Cheng et al. [6]) *Given a sink profile* $\mathbf{s}$ *on* I. *Let* n_1 *be the number of agents on* $\left[0, \frac{1}{2}\right]$ *and* n_2 *be the number of agents on* $\left(\frac{1}{2}, 1\right]$. *If* $n_1 \leq n_2$, *then output* 0; *otherwise output* 1.

Theorem 6 *Let* $u(\cdot)$ *be any quasi-convex and symmetric utility function. Assume all agents have the same utility function, namely,* $u_i(\cdot) = u(\cdot)$ *for all* $i = 1, \ldots, n$. *Denote*

$$\gamma = 1 + \frac{u(1)}{u\left(\frac{1}{2}\right)}.$$

Then

(i) Mechanism *1 is a group strategy-proof* γ*-efficient mechanism.*

(ii) Moreover, if $u(\cdot)$ *is log-concave, then this ratio is best possible, meaning that no deterministic strategy-proof mechanism on the line is* γ'*-efficient for any* $\gamma' < \gamma$.

Proof. Group strategy-proofness of Mechanism 1 follows from Corollary 4. We now prove the efficiency ratio in (i). Due to symmetry, we only consider $n_1 \leq n_2$, where Mechanism 1 outputs the facility location 0, implying that the social welfare is

$$\textsc{sw}(f(\mathbf{s})) = \sum_{i=1}^{n} u(-s_i) = \sum_{s_i \leq \frac{1}{2}} u(-s_i) + \sum_{s_i > \frac{1}{2}} u(-s_i) \geq \sum_{s_i > \frac{1}{2}} u(-s_i) \geq n_2 u\left(-\frac{1}{2}\right) = n_2 u\left(\frac{1}{2}\right),$$

where the last two inequalities follow from the quasi-convexity and symmetry of $u(\cdot)$, respectively. We only need to prove the ratio when the optimal location for $\mathbf{s}$ is at 1. So

$$\textsc{opt}(\mathbf{s}) = \sum_{i=1}^{n} u(1-s_i) = \sum_{s_i \leq \frac{1}{2}} u(1-s_i) + \sum_{s_i > \frac{1}{2}} u(1-s_i) \leq n_1 u(1) + n_2 u\left(\frac{1}{2}\right),$$

implying that

$$\frac{\text{OPT}(\mathbf{s})}{\text{SW}(f(\mathbf{s}))} \le 1 + \frac{n_1}{n_2}\frac{u(1)}{u\left(\frac{1}{2}\right)} \le 1 + \frac{u(1)}{u\left(\frac{1}{2}\right)}.$$

We now prove (ii). We apply Corollary 2 to prove this result when there are two agents. Let γ_f be the approximate ratio for any given strategy-proof mechanism f. Then obviously, $\gamma_f = \infty$ for (F-1) in Corollary 2. For (F-2) and (F-3) in Corollary 2, let ℓ, r, v be as defined therein, where $0 \le \ell < v < r \le 1$ and $v = (\ell + r)/2$.

Consider the sink profile $\mathbf{s}' = (\ell, v)$. Then Corollary 2 implies that $f(\mathbf{s}') = \ell$ or r. Now we have

$$\gamma_f \;=\; \max_{\mathbf{s}} \tfrac{\text{OPT}(\mathbf{s})}{\text{SW}(f(\mathbf{s}))} \ge \tfrac{\text{OPT}(\mathbf{s}')}{\text{SW}(\ell)} = \max\left\{ \tfrac{u(\ell)+u(v)}{u(\ell-v)}, \tfrac{u(1-\ell)+u(1-v)}{u(\ell-v)} \right\}.$$

Consider another sink profile $\mathbf{s}'' = (v, r)$. Then Corollary 2 implies that $f(\mathbf{s}'') = \ell$ or r. Now we have

$$\gamma_f \;=\; \max_{\mathbf{s}} \tfrac{\text{OPT}(\mathbf{s})}{\text{SW}(f(\mathbf{s}))} \ge \tfrac{\text{OPT}(\mathbf{s}'')}{\text{SW}(r)} = \max\left\{ \tfrac{u(r)+u(v)}{u(r-v)}, \tfrac{u(1-r)+u(1-v)}{u(r-v)} \right\}.$$

Together we have

$$\begin{aligned} \gamma_f \;&\ge\; \max\left\{ \tfrac{u(\ell)+u(v)}{u(\ell-v)}, \tfrac{u(1-\ell)+u(1-v)}{u(\ell-v)}, \tfrac{u(r)+u(v)}{u(r-v)}, \tfrac{u(1-r)+u(1-v)}{u(r-v)} \right\} \\ &=\; \max\left\{ \tfrac{u(1-\ell)+u(1-v)}{u(\ell-v)}, \tfrac{u(r)+u(v)}{u(r-v)} \right\} = \max\left\{ \tfrac{u(1-\ell)+u(1-v)}{u(r-v)}, \tfrac{u(r)+u(v)}{u(v-\ell)} \right\}. \end{aligned}$$

where the second equality follows from $\ell < v < r$ and the quasi-convexity of $u(\cdot)$, and the last inequality follows from symmetry of $u(\cdot)$ and $u(\ell - v) = u(r - v)$.

