Semi-continuous network flow problems

Gustavo Angulo, Shabbir Ahmed, Santanu S. Dey
H. Milton Stewart School of Industrial and Systems Engineering,
Georgia Institute of Technology, Atlanta, GA, USA.
765 Ferst Drive NW, GA, USA
gangulo@gatech.edu, sahmed@isye.gatech.edu, sdey@isye.gatech.edu

April 26, 2012

Abstract

We consider semi-continuous network flow problems, that is, a class of network flow problems where some of the variables are restricted to be semi-continuous. We introduce the semi-continuous inflow set with variable upper bounds as a relaxation of general semi-continuous network flow problems. Two particular cases of this set are considered, for which we present complete descriptions of the convex hull in terms of linear inequalities and extended formulations. We consider a class of semi-continuous transportation problems where inflow systems arise as substructures, for which we investigate complexity questions. Finally, we study the computational efficacy of the developed polyhedral results in solving randomly generated instances of semi-continuous transportation problems.

Keywords: Mixed-integer programming, network flow problems, semi-continuous variables.

1 Introduction

A variable x is said to be semi-continuous if x is required to belong to a set of the form $\{0\} \cup [l, u]$ for some $0 \leq l \leq u$. We call l and u lower and upper bounds of x, respectively. A semi-continuous variable can be regarded as a generalization of a binary variable. In fact, by setting $l = u = 1$ in the above definition, we have that x is binary. As such, the presence of these variables may lead to hard optimization problems.

Semi-continuous variables appear in inventory management models where shippings from a given supplier are required to be between prestablished minimum and maximum quantities whenever an order is placed [13]. In portfolio optimization, semi-continuous constraints are known as minimum transaction levels, and are studied in [2] and [11]. Semi-continuous variables are also common when modeling petrochemical processes as described in [8] and [9]. Furthermore, as [9] and [13] suggest, supply chain models may involve network flow structures with semi-continuity constraints on flow variables whenever production, purchases and shipping in low quantities are undesirable from the operational point of view.

Although semi-continuity can be modeled by means of introducing additional binary variables and constraints, this approach may have some drawbacks. We increase the size of the problem at hand, which can already be large-scale. Additionally, the presence of binary variables may lead to unnecessary branching decisions and large LP relaxations in a branch-and-bound procedure. On the other hand, models that incorporate auxiliary binary variables may benefit from presolve and bound tightening procedures available in state-of-the-art MIP solvers such as CPLEX and may be solved efficiently. To overcome difficulties with auxiliary binary variables, branching rules and cuts without the use of binary variables for some combinatorial problems have been studied in [4] and [5]. In particular, in [6] and [7] the semi-continuous knapsack problem is introduced and cutting-planes are presented.

In the present work, we study some particular semi-continuous sets. Specifically, given their wide applicability, we focus on network flow problems having semi-continuous flow variables. Our main contributions are complete descriptions of the convex hull of two particular cases of a semi-continuous inflow set with variable upper bounds and a computational study of the effectiveness of the derived inequalities on a class of semi-continuous transportation problems. We observe that the polyhedral results derived from the semi-continuous sets can significantly improve the performance of standard mixed integer formulations involving auxiliary binary variables. The rest of the paper is organized as follows. In Section 2 we introduce the semi-continuous inflow set along with some basic properties. In Sections 3 and 4 we present polyhedral studies of two particular cases of this set. Then, in Section 5 we introduce a class of semi-continuous transportation problems for which we give complexity results. We devote Section 6 to computational results regarding the performance of the polyhedral results when solving semi-continuous transportation problems. Finally, in Section 7 we conclude with some remarks.

2 The semi-continuous inflow set

Consider the network substructure shown in Figure 1. Let $N := \{1, \ldots, n\}$ be a set of nodes, where $n \geq 2$, and let $d > 0$ be the required total flow from nodes in N to another node 0. For $i \in N$, let y_i be flow from node i to node 0, and x_i be the flow into node i. Let l_i and h_i be the lower bounds on flows x_i and y_i whenever these variables are positive. Let t_i be the exogenous supply into node i. The semi-continuous inflow set with variable upper bounds is a set $S(t,h) \subseteq \mathbb{R}^n \times \mathbb{R}^n$ defined as

$$S(t,h) := \left\{ (x,y) \in \mathbb{R}^n \times \mathbb{R}^n : \begin{array}{rlr} \displaystyle\sum_{i \in N} y_i \geq d & & (1) \\ y_i \leq t_i + x_i & \forall i \in N & (2) \\ x_i \in \{0\} \cup [l_i, \infty) & \forall i \in N & (3) \\ y_i \in \{0\} \cup [h_i, \infty) & \forall i \in N & (4) \end{array} \right\}.$$

Constraint (1) ensures that the minimum total inflow into the node 0 is met. Constraints (2) bound y_i by the total available inflow $t_i + x_i$ for node $i \in N$. Finally, constraints (3) and (4) are semi-continuity requirements on x and y, respectively.

Next we discuss how the set $S(t,h)$ arises as a substructure in general semi-continuous network flow problems. Consider a network represented by a directed graph $G = (V,E)$, where each node $v \in V$ satisfies a constraint of the form

$$\sum_{u \in V^+(v)} f_{vu} - \sum_{u \in V^-(v)} f_{uv} = d_v, \tag{5}$$

where variable $f_{vu} \geq 0$ is the flow through the arc $(v,u) \in E$, $V^+(v) := \{u \in V : (v,u) \in E\}$, $V^-(v) := \{u \in V : (u,v) \in E\}$, and d_v is a given real parameter. Suppose that $f_{uv} \in \{0\} \cup [l_{uv}, u_{uv}]$, that is, f_{uv} is semi-continuous. We refer to such problems as semi-continuous network flow problems. We obtain $S(t,h)$ as a relaxation as follows.

Consider a node $v \in V$ with $d_v < 0$ as depicted in Figure 2. Since the first sum in (5) is nonnegative,

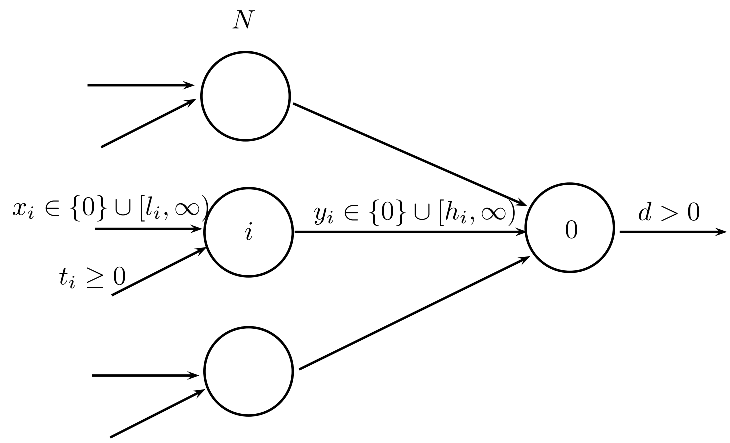

Figure 1: Inflow relaxation.

we have

$$\sum_{u \in V^-(v)} f_{uv} \geq -d_v = |d_v|,$$

which has the form of the semi-contiuous knapsack set introduced in [6]. However, since we are dealing with a network flow problem, there is more structure to be exploited when looking for tighter relaxations. Indeed, consider $(u, v) \in E$. Then $v \in V^+(u)$ and (5) applied to u can be written as

$$\sum_{w \in V^+(u) \setminus \{v\}} f_{uw} + f_{uv} - \sum_{w \in V^-(u)} f_{wu} = d_u. \tag{6}$$

As before, since the first sum in (6) is nonnegative, we arrive at

$$f_{uv} \leq \sum_{w \in V^-(u)} f_{wu} + d_u \leq \sum_{w \in V^-(u)} f_{wu} + \max\{d_u, 0\}. \tag{7}$$

Note that $f_u := \sum_{w \in V^-(u)} f_{wu}$ is a semi-continuous variable taking values in $\{0\} \cup [l_u, u_u]$, where $l_u := \min_{w \in V^-(u)}\{l_{wu}\}$ and $u_u := \sum_{w \in V^-(u)} u_{wu}$. We obtain the system

$$\begin{aligned}
\sum_{u \in V^-(v)} f_{uv} &\geq |d_v| & \\
f_{uv} &\leq f_u + \max\{d_u, 0\} & \forall u \in V^-(v) \\
f_u &\in \{0\} \cup [l_u, u_u] & \forall u \in V^-(v) \quad (8)\\
f_{uv} &\in \{0\} \cup [l_{uv}, u_{uv}] & \forall u \in V^-(v), \quad (9)
\end{aligned}$$

which is a relaxation for the original network flow set. Finally, removing the upper bounds from (8) and (9) we arrive at a relaxation having the form of $S(t, h)$.

A similar approach can be followed when $d_v > 0$, in which case we drop the second sum in (5) and relax the balance equation for nodes in $V^+(v)$. In either case, by appropriately manipulating (5) applied to $v \in V$ and $u \in V^+(v) \cup V^-(v)$, we obtain the set $S(t, h)$ as a relaxation.

We omit the case $d = 0$ since (1) becomes redundant and then $S(t, h)$ is the product of n simple 2-dimensional sets.

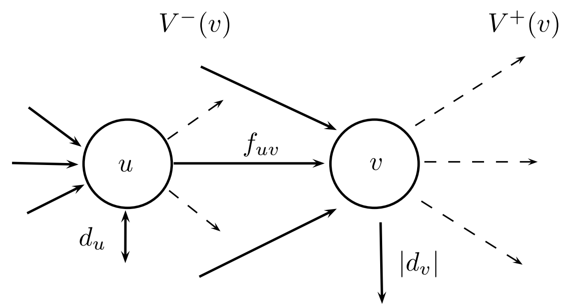

Figure 2: Nodes v and u, where $u \in V^-(v)$ and $d_v < 0$.

2.1 Complexity of optimization

It is not difficult to verify that having finite upper bounds as in (8) and (9) would yield a set that is already hard to deal with. On the other hand, (3) and (4) are not MIP-representable, and thus the complexity of optimization over $S(t, h)$ is not obvious at first glance. We show that optimization over this set is intractable.

Proposition 1. *Optimizing a linear function over $S(t, h)$ is $\mathcal{NP}$-hard, even if $l = 0$.*

Proof. We will show that the Binary Knapsack problem, which is $\mathcal{NP}$-hard, can be reduced to optimization of a linear function over $S(t, h)$.

We start with a feasible instance of the Binary Knapsack problem of the form

$$\begin{aligned} \min \quad & \sum_{i \in N} f_i z_i \\ \text{s.t.} \quad & \sum_{i \in N} w_i z_i \geq d \\ & z_i \in \{0, 1\} \qquad \forall i \in N, \end{aligned}$$

where $d \in \mathbb{Z}_+$, $w \in \mathbb{Z}_+^n$, and $f \in \mathbb{Z}_+^n$. Consider the change of variables $y_i = w_i z_i$ for all $i \in N$. Given that $z_i \in \{0, 1\}$, we have that $y_i \in \{0, w_i\}$. Furthermore, this is equivalent to requiring $y_i \in \{0\} \cup [w_i, \infty)$ and $y_i \leq w_i$. Thus, the optimal value of the instance is the same as that of

$$\begin{aligned} \min \quad & \alpha^\top y \\ \text{s.t.} \quad & \sum_{i \in N} y_i \geq d \\ & y_i \leq w_i && \forall i \in N \\ & y_i \in \{0\} \cup [w_i, \infty) && \forall i \in N, \end{aligned}$$

where $\alpha_i = \frac{f_i}{w_i}$ for each $i \in N$. Now, consider the problem

$$\begin{aligned} \min \quad & c^\top x + \alpha^\top y \\ \text{s.t.} \quad & \sum_{i \in N} y_i \geq d \\ & y_i \leq w_i + x_i && \forall i \in N \\ & x_i \geq 0 && \forall i \in N \\ & y_i \in \{0\} \cup [w_i, \infty) && \forall i \in N, \end{aligned}$$

where $c_i = M > 0$ for all $i \in N$. Let (x^*, y^*) be an optimal solution and let $N^* := \{i \in N : \ y_i^* > 0\}$. Given that $c > 0$, we must have $y_i^* = w_i + x_i^*$ for all $i \in N^*$. If $0 < \sum_{i \in N} x_i^* < 1$, then we have

$$d \le \sum_{i \in N} y_i^* = \sum_{i \in N^*} y_i^* = \sum_{i \in N^*} (w_i + x_i^*)$$

$$\Longrightarrow d \le \left\lfloor \sum_{i \in N} y_i^* \right\rfloor = \sum_{i \in N^*} w_i = \sum_{i \in N} \lfloor y_i^* \rfloor.$$

Thus, given that $\alpha > 0$, rounding down each component of y^* improves the solution. Hence, either $x^* = 0$ or $\sum_{i \in N} x_i^* \ge 1$. However, if M is sufficiently large, say $M = \sum_{i \in N} \alpha_i w_i = \sum_{i \in N} f_i$, then we must have $x^* = 0$. Therefore, the optimal values of this problem, which is an instance of linear optimization over $S(t, h)$, and the instance of the Binary Knapsack problem we started with are the same. Given that the transformation is polynomial in the original input size, the proof is complete. □

Despite the general complexity result in Proposition 1, there are at least two situations where $S(t, h)$ is tractable, namely when $t_i = 0$ for all $i \in N$ and when $h_i = 0$ for all $i \in N$. Note that the first case is a restriction. The second one is a relaxation as y becomes continuous. These cases will be discussed in Sections 3 and 4, respectively.

