On the bilinearity rank of a proper cone and Lyapunov-like transformations

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Abstract

A real square matrix Q is a *bilinear complementarity relation* on a proper cone K in R^n if

$$x \in K, s \in K^*, \text{ and } \langle x, s\rangle = 0 \Rightarrow x^T Q s = 0,$$

where K^* is the dual of K [14]. The *bilinearity rank* of K is the dimension of the space of all bilinear complementarity relations on K. In this article, we continue the study initiated in [14] by Rudol et al. We show that bilinear complementarity relations are related to Lyapunov-like transformations that appear in dynamical systems and in complementarity theory and further show that the bilinearity rank of K is the dimension of the Lie algebra of the automorphism group of K. In addition, we correct a result of [14], compute the bilinearity ranks of symmetric and completely positive cones, and state Schur-type results for Lyapunov-like transformations.

1 Introduction

For a proper cone K with dual K^* in R^n, the *complementarity set* of K is

$$C(K) := \{(x, s) : x \in K, s \in K^*, \langle x, s\rangle = 0\}.$$

It is known that this is an n-dimensional manifold homeomorphic to R^n [14]. Such a set arises, for example, in the context of primal and dual linear programming problems over a cone and in complementarity problems. In various strategies for solving the primal-dual cone-LP problems, one tries to write the optimality conditions in the form of a square system by replacing the complementarity constraints $x \in K, s \in K^*, \langle x, s\rangle = 0$ by n linearly independent bilinear relations. This idea is expounded in a recent article [14] where the following are introduced:
For the given proper cone K in R^n, an $n\times n$ (real) matrix Q is said to be a *bilinear complementarity relation* if

$$(x, s) \in C(K) \Rightarrow x^T Qs = 0$$

and the corresponding *bilinearity rank of* K is defined by

$$\beta(K) := dim\, Q(K),$$

where $Q(K)$ is the set of all bilinear complementarity relations for K. In [14], the bilinearity rank is computed for polyhedral cones, the cone of positive polynomials over R, and certain other related cones. In this paper, we relate the concepts of bilinear complementarity relation and bilinearity rank to the known concepts of 'Lyapunov-like transformation' and to the dimension of the Lie algebra of the automorphism group of K. To elaborate, consider a closed cone K (not necessarily convex) in a finite dimensional real Hilbert space H; let K^* denote the dual of K. We say that a linear transformation $L : H \to H$ is a *Lyapunov-like transformation* on K [7] if

$$x \in K, y \in K^*, \langle x, y\rangle = 0 \Rightarrow \langle L(x), y\rangle = 0.$$

Thus, when $H = R^n$ and K is a proper cone in R^n, a matrix Q is a bilinear complementarity relation if and only if Q^T is Lyapunov-like with respect to K. As we see below, Lyapunov-like transformations arise in different settings and have numerous properties.

(1) Let $H = \mathcal{S}^n$ (the space of all real $n \times n$ symmetric matrices with trace inner product) and $K = \mathcal{S}^n_+$ (the positive semidefinite cone). Then for any matrix $A \in R^{n\times n}$, the Lyapunov transformation L_A defined by

$$L_A(X) := AX + XA^T \quad (X \in \mathcal{S}^n)$$

is Lyapunov-like on $\mathcal{S}^n_+$ [9]. In fact, every Lyapunov-like transformation on $\mathcal{S}^n_+$ arises this way [4].

(2) In the case of $H = R^n$ and $K = R^n_+$, Lyapunov-like transformations are nothing but diagonal matrices.

(3) Given a proper cone K in H, a linear transformation L on H is said to have the **Z**-property on K if

$$x \in K, y \in K^*, \langle x, y\rangle = 0 \Rightarrow \langle L(x), y\rangle \leq 0.$$

This **Z**-property is a generalization of the **Z**-property of a matrix [2] and was introduced in [15] in the form of cross-positive matrices. We note that L is Lyapunov-like on K if and only if both L and $-L$ have the **Z**-property on K.

(4) For a proper cone K in H and a linear transformation L on H,

$$L \text{ is Lyapunov-like on } K \Leftrightarrow e^{tL} \in Aut(K) \text{ for all } t \in R \Leftrightarrow L \in Lie(Aut(K)),$$

where $Aut(K)$ denotes the automorphism group of K (these are invertible linear transformations that keep K invariant) and $Lie(Aut(K)$ denotes the corresponding Lie algebra, see Theorem 3 in [15] for the first equivalence and and Section 7.6 in [1] for the second equivalence.

