ON THE NON-HOMOGENEITY OF COMPLETELY POSITIVE CONES *

M. SEETHARAMA GOWDA† AND R. SZNAJDER‡

Abstract. For a closed cone $\mathcal{C}$ in $\mathbb{R}^n$, the completely positive cone of $\mathcal{C}$ is the convex cone $\mathcal{K}$ in $\mathcal{S}^n$ generated by $\{uu^T : u \in \mathcal{C}\}$. Completely positive cones arise, for example, in the conic LP reformulation of a nonconvex quadratic minimization problem over an arbitrary set with linear and binary constraints. Motivated by the useful and desirable properties of the nonnegative orthant and the positive semidefinite cone (and more generally of symmetric cones in Euclidean Jordan algebras), in this paper, we investigate when (or whether) $\mathcal{K}$ can be self-dual, irreducible, or homogeneous.

Key words. copositive and completely positive cones, self-dual, irreducible cone, homogeneous cone

1. Introduction. Consider $\mathbb{R}^n$ with the usual inner product. Given a closed cone $\mathcal{C}$ in $\mathbb{R}^n$ that is not necessarily convex, we consider two related cones in the space $\mathcal{S}^n$ of all $n \times n$ real symmetric matrices:

The *completely positive cone of* $\mathcal{C}$ defined by

$$\mathcal{K} := \left\{ \sum uu^T : u \in \mathcal{C} \right\}, \tag{1.1}$$

where the sum denotes a finite sum, and the *copositive cone of* $\mathcal{C}$ given by

$$\mathcal{E} := \{A \in \mathcal{S}^n : A \text{ is copositive on } \mathcal{C}\}. \tag{1.2}$$

When $\mathcal{C} = \mathbb{R}^n$, these two cones reduce to $\mathcal{S}^n_+$, which is the underlying cone in semidefinite programming [16] and semidefinite linear complementarity problems [13], [12]. In the case of $\mathcal{C} = \mathbb{R}^n_+$, these cones reduce, respectively, to the cones of completely positive matrices and copositive matrices which have appeared prominently in statistical and graph theoretic literature [3] and in copositive programming [7]. In a path-breaking work, Burer [4] showed that a nonconvex quadratic minimization problem over the nonnegative orthant with some additional linear and binary constraints can be reformulated as a linear program over the cone of completely positive matrices. Since then, a number of authors have investigated the properties of the cone of

*Research Report, November 3, 2011.

†Department of Mathematics and Statistics, University of Maryland Baltimore County, Baltimore, Maryland 21250, USA; E-mail: gowda@math.umbc.edu, URL: http://www.math.umbc.edu/∼gowda

‡Department of Mathematics, Bowie State University, Bowie, MD 20715, USA; rsznajder@bowiestate.edu

completely positive matrices, specifically describing the interior and facial structure of the cones of completely positive and copositive matrices, see [8], [5], [6].

The work of Burer has been recently extended to case of an arbitrary set (in place of the nonnegative orthant) by Eichfelder and Povh [10]. To elaborate, let $M \in \mathcal{S}^n$, $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, S be an arbitrary nonempty set in $\mathbb{R}^n$, and let $J \subseteq \{1, 2, \ldots, n\}$. It is shown in [10] that the quadratic optimization problem

$$\begin{gathered} \min\, x^T M x + 2c^T x \\ \text{such that} \\ Ax = b, \\ x_j \in \{0,1\} \text{ for all } j \in J, \\ x \in S, \end{gathered}$$

under a mild assumption, can be reformulated as linear programming problem over a cone in $\mathcal{S}^{n+1}$:

$$\begin{gathered} \min\, \langle \widehat{M}, Y\rangle \\ L(Y) = B \\ Y \in \mathcal{K}, \end{gathered}$$

where $\widehat{M} = \begin{bmatrix} 0 & c^T \\ c & M \end{bmatrix}$ and the closed convex cone $\mathcal{K}$ is given by

$$\mathcal{K} = closure \left\{ \sum_k \lambda_k \begin{pmatrix} 1 \\ x_k \end{pmatrix} \begin{pmatrix} 1 \\ x_k \end{pmatrix}^T : \lambda_k \geq 0\,,\, x_k \in S \right\}. \tag{1.3}$$

Since this cone is, see Lemma 1.4 in [10],

$$\left\{ \sum uu^T : u \in \overline{cone(\{1\} \times S)} \right\}$$

(with $\overline{cone(\{1\} \times S)} = \mathbb{R}_+ \times S$ when S is a closed cone), we see that the linear program defined above is over a completely positive cone corresponding to the closed (possibly nonconvex) cone $\mathcal{C} = \overline{cone(\{1\} \times S)}$.