2.2 Basic polyhedral results

For a set C of real vectors, let *conv*(C) denote its convex hull.

In [6], the semi-continuous knapsack is introduced. This set is of the form

$$K = \left\{ x \in \mathbb{R}^n : \begin{array}{ll} \sum_{i \in N} w_i x_i \le r & \\ x_i \in [0, p_i] \cup [l_i, u_i] & \forall i \in N^+ \\ x_i \in [0, p_i] \cup [l_i, \infty) & \forall i \in N^+_\infty \cup N^- \end{array} \right\},$$

where N^+, N^+_∞, N^- constitute a partition of N, $w_i > 0$ for all $i \in N^+ \cup N^+_\infty$, and $w_i < 0$ for all $i \in N^-$. Several classes of valid inequalities are presented along with lifting procedures. Note that when $r < 0$ and $N^- = N$, this set is a relaxation of $S(t, h)$ as we can aggregate constraints and arrive at a system having the above form. Thus, valid inequalities for K give rise to valid inequalities for $S(t, h)$. In some cases, a complete description of *conv*(K) can be found. In particular, if $N = N^-$, $p_i = 0$ for each $i \in N$, and $r < 0$, then

$$conv(K) = \left\{ x \in \mathbb{R}^n : \begin{array}{ll} \sum_{i \in N} \frac{w_i}{\min\{r, w_i\}} x_i \ge 1 & \\ 0 \le x_i & \forall i \in N \end{array} \right\}.$$

As we shall see, an exponential family of inequalities similar to the one above will suffice to describe *conv*$(S(t, h))$ when $t = 0$ or $h = 0$. We first establish some fundamental results regarding $S(t, h)$.

Proposition 2. *$S(t, h)$ is full-dimensional.*

Proof. Consider the point $(\bar{x}, \bar{y}) \in \mathbb{R}^n \times \mathbb{R}^n$ given by $\bar{x}_i = \max\{d, l_i, h_i\} + 1$ and $\bar{y}_i = \max\{d, h_i\}$ for all $i \in N$. We have that $(\bar{x}, \bar{y})$ belongs to $S(t, h)$, and adding any canonical vector from $\mathbb{R}^n \times \mathbb{R}^n$ to $(\bar{x}, \bar{y})$ yields another point that is also feasible to $S(t, h)$. The collection of such $2n$ points along with $(\bar{x}, \bar{y})$ is an affinely independent set, and therefore $S(t, h)$ is of full dimension. □

Proposition 3. *$conv(S(t,h))$ is a polyhedron.*

Proof. Proposition 3 is a particular case of Proposition 23 given in the Appendix. □

We now proceed to identify the trivial facets of $conv(S(t,h))$.

Proposition 4. *For each each $i \in N$, $y_i \geq 0$ and $y_i \leq t_i + x_i$ are facet-defining for $conv(S(t,h))$. In addition, $x_i \geq 0$ is facet-defining if and only if $t_i > 0$.*

Proof. Let $i \in N$. Choose a point $\bar{x} \in \mathbb{R}^n$ satisfying $\bar{x}_j > \max\{d, l_j, h_j\}$ for all $j \in N$. Set $\bar{y}_i = 0$ and $\bar{y}_j = \bar{x}_j$ for all $j \in N$, $j \neq i$. We have that $(\bar{x}, \bar{y})$ belongs to $S(t,h)$. Now for each $j \in N$, $j \neq i$, consider the points $\left(x^j, y^j\right)$ and $\left(x^{n+j}, y^{n+j}\right)$ given by

$$\left(x_k^j, y_k^j\right) = \begin{cases} (\bar{x}_j + \epsilon, \bar{y}_j) & k = j \\ (\bar{x}_k, \bar{y}_k) & k \neq j, \end{cases}$$

$$\left(x_k^{n+j}, y_k^{n+j}\right) = \begin{cases} (\bar{x}_j, \bar{y}_j - \epsilon) & k = j \\ (\bar{x}_k, \bar{y}_k) & k \neq j. \end{cases}$$

Finally, let $\left(x^i, y^i\right) = (\bar{x}, \bar{y})$ and let $\left(x^{n+i}, y^{n+i}\right)$ be given by

$$\left(x_k^{n+i}, y_k^{n+i}\right) = \begin{cases} (\bar{x}_i + \epsilon, \bar{y}_i) & k = i \\ (\bar{x}_k, \bar{y}_k) & k \neq i. \end{cases}$$

For $\epsilon > 0$ sufficiently small, $\left\{\left(x^j, y^j\right), \left(x^{n+j}, y^{n+j}\right) : j \in N\right\}$ is contained in $S(t,h)$. Moreover, it is an affinely independent set, and since these $2n$ points satisfy $y_i \geq 0$ at equality, this constraint defines a facet of $conv(S(t,h))$. The proof for $y_i \leq t_i + x_i$ is analogous by setting $\bar{y}_i = t_i + \bar{x}_i$ and defining $\left(x^{n+i}, y^{n+i}\right)$ as

$$\left(x_k^{n+i}, y_k^{n+i}\right) = \begin{cases} (\bar{x}_i + \epsilon, \bar{y}_i + \epsilon) & k = i \\ (\bar{x}_k, \bar{y}_k) & k \neq i. \end{cases}$$

For the last part, if $t_i > 0$, set $\bar{x}_i = 0$ and $\bar{y}_i = t_i$. Again, the proof is similar by defining $\left(x^{n+i}, y^{n+i}\right)$ as

$$\left(x_k^{n+i}, y_k^{n+i}\right) = \begin{cases} (\bar{x}_i, \bar{y}_i - \epsilon) & k = i \\ (\bar{x}_k, \bar{y}_k) & k \neq i. \end{cases}$$

Finally, note that if $t_i = 0$, then $x_i \geq 0$ is dominated by $y_i \geq 0$. □

In the following two sections we turn our attention to polyhedral results for $S(0,h)$ and $S(t,0)$, respectively.

3 The case $t = 0$

In this section we assume that $t = 0$, and therefore $S(0,h) \subseteq \mathbb{R}^n \times \mathbb{R}^n$ is the set of vectors (x,y) satisfying

$$\sum_{i \in N} y_i \geq d \tag{10}$$

$$y_i \leq x_i \quad \forall i \in N \tag{11}$$

$$x_i \in \{0\} \cup [l_i, \infty) \quad \forall i \in N \tag{12}$$

$$y_i \in \{0\} \cup [h_i, \infty) \quad \forall i \in N. \tag{13}$$

3.1 Inequality description of $conv(S(0,h))$

Define the sets

$$L := \{i \in N : \max\{d, h_i\} < l_i\},$$
$$H := \{i \in N : h_i \geq d\},$$

and consider the family of inequalities given by

$$\sum_{i \in T} \frac{x_i}{l_i} + \sum_{i \in N \setminus T} \frac{y_i}{\max\{d, h_i\}} \geq 1 \ \forall T \subseteq L. \tag{14}$$

Recalling that $d > 0$, we have that $l_i = 0$ implies $i \in N \setminus L$. Thus, (14) is well-defined for all $T \subseteq L$.

Proposition 5. *For each $T \subseteq L$, (14) is valid and facet-defining for $conv(S(0,h))$.*

Proof. To show validity, consider $(x,y) \in S(0,h)$ and $T \subseteq L$. If for some $i \in T$ we have $x_i > 0$, then $\frac{x_i}{l_i} \geq 1$. If for some $i \in (N \setminus T) \cap H$ we have $y_i > 0$, then $\frac{y_i}{\max\{d,h_i\}} = \frac{y_i}{h_i} \geq 1$. In both cases (14) is satisfied. If none of them occur, then $y_i = 0$ for all $i \in T \cup [(N \setminus T) \cap H]$. Since $(x,y) \in S(0,h)$, we must have $\sum_{i \in N} y_i \geq d$, and therefore

$$\sum_{i \in (N \setminus T) \setminus H} \frac{y_i}{\max\{d, h_i\}} = \sum_{i \in (N \setminus T) \setminus H} \frac{y_i}{d} = \sum_{i \in N} \frac{y_i}{d} \geq 1.$$

Hence, (14) is satisfied in this case as well.

Now, given $T \subseteq L$, we will show that (14) is facet-defining by showing $2n$ affinely independent points in $S(0,h)$ that satisfy (14) at equality. Let $\left(x^i, y^i\right)$, $i = 1, \ldots, 2n$, be such points. In particular, $\left(x^i, y^i\right)$ and $\left(x^{n+i}, y^{n+i}\right)$ will be associated to $i \in N$ as follows:

If $i \in T$, then

$$\left(x^i_j, y^i_j\right) = \begin{cases} (l_i, \max\{d, h_i\}) & j = i \\ (0,0) & j \neq i, \end{cases}$$

$$\left(x^{n+i}_j, y^{n+i}_j\right) = \begin{cases} (l_i, \max\{d, h_i\} + \epsilon) & j = i \\ (0,0) & j \neq i. \end{cases}$$

If $i \in N \setminus T$, then

$$\left(x^i_j, y^i_j\right) = \begin{cases} (\max\{d, h_i, l_i\}, \max\{d, h_i\}) & j = i \\ (0,0) & j \neq i, \end{cases}$$
$$\left(x^{n+i}_j, y^{n+i}_j\right) = \begin{cases} (\max\{d, h_i, l_i\} + \epsilon, \max\{d, h_i\}) & j = i \\ (0,0) & j \neq i. \end{cases}$$

The points previously defined belong to $S(0,h)$ for $\epsilon > 0$ sufficiently small. Finally, $\left\{\left(x^i, y^i\right), \left(x^{n+i}, y^{n+i}\right) : i \in N\right\}$ is a linearly independent set of points satisfying (14) at equality. Thus this constraint defines a facet of $conv(S(0,h))$. $\square$

Theorem 6 below shows that all the non-trivial facets of $conv(S(0,h))$ are given by (14).

Theorem 6. *conv*$(S(0,h))$ *is given by the following facet-defining inequalities*

$$\sum_{i\in T}\frac{x_i}{l_i}+\sum_{i\in N\setminus T}\frac{y_i}{\max\{d,h_i\}}\geq 1 \quad \forall T\subseteq L$$

$$y_i \leq x_i \qquad \forall i\in N \tag{15}$$

$$y_i \geq 0 \qquad \forall i\in N. \tag{16}$$

Proof. We already showed that (14) is facet-defining for each $T\subseteq L$, and that (16) and (15) are also facet-defining for each $i\in N$. To show that (14)-(15) completely describe $conv(S(0,h))$, it suffices to show that if we optimize any non-zero linear function over $S(0,h)$, then there exists one inequality from (14)-(15) such that all optimal solutions, if one exists, belong to the facet defined by that inequality.

Let $(c,\alpha)\in\mathbb{R}^n\times\mathbb{R}^n$ be a non-zero vector and consider the problem

$$\min\left\{c^\top x+\alpha^\top y:\ (x,y)\in S(0,h)\right\}.$$

Assumption 1: $c\geq 0$ and $c+\alpha\geq 0$.

If for some $i\in N$ we have $c_i<0$ or $c_i+\alpha_i<0$, then the problem is unbounded. Thus, we may assume $c\geq 0,\ c+\alpha\geq 0$. $\Diamond$

In particular, Assumption 1 implies that the optimal value is nonnegative and that an optimal solution exists. Let (x^*,y^*) be any such solution.

Assumption 2: $\alpha\geq 0$.

If for some $i\in N$, $\alpha_i<0$, then $y_i^*=x_i^*$, that is, (15) is satisfied as an equality. To see this, suppose that $y_i^*<x_i^*$. If $y_i^*>0$, then we can increase it and get a better solution. If $y_i^*=0$, since $x_i^*>0$ and $c_i>0$ by Assumption 1 and $\alpha_i<0$, we can decrease x_i^* to zero and get a better solution. Thus, we may assume $\alpha\geq 0$. $\Diamond$

Assumption 3: $c+\alpha>0$.

Suppose that $c_i=\alpha_i=0$ for some $i\in N$. Then the optimal value is zero. Since $(c,\alpha)\neq(0,0)$, by Assumptions 1 and 2, there must exist $j\in N$, $j\neq i$, such that either $\alpha_j>0$ or $c_j>0$. By optimality, in the former case we must have $y_j^*=0$, while in the latter $x_j^*=0$ must hold. Therefore, either (16) or (15) must be satisfied at equality. Thus, we may assume $c+\alpha>0$. $\Diamond$

Claim 1: $y_i^*>0\Rightarrow c_ix_i^*+\alpha_iy_i^*>0$.

If $y_i^*>0$, then $x_i^*>0$, and by Assumption 3, $c_ix_i^*+\alpha_iy_i^*>0$ holds. $\Diamond$

Let $T=\{i\in L:\alpha_i=0\}$. Then $c_i>0$ for all $i\in T$ by Assumption 3, and $\alpha_i>0$ for all $i\in L\setminus T$. We claim that

$$\sum_{i\in T}\frac{x_i^*}{l_i}+\sum_{i\in N\setminus T}\frac{y_i^*}{\max\{d,h_i\}}=1.$$

We prove the claim by contradiction. Let $T^+=\{i\in T: x_i^*>0\}$ and $(N\setminus T)^+=\{i\in N\setminus T: y_i^*>0\}$. Then

$$\sum_{i\in T^+}\frac{x_i^*}{l_i}+\sum_{i\in(N\setminus T)^+}\frac{y_i^*}{\max\{d,h_i\}}>1. \tag{17}$$

Claim 2: $T^+ = \emptyset$.