(5) When K is the symmetric cone of a simple Euclidean Jordan algebra V, L is Lyapunov-like if and only if it is of the form $L_a + D$, where $a \in V$ with $L_a(x) := a \circ x$ and D is a derivation on V, see [10] and Prop. VIII.2.6 in [5].

(6) **Z** and Lyapunov-like transformations appear in the study of dynamical systems [13] and complementarity problems [7], [10].

In view of Item (4) above, we may redefine the bilinearity rank of K as

$$\beta(K) = dim\, LL(K) = dim\, Lie(Aut(K)),$$

where $LL(K)$ denotes the space of all Lyapunov-like transformations on K. (We may thus call $\beta(K)$, the *Lyapunov rank* of K.)

The objective of this paper is to continue the study initiated in [14], compute $\beta(K)$ for various K, and describe some properties of Lyapunov-like transformations. Among other things,

- We show that on a proper polyhedral cone K, a transformation L is Lyapunov-like if and only if every extreme vector of K is an eigenvector of L,
- We correct a result of [14] by showing that for proper polyhedral cones in R^n, $\beta(K)$ can be any number between (and including) 1 and n, except $(n-1)$,
- We compute the bilinearity rank of symmetric cones in simple Euclidean Jordan algebras,
- We compute the bilinearity rank of the cone of completely positive matrices, and
- We prove Schur-type results for Lyapunov-like transformations on (some particular) proper cones: If L is Lyapunov-like on a proper irreducible cone in H and $L(K) \subseteq K$, then L is a multiple of the identity transformation.

2 Some preliminaries

Throughout this paper, $(H, \langle\cdot,\cdot\rangle)$ denotes a finite dimensional real Hilbert space. When $H = R^n$ with the usual inner product, we sometimes write $x^T y$ in place of $\langle x, y\rangle$. A set K in H is said to be a cone if $\lambda x \in K$ whenever $x \in K$ and $\lambda \geq 0$ in R. In this paper, *K denotes a closed convex cone in H* with dual K^* given by

$$K^* := \{x \in H : \langle x, y\rangle \geq 0 \text{ for all } y \in K\}.$$

K is said to be *proper* if it pointed (which means $K \cap -K = \{0\}$) and has nonempty interior. For a closed convex cone K, a closed convex subcone F of K is said to be a *face* of K if

$$y, z \in K \text{ and } y + z \in F \Rightarrow y, z \in F.$$

A nonzero vector x in K is said to be an *extreme vector* of K if the ray $\{\lambda x : \lambda \geq 0\}$ is a face of K. We let $Ext(K)$ denote the set of all extreme vectors of K. Given a linear transformation L on H and a face F of K, the *principal subtransformation* induced by F is $L_F : Span(F) \to Span(F)$ defined by

$$L_F(x) = P_F(L(x)) \quad (x \in Span(F)),$$

where P_F denotes the orthogonal projection from H onto $Span(F) := F - F$.
We denote the space of all (continuous) linear transformations on H by $\mathcal{L}(H)$. A linear transformation L on H that is invertible and maps K onto K is an *automorphism* of K. We denote the group of all automorphisms of K by $Aut(K)$. Its corresponding Lie algebra is given by

$$Lie(Aut(K)) = \{L \in \mathcal{L}(H) : e^{tL} \in Aut(K)\ \forall t \in R\}.$$

We say that a closed convex cone K is *reducible* in H if there are closed convex cones K_1 and K_2 and subspaces S_1 and S_2 in H such that $K = K_1 + K_2$, $Span(K_1) \subseteq S_1$, $Span(K_2) \subseteq S_2$ $H = S_1 + S_2$, and $S_1 \cap S_2 = \{0\}$.

Throughout this paper, $e_1, e_2, \ldots, e_n$ denote the standard coordinate vectors in R^n.

3 The bilinearity rank of a proper polyhedral cone

As noted in the Introduction, on R^n_+, a matrix is Lyapunov-like if and only if it is a diagonal matrix. (This can be easily seen from Item (3) in the Introduction.) Thus, on R^n_+, a matrix A is Lyapunov-like if and only if every extreme vector of R^n_+ is an eigenvector of A. It turns out that a similar result holds for any proper polyhedral cone.