The above reformulation demonstrates the importance of studying completely positive cones $\mathcal{K}$ that come from (general) closed cones $\mathcal{C}$ in $\mathbb{R}^n$. Motivated by the useful and desirable properties (such as self-duality and homogeneity) of the nonnegative orthant and the positive semidefinite cone (and more generally of symmetric cones in Euclidean Jordan algebras [11]), in this paper, we address the questions of when or whether $\mathcal{K}$ can be self-dual, irreducible, or homogeneous. We show, for example,

- $\mathcal{K}$ is self-dual if and only if $\mathbb{R}^n = \mathcal{C} \cup -\mathcal{C}$,

- $\mathcal{K}$ is irreducible when $\mathcal{C}$ has nonempty interior, and
- $\mathcal{K}$ is non-homogeneous when $\mathcal{C}$ is a proper (convex) cone.

The non-homogeneity of $\mathcal{K}$ is proved via a recent result in [14] where it is shown that under certain conditions on $\mathcal{C}$ (for example, $\mathcal{C}$ is a proper cone), every automorphism of $\mathcal{K}$ is of the form

$$X \mapsto QXQ^T,$$

where Q is an automorphism of $\mathcal{C}$.

2. Preliminaries. Throughout this paper, H denotes either $\mathbb{R}^n$ or $\mathcal{S}^n$. In the case of $\mathbb{R}^n$, vectors are regarded as column vectors and the usual inner product is written as $\langle x, y\rangle$ or as x^Ty. The space $\mathcal{S}^n$ – consisting of all real $n \times n$ symmetric matrices – carries the trace inner product $\langle X, Y\rangle = trace(XY)$, where the trace of a matrix is the sum of its diagonal elements. $\mathbb{R}^n_+$ denotes the nonnegative orthant in $\mathbb{R}^n$ and $\mathcal{S}^n_+$ denotes the set of all positive semidefinite matrices in $\mathcal{S}^n$.

For a set K in H, $int(K)$, $\overline{K}$, and $K^\perp$ denote, respectively, the interior, closure, and orthogonal complement of K. The dual of K is given by

$$K^* := \{y \in H : \langle y, x\rangle \geq 0 \ \forall x \in K\}.$$

A nonempty set K is a cone if for all $\lambda \geq 0$ in R and $x \in K$, we have $\lambda x \in K$. A closed cone K is said to be *pointed* if $K \cap -K = \{0\}$; it is said to be a *proper cone* if $\mathcal{K}$ is *convex, pointed, and has nonempty interior.* For a convex cone K, we denote the set of all extreme vectors by $Ext(K)$. Recall that a nonzero vector x in K is an extreme vector if the equality $x = y + z$ with $y, z \in K$ holds only when y and z are nonnegative multiples of x.
With $\mathcal{L}(H)$ denoting the Banach space of all (bounded) linear transformations on H with operator norm, we let, for any set K in H,

- $\Pi(K) := \{L \in \mathcal{L}(H) : L(K) \subseteq K\}$.
- $Aut(K) := \{L \in \mathcal{L}(H) : L \,\text{is invertible and}\, L(K) = K\}$.

We denote the closure of $Aut(K)$ in $\mathcal{L}(H)$ by $\overline{Aut(K)}$.

Throughout this paper, we assume that

- *K is a generic nonempty set in H,*
- *$\mathcal{C}$ is a closed cone in $\mathbb{R}^n$ that is not necessarily convex, and*
- *the associated cones $\mathcal{K}$ and $\mathcal{E}$ in $\mathcal{S}^n$ are given, respectively, by (1.1) and (1.2).*

While all our results are valid for any proper cone $\mathcal{C}$ and the corresponding completely positive cone, in order to have some generality, we impose weaker conditions on the closed cone $\mathcal{C}$ such as: (i) $\mathcal{C}$ is pointed, (ii) $\mathcal{C}$ has interior, (iii) $\mathcal{C}^*$ has interior, (iv) $\mathcal{C}\backslash\{0\}$ is connected, (v) $int(\mathcal{C})$ is connected.

We begin by presenting some lemmas that are needed in the paper. The first lemma is well known and easy to prove (see the proof of Theorem 2.2 in [5]).

LEMMA 2.1. *Suppose $\mathcal{C}$ is a closed cone in $\mathbb{R}^n$ with nonempty interior and $A \in \mathcal{E}$. Let $u \in int(\mathcal{C})$ with $u^TAu = 0$. Then $A \in \mathcal{S}^n_+$ and $Au = 0$.*

LEMMA 2.2. *Suppose K is a closed pointed cone in H with interior and $L \in \Pi(K)$. If $L(u) = 0$ for some $u \in int(K)$, then $L = 0$.*

Proof. Let $x \in H$ and $u \in int(K)$ with $L(u) = 0$. Then for all small $\varepsilon > 0$, $u + \varepsilon\, x, u - \varepsilon\, x \in K$. Since $L \in \Pi(K)$, we must have $\varepsilon L(x) = L(u + \varepsilon\, x) \in K$ and $-\varepsilon L(x) = L(u - \varepsilon\, x) \in K$. Thus, $L(x) \in K \cap -K = \{0\}$. Since x is arbitrary, we see that $L = 0$. ∎

LEMMA 2.3. *Let K be closed cone in H such that K and K^* have nonempty interiors. Suppose $L \in \overline{Aut(K)}$ such that for some $d \in int(K)$, $L(d) \in int(K)$. Then $L \in Aut(K)$. In particular, this conclusion holds if K is a proper cone.*