Suppose $i \in T^+$, that is, $x_i^* \geq l_i > \max\{d, h_i\}$. Since $\alpha_i = 0$ and $\alpha_j > 0$ for all $j \in L \setminus T$, by optimality we must have $y_j^* = 0$ for all $j \in L \setminus T$. In addition, by Claim 1, we must have $y_j^* = 0$ for all $j \in N \setminus L$ as well. Thus $(N \setminus T)^+ = \emptyset$. Moreover, given that $c_j > 0$ for any $j \in T^+$, we must have $T^+ = \{i\}$. Then (17) takes the form $x_i^* > l_i$, a contradiction with optimality as $c_i > 0$. $\Diamond$

By Claim 2, we arrive at

$$\sum_{i \in (N \setminus T)^+} \frac{y_i^*}{\max\{d, h_i\}} > 1. \tag{18}$$

Claim 3: $(N \setminus T)^+ \cap H = \emptyset$.

Let $i \in (N \setminus T)^+$ be such that $h_i \geq d$. By Claim 1 and optimality, $(N \setminus T)^+ = \{i\}$. Then (18) implies $y_i^* > h_i \geq d$. If $i \in L \setminus T$, then $\alpha_i > 0$ and by optimality we have a contradiction. If $i \in N \setminus L$, then $l_i \leq \max\{h_i, d\} = h_i$. Since $c_i + \alpha_i > 0$, by optimality we must have $y_i^* = h_i$, a contradiction as well. $\Diamond$

By Claim 3, we arrive at

$$\sum_{i \in (N \setminus T)^+} y_i^* > d. \tag{19}$$

Claim 4: $|(N \setminus T)^+| \geq 2$.

Since $(N \setminus T)^+$ cannot be empty, suppose $(N \setminus T)^+ = \{i\}$. Then (19) and Claim 3 imply $y_i^* > d > h_i$. Again, if $i \in L \setminus T$, then $\alpha_i > 0$ and we have a contradiction. If $i \in N \setminus L$, then $l_i \leq \max\{h_i, d\} = d$. Since $c_i + \alpha_i > 0$, by optimality we must have $y_i^* = d$, a contradiction as well. $\Diamond$

Let

$$i_0 \in \arg\min \left\{ c_i + \alpha_i : \ i \in (N \setminus T)^+ \right\},$$

and let $i_1 \in (N \setminus T)^+$, $i_1 \neq i_0$, which exists by Claim 4. Recall that from Assumption 3, $c_{i_0} + \alpha_{i_0} > 0$. For $\epsilon > 0$ sufficiently small, define $(\bar{x}, \bar{y})$ as

$$(\bar{x}_i, \bar{y}_i) = \begin{cases} \left(x_{i_0}^* + y_{i_1}^* - \epsilon, y_{i_0}^* + y_{i_1}^* - \epsilon\right) & i = i_0 \\ (0, 0) & i = i_1 \\ (x_i^*, y_i^*) & i \neq i_0, \ i \neq i_1. \end{cases}$$

Certainly $\bar{x}_i \geq l_i$ whenever $\bar{x}_i > 0$, $\bar{y}_i \geq h_i$ whenever $y_i > 0$, and $\bar{y}_i \leq \bar{x}_i$ for all $i \in N$. Thus, given that $\sum_{i \in N} y_i^* > d$, we conclude that $(\bar{x}, \bar{y})$ is a feasible solution. Moreover,

$$\begin{aligned} \sum_{i \in N} c_i(x_i^* - \bar{x}_i) + \alpha_i(y_i^* - \bar{y}_i) &= -c_{i_0}(y_{i_1}^* - \epsilon) - \alpha_{i_0}(y_{i_1}^* - \epsilon) + c_{i_1} x_{i_1}^* + \alpha_{i_1} y_{i_1}^* \\ &= -\left(c_{i_0} + \alpha_{i_0}\right)(y_{i_1}^* - \epsilon) + c_{i_1} x_{i_1}^* + \alpha_{i_1} y_{i_1}^* \\ &> -\left(c_{i_0} + \alpha_{i_0}\right) y_{i_1}^* + c_{i_1} y_{i_1}^* + \alpha_{i_1} y_{i_1}^* \\ &\geq 0, \end{aligned}$$

where the two inequalities follow from $c_{i_0} + \alpha_{i_0} > 0$, $y_{i_1}^* > 0$, and $x_{i_1}^* \geq y_{i_1}^*$, and from the definition of i_0, respectively.

Hence, $(\bar{x}, \bar{y})$ improves upon (x^*, y^*) and we get the required contradiction. $\square$

3.2 Extreme points of $conv(S(0,h))$

Since by Theorem 6 an outer description of $conv(S(0,h))$ in terms of linear inequalities is available, we look for an inner description in terms of extreme points.

Proposition 7. *Let (x,y) be an extreme point of $conv(S(0,h))$. Then both x and y have exactly one non-zero entry.*

Proof. We claim that if $x_i > 0$, then $y_i > 0$. By contradiction, suppose $x_i > 0$ and $y_i = 0$. We can set

$$(x_i, y_i) = \frac{1}{2}\left[(2x_i, 0) + (0,0)\right].$$

Thus, (x,y) can be written as the average of two distinct points in $S(0,h)$.

Now, suppose that x has more than one non-zero entry, say $x_i > 0$ and $x_j > 0$. By the claim, $y_i > 0$ and $y_j > 0$. We can set

$$h_i \leq y_i = \lambda x_i,\ 0 < \lambda \leq 1$$
$$h_j \leq y_j = \mu x_i,\ 0 < \mu \leq 1.$$

Finally, we can write

$$\begin{aligned}(x_i, x_j, y_i, y_j) &= (x_i, x_j, \lambda x_i, \mu x_j)\\ &= \frac{\lambda x_i}{\lambda x_i + \mu x_j}(x_i + \frac{\mu}{\lambda} x_j, 0, \lambda x_i + \mu x_j, 0)\\ &\quad + \frac{\mu x_j}{\lambda x_i + \mu x_j}(0, x_j + \frac{\lambda}{\mu} x_i, 0, \lambda x_i + \mu x_j).\end{aligned}$$

Hence, (x,y) can be written as a strict convex combination of two distinct points in $S(0,h)$. $\square$

Combining Theorem 6 and Proposition 7, we have the following result.

Proposition 8. *If (x,y) is an extreme point of $conv(S(0,h))$, then the non-zero entries of (x,y) are one of the following:*

- $i \in N \setminus L \Rightarrow \quad x_i = \max\{d, h_i\}, \quad y_i = \max\{d, h_i\},$
- $i \in L \Rightarrow \begin{cases} x_i = l_i, & y_i = l_i \\ x_i = l_i, & y_i = \max\{d, h_i\}. \end{cases}$

Proof. Let (x,y) be an extreme point of $conv(S(0,h))$. From Proposition 7, (x,y) has exactly one pair of non-zero entries, say (x_i, y_i). From Theorem 6, (x_i, y_i) has to satisfy either $y_i \geq \max\{d, h_i\}$ if $i \in N \setminus L$, or both $x_i \geq l_i$ and $y_i \geq \max\{d, h_i\}$ if $i \in L$. From these inequalities together with $y_i \leq x_i$, at least two have to be satisfied at equality since $x_i > 0$, $y_i > 0$, and $y_j = x_j = 0$ for all $j \in N$, $j \neq i$. The possible solutions are exactly the combinations indicated above. $\square$

From Proposition 8, optimization over $S(0,h)$ can be done by enumeration in $\mathcal{O}(n)$ time.

3.3 Extended formulation for $conv(S(0,h))$

Now, let us consider the separation problem associated to (14). Given (x^*, y^*), let

$$T^* = \left\{ i \in L : \frac{x_i^*}{l_i} \leq \frac{y_i^*}{\max\{d, h_i\}} \right\}.$$

If (14) is satisfied for T^*, then it is satisfied for any $T \subseteq L$, and if in addition (16) and (15) hold, then (x^*, y^*) belongs to $conv(S(0,h))$. Otherwise, T^* gives the most violated inequality from (14), and therefore it can be used to separate (x^*, y^*) from $conv(S(0,h))$. Clearly, computing T^* and its corresponding inequality can be done in $\mathcal{O}(n)$ time.

Further note that (x, y) satisfies (14) for all $T \subseteq L$ if and only if

$$\sum_{i \in N \setminus L} \frac{y_i}{\max\{d, h_i\}} + \sum_{i \in L} \min\left(\frac{x_i}{l_i}, \frac{y_i}{\max\{d, h_i\}}\right) \geq 1.$$

If fact, this is the separation routine for (14) given a point (x, y). Now, the above condition holds if and only if there exists $\pi \in \mathbb{R}^{|L|}$ such that

$$\frac{x_i}{l_i} \geq \pi_i \ \forall i \in L$$
$$\frac{y_i}{\max\{d, h_i\}} \geq \pi_i \ \forall i \in L$$
$$\sum_{i \in N \setminus L} \frac{y_i}{\max\{d, h_i\}} + \sum_{i \in L} \pi_i \geq 1.$$

Thus, introducing variables π, we obtain an extended formulation W of $conv(S(0,h))$ in a space of higher dimension given by

$$W = \left\{ (x, y, \pi) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{|L|} : \begin{array}{cc} \displaystyle\sum_{i \in N \setminus L} \frac{y_i}{\max\{d, h_i\}} + \sum_{i \in L} \pi_i \geq 1 & \\ \displaystyle\frac{x_i}{l_i} \geq \pi_i & \forall i \in L \\ \displaystyle\frac{y_i}{\max\{d, h_i\}} \geq \pi_i & \forall i \in L \\ y_i \geq 0 & \forall i \in N \\ x_i - y_i \geq 0 & \forall i \in N \end{array} \right\}.$$

Let $proj_{x,y}(W)$ denote the projection of W onto the (x, y)-space.

Corollary 9. *$conv(S(0,h)) = proj_{x,y}(W)$.*

This extended formulation is compact in the sense that we have, at most, doubled the number of variables and constraints.

4 The case $h = 0$

In this section we assume that $h = 0$ and then $S(t,0) \subseteq \mathbb{R}^n \times \mathbb{R}^n$ takes the form

$$\sum_{i\in N} y_i \geq d \tag{20}$$

$$y_i \leq t_i + x_i \quad \forall i \in N \tag{21}$$

$$x_i \in \{0\} \cup [l_i, \infty) \quad \forall i \in N \tag{22}$$

$$y_i \geq 0 \quad \forall i \in N. \tag{23}$$

4.1 Inequality description of $conv(S(t,0))$

Proposition 10. *$\sum_{i\in N} y_i \geq d$ is facet-defining for $conv(S(t,0))$.*

Proof. Choose a point $\bar{x} \in \mathbb{R}^n$ satisfying $\bar{x}_i > \max\{d, l_i\}$ for all $i \in N$ and set $\bar{y}_i = \frac{d}{n}$ for all $i \in N$. We have that $(\bar{x}, \bar{y})$ belongs to $S(t,0)$ and satisfies $\sum_{i\in N} \bar{y}_i = d$. Now for each $j \in N$, $j < n$, consider the points $\left(x^j, y^j\right)$ and $\left(x^{n+j}, y^{n+j}\right)$ given by

$$\left(x_i^j, y_i^j\right) = \begin{cases} (\bar{x}_j + \epsilon, \bar{y}_j) & i = j \\ (\bar{x}_i, \bar{y}_i) & i \neq j, \end{cases}$$

$$\left(x_i^{n+j}, y_i^{n+j}\right) = \begin{cases} (\bar{x}_j, \bar{y}_j - \epsilon) & i = j \\ (\bar{x}_n, \bar{y}_n + \epsilon) & i = n \\ (\bar{x}_i, \bar{y}_i) & i \neq j,\ i \neq n. \end{cases}$$

Finally, let $\left(x^{2n}, y^{2n}\right) = (\bar{x}, \bar{y})$ and let (x^n, y^n) be given by

$$(x_i^n, y_i^n) = \begin{cases} (\bar{x}_n + \epsilon, \bar{y}_n) & i = n \\ (\bar{x}_i, \bar{y}_i) & i \neq n. \end{cases}$$

For $\epsilon > 0$ sufficiently small, $\left\{\left(x^j, y^j\right), \left(x^{n+j}, y^{n+j}\right) : j \in N\right\}$ is contained in $S(t,0)$. Moreover, it is an affinely independent set, and since these $2n$ points satisfy $\sum_{i\in N} y_i \geq d$ at equality, this constraint defines a facet of $conv(S(t,0))$. □

Definition 11. *A subset $R \subseteq N$ is a reverse cover if $d_R := d - \sum_{i\in R} t_i > 0$.*

Let $\mathcal{R} \subseteq 2^N$ be the set of all reverse covers. For a reverse cover $R \in \mathcal{R}$, consider the inequality

$$\sum_{i\in R} \frac{x_i}{\max\{l_i, d_R\}} + \sum_{i\in N\setminus R} \frac{y_i}{d_R} \geq 1. \tag{24}$$

Also, let $L_R := \{i \in R :\ l_i > d_R\}$. Note that if $R = \emptyset$, we recover (20).