Theorem 1 *Suppose K is a proper polyhedral cone in H. Then, a linear transformation L is Lyapunov-like on K if and only if every extreme vector of K is an eigenvector of L.*

Proof. Since K is polyhedral, it is generated by a finite number of extreme vectors. Let $d_1, d_2, \cdots, d_k$ denote a collection of distinct, non-proportional extreme vectors generating K. First suppose that L is Lyapunov-like on K. We show that d_1 is an eigenvector of L (with a similar argument for other extreme vectors).

By a result of Schneider and Vidyasagar [15] (mentioned in Item (4) of the Introduction), $e^{tL} \in Aut(K)$ for all $t \in R$ and hence extreme vectors are preserved under any e^{tL}. In particular, (as e^{tL} is always invertible) for every t, $e^{tL}(d_1)$ belongs to the disjoint union of rays $\{\lambda d_i : \lambda > 0\}$, $1 \leq i \leq k$. Since $\{e^{tL}(d_1) : t \in R\}$ is connected, it must be contained in one of these rays. As $e^{tL}(d_1) = d_1$ for $t = 0$, this ray must be $\{\lambda d_1 : \lambda > 0\}$. It follows that $e^{tL}(d_1) = \lambda(t) d_1$ for every t, where $\lambda(t)$ is a real valued function of t. It is easy to see that $\lambda(t)$ is differentiable. Differentiating both sides of the relation $e^{tL}(d_1) = \lambda(t) d_1$ and putting $t = 0$, we get $L(d_1) = \lambda'(0)\, d_1$ (with prime denoting the derivative). We see that d_1 is an eigenvector of L. This completes the proof of the 'only if' part.
Now suppose that every extreme vector of K is an eigenvector of L. Writing $L(d_i) = \mu_i\, d_i$ for every i, we get $e^{tL}(d_i) = e^{t\mu_i} d_i \in K$ for every i. As K is the cone-convex hull of its extreme vectors, $e^{tL}(K) \subseteq K$ for all t and consequently, $e^{tL} \in Aut(K)$. Once again, by the result of Schneider and Vidyasagar [15], L is Lyapunov-like on K. This completes the proof. □

We now deal with the computation of the bilinearity rank of a proper polyhedral cone in R^n. In doing so, we will correct a result given in [14], where it is claimed that the bilinearity rank of a proper polyhedral cone in R^n with more than n extreme vectors is one.

Theorem 2 *The following statements hold:*

(i) For any proper polyhedral cone K in R^n, $1 \leq \beta(K) \leq n$, $\beta(K) \neq n-1$.

(ii) For any natural number m with $1 \leq m \leq n$, $m \neq n-1$, there is a proper polyhedral cone K in R^n with $\beta(K) = m$.

Proof. (i) Let K be a proper polyhedral cone in R^n with (distinct, non-proportional) extreme vectors $f_1, f_2, \ldots, f_l$. As the cone is proper, $l \geq n$ and there are n linearly independent vectors among these vectors. By means of an invertible matrix, we can map K into another proper polyhedral cone whose extreme vectors are $e_1, e_2, \ldots, e_n$ and (possibly vacuous) $d_1, d_2, \ldots, d_{l-n}$. Since the bilinearity rank is preserved under an isomorphism, we may assume that our cone is generated by these (new) vectors. Now, by the previous theorem, any Lyapunov-like matrix on K is diagonal (as the standard coordinate vectors are eigenvectors). Since any such matrix is a linear combination of at most n linearly independent diagonal matrices, we see that $1 \leq \beta(K) \leq n$. Note that when the set $\{d_1, d_2, \ldots, d_{l-n}\}$ is vacuous, K (which is now R^n_+) has bilinearity rank n.
We now claim that $\beta(K) \neq n-1$. As this is obvious for $n = 1$ or 2 (in these cases, $K = R^n_+$ and $\beta(K) = n$), we assume that $n \geq 3$. Suppose if possible, $\beta(K) = n-1$. This implies that $\{d_1, d_2, \ldots, d_{l-n}\}$ is non-vacuous.
Now, every Lyapunov-like matrix on K is diagonal. By identifying such a matrix with its diagonal,