It follows from log-concavity of $u(\cdot)$ in (7) that the two terms in the last quantity are increasing and decreasing in terms of v, respectively. Therefore this quantity achieves its minimum when the two terms are equal, namely,

$$u(1-\ell) + u(1-v) = u(r) + u(v).$$

It follows from quasi-convexity of $u(\cdot)$ that the above implies that either $1 - v \le v \le r \le 1 - \ell$, or $v \le 1 - v \le 1 - \ell \le r$. Both cases lead to $v = 1/2$. Therefore we have

$$\gamma_f \ge \max\left\{ \frac{u(1-\ell) + u\left(\frac{1}{2}\right)}{u\left(\frac{1}{2} - \ell\right)}, \frac{u(r) + u\left(\frac{1}{2}\right)}{u\left(r - \frac{1}{2}\right)} \right\},$$

implying that

$$\gamma \ge \min_{f:\text{strategy-proof}} \gamma_f \ge \min_{0 \le \ell \le \frac{1}{2} \le r \le 1} \max\left\{ \frac{u(1-\ell) + u\left(\frac{1}{2}\right)}{u\left(\frac{1}{2} - \ell\right)}, \frac{u(r) + u\left(\frac{1}{2}\right)}{u\left(r - \frac{1}{2}\right)} \right\} = 1 + \frac{u(1)}{u\left(\frac{1}{2}\right)},$$

where the last equality follows since the two terms in the second-to-last quantity are increasing and decreasing in terms of l and r, respectively, due to the log-concavity of $u(\cdot)$ in (7). □

5.2 The obnoxious facility location game on the line We now apply Theorem 6 to resolve an open question left by Cheng et al. [6] for the obnoxious facility location game on the line.

Consider a set $N = \{1, \ldots, n\}$ of n agents located on the line I, whose true locations are unknown to the mechanism designer. Agents report their locations to form a location profile $\mathbf{s} = (s_1, \ldots, s_n) \in I^n$, which is then used by the mechanism designer to decide an obnoxious location $f(\mathbf{s})$ in the meaning that every agent wants to be as far from $f(\mathbf{s})$ (e.g., a garbage site to be deployed by the city) as possible. Therefore the utility of agent $i \in N$ is its distance to the obnoxious site, namely

$$u(f(\mathbf{s}) - s_i) = |f(\mathbf{s}) - s_i|. \tag{8}$$

Cheng et al. [6] obtain the following result for this game.

Theorem 7 *There exists a 3-efficient deterministic (group) strategy-proof mechanism for the obnoxious facility location game on the line. Moreover, no deterministic strategy-proof mechanism is γ-efficient for any $\gamma < 2$.*

Therefore they show that the efficiency ratio of any strategy-proof mechanism is within the interval $[2, 3]$, and left the open question on whether 2 or 3 is the best ratio one can expect for any deterministic strategy-proof mechanism. Note that the utility function $u(\cdot)$ in (8) satisfies the conditions in Theorem 6, implying that 3 is actually the best possible, since therein, we have

$$\gamma = 1 + \frac{u(1)}{u\left(\frac{1}{2}\right)} = 3.$$

Corollary 5 *No deterministic strategy-proof mechanism for the obnoxious facility location game on the line is γ-efficient for any $\gamma < 3$*

Moreover, Cheng et al. [6] extends the existence result to the tree domain by the Mechanism 4 therein:

Theorem 8 *There exists a 3-efficient deterministic (group) strategy-proof mechanism for the obnoxious facility location game on the tree.*

Since line is a special tree, therefore Corollary 5 implies that no deterministic strategy-proof mechanism for the obnoxious facility location game on the tree is γ-efficient for any $\gamma < 3$, and hence resolving another open question left by Cheng et al. [6].