Proposition 12. *For each reverse cover $R \in \mathcal{R}$, (24) is valid for $conv(S(t,0))$.*

Proof. Let $(x,y) \in S(t,0)$. If there exists $i \in L_R$ with $x_i > 0$, then (24) is satisfied. Otherwise, $x_i = 0$ for all $i \in L_R$. Then

$$d \leq \sum_{i\in N} y_i = \sum_{i\in L_R} y_i + \sum_{i\in R\setminus L_R} y_i + \sum_{i\in N\setminus R} y_i \leq \sum_{i\in L_R} t_i + \sum_{i\in R\setminus L_R} (t_i + x_i) + \sum_{i\in N\setminus R} y_i$$

$$\implies d_R = d - \sum_{i\in R} t_i \leq \sum_{i\in R\setminus L_R} x_i + \sum_{i\in N\setminus R} y_i.$$

Since $\max\{l_i, d_R\} = d_R > 0$ for each $i \in R \setminus L_R$, (24) is satisfied. $\square$

Definition 13. *A reverse cover $R \in \mathcal{R}$ is proper if*

1. $L_R \neq \emptyset$.
2. $t_i > 0$ *for all* $i \in R \setminus L_R$.

Proposition 14. *For each reverse cover $R \in \mathcal{R}$, (24) is facet-defining if and only if R is empty or if R is proper.*

Proof. The case $R = \emptyset$ follows from Proposition 10. Thus, let R be a proper reverse cover and let $i \in L_R$. For each $j \in N$, consider the points $\left(x^j, y^j\right)$ and $\left(x^{n+j}, y^{n+j}\right)$ defined as follows.

If $j \in R$,

$$\left(x_k^j, y_k^j\right) = \begin{cases} (\max\{l_j, d_R\}, t_j + d_R) & k = j \\ (0, t_k) & k \in R,\ k \neq j \\ (0,0) & k \in N \setminus R. \end{cases}$$

Then

$$\sum_{k\in N} y_k^j = \sum_{k\in R} t_k + d_R = d$$

and

$$\sum_{k\in R} \frac{x_k^j}{\max\{l_k, d_R\}} + \sum_{k\in N\setminus R} \frac{y_k^j}{d_R} = \frac{\max\{l_j, d_R\}}{\max\{l_j, d_R\}} = 1.$$

If $j \in L_R$,

$$\left(x_k^{n+j}, y_k^{n+j}\right) = \begin{cases} (l_j, t_j + d_R + \epsilon) & k = j \\ (0, t_k) & k \in R,\ k \neq j \\ (0,0) & k \in N \setminus R. \end{cases}$$

Then

$$\sum_{k\in N} y_k^{n+j} = \sum_{k\in R} t_k + d_R + \epsilon \geq d$$

and

$$\sum_{k\in R} \frac{x_k^{n+j}}{\max\{l_k, d_R\}} + \sum_{k\in N\setminus R} \frac{y_k^{n+j}}{d_R} = \frac{l_j}{\max\{l_j, d_R\}} = 1.$$

If $j \in R \setminus L_R$,

$$\left(x_k^{n+j}, y_k^{n+j}\right) = \begin{cases} (l_i, t_i + d_R + \epsilon) & k = i \\ (0, t_j - \epsilon) & k = j \\ (0, t_k) & k \in R,\ k \neq i,\ k \neq j \\ (0,0) & k \in N \setminus R. \end{cases}$$

Then

$$\sum_{k\in N} y_k^{n+j} = \sum_{k\in R} t_k - \epsilon + d_R + \epsilon = d$$

and

$$\sum_{k\in R} \frac{x_k^{n+j}}{\max\{l_k, d_R\}} + \sum_{k\in N\setminus R} \frac{y_k^{n+j}}{d_R} = \frac{l_i}{\max\{l_i, d_R\}} = 1.$$

If $j \in N \setminus R$,

$$\left(x_k^j, y_k^j\right) = \begin{cases} (\max\{l_j, d_R\}, d_R) & k = j \\ (0, t_k) & k \in R \\ (0,0) & k \in N \setminus R,\ k \neq j, \end{cases}$$

$$\left(x_k^{n+j}, y_k^{n+j}\right) = \begin{cases} (\max\{l_j, d_R\} + \epsilon, d_R) & k = j \\ (0, t_k) & k \in R \\ (0,0) & k \in N \setminus R,\ k \neq j. \end{cases}$$

Then

$$\sum_{k\in N} y_k^j = \sum_{k\in N} y_k^{n+j} = \sum_{k\in R} t_k + d_R = d$$

and

$$\sum_{k\in R} \frac{x_k^j}{\max\{l_k, d_R\}} + \sum_{k\in N\setminus R} \frac{y_k^j}{d_R} = \sum_{k\in R} \frac{x_k^{n+j}}{\max\{l_k, d_R\}} + \sum_{k\in N\setminus R} \frac{y_k^{n+j}}{d_R} = \frac{d_R}{d_R} = 1.$$

Given that $d_R < l_j$ for all $j \in L_R$ and $0 < t_j$ for all $j \in R \setminus L_R$, for $\epsilon > 0$ sufficiently small, we have that $\left\{\left(x^j, y^j\right),\ \left(x^{n+j}, y^{n+j}\right) : j \in N\right\}$ is contained in $S(t, 0)$. Moreover, it is an affinely independent set, and since these $2n$ points satisfy (24) at equality, this constraint defines a facet of $conv(S(t, 0))$.

For the converse, let R be a nonempty cover that is not proper, thus either $L_R = \emptyset$ or there exists $i \in R \setminus L_R$ having $t_i = 0$. In the former case, $\max\{l_i, d_R\} = d_R$ for all $i \in R$, and then (24) is generated as the sum of (20) and (21) for $i \in R$. In the latter, since $t_i = 0$, we have $d_{R\setminus\{i\}} = d_R$ and $y_i \le x_i$. Since $i \in R \setminus L_R$, we also have $\max\{l_i, d_R\} = d_R$. Thus

$$\begin{aligned} \sum_{j\in R} \frac{x_j}{\max\{l_j, d_R\}} + \sum_{j\in N\setminus R} \frac{y_j}{d_R} &= \sum_{j\in R\setminus\{i\}} \frac{x_j}{\max\{l_j, d_R\}} + \frac{x_i}{\max\{l_i, d_R\}} + \sum_{j\in N\setminus R} \frac{y_j}{d_R} \\ &\geq \sum_{j\in R\setminus\{i\}} \frac{x_j}{\max\{l_j, d_R\}} + \frac{y_i}{d_R} + \sum_{j\in N\setminus R} \frac{y_j}{d_R} \\ &= \sum_{j\in R\setminus\{i\}} \frac{x_j}{\max\{l_j, d_R\}} + \sum_{j\in N\setminus(R\setminus\{i\})} \frac{y_j}{d_R}. \end{aligned}$$

Hence, the inequality given by R is implied by the one given by $R \setminus \{i\}$, and therefore it cannot be facet-defining. □

We now present the main result of this section.

Theorem 15. *$conv(S(t,0))$ is given by the following inequalities*

$$\sum_{i\in R}\frac{x_i}{\max\{l_i,d_R\}}+\sum_{i\in N\setminus R}\frac{y_i}{d_R}\geq 1 \quad \forall R\in\mathcal{R}$$

$$y_i \leq x_i+t_i \quad \forall i\in N \tag{25}$$

$$x_i\geq 0 \quad \forall i\in N \tag{26}$$

$$y_i\geq 0 \quad \forall i\in N. \tag{27}$$

Proof. Let $(c,\alpha)\in\mathbb{R}^n\times\mathbb{R}^n$ be a non-zero vector and consider the problem

$$\min\left\{c^\top x+\alpha^\top y:\ (x,y)\in S(t,0)\right\}.$$

As in the proof of Theorem 6, we will show that if this problem has finite optimal value, then there exists one inequality from (24)-(27) that contains all optimal solutions.

Assumption 1: $c\geq 0$ and $c+\alpha\geq 0$.

If for some $i\in N$ we have $c_i<0$ or $c_i+\alpha_i<0$, then the problem is unbounded. Thus, we may assume $c\geq 0$ and $c+\alpha\geq 0$. $\Diamond$

In particular, Assumption 1 implies that the objective value is bounded and there exists an optimal solution. Let (x^*,y^*) be any such solution.

Assumption 2: $\alpha\geq 0$.

If for some $i\in N$ we have $\alpha_i<0$, then $y_i^*=t_i+x_i^*$ by optimality, that is, (25) is satisfied at equality. Thus, we may assume $\alpha\geq 0$. $\Diamond$

From Assumptions 1 and 2, we have that the optimal value is nonnegative.

Assumption 3: $c^\top x^*+\alpha^\top y^*>0$.

Suppose that the optimal value is zero. Since $(c,\alpha)\neq(0,0)$, by Assumptions 1 and 2, there must exist $i\in N$ such that either $\alpha_i>0$ or $c_i>0$. By optimality, in the former case we must have $y_i^*=0$, while in the latter $x_i^*=0$ must hold. Therefore, either (27) or (26) must be satisfied at equality. Thus, we may assume $c^\top x^*+\alpha^\top y^*>0$. $\Diamond$

Claim 1: $c+\alpha>0$.

If $c_i=\alpha_i=0$ for some $i\in N$, then the optimal value is zero, contradicting Assumption 3. $\Diamond$

Let $R:=\{i\in N:\ \alpha_i=0\}$. From Assumption 3 and the definition of R, we have $\sum_{i\in R}t_i<d$, since otherwise the optimal value is zero. Hence, R is a reverse cover. We also have $c_i>0$ for all $i\in R$ by Claim 1, and $\alpha_i>0$ for all $i\in N\setminus R$.

We claim that

$$\sum_{i\in R}\frac{x_i^*}{\max\{l_i,d_R\}}+\sum_{i\in N\setminus R}\frac{y_i^*}{d_R}=1.$$

Suppose not. Let $L_R^+:=\{i\in L_R: x_i^*>0\}$, $(R\setminus L_R)^+:=\{i\in R\setminus L_R: x_i^*>0\}$, and $(N\setminus R)^+:=\{i\in N\setminus R: y_i^*>0\}$. Then

$$\sum_{i\in L_R^+}\frac{x_i^*}{l_i}+\sum_{i\in(R\setminus L_R)^+}\frac{x_i^*}{d_R}+\sum_{i\in(N\setminus R)^+}\frac{y_i^*}{d_R}>1. \tag{28}$$

Claim 2: $L_R^+ = \emptyset$.

Suppose $i \in L_R^+$, that is, $i \in R$ and $x_i^* \geq l_i > d_R$. Note that since $\alpha_j = 0$ for all $j \in R$, we can set $y_j^* = t_j$ for each $j \in R$, $j \neq i$, and $y_i^* = t_i + d_R$ without affecting the feasibility and objective value of the solution. Recalling that $c_i > 0$ for all $i \in R$ and $\alpha_i > 0$ for all $\in N \setminus R$, from (28) and optimality we have $(R \setminus L_R)^+ = (N \setminus R)^+ = \emptyset$ and $L_R^+ = \{i\}$. Then (28) implies $x_i^* > l_i > d_R$, contradicting optimality since setting $x_i^* = l_i$ improves the objective value. $\Diamond$

Now, we have

$$\sum_{i \in (R \setminus L_R)^+} x_i^* + \sum_{i \in (N \setminus R)^+} y_i^* > d_R. \tag{29}$$

Claim 3: $(N \setminus R)^+ = \emptyset$.

From (29) and Claim 2, we have

$$d < \sum_{i \in (R \setminus L_R)^+} x_i^* + \sum_{i \in R} t_i + \sum_{i \in (N \setminus R)^+} y_i^* = \sum_{i \in R} (x_i^* + t_i) + \sum_{i \in (N \setminus R)^+} y_i^*.$$

If $(N \setminus R)^+$ is nonempty, we can set $y_i^* = t_i + x_i^*$ for each $i \in R$ without changing the objective value, and then decrease y_i^* for some $i \in (N \setminus R)^+$, contradicting optimality as $\alpha_i > 0$ for all $i \in (N \setminus R)^+$. $\Diamond$

We arrive at

$$\sum_{i \in (R \setminus L_R)^+} x_i^* > d_R.$$

Then we can improve upon (x^*, y^*) by taking $i \in \arg\min\{c_j : j \in (R \setminus L_R)^+\}$ and defining $(\bar{x}, \bar{y})$ by

$$(\bar{x}_j, \bar{y}_j) = \begin{cases} (d_R, t_j + d_R) & j = i \\ (0, t_j) & j \in R,\ j \neq i \\ (0, 0) & j \in N \setminus R. \end{cases}$$

□

4.2 Extended formulation for $conv(S(t, 0))$

At first sight, it is not clear how to separate the inequalities given by (24). We will show that this can be done using an extended formulation. We first state a result similar to Proposition 7.

Proposition 16. *If (x, y) is an extreme point of $conv(S(t, 0))$, then x has at most one non-zero entry.*

Proof. We claim that if $x_i > 0$, then $y_i > t_i$. By contradiction, suppose $x_i > 0$ and $y_i \leq t_i$. We can set

$$(x_i, y_i) = \frac{1}{2}\left[(2x_i, y_i) + (0, y_i)\right].$$

Thus, (x, y) can be written as the average of two distinct points in $S(t, 0)$.