we may regard the space $LL(K)$ as a subspace S of R^n of dimension $(n-1)$. Now, by taking a nonzero vector $r=(r_1,r_2,\ldots,r_n)\in R^n$ that is orthogonal to S, we may write

$$S=\{x=(x_1,x_2,\ldots,x_n):\sum r_ix_i=0\}.$$

As $r\neq 0$, we may solve for one of the variables x_i in terms of the others; we see that there is a vector in S with $(n-1)$ or more components nonzero and distinct. Thus (by renaming the coordinate vectors, if necessary), we may assume that there is a Lyapunov-like (diagonal) matrix on K given by $L=diag(\lambda_1,\lambda_2,\ldots,\lambda_{n-1},\lambda_n)$, where the first $n-1$ entrees are all nonzero and distinct, and the last entry is either different from the previous ones or equal to λ_{n-1}. Now let $v\in\{d_1,d_2,\ldots,d_{l-n}\}$ and $L(v)=\mu v$ for some real μ. Let $v_1,v_2,\ldots,v_n$ be the components of v. Then $L(v)=\mu v$ implies that $(\lambda_i-\mu)v_i=0$ for all i. As v is nonzero, $\mu=\lambda_i$ for some i. Let

$$I=\{i:\lambda_i=\mu\}.$$

If all the λs are distinct, then I contains only one element. In this case, v has only one nonzero component and v must be a multiple of some, say, e_k. This would contradict either the pointedness of K or the assumption that $v\in\{d_1,d_2,\ldots,d_{l-n}\}$. Now consider the case when $\lambda_1,\lambda_2,\ldots,\lambda_{n-1}$ are distinct and $\lambda_{n-1}=\lambda_n$. In this case, I cannot intersect both $\{1,2,\ldots,n-2\}$ and $\{n-1,n\}$. If $I\subseteq\{1,2,\ldots,n-2\}$, then I is a singleton set, v is a multiple of some e_k and a contradiction ensues. Now consider the last case $I=\{n-1,n\}$, that is, $\mu=\lambda_{n-1}=\lambda_n$. Then all components of v except the last two components of v vanish. As v is not a multiple of either e_{n-1} or e_n, these last two components are nonzero. They cannot be both positive, as that would say that v is a nonnegative combination of e_{n-1} and e_n (contradicting the assumption that v is an extreme vector). If these two components have opposite signs, say $v_{n-1}>0$ and $v_n<0$, then we can write $e_{n-1}=\frac{1}{v_{n-1}}v+\frac{-v_n}{v_{n-1}}e_n$. This is a contradiction as e_{n-1} is an extreme vector of K. Thus, we can never have $\beta(K)=n-1$.

(ii) Let m be a natural number with $1\leq m\leq n$, $m\neq n-1$. For $n=1$ or 2, proper polyhedral cones in R^n are isomorphic to R^n_+ and so $\beta(K)=n$. For $m=n$, we take $K=R^n_+$; In this case, $\beta(K)=n$. Suppose $n\geq 3$, $1\leq m\leq n-2$ and let $k:=m-1$ so that $0\leq k\leq n-3$. We define a vector d in R^n by specifying its components in the following way: We let $d_i=0$ for $1\leq i\leq k$, $d_i=1$ for $k+1\leq i\leq n-1$ and $d_n=-1$. Note that in d there are at least two ones and exactly one negative one. When $k=0$, all components of d are nonzero. Also, when $k\geq 1$, d has at least one zero component. We let K be the convex cone generated by $\{e_1,e_2,\ldots,e_n,d\}$. Clearly, K has nonempty interior. Before we show that K is pointed and $e_1,e_2,\ldots,e_n,d$ are its extreme vectors, we verify the following: Suppose we have numbers $x_1,x_2,\ldots,x_n,x_{n+1}$ such that

$$x_1e_1+x_2e_2+\cdots+x_ne_n+x_{n+1}d=0.$$

Then $x_i=0$ for $1\leq i\leq k$, $x_i+x_{n+1}=0$ for $k+1\leq i\leq n-1$ and $x_n-x_{n+1}=0$. Note that the case of exactly one x_i negative and others nonnegative is not possible. Also, all the x_is reduce to zero when every x_i is nonnegative. These imply that K is pointed and $e_1,e_2,\ldots,e_n,d$ are its extreme vectors. This shows that K is a proper polyhedral cone in R^n. Now, suppose L is Lyapunov-like on