5.3 The obnoxious facility location game on the line with the l_p-norm utility We now consider a more general social welfare function in the obnoxious location game considered by Cheng et al. [6]. This new social welfare function is defined as the l_p-norm ($p \geq 1$) of the agents utility function $(u_1, \ldots, u_n)$ when the sink profile is given as $\mathbf{s} = (s_1, \ldots, s_n)$ and the mechanism is f:

$$\left[\sum_{i=1}^{n} u^p(f(\mathbf{s}) - s_i)\right]^{\frac{1}{p}}.$$

Evidently this utility function $|f(\mathbf{s}) - s_i|^p$ is log-concave, which satisfies the condition of Theorem 6, implying that the best possible efficiency ratio is given by

$$\left(1 + \frac{u(1)}{u\left(\frac{1}{2}\right)}\right)^{\frac{1}{p}} = \left(1 + \frac{1}{\frac{1}{2^p}}\right)^{\frac{1}{p}} = (1 + 2^p)^{\frac{1}{p}}.$$

This result therefore extends that by Cheng et al. [6] where the utility is the special l_1-norm.

5.4 The obnoxious game on the line with maxmin utility In this section, for another natural social welfare function, we show that the efficiency ratio can be unbounded. Assume the social welfare is defined as

$$\min_i \{u(f(\mathbf{s}) - s_i)\}.$$

Note that for any sink profile $\mathbf{s}$, $\text{OPT}(\mathbf{s}) > 0$. For any strategy-proof mechanism, there exists a profile $\mathbf{s}$ such that $\text{SW}(f(\mathbf{s})) = 0$ because we can always choose a profile $\mathbf{s}$ with one component equal to ℓ (or r) such that whenever $f(\mathbf{s}) = \ell$ (or r, respectively). Then this player's utility is zero, implying that the social welfare is also zero. Now we can conclude that there is no γ-efficient deterministic strategy-proof mechanism with bounded γ, namely any strategy-proof mechanism has an unbounded approximation ratio.

5.5 The obnoxious facility location game on the cycle This game is similar to the one on the line. But the policy domain becomes a cycle C. Without loss of generality, we normalize the cycle length to be one and parametrize the cycle such that any point $x \in C$ can be viewed as a number $x \in I$. Point 1 coincides with point 0. For any two points $x, y \in C$, The distance $d(x, y)$ between x and y is the shorter arc length on the cycle. Therefore the utility of agent $i \in N$ is its distance to the obnoxious site, namely

$$u(f(\mathbf{s}) - s_i) = d(f(\mathbf{s}), s_i). \tag{9}$$

Mechanism 2. (Cheng et al. [6]) *Given a sink profile* $\mathbf{s}$ *on the cycle* C*. Let* n_1 *be the number of agents on* $\left[0, \frac{1}{2}\right]$ *and* n_2 *be the number of agents on* $\left(\frac{1}{2}, 1\right]$*. If* $n_1 \leq n_2$*, then output* $\frac{1}{4}$*, otherwise output* $\frac{3}{4}$*.*

Cheng et al. [6] proposed the above group strategy-proof mechanism for this game and prove the following result.

Theorem 9 Mechanism *2 is a 3-efficient deterministic (group) strategy-proof mechanism for the obnoxious facility location game on the cycle.*

However, they did not provide any lower bound on the efficiency ratio. Now we apply Theorem 3 to show that the bound 3 is actually the best possible. In particular, we use the special case where there are two agents. For ease of proof later on, we restate Theorem 3 in this special two-agent case by choosing the coordinate as follows. For given A and B, let O and O' be located at $0(1)$ and $\frac{1}{2}$, respectively. Under this coordinate, let $A = \ell$, implying that $B = 1 - \ell$.

Corollary 6 *Assume there are two agents and their utility functions are quasi-convex and symmetric. A deterministic mechanism* f *on the cycle is strategy-proof if and only if it assumes one of the following three forms:*

(F-1): there exists a constant $c \in C$ *such that* $f(i, j) = c$, $\forall i, j \in C$*;*

(F-2): there exist ℓ *satisfying* $0 < \ell < \frac{1}{2}$*, such that for any* $i \in C$*,*

$$f(i,j) = \begin{cases} 1 - \ell, & \text{if} \quad 0 < j < \frac{1}{2}, \\ \ell, & \text{if} \quad \frac{1}{2} < j < 1, \\ \ell \text{ or } 1 - \ell, & \text{if} \quad j = 0, \frac{1}{2}, 1 \end{cases}$$

or for $j \in C$,

$$f(i,j) = \begin{cases} 1-\ell, & \text{if } \ 0 < i < \frac{1}{2}, \\ \ell, & \text{if } \ \frac{1}{2} < i < 1, \\ \ell \text{ or } 1-\ell, & \text{if } \ i = 0, \frac{1}{2}, 1. \end{cases}$$