Proof. Let $L_k \in Aut(K)$ such that $L_k \to L$ in $\mathcal{L}(H)$. Then $L_k^T \to L^T$. Note that $L_k \in Aut(K) \Rightarrow L_k^T \in Aut(K^*)$. Fix $u \in int(K^*)$ and let $x_k := (L_k^T)^{-1}(u)$ so that $x_k \in K^*$ and $L_k^T(x_k) = u$ for all $k = 1, 2, \ldots$. We claim that the sequence x_k is bounded. Assuming the contrary, let, without loss of generality, $||x_k|| \to \infty$ and $\lim \frac{x_k}{||x_k||} = y \in K^*$. Then $L^T(y) = 0$ and $0 = \langle L^T(y), d\rangle = \langle y, L(d)\rangle > 0$ (the last inequality holds since K is a closed cone, $0 \neq y \in K^*$ and $L(d) \in int(K)$), which is a contradiction. Now, as x_k is bounded, we may assume that $x_k \to x \in K^*$. Then $L^T(x) = u$. This means that the range of L^T contains an open set and hence L^T is invertible. It follows that L is invertible. From $L_k \to L$ and $L_k^{-1} \to L^{-1}$, we see that $L \in Aut(K)$. ∎

As a simple consequence, we have

COROLLARY 2.4. *Suppose $\mathcal{C}$ is a closed cone in $\mathbb{R}^n$ $(n > 1)$, such that $\mathcal{C}$ and $\mathcal{C}^*$ have nonempty interiors. Then for any $v \in int(\mathcal{C})$, $vv^T \not\in \overline{Aut(\mathcal{C})}$.*

Remark. The above Lemma and its corollary may not hold for a closed cone whose dual has empty interior. For example, when K is the closed upper-half plane in $\mathbb{R}^2$, every element of $Aut(K)$ is of the form (see Example 4 in [14])

$$A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix},$$

where $a \neq 0$ and $c > 0$. Clearly the coordinate vector $e_2 \in int(K)$ and $e_2 e_2^T \in \overline{Aut(K)}$, but not invertible.

3. Some general properties. In this section, we collect some elementary properties of the completely positive cone $\mathcal{K}$.

PROPOSITION 3.1. *Let $\mathcal{C}$ be a closed cone in $\mathbb{R}^n$. Then the following statements hold:*

- *(i) $\mathcal{E}$ and $\mathcal{K}$ are closed convex cones in $\mathcal{S}^n$, and $\mathcal{K} \subseteq \mathcal{S}^n_+ \subseteq \mathcal{E}$.*
- *(ii) $\mathcal{K}$ is pointed.*
- *(iii) $\mathcal{E}$ is the dual of $\mathcal{K}$.*
- *(iv) If $\mathcal{C}$ has nonempty interior, then $\mathcal{K}$ and $\mathcal{E}$ are proper cones. Converse holds if $\mathcal{C}$ also convex.*

Proof. All four statements except the converse in (iv) are covered in [14], Prop. 5. Suppose $\mathcal{C}$ is convex and let $\mathcal{E}$ be proper. If $\mathcal{C}$ does not have any interior, then (the subspace) $\mathcal{C} - \mathcal{C} \neq \mathbb{R}^n$. Let $0 \neq v \in (\mathcal{C} - \mathcal{C})^\perp$ and $A := vv^T$. Then A is nonzero and $\langle A, uu^T \rangle = 0$ for all $u \in \mathcal{C}$. This implies that $A, -A \in (\mathcal{K})^* = \mathcal{E}$ contradicting the properness of $\mathcal{E}$. Hence $\mathcal{C}$ must have interior. ∎

The proof of the following result is similar to the one given for the completely positive cone of $\mathbb{R}^n_+$, see [3], [8], [5]. For the sake of completeness, we include a proof.

THEOREM 3.2. *Let $\mathcal{C}$ be a closed cone. Then*

$$Ext(\mathcal{K}) = \left\{ uu^T : 0 \neq u \in \mathcal{C} \right\}.$$

Moreover, if $\mathcal{C}$ has nonempty interior, then $int(\mathcal{K}) = \mathcal{M}$, where

$$\mathcal{M} = \left\{ \sum_1^N u_i u_i^T \,:\, span\{u_1, \ldots, u_N\} = \mathbb{R}^n, u_i \in \mathcal{C} \;\forall\, i \text{ and } u_j \in int(\mathcal{C}) \text{ for some } j \right\}.$$