Now, suppose that x has more than one non-zero entry, say $x_i > 0$ and $x_j > 0$. By the claim, $y_i > t_i$ and $y_j > t_j$. Thus, there exist $\lambda, \mu \in (0, 1]$ such that $y_i = t_i + \lambda x_i$ and $y_j = t_j + \mu x_j$. Then we can write

$$\begin{aligned}(x_i, x_j, y_i, y_j) &= (x_i, x_j, t_i + \lambda x_i, t_j + \mu x_j) \\ &= \frac{\lambda x_i}{\lambda x_i + \mu x_j}(x_i + \frac{\mu}{\lambda} x_j, 0, t_i + \lambda x_i + \mu x_j, t_j) \\ &\quad + \frac{\mu x_j}{\lambda x_i + \mu x_j}(0, x_j + \frac{\lambda}{\mu} x_i, t_i, t_j + \lambda x_i + \mu x_j).\end{aligned}$$

Also, notice that

$$t_i + \lambda x_i + \mu x_j = t_i + \lambda\left(x_i + \frac{\mu}{\lambda} x_j\right) \le t_i + \left(x_i + \frac{\mu}{\lambda} x_j\right),$$
$$t_j + \lambda x_i + \mu x_j = t_j + \mu\left(\frac{\lambda}{\mu} x_i + x_j\right) \le t_j + \left(\frac{\lambda}{\mu} x_i + x_j\right).$$

Hence, (x, y) can be written as a strict convex combination of two distinct points in $S(t, 0)$. $\square$

Now consider the polyhedra

$$S_0 := \{(x, y) \in S(t, 0) : \; x_j = 0 \; \forall j \in N\} = \left\{ (x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ : \begin{array}{ll} \sum_{j \in N} y_j \ge d & \\ -y_j \ge -t_j & \forall j \in N \\ -x_j \ge 0 & \forall j \in N \end{array} \right\},$$

$$S_i := \{(x, y) \in S(t, 0) : \; x_i \ge l_i, \; x_j = 0 \, \forall j \ne i\} = \left\{ (x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ : \begin{array}{ll} \sum_{j \in N} y_j \ge d & \\ x_i - y_i \ge -t_i & \\ -y_j \ge -t_j & \forall j \ne i \\ x_i \ge l_i & \\ -x_j \ge 0 & \forall j \ne i \end{array} \right\}, \; i \in N.$$

Note that S_i is nonempty for each $i \in N$, while S_0 is nonempty if and only if $\sum_{j \in N} t_j \ge d$. Set $\bar{N} = \{0\} \cup N$.

For a set C, let $\overline{conv}(C)$ denote the closure of its convex hull. If C is convex, let $ext(C)$ and $rec(C)$ denote the set of extreme points and the recession cone of C, respectively.

Proposition 17. *$conv(S(t, 0)) = \overline{conv}(\cup_{i \in \bar{N}} S_i)$.*

Proof. The reverse inclusion is easy as $S_i \subseteq S(t, 0)$ for all $i \in \bar{N}$ and $conv(S(t, 0))$ is closed by Proposition 3.

For the forward inclusion, let $(x, y) \in ext(conv(S(t, 0)))$. From Proposition 16, (x, y) belongs to some S_i, $i \in \bar{N}$, thus $ext(conv(S(t, 0))) \subseteq conv(\cup_{i \in \bar{N}} S_i)$. It remains to show that $rec(conv(S(t, 0))) \subseteq rec(\overline{conv}(\cup_{i \in \bar{N}} S_i))$. Let $(x, y) \in rec(conv(S(t, 0)))$. From Theorem 15, we can conclude that $x \ge 0$, $y \ge 0$, and $x \ge y$. Write $(x, y) = \sum_{i \in N} (x_i e^i, y_i e^i)$, where e^i is the i-th canonical vector in $\mathbb{R}^n$. On the other hand, from a result in disjuctive programming [1], we have $rec(\overline{conv}(\cup_{i \in \bar{N}} S_i)) = conv(\cup_{i \in \bar{N}} rec(S_i))$. Since $rec(S_i)$ is a convex cone for each $i \in \bar{N}$, we also have $conv(\cup_{i \in \bar{N}} rec(S_i)) = \sum_{i \in \bar{N}} rec(S_i)$. Given that $(x_i e^i, y_i e^i) \in rec(S_i)$ for each $i \in N$, we have that $(x, y) \in rec(\overline{conv}(\cup_{i \in \bar{N}} S_i))$, which completes the proof. $\square$

From Proposition 17, $conv(S(t,0))$ admits a compact representation as the projection onto (x, y) of a higher dimensional polyhedron which can be used to find violated inequalities. Specifically, given $(\bar{x}, \bar{y}) \in \mathbb{R}^n \times \mathbb{R}^n$, let $P \subseteq \mathbb{R}_+^{(n+1)n} \times \mathbb{R}_+^{(n+1)n} \times \mathbb{R}_+^{n+1}$ be the set of vectors (x, y, λ) satisfying

$$
\begin{aligned}
\sum_{j\in N} y_j^0 - d\lambda^0 &\geq 0 && && (\alpha_0)\\
-y_j^0 + t_j\lambda^0 &\geq 0 && \forall j \in N && (\beta_{0j})\\
-x_j &\geq 0 && \forall j \in N &&\\
\sum_{j\in N} y_j^i - d\lambda^i &\geq 0 && \forall i \in N && (\alpha_i)\\
x_i^i - y_i^i + t_i\lambda^i &\geq 0 && \forall i \in N && (\beta_{ii})\\
-y_j^i + t_j\lambda^i &\geq 0 && \forall i \in N,\ \forall j \neq i && (\beta_{ij})\\
x_i^i - l_i\lambda^i &\geq 0 && \forall i \in N && (\gamma_i)\\
-x_j^i &\geq 0 && \forall i \in N,\ \forall j \neq i &&\\
x_i^i &= \bar{x}_i && \forall i \in N && (\nu_i)\\
\sum_{j\in \bar{N}} y_i^j &= \bar{y}_i && \forall i \in N && (\eta_i)\\
\sum_{j\in \bar{N}} \lambda^j &= 1 && && (\pi).
\end{aligned}
$$

Thus, $(\bar{x}, \bar{y})$ belongs to $\overline{conv}(\cup_{i\in\bar{N}} S_i)$, and therefore to $conv(S(t,0))$, if and only if P is nonempty. Let $Q \subseteq \mathbb{R}_+^{n+1} \times \mathbb{R}_+^{(n+1)n} \times \mathbb{R}_+^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ be the set of vectors $(\alpha, \beta, \gamma, \eta, \nu, \pi)$ such that

$$
\begin{aligned}
\alpha_0 - \beta_{0j} + \nu_j &\leq 0 && \forall j \in N\\
-d\alpha_0 + \sum_{j\in N} t_j\beta_{0j} + \pi &\leq 0 &&\\
\alpha_i - \beta_{ij} + \nu_j &\leq 0 && \forall i \in N,\ \forall j \in N\\
\beta_{ii} + \gamma_i + \eta_i &\leq 0 && \forall i \in N\\
-d\alpha_i + \sum_{j\in N} t_j\beta_{ij} - l_i\gamma_i + \pi &\leq 0 && \forall i \in N\\
\pi + \sum_{i\in N} \eta_i\bar{x}_i + \sum_{i\in N} \nu_i\bar{y}_i &> 0. &&
\end{aligned}
$$

After removing unnecessary variables and constraints from P, by Farkas' Lemma, P is nonempty if and only if Q is empty. Moreover, given $(\bar{x}, \bar{y})$ in the continuous relaxation of (20)-(23), there is a violated inequality from (24) if and only if the problem

$$
\begin{aligned}
\min \quad & \sum_{i\in N} \eta_i \bar{x}_i + \sum_{i\in N} \nu_i \bar{y}_i - \pi && && (30)\\
\text{s.t.} \quad & \alpha_0 - \beta_{0j} - \nu_j \le 0 && \forall j \in N \\
& -d\alpha_0 + \sum_{j\in N} t_j \beta_{0j} + \pi \le 0 && \\
& \alpha_i - \beta_{ij} - \nu_j \le 0 && \forall i \in N,\ \forall j \in N \\
& \beta_{ii} + \gamma_i - \eta_i \le 0 && \forall i \in N \\
& -d\alpha_i + \sum_{j\in N} t_j \beta_{ij} - l_i \gamma_i + \pi \le 0 && \forall i \in N \\
& \sum_{i\in N} \eta_i + \sum_{i\in N} \nu_i + \pi = 1 && \\
& \alpha, \beta, \gamma, \eta, \nu, \pi \ge 0 &&
\end{aligned}
$$

has negative optimal value. In such case, any optimal solution to (30) yields a valid inequality for $conv(S(t,0))$ that is not satisfied by $(\bar{x}, \bar{y})$.

5 A semi-continuous transportation problem

5.1 The problem and its complexity

Consider now the case where we intersect $m \ge 1$ sets of the form $S(t,h)$. Specifically, let $M := \{1, \ldots, m\}$ be a set of nodes that receive flow from nodes in N, where each $j \in M$ has a demand $d_j > 0$ to be met. In this context, we refer to N and M as suppliers and customers, respectively. In this setting, $l \in \mathbb{R}^n_+$ is a vector of lower bounds for supplier capacities, $h \in \mathbb{R}^{nm}_+$ is a vector of lower bounds for arc flows, and $t \in \mathbb{R}^n_+$ is a vector of initial supplier capacities.

Let $S_* \subseteq \mathbb{R}^n \times \mathbb{R}^{nm}$ be the set of vectors (x, y) such that

$$\sum_{i\in N} y_{ij} \ge d_j \qquad \forall j \in M \tag{31}$$

$$\sum_{j\in M} y_{ij} \le t_i + x_i \qquad \forall i \in N \tag{32}$$

$$x_i \in \{0\} \cup [l_i, \infty) \qquad \forall i \in N \tag{33}$$

$$y_{ij} \in \{0\} \cup [h_{ij}, \infty) \qquad \forall i \in N,\ \forall j \in M. \tag{34}$$

Constraints (31), (33), and (34) are analogous to (1), (3), and (4) of $S(t,h)$, respectively. In addition, constraints (32) ensure that the total outflow from any supplier does not exceed its available capacity. As with the inflow set, a graphical interpretation is given in Figure 3.

Now we address the complexity of optimization over S_*.

Proposition 18. *Optimizing a linear function over S_* is $\mathcal{NP}$-hard, even if $t = 0$ and $h = 0$.*

Proof. We will show that the Uncapacitated Facility Location Problem (UFLP), which is $\mathcal{NP}$-hard, can be reduced to optimization of a linear function over S_*. An instance of UFLP is defined by a set of

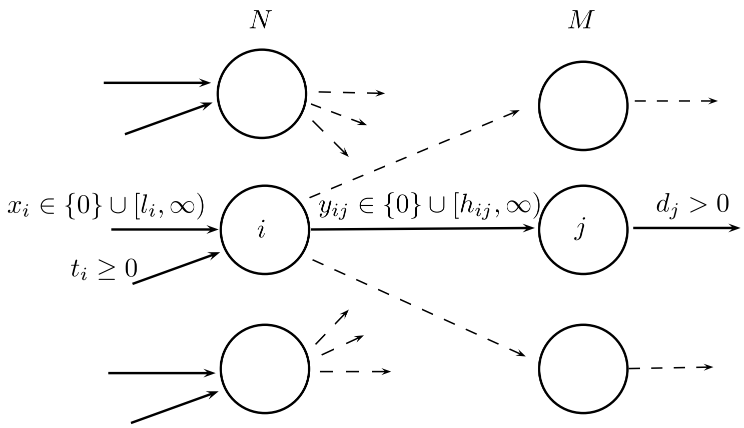

Figure 3: Semi-continuous transportation problem.

potential facilities N, a set of customers M, and cost functions $f : N \to \mathbb{R}_+$ and $e : N \times M \to \mathbb{R}_+$. The objective is to compute

$$\min_{N' \subseteq N} \left\{ \sum_{i \in N'} f_i + \sum_{j \in M} \min_{i \in N'} e_{ij} \right\}.$$

We can formulate UFLP as an integer programming problem. Let $z_i = 1$ if and only if facility i is open, and $w_{ij} = 1$ if and only if customer j is assigned to facility i. The corresponding formulation is

$$\begin{aligned} z_1 = \min \quad & \sum_{i \in N} f_i z_i + \sum_{j \in M} \sum_{i \in N} e_{ij} w_{ij} & \\ \text{s.t.} \quad & w_{ij} \leq z_i & \forall i \in N,\ \forall j \in M \\ & \sum_{i \in N} w_{ij} = 1 & \forall j \in M \\ & w_{ij} \in \{0, 1\} & \forall i \in N,\ \forall j \in M \\ & z_i \in \{0, 1\} & \forall i \in N. \end{aligned}$$

Given an instance π_1 of UFLP, we want to construct an instance π_2 of linear optimization over S_* with the same objective value. We identify N with the set of supply nodes and M with the set of customers. Let $l_i = m+1$ for all $i \in N$, $d_j = 1$ for all $j \in M$, $c_i = \frac{f_i}{m+1}$ for all $i \in N$, and $\alpha_{ij} = e_{ij}$ for all $i \in N$ and $j \in M$. We also set $t_i = 0$ for each $i \in N$, and $h_{ij} = 0$ for each $i \in N$ and $j \in M$. The corresponding instance π_2 is then

$$\begin{aligned} z_2 = \min \quad & \sum_{i \in N} \frac{f_i}{m+1} x_i + \sum_{i \in N} \sum_{j \in M} e_{ij} y_{ij} & \\ \text{s.t.} \quad & \sum_{j \in M} y_{ij} \leq x_i & \forall i \in N \\ & \sum_{i \in N} y_{ij} \geq 1 & \forall j \in M \\ & y_{ij} \geq 0 & \forall i \in N,\ \forall j \in M \\ & x_i \in \{0\} \cup [m+1, \infty) & \forall i \in N. \end{aligned}$$

Let (z^*, w^*) be an optimal solution to π_1. If we set $x_i = l_i$ if $z_i^* = 1$ and 0 otherwise, and $y_{ij} = d_j$ if $w_{ij}^* = 1$ and 0 otherwise, then we get a feasible solution (x, y) to π_2 with cost z_1. Hence, $z_2 \leq z_1$.