K. Then, by the previous Theorem, L must be a diagonal matrix and $L(d) = \mu d$ for some $\mu \in R$. If $L = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$, then $\lambda_i = \mu$ for every index i for which $d_i \neq 0$ and with no relation among the other λ_is. It follows that

- When every component of d is nonzero (which corresponds to $k = 0$), L is a multiple of the Identity matrix. In this case, $LL(K)$ is one-dimensional and hence $\beta(K) = 1$.
- If k (> 0) components of d are zero, then $k+1$ diagonal entries in L can be chosen arbitrarily; hence in this case, $\beta(K) = k + 1 = m$.

Thus, we have constructed a proper polyhedral cone K in R^n, for which $\beta(K)$ is m or any specified number m between (and including) 1 and n, but not equal to $n - 1$. This completes the proof. □

Remarks. The (last part of the) proof of the above theorem reveals the following: Let K be a proper polyhedral cone K with extreme vectors $f_1, f_2, \ldots, f_l$, where $l > n$ and $f_1, f_2, \ldots, f_n$ are linearly independent. Suppose there is an extreme vector $f \in \{f_{n+1}, f_{n+2}, \ldots, f_l\}$ such that if $f = \sum_1^n \alpha_i f_i$, then $\alpha_i \neq 0$ for all i. Then $\beta(K) = 1$.
It is possible to extend this result somewhat. Write any $f \in \{f_{n+1}, f_{n+2}, \ldots, f_l\}$ as $f = \sum_1^n \alpha_i f_i$, and let $I(f) := \{i : \alpha_i \neq 0\}$. Suppose we can find a collection of these index sets, say, $I_1, I_2, \ldots, I_m$ ($m \leq l - n$) such that $\bigcup_1^m I_k = \{1, 2, \ldots, n\}$ and $I_k \cap I_{k+1} \neq \emptyset$ for all $k = 1, 2, \ldots, m - 1$. Then $\beta(K) = 1$. This can be seen by assuming (without loss of generality) that $f_i = e_i$ for $1 \leq i \leq n$, looking at a diagonal matrix for which every f_i ($n + 1 \leq i \leq l$) is an eigenvector.

4 The bilinearity rank of a symmetric cone

In this section, we compute the bilinearity rank of a symmetric cone. Since the bilinearity rank is additive on Cartesian products (see Proposition 9 in [14]), it is enough to describe it for irreducible symmetric cones. We recall that there are five irreducible symmetric cones, each being the cone of squares in a simple Euclidean Jordan algebra [5]. Below, we list these irreducible cones and their bilinearity rank. First, we recall a result from [10]:

Theorem 3 *Let V be a Euclidean Jordan algebra with the corresponding symmetric cone K. Then the following are equivalent for a linear transformation L on V:*

(i) L is Lyapunov-like on K.

(ii) $L \in Lie(Aut(K))$.

(iii) $L = L_a + D$, where $a \in V$ and D is a inner derivation.

Here, a linear transformation D on V is said to be a **derivation** if for all $x, y \in V$,

$$D(x \circ y) = D(x) \circ y + x \circ D(y).$$

It is said to be *inner* if it is a linear combination of commutators of the form

$$L_a L_b - L_b L_a$$

for some a, b in V. It is known (see [5], Prop. VIII.2.6) that D is a derivation if and only if it is in $Lie(Aut(K))$ and either skew-hermitian (that is, $D + D^T = 0$) or $D(e) = 0$, where e is the unit element in V. In view of this, $\beta(K) = dim\,\{L_a : a \in V\} + dim\,Der(V)$, where $Der(V)$ denotes the space of all derivations on V. Since $a \to L_a$ is one-to-one on V, we see that

$$\beta(K) = dim(V) + dim\,Der(V).$$

Since $dim(V)$ and $dim\,Der(V)$ are known for simple algebras (see [5], pages 6 and 97), we get the following:

(i) In $Herm(R^{n\times n})$ (which is $\mathcal{S}^n$), $\beta(K) = n^2$.

(ii) In $Herm(C^{n\times n})$, $\beta(K) = 2n^2 - 1$.

(iii) In $Herm(Q^{n\times n})$, $\beta(K) = 4n^2$.