(F-3): there exist ℓ satisfying $0 < \ell < \frac{1}{2}$, such that

$$f(i,j) = \begin{cases} 1-\ell, & \text{if } \ 0 < i < \frac{1}{2}, \ 0 < j < \frac{1}{2}, \\ \ell, & \text{if } \ \frac{1}{2} < i < 1, \ 0 < j < \frac{1}{2}, \ \text{or} \ \ 0 \le i \le 1, \frac{1}{2} < j < 1, \\ \ell \text{ or } 1-\ell, & \text{if } \ i = 0, \frac{1}{2}, 1, \ 0 \le j \le \frac{1}{2}, \ \text{or} \ \ 0 \le i \le \frac{1}{2}, \ j = 0, \frac{1}{2}, 1, \end{cases}$$

or

$$f(i,j) = \begin{cases} 1-\ell, & \text{if } \ 0 < i < \frac{1}{2}, 0 \le j \le 1, \ \text{or} \ \ \frac{1}{2} < i < 1, 0 < j < \frac{1}{2}, \\ \ell, & \text{if } \ \frac{1}{2} < i < 1, \frac{1}{2} < j < 1 \\ \ell \text{ or } 1-\ell, & \text{if } \ \frac{1}{2} \le i \le 1, j = 0, \frac{1}{2}, 1, \ \text{or} \ \ i = 0, \frac{1}{2}, 1, \frac{1}{2} \le j \le 1, \end{cases}$$

Note that the optimal social welfare when there are only two agents is given as follows. For any $\mathbf{s} = (i,j)$, where $i < j$, we have

$$\textsc{opt}(\mathbf{s}) = \max\{j-i, 1-(j-i)\}. \tag{10}$$

Theorem 10 *No deterministic strategy-proof mechanism for the obnoxious facility location game on the cycle is γ-efficient for any $\gamma < 3$.*

Proof. We apply Corollary 6 to prove this result when there are two agents. Let γ_f be the approximate ratio for any given strategy-proof mechanism f. Then obviously, $\gamma_f = \infty$ for (F-1) in Corollary 6. For (F-2) and (F-3) in Corollary 6, let $0 < \ell < \frac{1}{2}$, then $0 < \ell < \frac{1}{2} < 1-\ell < 1$.

Consider the sink profile $\mathbf{s}' = (\ell, \frac{1}{2})$. Then Corollary 6 implies that $f(\mathbf{s}') = \ell$ or $1-\ell$. Now we have

$$\gamma_f \ = \ \max_{\mathbf{s}} \tfrac{\textsc{opt}(\mathbf{s})}{\textsc{sw}(f(\mathbf{s}))} \ge \tfrac{\textsc{opt}(\mathbf{s}')}{\textsc{sw}(\ell)} = \tfrac{\frac{1}{2}+\ell}{\frac{1}{2}-\ell},$$

where the last equality follows from (10).

Consider another sink profile $\mathbf{s}' = (\ell, 1)$. Then Corollary 6 implies that $f(\mathbf{s}') = \ell$ or $1-\ell$. Now we have

$$\gamma_f \ = \ \max_{\mathbf{s}} \tfrac{\textsc{opt}(\mathbf{s})}{\textsc{sw}(f(\mathbf{s}))} \ge \tfrac{\textsc{opt}(\mathbf{s}')}{\textsc{sw}(\ell)} = \tfrac{1-\ell}{\ell},$$

where the last equality follows from (10).

Together we have

$$\gamma_f \ge \max\left\{\frac{1-\ell}{\ell}, \frac{\frac{1}{2}+\ell}{\frac{1}{2}-\ell}\right\} = \begin{cases} \frac{1-\ell}{\ell}, \ \text{if } \ell \le \frac{1}{4}, \\ \frac{\frac{1}{2}+\ell}{\frac{1}{2}-\ell}, \ \text{otherwise.} \end{cases}$$

Therefore

$$\gamma \ge \min_{f:\text{strategy-proof}} \gamma_f \ge \min\left\{\min_{0<\ell\le\frac{1}{4}} \frac{1-\ell}{\ell}, \min_{\frac{1}{4}\le\ell<\frac{1}{2}} \frac{\frac{1}{2}+\ell}{\frac{1}{2}-\ell}\right\} = 3.$$

□

Theorem 10 resolves yet another open question left in Cheng et al. [6].

6. Conclusion This work has been exclusively on deterministic mechanisms and an equally important question is to characterize randomized strategy-proofness.

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