Now, let (x^*, y^*) be an optimal solution to π_2. Since $c \geq 0$, $\alpha \geq 0$, and $l_i \geq m+1$, we may assume that $x_i^* \in \{0, l_i\}$ for all $i \in N$. In addition, by integrality property of networks, we may also assume that $y_{ij}^* \in \{0, d_j\}$ for any $i \in N$ and $j \in M$. Setting $z_i = 1$ if $x_i^* = l_i$ and 0 otherwise, and $w_{ij} = 1$ if $y_{ij}^* = d_j$ and 0 otherwise, we get a feasible solution (z, w) to π_1 with cost z_2. Hence, $z_1 \leq z_2$. □

5.2 Analysis of a relaxation of S_*

A special case of S_* arises when $h = 0$, which constitutes a relaxation for this class of problems. In such a case, we shall present structural characteristics of the convex hull of this set that will give us some insight into the complexity of optimization over it. In fact, we will show some results for a slightly more general set.

For lower bounds $l \in \mathbb{R}^n_+$, demands $d \in \mathbb{R}^m_+$, not necessarily positive, and initial capacities $t \in \mathbb{R}^n_+$, we define

$$S_*(l,d,t) := \left\{ (x,y) \in \mathbb{R}^n \times \mathbb{R}^{nm} : \begin{array}{ll} \sum_{i \in N} y_{ij} \geq d_j & \forall j \in M \\ \sum_{j \in M} y_{ij} \leq t_i + x_i & \forall i \in N \\ y_{ij} \geq 0 & \forall i \in N,\ j \in M \\ x_i \in \{0\} \cup [l_i, \infty) & \forall i \in N \end{array} \right\}.$$

Once more, we begin with a result in the spirit of Propositions 7 and 16.

Proposition 19. *If (x, y) is an extreme point of conv$(S_*(l,d,t))$, then $\sum_{j \in M} y_{ij} > t_i$ for all $i \in N$ such that $x_i > 0$.*

Proof. Suppose that $x_i > 0$ and $\sum_{j \in M} y_{ij} \leq t_i$ for some $i \in N$. Then we can write

$$(x_i, y_{i1}, \ldots, y_{im}) = \frac{1}{2}[(2x_i, y_{i1}, \ldots, y_{im}) + (0, y_{i1}, \ldots, y_{im})],$$

that is, (x, y) is the strict convex combination of two distinct points in $S_*(l,d,t)$, and thus it cannot be an extreme point of *conv*$(S_*(l,d,t))$. □

For $(\bar{x}, \bar{y}) \in S_*(l,d,t)$, we define the support $\sigma(\bar{x})$ of $\bar{x}$ as the subset of suppliers with positive production, that is

$$\sigma(\bar{x}) := \{i \in N : \bar{x}_i > 0\}.$$

We will prove that if $(\bar{x}, \bar{y})$ is an extreme point of *conv*$(S_*(l,d,t))$, then $|\sigma(\bar{x})| \leq m$. We need the following key lemma.

Lemma 20. *If $t = 0$ and $(\bar{x}, \bar{y})$ is an extreme point of conv$(S_*(l,d,0))$, then $|\sigma(\bar{x})| \leq m$.*

Proof. For a contradiction, suppose that for some positive integers $n > m$ the claim does not hold. Choose n and m so that $n + m$ is minimum among all such instances. Note that by Proposition 16, $m > 1$. Let $(\bar{x}, \bar{y})$ be an extreme point of $S_*(l,d,0)$ having $|\sigma(\bar{x})| > m$, where $l \in \mathbb{R}^n_+$ and $d \in \mathbb{R}^m_+$.

By minimality of $n + m$, we may assume that $|\sigma(\bar{x})| = n$, since otherwise $x_i = 0$ for some $i \in N$, and removing this supplier from the instance would yield a smaller counterexample.

Claim 1: $n = m + 1$.

If $n > m + 1$, let $\widehat{N} := N \setminus \{n\}$. We define $\widehat{d} \in \mathbb{R}^m_+$ by

$$\widehat{d}_j := \sum_{i \in \widehat{N}} \bar{y}_{ij} \ \forall j \in M.$$

Let $(\widehat{x}, \widehat{y}) \in \mathbb{R}^{n-1}_+ \times \mathbb{R}^{(n-1)m}_+$ and $\widehat{l} \in \mathbb{R}^{n-1}_+$ be the restrictions of $(\bar{x}, \bar{y})$ and l with respect to $\widehat{N}$, respectively. We have that $(\widehat{x}, \widehat{y})$ is feasible for $S_*(\widehat{l}, \widehat{d}, 0)$ and $|\sigma(\widehat{x})| = n - 1 \geq m + 1$. By minimality of $n + m$, $(\widehat{x}, \widehat{y})$ cannot be an extreme point of $conv(S_*(\widehat{l}, \widehat{d}, 0))$. Thus, we can write

$$(\widehat{x}, \widehat{y}) = \sum_{p=1}^{q} \lambda_p (x^p, y^p),$$

where $q \geq 2$, $\{(x^p, y^p) : p = 1, \ldots, q\}$ are distinct points in $S_*(\widehat{l}, \widehat{d}, 0)$, $\lambda_p > 0$ for all $p = 1, \ldots, q$, and $\sum_{p=1}^{q} \lambda_p = 1$. For each $p = 1, \ldots, q$, we extend (x^p, y^p) to $(\widetilde{x}^p, \widetilde{y}^p) \in \mathbb{R}^n \times \mathbb{R}^{nm}$ by setting $\widetilde{x}^p_n = \bar{x}_n$ and $\widetilde{y}^p_{nj} = \bar{y}_{nj}$ for all $j \in M$. Since $x^p \geq \widehat{l}$ and $\bar{x}_n \geq l_n$, we have $\widetilde{x}^p \geq l$. In addition, for each $j \in M$, we have

$$\sum_{i \in N} \widetilde{y}^p_{ij} = \sum_{i \in \widehat{N}} y^p_{ij} + \bar{y}_{nj} \geq \widehat{d}_j + \bar{y}_{nj} \geq d_j.$$

Thus, $\{(\widetilde{x}^p, \widetilde{y}^p) : p = 1, \ldots, q\}$ are distinct points in $S_*(l, d, 0)$. We can see that

$$(\bar{x}, \bar{y}) = \sum_{p=1}^{q} \lambda_p (\widetilde{x}^p, \widetilde{y}^p)$$

and therefore $(\bar{x}, \bar{y})$ cannot be an extreme point of $conv(S_*(l, d, 0))$. The claim is thus proved. $\Diamond$

Let $G = (N \cup M, E)$ be a bipartite graph where $i \in N$ is adjacent to $j \in M$ if and only if $\bar{y}_{ij} > 0$. Notice that since $\sigma(\bar{x}) = N$, by Proposition 19 we have that for each $i \in N$, there exists $j \in M$ having $\bar{y}_{ij} > 0$, and therefore $deg(i) \geq 1$ for all $i \in N$. Furthermore, we may assume $deg(j) \geq 1$ for all $j \in M$, since if $deg(j) = 0$, then $d_j = 0$ and removing this customer from the instance yields a smaller counterexample. Therefore, given that $n = m + 1$, there must exist some component of G having more suppliers than customers. Hence, we may assume that G is connected, since otherwise some component of G induces a smaller counterexample. We may also assume that G is acyclic, since otherwise we can modify $\bar{y}$ along the arcs in a cycle and write $(\bar{x}, \bar{y})$ as the average of two different solutions in $S_*(l, d, 0)$. Thus, we may assume that G is a tree.

Claim 2: $deg(j) = 2 \ \forall j \in M$.

We first argue that $deg(j) \geq 2$ for all $j \in M$. By contradiction, we may assume that $deg(m) = 1$ and that m is supplied by n. As before, let $\widehat{N} := N \setminus \{n\}$ and $\widehat{M} := M \setminus \{m\}$. We define $\widehat{d} \in \mathbb{R}^{m-1}_+$ by

$$\widehat{d}_j := \sum_{i \in \widehat{N}} \bar{y}_{ij} \ \forall j \in \widehat{M}.$$

Taking the restrictions of $(\bar{x}, \bar{y})$ and l with respect to $\widehat{N}$ and $\widehat{M}$, and proceeding as in the proof of Claim 1, we conclude that $(\bar{x}, \bar{y})$ cannot be an extreme point of $conv(S_*(l, d, 0))$. Hence, $deg(j) \geq 2 \ \forall j \in M$. However, since G is a tree, we have $|E| = |N \cup M| - 1 = m + 1 + m - 1 = 2m$, and thus $deg(j) = 2$ for each $j \in M$. The claim is thus proved. $\Diamond$

Now, for each $i \in N$, let

$$M(i) := \{j \in M : \ (i,j) \in E\},$$
$$N(i) := \{l \in N \setminus \{i\} : \ \exists j \in M \text{ such that } (i,j),(l,j) \in E\}.$$

In other words, $M(i)$ are the customers served by i, while $N(i)$ are the suppliers that share a customer with i, which we refer to as its neighbors. Clearly $l \in N(i)$ if and only if $i \in N(l)$. Note that since G is acyclic, any two suppliers can have at most one common customer. Thus, given neighbors i and l in N, there exists a unique $j =: j(i,l) \in M$ connecting them in G.

Let $(c,\alpha) \in \mathbb{R}^n \times \mathbb{R}^{nm}$ be such that $(\bar{x},\bar{y})$ is the unique minimizer in $S_*(l,d,0)$ with respect to this function. For each $i \in N$, consider the solution (x^i,y^i) given by

$$x^i_l = \begin{cases} 0 & l = i \\ \bar{x}_l + \bar{y}_{ij(i,l)} & l \in N(i) \\ \bar{x}_l & \text{otherwise,} \end{cases} \qquad y^i_{lj} = \begin{cases} 0 & l = i \\ \bar{y}_{lj} + \bar{y}_{ij(i,l)} & l \in N(i),\ j = j(i,l) \\ \bar{y}_{lj} & \text{otherwise.} \end{cases}$$

Thus, we obtain (x^i,y^i) from $(\bar{x},\bar{y})$ by moving the production from i to its neighbors and removing i from the solution. It is straightforward to verify that (x^i,y^i) is feasible to $S_*(l,d,0)$. However, since $(\bar{x},\bar{y})$ is the unique minimizer for (c,α), we have that the cost incurred by $(\bar{x},\bar{y})$ is less than the cost incurred by (x^i,y^i). Since these solutions only differ in the variables associated to i and its neighbors, we have

$$c_i\bar{x}_i + \sum_{j \in M(i)} \alpha_{ij}\bar{y}_{ij} < \sum_{l \in N(i)} (c_l + \alpha_{lj(i,l)})\bar{y}_{ij(i,l)}.$$

Recalling that $\sum_{j \in M(i)} \bar{y}_{ij} \leq \bar{x}_i$, we have

$$\sum_{j \in M(i)} (c_i + \alpha_{ij})\bar{y}_{ij} < \sum_{l \in N(i)} (c_l + \alpha_{lj(i,l)})\bar{y}_{ij(i,l)}.$$

Rewriting the left-hand-side in the last inequality, we obtain

$$\sum_{l \in N(i)} (c_i + \alpha_{ij(i,l)})\bar{y}_{ij(i,l)} < \sum_{l \in N(i)} (c_l + \alpha_{lj(i,l)})\bar{y}_{ij(i,l)}.$$

Hence, there must exist some $l \in N(i)$ such that

$$c_i + \alpha_{ij(i,l)} < c_l + \alpha_{lj(i,l)}.$$

For neighbors i and l, we say that i dominates l if the above inequality holds. Thus, we have that any supplier has to dominate at least one of its neighbors.

Let $G' = (N,E')$ be a graph where $(i,j) \in E'$ if and only if i and j are neighbors in G, and note that G' is also a tree. Let $L \subseteq N$ be the set of leaves of G'. Since $n = m+1 \geq 3$, L has at least two elements and $N \setminus L$ is nonempty. Note that any leaf dominates its unique neighbor. Now, pick some $r \in L$ as a root of G', and let $i \in N \setminus L$ be such that all of its children are leaves of G'. Since i is dominated by its children, it must dominate its parent. Reasoning by induction, we have that any supplier has to dominate its parent. In particular, we conclude that r is dominated by its child, a contradiction since r is a leaf. This completes the proof. $\square$

With Proposition 19 and Lemma 20 at hand, we can prove the main result of this section.