Proof. The statement about the extreme vectors of $\mathcal{K}$ is covered in [14], Prop. 7. Now suppose that $\mathcal{C}$ has nonempty interior. Note that $\mathcal{M} \subseteq \mathcal{K}$. Consider any nonzero $A \in \mathcal{E}$. Then $\langle A, X \rangle \geq 0$ for all $X \in \mathcal{M}$. If $\langle A, X \rangle = 0$, say, for some $X = \sum_1^N u_i u_i^T \in \mathcal{M}$, then $u_i^T A u_i = 0$ for all i; As some $u_j \in int(\mathcal{C})$, by Lemma 2.1, $A \in \mathcal{S}^n_+$ and hence (whether u_i belongs to $int(\mathcal{C})$ or not), $Au_i = 0$ for all i. Since the vectors u_i span $\mathbb{R}^n$, we must have $A = 0$, contradicting our choice of A. Thus, for any nonzero $A \in \mathcal{E}$, $\langle A, X \rangle > 0$ for all $X \in \mathcal{M}$. As $\mathcal{K}^* = \mathcal{E}$ and both cones are proper (by the previous result) , we must have (see [2], page 3)

$$int(\mathcal{K}) = \{ Z : \langle A, Z \rangle > 0 \;\forall\; A \in \mathcal{E} \setminus \{0\} \}$$

and so $\mathcal{M} \subseteq int(\mathcal{K})$. Now to see the reverse inclusion, let $Y \in int(\mathcal{K})$, $X \in \mathcal{M}$, $X \neq Y$. Since $\mathcal{K}$ is convex, we can extend the line segment joining X and Y (slightly) beyond Y to get $Z \in \mathcal{K}$ such that Y is a convex combination of X and Z. Because of the form of X and Z, this convex combination is in $\mathcal{M}$. This completes the proof. ∎

Remark. By using Lemma 3.7 in [5], we can actually state the following.
When the closed cone $\mathcal{C}$ has nonempty interior,

$$int(\mathcal{K}) = \left\{ \sum_1^N u_i u_i^T : span\{u_1, u_2, \ldots, u_N\} = \mathbb{R}^n,\ u_i \in int(\mathcal{C}) \text{ for all } i \right\}.$$

4. Irreducibility. A closed cone K in H is said to be *reducible* in H if there exist nontrivial (i.e., nonzero) closed cones K_1 and K_2 and subspaces H_1 and H_2 in H such that $K_1 \subseteq H_1$, $K_2 \subseteq H_2$, with

$$K = K_1 + K_2, \quad H = H_1 + H_2, \quad \text{and} \quad H_1 \cap H_2 = \{0\}.$$

If K is not reducible, we say that it is *irreducible.*

THEOREM 4.1. *Let $\mathcal{C}$ be a closed cone in $\mathbb{R}^n$. If $\mathcal{C}$ has nonempty interior, then $\mathcal{K}$ is irreducible.*

Proof. Suppose int($\mathcal{C}$) is nonempty and, if possible, $\mathcal{K}$ is reducible. Let K_i and H_i be as in the definition of reducibility with $H = \mathcal{S}^n$. For any $0 \neq u \in \mathcal{C}$, $uu^T \in \mathcal{K}$. If $uu^T = x_1 + x_2$, where $x_i \in K_i \subseteq \mathcal{K}$ for $i = 1, 2$, then, because uu^T is an extreme vector of $\mathcal{K}$ and $K_1 \cap K_2 = \{0\}$, we must have $x_1 = uu^T$ (say) and $x_2 = 0$. This shows that each uu^T belongs to K_1 or to K_2. Let

$$\mathcal{C}_1 := \{u \in \mathcal{C} : uu^T \in K_1\} \quad \text{and} \quad \mathcal{C}_2 := \{u \in \mathcal{C} : uu^T \in K_2\}.$$

It is clear that $\mathcal{C}_1$ and $\mathcal{C}_2$ are closed and $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$. Since $\mathcal{C}$ has nonempty interior, one of the sets, say, $\mathcal{C}_1$ has nonempty interior in $\mathcal{C}$ as well as in $\mathbb{R}^n$. (This follows from, for example, Baire category Theorem.) Then the completely positive cone $\mathcal{K}_1$ generated by $\mathcal{C}_1$ within $\mathcal{S}^n$ is proper (by Prop. 3.1), and in particular, $\mathcal{K}_1 - \mathcal{K}_1 = \mathcal{S}^n$. This implies that $K_1 - K_1 = \mathcal{S}^n$, $H_1 = \mathcal{S}^n$, and $H_2 = \{0\}$. This contradiction completes the proof. ∎

Remark. The result may not hold if $\mathcal{C}$ has empty interior: In $\mathbb{R}^2$, consider the standard unit vectors e_1 and e_2 and let $\mathcal{C} = \{\lambda\, e_1, \mu\, e_2 : \lambda, \mu \geq 0\}$ so that the corresponding completely positive cone is given by $\mathcal{K} = \{\lambda\, e_1 e_1^T + \mu\, e_2 e_2^T : \lambda, \mu \geq 0\}$. Clearly, K is reducible.