(iv) In $Herm(O^{3\times 3})$, $\beta(K) = 79$.

(v) In $\mathcal{L}^n$, $\beta(K) = \frac{n^2-n+2}{2}$.

In the above list, $Herm(\mathcal{F}^{n\times n})$ denotes the set of all Hermitian matrices with entries from $\mathcal{F}$, where F denotes one of the following sets: real numbers R, complex numbers C, quaternions Q, and octonions O. The symbol $\mathcal{L}^n$ denotes the (Jordan) spin algebra.

5 The bilinearity rank of a completely positive cone

Let C be a proper cone in R^n. Then the corresponding completely positive cone in $\mathcal{S}^n$ is defined by

$$K := \left\{\sum uu^T : u \in C\right\},$$

with $\sum uu^T$ denoting a finite sum. As $LL(C) = Lie(Aut(C))$ and $LL(K) = Lie(Aut(K))$ (see the Introduction), a recent result of Gowda, Sznajder, and Tao [12] says that $LL(C)$ and $LL(K)$ are isomorphic. This implies that

$$\beta(K) = \beta(C).$$

As an illustration, let $C = R^n_+$. In this case, K is the cone of completely positive matrices. Since a matrix is Lyapunov-like on R^n_+ if and only if it is a diagonal matrix, it follows that $\beta(R^n_+) = n$. Thus, the bilinearity rank of the cone of completely positive matrices is n.

6 Some Schur-type results

In various algebraic settings, Schur-type results say that under certain conditions, a linear transformation is a multiple of the Identity transformation. For example, in Lie algebra theory, we have the following result (see Lemma 7.13 in [16]): Let V be a complex Lie algebra and S be a finite dimensional irreducible V-module. A map $L : S \to S$ is a V-module homomorphism if and only if L is a scalar multiple of the Identity transformation.

We now present a Schur-type result for Lyapunov-like transformations.

Theorem 4 *Suppose K is an irreducible proper cone in H, L is Lyapunov-like on K, and $L(K) \subseteq K$. Then under each of the following conditions, L is a multiple of the Identity transformation.*

(i) Every principal subtransformation of L is Lyapunov-like.

(ii) K is a symmetric cone in a simple Euclidean Jordan algebra.

(iii) $H = \mathcal{S}^n$ and K is the completely positive cone corresponding to a proper cone in R^n.

We begin with the following Lemma.

Lemma 1 *Suppose K is a closed convex cone with interior in the (real finite dimensional) Hilbert space H and L is a linear transformation on H. Suppose further that every extreme vector of K is an eigenvector of L. Then*

$$K = K_1 \oplus K_2 \oplus \cdots \oplus K_m,$$

where each K_i is a closed convex cone contained in an eigenspace of L. In particular, if K is irreducible, then L multiple of the identity transformation.

Proof of the Lemma. Assume that every extreme vector of K is an eigenvector of L. Let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be the distinct real eigenvalues of L that correspond to various extreme vectors of K. (Note that H is finite dimensional and K may have infinitely many extreme vectors.) Let for each $i \in \{1, 2, \ldots, N\}$,

$$E_i := \{x \in H : L(x) = \lambda_i x\} \quad \text{and} \quad K_i := K \cap E_i.$$

Clearly, each K_i is a closed convex cone in H, $\sum_1^N K_i \subseteq K$, and $K_i \cap K_j = \{0\}$ for all $i \neq j$. We show that $\sum_1^N K_i = K$, proving the lemma. To see this equality, let $x \in K$. By Krein-Milman Theorem (see Theorem 1.33 in [3]), K is the closure of the convex-conic hull of $Ext(K)$. Thus, we may write $x = \lim y_m$, where each y_m is given by, thanks to Carathéodory's Theorem (see Theorem 1.34 in [3]) and grouping,