Theorem 21. *For any $t \geq 0$, if $(\bar{x}, \bar{y})$ is an extreme point of $conv(S_*(l,d,t))$, then $|\sigma(\bar{x})| \leq m$.*

Proof. For a contradiction, suppose that for some positive integers $n > m$ the claim does not hold. Let $(\bar{x}, \bar{y})$ be an extreme point of $S_*(l,d,t)$ having $|\sigma(\bar{x})| > m$, where $l, t \in \mathbb{R}^n_+$ and $d \in \mathbb{R}^m_+$.

For each $i \in N$, let $j(i) \in M$ be such that $\sum_{j \in M,\ j < j(i)} \bar{y}_{ij} \leq t_i$ and $\sum_{j \in M,\ j \leq j(i)} \bar{y}_{ij} > t_i$. Since $(\bar{x}, \bar{y})$ is an extreme point of $conv(S_*(l,d,t))$, by Proposition 19, $j(i)$ is well defined for all $i \in N$. We define $\widehat{y} \in \mathbb{R}^{nm}_+$ and $\widehat{d} \in \mathbb{R}^m_+$ by

$$\widehat{y}_{ij} = \begin{cases} 0 & j < j(i) \\ \displaystyle\sum_{k \in M,\ k \leq j(i)} \bar{y}_{ik} - t_i & j = j(i) \\ \bar{y}_{ij} & j > j(i), \end{cases}$$

$$\widehat{d}_j = \sum_{i \in N} \widehat{y}_{ij} \ \forall j \in M.$$

Also, let $\widehat{x} = \bar{x}$. Then $\sum_{j \in M} \widehat{y}_{ij} = \sum_{j \in M} \bar{y}_{ij} - t_i \leq \bar{x}_i = \widehat{x}_i$ for all $i \in N$. Moreover, $(\widehat{x}, \widehat{y})$ is feasible to $S_*(l, \widehat{d}, 0)$. Since $|\sigma(\widehat{x})| = |\sigma(\bar{x})| > m$, by Lemma 20, $(\widehat{x}, \widehat{y})$ cannot be an extreme point of $conv(S_*(l, \widehat{d}, 0))$. Thus, we can write

$$(\widehat{x}, \widehat{y}) = \sum_{p=1}^{q} \lambda_p (x^p, y^p),$$

where $q \geq 2$, $\{(x^p, y^p) : p = 1, \ldots, q\}$ are distinct points in $S_*(l, \widehat{d}, 0)$, $\lambda_p > 0$ for all $p = 1, \ldots, q$, and $\sum_{p=1}^{q} \lambda_p = 1$. Notice that for each $p = 1, \ldots, q$ and $i \in N$, $y^p_{ij} = 0$ for all $j < j(i)$. Then we can define $w \in \mathbb{R}^{nm}$ by

$$w_{ij} = \begin{cases} \bar{y}_{ij} & j < j(i) \\ \bar{y}_{ij} - \widehat{y}_{ij} & j = j(i) \\ 0 & j > j(i), \end{cases}$$

and set $\widetilde{x}^p = x^p$ and $\widetilde{y}^p = y^p + w$. Notice that for all $i \in N$,

$$w_{ij(i)} = \bar{y}_{ij(i)} - \widehat{y}_{ij(i)} = \bar{y}_{ij(i)} - \sum_{j \in M,\ j \leq j(i)} \bar{y}_{ij} + t_i = - \sum_{j \in M,\ j < j(i)} \bar{y}_{ij} + t_i \geq 0.$$

Thus, $w \geq 0$ and $\widetilde{y}^p$ is nonnegative for all $p = 1, \ldots, q$. Also, for all $i \in N$ we have

$$\sum_{j \in M} \widetilde{y}^p_{ij} = \sum_{j \in M} y^p_{ij} + \sum_{j \in M,\ j \leq j(i)} \bar{y}_{ij} - \widehat{y}_{ij(i)} = \sum_{j \in M} y^p_{ij} + t_i \leq x^p_i + t_i = \widetilde{x}^p_i + t_i.$$

Finally, for all $j \in M$ we have

$$\begin{aligned}
\sum_{i \in N} \widetilde{y}^p_{ij} &= \sum_{i \in N} y^p_{ij} + \sum_{i \in N:\ j \leq j(i)} \bar{y}_{ij} - \sum_{i \in N:\ j = j(i)} \widehat{y}_{ij} \\
&\geq \widehat{d}_j + \sum_{i \in N:\ j \leq j(i)} \bar{y}_{ij} - \sum_{i \in N:\ j = j(i)} \widehat{y}_{ij} \\
&= \sum_{i \in N} \widehat{y}_{ij} + \sum_{i \in N:\ j \leq j(i)} \bar{y}_{ij} - \sum_{i \in N:\ j = j(i)} \widehat{y}_{ij} \\
&= \sum_{i \in N:\ j = j(i)} \widehat{y}_{ij} + \sum_{i \in N:\ j > j(i)} \widehat{y}_{ij} + \sum_{i \in N:\ j \leq j(i)} \bar{y}_{ij} - \sum_{i \in N:\ j = j(i)} \widehat{y}_{ij} \\
&= \sum_{i \in N} \bar{y}_{ij} \\
&\geq d_j.
\end{aligned}$$

Thus, $(\widetilde{x}^p, \widetilde{y}^p) \in S_*(l, d, t)$ for all $p = 1, \ldots, q$ and are all distinct by the definition of $\widetilde{y}^p$. Furthermore, it is straightforward to verify that $\sum_{p=1}^{q} \lambda_p (\widetilde{x}^p, \widetilde{y}^p) = (\bar{x}, \bar{y})$. Hence $(\bar{x}, \bar{y})$ cannot be an extreme point of $conv(S_*(l, d, t))$, yielding the required contradiction. $\square$

Corollary 22. *Minimizing a linear function over $S_*(l, d, t)$ can be done by solving $\mathcal{O}(n^m)$ linear programming problems.*

In other words, optimization over $S_*(l, d, t)$ can be done in polynomial time when m is fixed.

As an algorithmic implication, we can tweak the branch-and-bound procedure when we optimize over $S_*(l, d, t)$: whenever a node of the search-tree has m bounds of the form $x_i \geq l_i$, we can fix the production of the remaining suppliers to 0. However, our experimental experience indicates that a standard branch-and-cut solver does not need to branch that many times, rendering this approach inapplicable for practical purposes.

On the other hand, we can construct relaxations of S_* by considering the subsystem defined by a few customers, say two, and taking $h = 0$. By Theorem 21 and an argument similar to Proposition 17, a compact extended formulation is available for its convex hull from which strong valid inequalities for $conv(S_*)$ may be devised.

6 Computation

We test the performance of the inequalities presented in Sections 3 and 4 on instances of the semi-continuous transportation problem described in Section 5. We address the effectivity of the cuts used alone or combined with CPLEX cuts, and the differences between semi-continuous and binary formulations.

Each instance is formulated in CPLEX either declaring all variables as semi-continuous or using auxiliary binary variables to enforce semi-continuity. Even though the description of the transportation problem given in Section 5 involves unbounded semi-continuous variables, we use a constant $M > 0$ as an upper bound in the binary formulation. Letting $\bar{d} := \sum_{j \in M} d_j$, $\bar{l} := \max_{i \in N}\{l_i\}$, $\bar{h} := \max_{i \in N,\ j \in M}\{h_{ij}\}$, and $\bar{t} := \max_{i \in N}\{t_i\}$, we set $M = \max\{\bar{d}, \bar{l}, \bar{h}\} + \bar{t}$.

Also, we consider the cases $t = 0$ and $t > 0$ separately. In the first case, we ignore the initial capacities and therefore cuts of the form (14) may be generated. In the second case, valid cuts may be generated using the extended formulation (30). In both cases, to separate a fractional solution $(\bar{x}, \bar{y})$, we consider the inflow set corresponding to each customer $j \in M$ and we try to find a cut violated by $(\bar{x}, \bar{y}_j)$. Thus, we may add up to m cuts in a single round. Cuts are added only at the root node. In addition, when $t = 0$, we also test an extended formulation where a vector π^j is appended for each $j \in M$. Adding the constraints that define W in Corollary 9 for each $j \in M$, we obtain an extended formulation where all the inequalities describing the inflow relaxation for each customer are already implied, and therefore there is no need to generate cuts on-the-fly. Even though an extended formulation is also available when $t > 0$, its size becomes a bottleneck even when solving the root relaxation, and thus it is not considered in our experimental setup.

In our experiments, we use $n \in \{30, 50, 80\}$ and $m \in \{30, 50, 80\}$. For each combination of these parameters, with the exception of $(n, m) = (80, 80)$ due to time limits, we generate 10 instances as follows:

- $l_i \sim \mathcal{U}[100, 500]\ \forall i \in N$
- $h_{ij} \sim \mathcal{U}\left[0, \frac{2}{m} l_i\right]\ \forall i \in N,\ \forall j \in M$
- $t_i \sim \mathcal{U}[10, 50]\ \forall i \in N$
- $d_j \sim \mathcal{U}\left[10\frac{n}{m}, 50\frac{n}{m}\right]\ \forall j \in M$
- $c_i \sim \mathcal{U}\left[40, 40 + \frac{1000}{l_i}\right]\ \forall i \in N$
- $\alpha_{ij} \sim \mathcal{U}\left[-10, 90\right]\ \forall i \in N,\ \forall j \in M,$

where $X \sim \mathcal{U}[a, b]$ means that X is a random variable following a uniform distribution on the interval $[a, b]$. Then, for each instance and for each formulation, we solve using CPLEX 12.2 default branch-and-cut (C), using only our cuts within branch-and-cut (U), using both CPLEX and user cuts (C+U), and solving the extended formulation (E) in the case $t = 0$. All experiments were carried out on a personal computer on a single thread running at 3.33 Ghz with 4 GB of RAM under Linux environment. A time limit of 1800 CPU seconds per instance is enforced.

6.1 The case $t = 0$

Table 1 shows the number of instances solved within the time limit, Table 2 shows the average number of explored nodes needed to reach optimality within CPLEX's default tolerance, and Table 3 shows the average time in CPU seconds required by such task. In all cases, columns n and m denote the size of the problem, columns *Semi-continuous* and *Binary* denote the type of formulation being considered, and columns C, U, C+U, and E denote the procedure being used, as explained above. All the averages are with respect to the number of instances that were solved. If no instance was solved for a particular combination of n and m, a dash "-" appears in the corresponding cell.

Table 1 shows that not all instances were solved within the time limit. This may be a bit surprising, as the underlying problem structure is fairly simple and the number of variables does not exceed a few thousands. Adding our cuts alone and the extended formulation have the best performance in this sense, specially in the binary formulation where all instances where solved by both methods. As we can see from Table 2, the node count of the extended formulation is roughly one or two orders of magnitude smaller when compared to the other procedures in both formulations. If we consider adding cuts on-the-fly, combining CPLEX and user cuts shows the best performance in the semi-continuous formulation, while CPLEX default cuts seem to be the best option for the largest instances with the binary formulation. Regarding time, from Table 3 we observe that the extended formulation is the best

n	m	*Semi-continuous*				*Binary*			
		C	U	C+U	E	C	U	C+U	E
30	30	10	10	10	10	10	10	10	10
30	50	10	10	10	10	10	10	10	10
30	80	10	10	10	8	10	10	10	10
50	30	10	10	10	10	10	10	10	10
50	50	10	10	10	9	9	10	10	10
50	80	4	10	5	10	1	10	4	10
80	30	5	10	4	10	10	10	10	10
80	50	0	5	0	10	4	10	2	10

Table 1: Number of solved instances when $t = 0$.

method in most cases when the semi-continuous formulation is used, whereas this approach is the best only in the largest instances when the binary formulation is considered. Among cutting procedures, adding only user cuts performs better than the rest in both formulations and is the only way to solve the largest instances within the time limit, with time reductions of up to one order of magnitude. Again, this can be somewhat surprising in the case of the binary formulation, as these cuts were not developed with binary variables in mind, and in this case we expected the presolve routines and flow covers to be particulary effective. On the other hand, combining these and CPLEX cuts decrease the overall performance and is comparable to the default solver.

n	m	*Semi-continuous*				*Binary*			
		C	U	C+U	E	C	U	C+U	E
30	30	3936.2	3266.5	2919.2	**72.8**	313.3	808.4	268.1	**32.7**
30	50	6246.6	4940.7	3653.6	**213.0**	493.3	731.9	618.7	**61.7**
30	80	11764.3	9330.0	6232.5	**930.3**	1142.3	1042.6	840.3	**206.1**
50	30	24045.9	23725.8	20548.9	**297.5**	1501.4	4545.5	1248.5	**84.8**
50	50	49407.0	40399.5	54556.6	**145.1**	3433.0	7446.9	2382.6	**135.1**
50	80	81456.8	159338.0	55918.8	**1019.9**	2086.0	23129.2	2621.3	**470.3**
80	30	56262.8	210466.0	48761.0	**192.8**	4049.3	30828.1	4731.6	**67.6**
80	50	-	438369.0	-	**332.3**	12265.2	114003.0	17426.5	**287.5**

Table 2: Number of nodes needed to optimality when $t = 0$.

n	m	*Semi-continuous*				*Binary*			
		C	U	C+U	E	C	U	C+U	E
30	30	17.1	4.0	17.7	**3.4**	23.6	**2.0**	22.9	4.8
30	50	54.7	**12.9**	45.2	16.9	98.7	**4.6**	126.2	20.3
30	80	129.7	**33.9**	117.8	126.5	409.4	**9.3**	332.0	89.1
50	30	256.1	28.9	244.0	**10.0**	148.1	**11.3**	151.9	12.1
50	50	609.0	59.6	724.1	**13.6**	597.4	**21.7**	586.0	37.6
50	80	1399.7	316.7	1155.6	**135.5**	578.2	**98.5**	1144.6	165.3
80	30	924.7	168.2	1018.3	**8.2**	264.5	48.1	234.2	**8.2**
80	50	-	746.4	-	**28.5**	1438.7	354.6	1409.4	**45.7**

Table 3: CPU time needed to optimality when $t = 0$.