5. Self-duality. Recall that a cone K is *self-dual* in H if $K^* = K$.

THEOREM 5.1. *Let $\mathcal{C}$ be a closed cone in $\mathbb{R}^n$. Then $\mathcal{K}$ is self-dual if and only if $\mathbb{R}^n = \mathcal{C} \cup -\mathcal{C}$.*

Proof. Suppose $\mathcal{K}$ is self-dual. Then the inclusions $\mathcal{K} \subseteq \mathcal{S}^n_+ \subseteq \mathcal{E}$ imply that $\mathcal{K} = \mathcal{S}^n_+ = \mathcal{E}$. Now consider any nonzero $x \in \mathbb{R}^n$. Then $xx^T \in Ext(\mathcal{S}^n_+) = Ext(\mathcal{K})$. By a known characterization of $Ext(\mathcal{K})$, see Prop. 7 in [14], $xx^T = uu^T$ for some $0 \neq u \in \mathcal{C}$. By Prop. 6 in [14], $x = \pm u \in \mathcal{C}$. Thus, every x in $\mathbb{R}^n$ belongs to $\mathcal{C} \cup -\mathcal{C}$.
Now conversely, suppose $\mathbb{R}^n = \mathcal{C} \cup -\mathcal{C}$. Then the completely positive cone of $\mathcal{C}$, by the spectral theorem for real symmetric matrices, is $\mathcal{S}^n_+$ which is self-dual. ∎

COROLLARY 5.2. *Suppose $n > 1$ and $\mathcal{C}$ is a closed pointed cone. Then $\mathcal{K}$ cannot be self-dual.*

Proof. If $\mathcal{K}$ is self-dual, then $\mathbb{R}^n = \mathcal{C} \cup -\mathcal{C}$ and so $\mathbb{R}^n \setminus \{0\} = \mathcal{C} \setminus \{0\} \cup -(\mathcal{C} \setminus \{0\})$. As the sets $\mathcal{C} \setminus \{0\}$ and $-(\mathcal{C} \setminus \{0\})$ are separated (in the sense that each set is disjoint from the closure of the other) and $\mathbb{R}^n \setminus \{0\}$ is connected, we reach a contradiction. ∎

Remark. Suppose $\mathcal{C}$ is a closed convex cone. Then $\mathbb{R}^n = \mathcal{C} \cup -\mathcal{C}$ if and only if $\mathcal{C} = \mathbb{R}^n$ or $\mathcal{C}$ is a closed half-space. This statement is well-known and easy to prove: If the origin is an interior point of $\mathcal{C}$, then $\mathcal{C} = \mathbb{R}^n$. Now suppose that the origin is a boundary point of $\mathcal{C}$ so that there is a supporting hyperplane induced by a nonzero vector $d \in \mathbb{R}^n$, that is, $\mathcal{C} \subseteq \{x \in \mathbb{R}^n : \langle x, d \rangle \geq 0\}$. Now for any $y \in \{x \in \mathbb{R}^n : \langle x, d \rangle \geq 0\} \setminus \mathcal{C}$ we have $-y \in \mathcal{C}$ and so $-\langle y, d \rangle \geq 0$. For any such y, $\langle y, d \rangle = 0$. Now, the sequence $y + \frac{1}{k}d$ belongs to $\{x \in \mathbb{R}^n : \langle x, d \rangle \geq 0\}$ and is not orthogonal to d. Thus, $y + \frac{1}{k}d \in \mathcal{C}$ for all k and taking limits, we see that $y \in \mathcal{C}$, which is a contradiction. Thus, $\mathcal{C} = \{x \in \mathbb{R}^n : \langle x, d \rangle \geq 0\}$.

6. Homogeneity. Recall that a closed convex cone K in H with a nonempty interior is said to be *homogeneous* if for every $x, y \in int(K)$, there exists $A \in Aut(K)$ such that $Ax = y$. For a detailed study of homogeneous cones, see [17]. When $\mathcal{C}$ is a closed cone with $\mathcal{C} \cup -\mathcal{C} = \mathbb{R}^n$, the corresponding completely positive cone (namely, $\mathcal{S}^n_+$) is homogeneous and self-dual (that is, a symmetric cone). Theorem 5.1 shows that the converse of this statement holds.

We now address the question of (non)homogeneity of the completely positive cone $\mathcal{K}$.

THEOREM 6.1. *Suppose $\mathcal{C}$ is a closed pointed cone in $\mathbb{R}^n$ ($n > 1$) such that $\mathcal{C}$ and $\mathcal{C}^*$ have nonempty interiors and $\mathcal{C}\setminus\{0\}$ is connected. Then $\mathcal{K}$ cannot be homogeneous. In particular, this conclusion holds if $\mathcal{C}$ is a proper cone.*

Proof. Suppose that $\mathcal{K}$ is homogeneous. Pick $u_1, u_2, \ldots, u_n$ and v in $int(\mathcal{C})$ such that $\{u_1, u_2, \ldots, u_n\}$ and $\{v, u_2, \ldots, u_n\}$ are bases in $\mathbb{R}^n$. Put $X := u_1u_1^T + u_2u_2^T + \cdots + u_nu_n^T$ and for any natural number k, $Y_k := vv^T + \frac{1}{k}(u_2u_2^T + \cdots + u_nu_n^T)$. Then, by Theorem 3.2, X and Y_k are in $int(\mathcal{K})$. By assumption, there exists $L_k \in Aut(\mathcal{K})$ such that $L_k(X) = Y_k$ for all k. Since $\mathcal{C}$ is a closed pointed cone such that $int(\mathcal{C})$ is nonempty and $\mathcal{C}\backslash\{0\}$ is connected, by Theorem 2 in [14], there exists $Q_k \in Aut(\mathcal{C})$ such that $L_k(Z) = Q_kZQ_k^T$ for all $Z \in \mathcal{S}^n$; in particular, $L_k(X) = Q_kXQ_k^T$. This implies