$$y_m = x_1^{(m)} + x_2^{(m)} + \cdots + x_N^{(m)},$$

and $x_i^{(m)} \in K_i$ for $m = 1, 2, \ldots$. Since the spaces E_i (which correspond to distinct eigenvalues) are linearly independent, for each i, the projection mapping from $E_1 \oplus E_2 \oplus \cdots \oplus E_N$ to E_i is well defined and linear, hence continuous. Since the sequence y_m is Cauchy, the sequence $x_i^{(m)}$ is Cauchy in K_i for each i. Letting $x_i^{(m)} \to x_i$ in K_i, we see that $x = x_1 + x_2 + \cdots + x_N$. Thus, $K \subseteq \sum_1^N K_i$, proving the required equality. Finally, $H = K - K = (K_1 - K_1) + (K_2 - K_2) + \cdots + (K_N - K_N) \subseteq E_1 + E_2 + \cdots + E_N$ shows that K is the direct sum of K_i, $i = 1, 2, \ldots, N$. When K is irreducible, there is only one K_i

and thus L is a multiple of the Identity transformation. This completes the proof of the Lemma. $\square$

Proof of the Theorem. We assume that all (general) conditions on L and K are in place.

(i) Assume that every principal subtransformation of L is Lyapunov-like. Let $d \in K^*$ and consider the face $F := d^\perp \cap K$ in K. Then for any $x \in F$, $\langle L(x), d\rangle = 0$ (by the Lyapunov-like property of L on K); hence $L(x) \in d^\perp$ and $L(x) \in K$ (as $L(K) \subseteq K$). It follows that

$$L(d^\perp \cap K) \subseteq d^\perp \cap K. \tag{1}$$

In view of this inclusion, $L(F) \subseteq F$ and the given assumption says that (restricted) $L : F - F \to F - F$ is Lyapunov-like on F.
We now show that each extreme vector of K is an eigenvector of L. Let $u \in Ext(K)$. As u is on the boundary of K, there is a supporting hyperplane at u: There exists nonzero d_1 in H such that $\langle u, d_1\rangle = 0$ and $K \subseteq \{x : \langle x, d_1\rangle \geq 0\}$. This means that $d_1 \in K^*$ and $\langle u, d_1\rangle = 0$. Hence, $u \in F_1 := d_1^\perp \cap K$ and by (1), $L(F_1) \subseteq F_1$. Note that the cone F_1 is proper in $H_1 := F_1 - F_1$ and L restricted to H_1 is Lyapunov-like. Also, $dim(H_1) < dim(H)$. As u is an extreme vector of F_1, we can repeat the above argument by replacing K and H by F_1 and H_1. This results in the vector d_2, face $F_2 = d_2^\perp \cap F_1$ and the subspace $H_2 = F_2 - F_2$. Note that F_2 is a face of K and $L(F_2) \subseteq F_2$. Furthermore, $dim(H_2) < dim(H_1) < dim(H)$. We continue this procedure till we get a one-dimensional subspace H_m. At this stage, $u \in F_m$ and L maps H_m to itself. It follows that $L(u) = \mu u$ for some μ proving the assertion that u is an eigenvector of L. Now, the conclusion that L is a multiple of the Identity transformation follows from the previous Lemma.

(ii) We assume that K is a symmetric cone in a simple Euclidean Jordan algebra V. Note that K is irreducible [5]. We show that for any face F of K, the principal subtransformation L_F is Lyapunov-like. Fix any face F of K. Then there is an idempotent c in V such that F is the symmetric cone in the (sub)algebra $V(c, 1) := \{x \in V : x \circ c = x\}$ (see [6]). Since F is self-dual in $V(c, 1)$, to verify the Lyapunov-like property of L_F, take $x, y \in F$ with $\langle x, y\rangle = 0$. As x and y belong to K (which is self-dual), we have $\langle L(x), y\rangle = 0$. Denoting the projection (mapping) of V onto $V(c, 1)$ by P_c, we see that $\langle L_F(x), y\rangle = \langle P_c(L(x)), y\rangle = \langle L(x), P_c(y)\rangle = \langle L(x), y\rangle = 0$ (as the projection mapping is self-adjoint). Thus, L_F is Lyapunov-like on F. We can now apply Item (i) and get the conclusion that L is a multiple of the Identity transformation.

