Einstein-Hessian barriers on convex cones

Roland Hildebrand *

May 21, 2012

Abstract

On the interior of a regular convex cone $K \subset \mathbb{R}^n$ there exist two canonical Hessian metrics, the one generated by the logarithm of the characteristic function, and the Cheng-Yau metric. The former is associated with a self-concordant logarithmically homogeneous barrier on K with parameter of order $O(n)$, the universal barrier. This barrier is invariant with respect to the unimodular automorphism subgroup of K, is compatible with the operation of taking product cones, but in general it does not behave well under duality. In this contribution we introduce the barrier associated with the Cheng-Yau metric, the Einstein-Hessian barrier. It shares with the universal barrier the invariance, existence and uniqueness properties, is compatible with the operation of taking product cones, but in addition is invariant under duality. The Einstein-Hessian barrier can be characterized as the convex solution of the partial differential equation $\log\det F'' = 2F$ with boundary condition $F|_{\partial K} = +\infty$. Its barrier parameter does not exceed the dimension n of the cone. On homogeneous cones both barriers essentially coincide.

MSC: 90C51, 90C25
Keywords: interior-point methods, self-concordant barriers, convex cones

1 Introduction

Self-concordant logarithmically homogeneous barriers play a paramount role in the theory of interior-point methods for solving convex conic programs. Let $K \subset \mathbb{R}^n$ be a regular (with nonempty interior, containing no lines) convex cone. A self-concordant logarithmically homogeneous barrier on K is a smooth real-valued locally strongly convex function $F : K^o \to \mathbb{R}$ on the interior of the cone, satisfying the following properties [16, Section 2.3]. The second and third derivative have to satisfy the self-concordance relation

$$F'''(x)[h,h,h] \leq 2(F''(x)[h,h])^{3/2} \qquad \forall\ x \in K^o,\ h \in T_xK^o, \tag{1}$$

with h running through the tangent space at x. The function F has to tend to infinity as its argument tends to the boundary of the cone,

$$\lim_{x\to\partial K} F(x) = +\infty, \tag{2}$$

and it has to satisfy the logarithmic homogeneity condition

$$F(\alpha x) = -\nu \log\alpha + F(x) \qquad \forall\ \alpha > 0,\ x \in K^o. \tag{3}$$

The real constant ν is called the *barrier parameter* of the barrier F. The lower the barrier parameter of a barrier, the faster are the interior point algorithms based on this barrier. For conic optimization problems over a cone K, it is therefore desirable to have barriers on K with a barrier parameter as small as possible.

In [16, Section 2.5] Nesterov and Nemirovski introduce the *universal barrier*. This is a self-concordant logarithmically homogeneous barrier which exists and is unique, up to an additive constant[1], for any regular convex cone $K \subset \mathbb{R}^n$. Its barrier parameter is of order $O(n)$, i.e., there exists a

*LJK, Université Grenoble 1 / CNRS, 51 rue des Mathématiques, BP53, 38041 Grenoble cedex, France (roland.hildebrand@imag.fr).

[1] This constant is determined by the choice of the volume form on K^* when integrating (4).

constant $C > 0$, independent of K and n, such that $\nu = Cn$ [16, Theorem 2.5.1, Remark 2.5.1, p.50]. In [8, Cor. 4.1, p.868] Güler showed that the universal barrier on a regular convex cone K is, up to an additive constant, equal to C times the logarithm of the *characteristic function* of K. The latter object was introduced by Koranyi in [12] and is given by the expression

$$\varphi(x) = \int_{K^*} e^{-\langle x,y\rangle} dy \tag{4}$$

for $x \in K^o$. Here K^* is the dual cone of K. A similar function was introduced by Köcher in [11] on so-called *domains of positivity*, which are self-dual cones K[2] such that the isomorphism between K and K^* is a self-adjoint map. Some interesting properties of a particular level surface of the characteristic function, the *constant volume envelope*, have been deduced in [6, Sections 3,4]. From definition (4) it follows that the universal barrier on a product cone is the sum of the universal barriers on the factor cones, thus it is compatible with the operation of taking product cones. Moreover, it is invariant with respect to any automorphism of K with determinant 1, i.e., the unimodular automorphism subgroup. Note that this property ensures that for homogeneous cones, the level surfaces of the universal barrier are the orbits of the unimodular subgroup of automorphisms. The universal barrier does not behave well with respect to duality, however: in [21] an example of a self-dual cone was given where the universal barrier is not equal to its dual barrier.