$$Q_k(u_1u_1^T + u_2u_2^T + \cdots + u_nu_n^T)Q_k^T = vv^T + \frac{1}{k}(u_2u_2^T + \cdots + u_nu_n^T)$$

for all k. We now consider two cases.
Case (i) : The sequence Q_k is unbounded.
In this case, we may let $||Q_k|| \to \infty$ and $\frac{Q_k}{||Q_k||} \to Q \in \overline{Aut(\mathcal{C})}$. This leads to

$$Q(u_1u_1^T + u_2u_2^T + \cdots + u_nu_n^T)Q^T = 0$$

and, upon simplification, to $Qu_i = 0$ for all i. As $\{u_1, u_2, \ldots, u_n\}$ spans $\mathbb{R}^n$, we see that $Q = 0$ leading to a contradiction (as norm of Q is one). Thus, this case cannot happen.
Case (ii)*:* The sequence Q_k is bounded.
In this case, we may assume that $Q_k \to Q \in \overline{Aut(\mathcal{C})}$. This leads to

$$Q(u_1u_1^T + u_2u_2^T + \cdots + u_nu_n^T)Q^T = vv^T.$$

Now, $vv^T \in Ext(\mathcal{K})$ (see Theorem 3.2) and $Qu_i \in \mathcal{C}$ for every i. (Note that $Q_k(u_i) \in \mathcal{C}$ for each i.) Thus, by definition of extreme vector, Qu_i is a multiple of v for each i. Since $Q \neq 0$ and $\{u_1, u_2, \ldots, u_n\}$ spans $\mathbb{R}^n$, the range of Q is one-dimensional and so Q is of rank one. Let $Qu_1 = \lambda v$. Then $\lambda \neq 0$ by Lemma 2.2 (applied to $\mathcal{C}$ and Q in place of K and L). Also, the pointedness of $\mathcal{C}$ implies that λ cannot be negative. Thus, $Qu_1 \in int(\mathcal{C})$ and $Q \in \overline{Aut(\mathcal{C})}$. As $u_1 \in int(\mathcal{C})$, by Lemma 2.3 (applied to $\mathcal{C}$ and Q in place of K and L), Q is invertible. But this cannot happen as Q has rank one and $n > 1$. Thus, even this case cannot happen. We conclude that $\mathcal{K}$ is not homogeneous. ■

To illustrate the above result, we consider the following example, where the underlying cone $\mathcal{C}$ is not convex.

Example 1. Let S (inside $\mathbb{R}^2_+$) be the union of two closed convex cones S_1 and S_2, where S_1 is generated by $(1,0)$ and $(2,1)$, and S_2 is generated by $(0,1)$ and $(1,2)$. Then $\mathcal{C} = \overline{\{1\} \times S} = \mathbb{R}_+ \times S$ (that appears in the Introduction) is pointed, has nonempty interior, $\mathcal{C}\backslash\{0\}$ is connected, and $int(\mathcal{C}^*) = \mathbb{R}_{++} \times int(S^*)$ is nonempty.

The following corollary is immediate from the above theorem. However, we give an independent and slightly different proof.

COROLLARY 6.2. *For any $n > 1$, the completely positive cone of $\mathbb{R}^n_+$ is not homogeneous.*

Proof. Let $X = [x_{ij}]$ and $Y = [y_{ij}]$ be in $int(\mathcal{K})$, where $\mathcal{K}$ is the completely positive cone of $\mathbb{R}^n_+$ and assume that there is an automorphism $L \in Aut(\mathcal{K})$ such that $L(X) = Y$. By Theorem 2 in [14], there is a $Q \in Aut(\mathbb{R}^n_+)$ such that $L(Z) = QZQ^T$ for all $Z \in \mathcal{S}^n$; in particular, $Y = L(X) = QXQ^T$. Since every element of $Aut(\mathbb{R}^n_+)$ is a product of a permutation and a diagonal matrix with positive diagonals, we must have, for some $i \neq j$ and positive numbers r_i and r_j,

$$\begin{bmatrix} r_i^2 x_{ii} & r_i r_j x_{ij} \\ r_i r_j x_{ij} & r_j^2 x_{jj} \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{12} & y_{22} \end{bmatrix}.$$

This implies (as all entries of X and Y are positive) that

$$\frac{y_{11}y_{22}}{y_{12}^2} \in \left\{ \frac{x_{ii}x_{jj}}{x_{ij}^2} : i \neq j \right\}. \tag{6.1}$$