(iii) Now suppose that $H = \mathcal{S}^n$ and K is the completely positive cone corresponding to a proper cone C in R^n. This means, see Section 5, that

$$K := \left\{\sum uu^T : u \in C\right\}.$$

It is known that K is irreducible [8]. Since K is a proper cone, see [12], and L is Lyapunov-like on K, it follows from the result of Schneider and Vidyasagar (mentioned in the Introduction) that $L \in Lie(Aut(K))$. By a recent result of Gowda, Sznajder, and Tao, see [12], we can write $L = L_A$, where $A \in Lie(Aut(C))$ and L_A is defined on $\mathcal{S}^n$ by $L_A(X) := AX + XA^T$. We now show that every extreme vector of K is an eigenvector of L_A. Then, by an application of the previous Lemma, we get the required result. Now, the extreme vectors of the completely positive cone K are known: They are of the form uu^T for some nonzero $u \in C$ (see Proposition 7, [12]). As $L(K) \subseteq K$, we have

$$Auu^T + uu^T A^T = \sum_1^N v_i v_i^T,$$

for some $v_i \in C$. Now for any vector $x \perp u$ in R^n,

$$0 = x^T(Auu^T + uu^T A^T)x = \sum_1^N (x^T v_i)^2.$$

This implies that $x \perp v_i$ for all i. Since x is arbitrary, we conclude that each v_i is a multiple of u. Thus, $L_A(uu^T) = \mu\, uu^T$ for some $\mu \geq 0$. This implies that uu^T is an eigenvector of L_A with corresponding eigenvalue μ. $\square$

Remarks. While proving Item (ii) in the above theorem, we appealed to Item (i). Direct proofs avoiding Item (i) can be given.

In the first proof, we show that when L is Lyapunov-like on a symmetric cone K (in a simple algebra) and $L(K) \subseteq K$, every extreme vector of K is an eigenvector of L and then appeal to the above Lemma. In the second proof, we prove the result directly without even using the above Lemma.

Assume that K is a symmetric cone in a simple Euclidean Jordan algebra and L is Lyapunov-like on K. Let u be an extreme vector of K. It is known that such a u must be a positive multiple of a primitive idempotent, say, c in V. We assume that V carries the canonical/trace inner product (see [5] and Proposition 2, [10]) so that the norm of any primitive idempotent is one.

Proof 1. Since L is Lyapunov-like on the symmetric cone K, we may write (see Section 4) $L = L_a + D$, where $a \in V$ and D is a derivation on V. Since $D(e) = 0$, we see that $L(e) = a \in K$. For any primitive idempotent c in V, let $c = e_1$, where $\{e_1, e_2, \ldots, e_r\}$ is a Jordan frame in V. Let the Peirce decomposition of $L(e_1)$ be given by

$$L(e_1) = \sum_{i \leq j} x_{ij}.$$

Since $\langle L(e_1), e_j \rangle = 0$ for all $j \neq 1$ and $L(e_1) \in K$, we see that $x_{ij} = 0$ except for $x_{11} = \lambda e_1$ for some $\lambda \geq 0$. Thus, for each primitive idempotent c, $L(c) = \lambda c$ for some $\lambda \geq 0$. $\square$

Proof 2. As in the previous proof, $L = L_a + D$ so that $a = L(e)$. We know from the previous proof that for each primitive idempotent c, there is a $\lambda \geq 0$ such that $L(c) = \lambda c$. Then $\lambda = \langle L(c), c \rangle = \langle L_a(c) + D(c), c \rangle = \langle a \circ c, c \rangle = \langle a, c^2 \rangle = \langle a, c \rangle$, where we have used the facts that D is skew-symmetric. Hence, $L(c) = \langle a, c \rangle\, c$ for every primitive idempotent c.
Now, since L is Lyapunov-like and K is self-dual, L^T is also Lyapunov-like on K; hence $L + L^T$ is Lyapunov-like on K. Moreover, $L(K) \subseteq K$ implies that $L^T(K) \subseteq K$ and $(L + L^T)(K) \subseteq K$.

Since $L + L^T = 2L_a$, we see that $L_a(K) \subseteq K$. By Corollary 3.2 in [11], $L_a = \mu\, Id$ on V for some $\mu \in R$. This implies that $2\mu||c||^2 = \mu\langle (L + L^T)(c), c\rangle = 2\langle L(c), c\rangle = 2\langle a, c\rangle ||c||^2$. Thus, $\langle a, c\rangle = \mu$ and $L(c) = \mu c$ for any primitive idempotent c. Finally for any $x \in V$, we write the spectral decomposition $x = \sum x_i f_i$ to get $L(x) = \sum x_i(\mu f_i) = \mu x$. This gives the stated conclusion. □

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