Table 4 shows information regarding cuts. Column headers n, m, *Semi-continuous*, *Binary*, U, and C+U have the same meaning as in the previous tables. In addition, columns *Gen* denote the average number of user cuts that were generated, while columns *Appl* denote the average number of cuts that were actually applied. As we let CPLEX decide whether or not to apply user cuts that are generated by our separation routine, the numbers in these columns are different in general.

n	m	*Semi-continuous*				*Binary*			
		U		C+U		U		C+U	
		Gen	*Appl*	*Gen*	*Appl*	*Gen*	*Appl*	*Gen*	*Appl*
30	30	96.0	24.3	96.0	76.4	60.0	15.9	51.0	10.3
30	50	154.6	69.4	154.6	132.8	100.0	45.7	70.0	29.1
30	80	262.0	137.0	262.0	218.3	128.0	69.9	120.0	79.3
50	30	87.4	7.4	87.4	49.6	67.0	6.7	102.6	37.9
50	50	147.6	18.8	147.6	101.9	123.4	15.7	180.5	98.5
50	80	239.9	45.5	239.8	181.2	231.9	43.9	316.5	178.5
80	30	88.3	5.9	87.0	42.8	58.8	4.7	94.0	23.2
80	50	147.2	9.0	-	-	101.3	6.9	173.5	75.0

Table 4: Number of cuts when $t = 0$.

First, note that more cuts are generated and applied in the semi-continuous formulation than in the binary formulation. Now, in both cases, the proportion of applied cuts with respect to the number of generated cuts is smaller when CPLEX cuts are turned off. Given the results in Table 3, just a few cuts are required to get a non-trivial improvement over the default solver, and the generation of more user cuts than needed seems to increase the running times.

6.2 The case $t > 0$

Tables 5, 6, and 7 are analogous to Tables 1, 2, and 3, respectively, with the difference that there is no column E as no extended formulation was tested in this case.

n	m	*Semi-continuous*			*Binary*		
		C	U	C+U	C	U	C+U
30	30	10	10	10	10	10	10
30	50	7	10	8	10	10	10
30	80	2	9	8	10	10	10
50	30	0	10	3	10	10	10
50	50	0	9	1	9	10	8
50	80	0	0	0	1	10	4
80	30	0	5	0	4	10	6
80	50	0	0	0	0	10	0

Table 5: Number of solved instances when $t > 0$.

From Table 5, we see that when $t > 0$, the instances become much harder than in the case $t = 0$. The performance of the semi-continuous formulation is quite poor in general. In contrast, the binary formulation is able to solve all small instances with any procedure, but only when CPLEX cuts are turned off it is possible to solve all large instances as well. Regarding explored nodes, Table 6 shows that combining CPLEX and user cuts in the binary formulation gives the best results. With the semi-continuous formulation, the picture is not that clear, but the presence of user cuts still helps. With respect to computation times, we have that user cuts alone in the binary formulation outperforms all other methods, as shown in Table 7. This procedure is also the best with the semi-continuous formulations. Once again, combining CPLEX and user cuts is comparable to the default solver.

Finally, Table 8 shows information regarding cuts, and it is analogous to Table 4. As in the case $t = 0$, when CPLEX and user cuts are combined, the solver attemps to generate and apply more cuts than needed, decreasing the overall performance as follows from Table 7.

n	m	*Semi-continuous*			*Binary*		
		C	U	C+U	C	U	C+U
30	30	120194.0	**17069.0**	53748.7	603.1	852.3	**433.5**
30	50	137858.0	52383.7	**40540.5**	651.2	883.4	**476.3**
30	80	83006.5	112944.0	**33035.1**	1153.6	1103.7	**901.6**
50	30	-	**106912.0**	133777.0	3555.5	5927.4	**2596.4**
50	50	-	216427.0	**121396.0**	4991.4	10361.9	**3013.5**
50	80	-	-	-	7998.0	22166.4	**2496.8**
80	30	-	**714998.0**	-	17143.5	77894.5	**16998.0**
80	50	-	-	-	-	104097.0	-

Table 6: Number of nodes needed to optimality when $t > 0$.

n	m	*Semi-continuous*			*Binary*		
		C	U	C+U	C	U	C+U
30	30	343.5	**22.1**	211.7	26.6	**5.3**	26.1
30	50	759.3	**116.5**	311.0	74.0	**9.1**	66.4
30	80	945.6	**413.5**	488.8	224.4	**18.6**	202.4
50	30	-	**126.4**	962.1	168.8	**38.7**	157.5
50	50	-	**406.2**	1283.7	601.8	**93.8**	469.9
50	80	-	-	-	1427.6	**256.6**	872.4
80	30	-	**838.1**	-	542.2	**295.8**	804.0
80	50	-	-	-	-	**762.9**	-

Table 7: CPU time needed to optimality when $t > 0$.

As we have seen, the proposed valid inequalities, either in their original form or through an extended formulation when possible, are quite useful in solving this class of semi-continuous network flow problems. Although these cuts involve only the original variables of the problem, the introduction of binary variables seems to improve the overall performance, probably due to advanced bound tightening techniques in CPLEX.

7 Conclusions

In this work we have considered semi-continuous network flow problems. In particular, we introduced the semi-continuous inflow set with variable upper bounds as a relaxation. Two particular cases of this set were considered, for which we presented complete descriptions of the convex hull in terms of linear inequalities and extended formulations. These inequalities proved to be quite efficient in solving a class of semi-continuous transportation problems. In fact, applying these cuts to a binary formulation of such problems turned out to be the most effective method.

We envision at least two possible venues of future research, mainly based on the semi-continuous inflow set. The first one is to consider finite upper bounds on semi-continuous variables. In this case, further connections with [6] may be established. Another direction is to consider semi-continuous inflows and outflows simultaneously. This would lead to a more general set that can be a better relaxation for appropriate problems.

Acknowledgements: The authors acknowledge ExxonMobil for support, and Myun-Seok Cheon and Ahmet Keha for useful discussions.

n	m	*Semi-continuous*				*Binary*			
		U		C+U		U		C+U	
		Gen	*Appl*	*Gen*	*Appl*	*Gen*	*Appl*	*Gen*	*Appl*
30	30	101.7	51.2	98.7	86.6	48.0	25.6	42.0	21.1
30	50	159.0	92.1	161.9	143.1	74.9	43.8	94.9	51.8
30	80	253.0	169.4	265.0	235.1	120.0	92.4	152.0	104.7
50	30	89.8	29.5	94.3	87.7	84.0	29.1	101.2	69.4
50	50	147.7	68.8	150.0	144.0	147.9	70.4	182.9	136.5
50	80	-	-	-	-	239.3	102.2	308.8	222.5
80	30	86.4	30.0	-	-	79.0	25.1	111.2	85.3
80	50	-	-	-	-	130.6	37.9	-	-

Table 8: Number of cuts when $t > 0$.

Appendix

Given an integer $t \geq 1$, let $T := \{1, \ldots, t\}$. For each $r \in T$, consider $\pi^r \in \mathbb{R}^n$ and $\pi_0^r, \pi_1^r \in \mathbb{R}$. We are mainly interested in the case $\pi_0^r < \pi_1^r$, although this is not required in what follows. Given a closed convex set $C \subseteq \mathbb{R}^n$, for each $Q \in \mathcal{T} := 2^T$, consider the set

$$C^Q := \{x \in C : \ \pi^r x \leq \pi_0^r \ \forall r \in Q, \ \pi^r x \geq \pi_1^r \ \forall r \notin Q\}.$$

We call the set $\cup_{Q \in \mathcal{T}} C^Q$ a t-branch split disjunction as defined in [10]. Let

$$C^{\pi,\pi_0,\pi_1} := conv\left(\cup_{Q\in\mathcal{T}} C^Q\right).$$

When $t = 1$, the closedness of C^{π,π_0,π_1} was addressed in [3]. We extend this result for any $t \geq 1$.

Proposition 23. *C^{π,π_0,π_1} is a closed convex set. Moreover, if C is a polyhedron, so is C^{π,π_0,π_1}.*

Proof. Let C_∞ be the recession cone of C, and for each $Q \in \mathcal{T}$, let $C_\infty^Q := C^Q + C_\infty$. Also, let $\mathcal{T}^* := \{Q \in \mathcal{T} : \ C^Q \neq \emptyset\}$. If $\mathcal{T}^*$ is empty, then the result holds. Thus, assume $\mathcal{T}^*$ is nonempty.

Claim: $C^{\pi,\pi_0,\pi_1} = conv\left(\cup_{Q\in\mathcal{T}^*} C_\infty^Q\right)$.

The forward inclusion is easy as $\cup_{Q\in\mathcal{T}} C^Q \subseteq \cup_{Q\in\mathcal{T}^*} C_\infty^Q$.

For the reverse inclusion, consider $x \in conv\left(\cup_{Q\in\mathcal{T}^*} C_\infty^Q\right)$. We can write $x = \sum_{Q\in\mathcal{T}^*} \lambda^Q (x^Q + y^Q)$, where $x^Q \in C^Q$, $y^Q \in C_\infty$, and $\lambda^Q \geq 0$ for each $Q \in \mathcal{T}^*$, and $\sum_{Q\in\mathcal{T}^*} \lambda^Q = 1$. If we show that for any $Q \in \mathcal{T}^*$, $x^Q + y^Q$ belongs to C^{π,π_0,π_1}, then the result follows. To that end, fix $Q \in \mathcal{T}^*$ and let

$$R^- := \{r \in T : \ \pi^r y^Q < 0\},$$

$$R^+ := \{r \in T : \ \pi^r y^Q > 0\},$$

$$R^= := \{r \in T : \ \pi^r y^Q = 0\}.$$

Note that there exists finite $\lambda \geq 1$ such that $\pi^r(x^Q + \lambda y^Q) \leq \pi_0^r$ for all $r \in R^-$ and $\pi^r(x^Q + \lambda y^Q) \geq \pi_1^r$ for all $r \in R^+$. Also, recall that x^Q satisfies $\pi^r x^Q \leq \pi_0^r$ for all $r \in Q$ and $\pi^r x^Q \geq \pi_1^r$ for all $r \notin Q$. Thus $x^Q + \lambda y^Q$ belongs to $C^{Q'}$, where $Q' := R^- \cup (R^= \cap Q)$. Finally, note that $x^Q + y^Q \in conv(\{x^Q, x^Q + \lambda y^Q\})$, which implies $x^Q + y^Q \in C^{\pi,\pi_0,\pi_1}$ as desired. $\diamondsuit$

By the claim, C^{π,π_0,π_1} is the convex hull of the union of nonempty closed convex sets having the same recession cone. By Corollary 9.8.1 of [12], C^{π,π_0,π_1} is a closed convex set. Moreover, if C is a polyhedron, then C^{π,π_0,π_1} is the convex hull of the union of nonempty polyhedra having the same recession cone, which is a polyhedron [1]. $\square$

References

[1] E. Balas. Disjunctive programming. Annals of Discrete Mathematics 5:3–51, 1979.

[2] D. Bienstock. Computational study of a family of mixed-integer quadratic programming problems. Mathematical Programming 74, 2:121–140, 1996.

[3] D. Dadush, S.S. Dey, and J.P. Vielma. The split closure of a strictly convex body. Operations Research Letters 39, 2:121–126, 2011.

[4] I.R. de Farias, E.L. Johnson, and G.L. Nemhauser. Branch-and-Cut for Combinatorial Optimization Problems without Auxiliary Binary Variables. Knowledge Engineering Review 16, 1:25–39, 2001.

[5] I.R. de Farias and G.L. Nemhauser. A Polyhedral Study of the Cardinality Constrained Knapsack Problem. Mathematical Programming 96, 3:439–467, 2003.

[6] I.R. de Farias. Semi-continuous Cuts for Mixed-Integer Programming. Integer Programming and Combinatorial Optimization, 2004, 163–177.

[7] I.R. de Farias and M. Zhao. A Polyhedral Study of the Semi-Continuous Knapsack Problem. www.optimization-online.com, 2011.

[8] J. Kallrath. Mixed Integer Optimization in the Chemical Process Industry: Experience, Potential and Future Perspectives. Chemical Engineering Research and Design 78, 6:809–822, 2000.

[9] J. Kallrath. Combined strategic and operational planning - an MILP success story in chemical industry. OR Spectrum 24, 3:315–341, 2002.

[10] Y. Li and J.-P.P. Richard. Cook, Kannan and Schrijver's example revisited. Discrete Optimization 5, 4:724–734, 2008.

[11] A.F. Perold. Large-Scale Portfolio Optimization. Management Science 30, 10:1143-1160, 1984.

[12] R.T. Rockafellar. Convex Analysis. Princeton University Press, 1996.

[13] C.H. Timpe and J. Kallrath. Optimal planning in large multi-site production networks. European Journal of Operational Research 126, 2:422–435, 2000.