Now, we construct specific X and Y violating this property.
Let $e_1, e_2, \ldots, e_n$ denote the standard coordinate vectors in $\mathbb{R}^n$, and let e be the vector of all ones. Let

$$X = ee^T + e_1e_1^T + e_2e_2^T + \ldots + e_{n-1}e_{n-1}^T$$

so that $X \in int(\mathcal{K})$ and $\{\frac{x_{ii}x_{jj}}{x_{ij}^2} : i \neq j\} = \{2, 4\}$. With $f = [1\,2\,3 \cdots n]^T$, let

$$Y = ff^T + e_1e_1^T + e_2e_2^T + \ldots + e_{n-1}e_{n-1}^T$$

so that $\frac{y_{11}y_{22}}{y_{12}^2} = \frac{5}{2}$. For the above X and Y, (6.1) is violated and hence X cannot be mapped onto Y by any automorphism of $\mathcal{K}$. Thus, $\mathcal{K}$ is not homogeneous. ∎

Remark. That the completely positive cone of $\mathcal{C} = \mathbb{R}^2_+$ is not homogeneous can also be seen by Vinberg's classification of homogeneous cones [17]: In R^3, there are two non-isomorphic homogeneous cones, namely, R^3_+ and the positive semidefinite cone $\mathcal{S}^2_+$ (or the second order cone $\mathcal{L}^3_+$). By a comparison of the extreme vectors, we conclude that the completely positive cone of $\mathcal{C} = \mathbb{R}^2_+$ is not isomorphic to one of these.

Consider a closed cone K in H with interior. For each $x \in int(K)$, let

$$[x] := \{L(x) : L \in Aut(K)\} \subseteq int(K)$$

denote the orbit of x under the automorphism group $Aut(K)$. Note that $int(K)$ is a disjoint union of such orbits and K is homogeneous if and only if there is only one orbit in $int(K)$. The following result, perhaps known, sheds some light on the nature and number of orbits.

PROPOSITION 6.3. *Let K be a closed pointed cone in H such that K and K^* have nonempty interiors. Then, for any $x \in int(K)$, $[x]$ is a closed subset of $int(K)$. Moreover, if K is not homogeneous and $int(K)$ is connected, then there are an uncountable number of orbits in $int(K)$. In particular, this conclusion holds when K is a proper cone.*

Proof. Fix $x \in int(K)$ and a sequence $x_k \in [x]$ with $\lim x_k = y \in int(K)$. We show that $y \in [x]$.
By definition, there exist $L_k \in Aut(K)$ such that $L_k(x) = x_k$ and so $y = \lim\, L_k(x)$. We consider two cases.
Case 1: Assume that the sequence L_k is bounded and let $L_k \to L \in \overline{Aut(K)}$. Then, $y = L(x)$ with $x, y \in int(K)$ and $L \in \overline{Aut(K)}$. By Lemma 2.3, $L \in Aut(K)$. Thus, $y \in [x]$.
Case 2: Suppose that the sequence L_k is unbounded.
Then we may assume that $||L_k|| \to \infty$ and $\frac{L_k}{||L_k||} \to L \in \overline{Aut(K)} \subseteq \Pi(K)$. Then $L(x) = 0$. By Lemma 2.2, $L = 0$, which is clearly a contradiction. Thus, this case is not possible, and hence $[x]$ is closed in $int(K)$.
Now, suppose that K is not homogeneous and there are a countable number of orbits. Then, $int(K)$ can be written as a disjoint union of countable number (more than one) closed sets (orbits) within $int(K)$. Since $int(K)$ is locally compact, by Baire category Theorem (see [15], Theorem 2.2), there is one orbit whose interior is nonempty in $int(K)$. By considering the union of images of this interior under various automorphisms, we conclude that this orbit is also open in $int(K)$. Thus, this orbit is both open and closed, contradicting the connectedness of $int(K)$. This proves that there must be an uncountable number of orbits in $int(K)$. Finally, when K is a proper cone which is not homogeneous, all the conditions listed in the proposition hold and the result follows. ∎

The following corollary is immediate.

COROLLARY 6.4. *For any proper cone $\mathcal{C}$ in $\mathbb{R}^n$ $(n > 1)$, the number of orbits in $int(\mathcal{K})$ (induced by $Aut(\mathcal{K})$) is uncountable.*

7. The copositive cone of $\mathcal{C}$. Based on the results we have obtained so far, we can record some properties of the copositive cone $\mathcal{E}$ corresponding to a closed cone $\mathcal{C}$.

THEOREM 7.1.

(i) $\mathcal{E}$ *is self-dual if and only if* $\mathbb{R}^n = \mathcal{C} \cup -\mathcal{C}$.

(ii) *If* $\mathcal{C}$ *has nonempty interior, then* $\mathcal{E}$ *is irreducible.*

(iii) *If* $\mathcal{C}$ *is a proper cone in* $\mathbb{R}^n$ $(n > 1))$, *then* $\mathcal{E}$ *is not homogeneous and* $int(\mathcal{E})$ *contains uncountable number of orbits* (*induced by* $Aut(\mathcal{E})$).

Proof. (i) Since $\mathcal{E}$ is self-dual if and only if $\mathcal{K}$ is self-dual, the result follows from Theorem 5.1.
(ii) Suppose $\mathcal{C}$ is a closed cone with nonempty interior. Then $\mathcal{K}$ is irreducible from Theorem 4.1. Hence its dual $\mathcal{E}$ is also irreducible ([2], Page 20).
(iii) Suppose $\mathcal{C}$ is a proper cone. Then, by Theorem 6.1, $\mathcal{K}$ is not homogeneous. Then its dual $\mathcal{E}$ is also not homogeneous, by a result of Vinberg (Prop. 9 in [17]). The uncountability of the orbits come from the previous proposition. ∎

Concluding Remarks. In this paper, we studied some properties of a completely positive cone in $\mathcal{S}^n$ that arises from a closed cone in $\mathbb{R}^n$. We discussed self-duality, irreducibility, and homogeneity properties of such a cone. Our results are, in particular, applicable to the cone $\mathcal{K}$ that comes from $\mathcal{C} = \overline{\{1\} \times S}$ (mentioned in the Introduction). Note that when S is a closed cone, $\mathcal{C} = \overline{\{1\} \times S} = \mathbb{R}_+ \times S$ is a closed cone. In this setting, one can show that $\mathcal{C}$ inherits certain properties of S. For example, if S is pointed, so is $\mathcal{C}$; if S (S^*) has nonempty interior, so does $\mathcal{C}$ (respectively, $\mathcal{C}^*$), etc. Also, $\mathcal{C}\backslash\{0\}$ is always (path) connected. We end this paper with two examples.

Example 2. Let S in $\mathbb{R}^2$ be the union of closed unit disc (centered at the origin) and a finite set of points outside S. Then S looks like 'Sun with planets' and $\mathcal{C} = \overline{\{1\} \times S}$ looks like an 'ice-cream cone with whiskers'. In this case, $\mathcal{C}$ is pointed, has nonempty interior, $\mathcal{C}\backslash\{0\}$ is not connected, $\mathcal{C}^*$ has nonempty interior, and $int(\mathcal{C})$ is connected.

Example 3. Let S in $\mathbb{R}^2$ be the union of closed unit disc and a finite set of rays emanating from the origin. Then S looks like 'Sun with (some) rays' and $\mathcal{C} = \overline{\{1\} \times S}$ looks like an 'ice-cream cone with wings'. In this case, (depending on the rays of S) $\mathcal{C}$ can be pointed, has nonempty interior, $\mathcal{C}\backslash\{0\}$ is connected, $\mathcal{C}^*$ (which is just a ray) has empty interior, and $int(C)$ is connected.

REFERENCES

[1] A. Baker. *Matrix Groups.* Springer, London 2002.

[2] A. Berman and R. J. Plemmons. *Nonnegative Matrices in Mathematical Sciences.* SIAM, Philadelphia, 1994.

[3] A. Berman and N. Shaked-Monderer. *Completely Positive Matrices.* World Scientific, New Jersey, 2003.

[4] S. Burer. On the copositive representation of binary and continuous nonconvex quadratic programs. *Mathematical Programming, Series A,* 120: 479-495, 2009.

[5] P.J.C. Dickinson. An improved characterization of the interior of the completely positive cone. *Electronic Journal of Linear Algebra*, 20: 723-729, 2010.

[6] P.J.C. Dickinson. Geometry of copositive and completely positive cones. *J. Math. Anal. Appl.*, 380: 377-395, 2011.

[7] M. Dür. Copositive programming, a Survey. *www.optimization-online.org/DB_FILE/2009/11/2464.pdf*

[8] M. Dür and G. Still. Interior points of the completely positive cone. *Electronic Journal of Linear Algebra*, 17: 48-53, 2008.

[9] G. Eichfelder and J. Povh. On reformulations of nonconvex quadratic programs over convex cones by set-semidefinite constraints. *Research Report # 342, Department of Mathematics, University of Erlangen-Nuremberg, Erlangen, Germany*, December 8, 2010.

[10] G. Eichfelder and J. Povh. On the set-semidefinite representation of nonconvex quadratic programs over arbitrary feasible sets. *Research Report # 349, Department of Mathematics, University of Erlangen-Nuremberg, Erlangen, Germany*, June 30, 2011.

[11] J. Faraut and A. Korányi. *Analysis on symmetric cones.* Clarendon Press, Oxford, 1994.

[12] M.S. Gowda and T. Parthasarathy. Complementarity forms of theorems of Lyapunov and Stein, and related results. *Linear Algebra and Its Applications*, 320: 131–144, 2000.

[13] M.S. Gowda and Y. Song. On semidefinite linear complementarity problems. *Mathematical Programming*, Series A, 88: 575–587, 2000.

[14] M.S. Gowda, R. Sznajder and J. Tao. The automorphism group of a completely positive cone and its Lie algebra. To appear in Linear Algebra and its Applications. Available at URL www.math.umbc.edu/∼gowda/papers/papers.html.

[15] W. Rudin *Functional Analysis*. McGraw-Hill, New York 1973.

[16] L. Vandenberghe and S. Boyd. Semidefinite Programming. *SIAM Review*, 38(1): 49-95, 1996.

[17] E.B. Vinberg. The theory of convex homogeneous cones. *Trans. Moscow Math. Soc*, 12: 340-403, 1963.