In this contribution we present another self-concordant logarithmically homogeneous barrier which exists and is unique, up to an additive constant, for every regular convex cone. It can be obtained as the potential of a natural metric on the interior of K, the *Cheng-Yau metric*, which was first introduced in [5]. This metric is a so-called *Einstein-Hessian metric*, and for this reason we shall call the barrier *Einstein-Hessian barrier*. The Einstein-Hessian barrier shares with the universal barrier all invariance properties, but in addition it behaves well under duality. In particular, its level surfaces equal those of the universal barrier on homogeneous cones, and on this class of cones the two barriers are essentially the same object.

We define the Cheng-Yau metric and list its properties in Section 2. In Section 3 we will describe a close relation between the Einstein-Hessian barrier and *affine hyperspheres*, which are hypersurfaces in $\mathbb{R}^n$ studied for more than a century in affine differential geometry. In Sections 2, 3 we mostly recollect known results from the literature on differential geometry and also prove the invariance properties of the Einstein-Hessian barrier. As with the universal barrier, which can easily be obtained as the logarithm of the long-known characteristic function of K, the difficulty with the Einstein-Hessian barrier is to show that it is indeed a barrier, in particular, that it satisfies the self-concordance condition (1). The proof of this fact is the main result of the paper and will be accomplished in Section 4. It turns out that the barrier parameter of the Einstein-Hessian barrier equals the dimension n of K. Like with the universal barrier, it may be possible to lower the barrier parameter of the Einstein-Hessian barrier by multiplying it by a positive constant smaller than 1. The results of Sections 2, 3, 4 are summarized in Theorem 4.5. In addition, the Einstein-Hessian barrier has a transparent geometric interpretation as a minimal submanifold of a natural pseudo-Riemannian space form associated to the ambient space $\mathbb{R}^n$. This relation will be the subject of Section 5. Finally, in Section 6 we compute the Einstein-Hessian barrier on a class of cones including the 3-dimensional power cone in order to provide a nontrivial example.

2 The Cheng-Yau metric

In this section we introduce the *Cheng-Yau metric* on regular convex domains in $\mathbb{R}^n$ and collect some of its elementary properties. In order to make clear the origins of this metric, we will need to undertake an excursion into complex analysis. The use of complex variables will be limited to this section.

A *domain* $\Omega \subset \mathbb{R}^n$ or $\Omega \subset \mathbb{C}^n$ is an open connected set. A bijective holomorphic map $f : \Omega \to \Omega'$ between domains $\Omega, \Omega' \subset \mathbb{C}^n$ is called a *biholomorphism*. A convex domain $\Omega \subset \mathbb{R}^n$ is *regular* if it does not contain any line. The *tube domain* over a regular convex domain $\Omega \subset \mathbb{R}^n$ is the set $T_\Omega = \{z = x + iy \,|\, x \in \Omega,\ y \in \mathbb{R}^n\} = \Omega + i\mathbb{R}^n$. If Ω is the interior of a regular convex cone, then the tube domain over Ω is called *Siegel domain*.

[2] Here we understand self-dual in the wider sense that K is linearly isomorphic to its dual cone K^*.

Let $z^k = x^k + iy^k$, $k = 1, \dots, n$, be coordinates in $\mathbb{C}^n$. Since $\bar{z}^k = x^k - iy^k$, we have $\frac{\partial(z^k,\bar{z}^k)}{\partial(x^k,y^k)} = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$, $\frac{\partial(x^k,y^k)}{\partial(z^k,\bar{z}^k)} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$. On a complex domain $\Omega \subset \mathbb{C}^n$ it is more convenient to work with the differential operators $\frac{\partial}{\partial z^k} = \frac{1}{2}\left(\frac{\partial}{\partial x^k} - i\frac{\partial}{\partial y^k}\right)$, $\frac{\partial}{\partial \bar{z}^k} = \frac{1}{2}\left(\frac{\partial}{\partial x^k} + i\frac{\partial}{\partial y^k}\right)$ [19, p.18] instead of $\frac{\partial}{\partial x^k}, \frac{\partial}{\partial y^k}$. A map $\mathbb{C}^n \supset \Omega \ni z \mapsto w \in \tilde{\Omega} \subset \mathbb{C}^n$ is holomorphic if and only if $\frac{\partial w}{\partial \bar{z}} = 0$.

A Riemannian manifold M is called *complete* if every geodesic can be prolonged infinitely in both directions. A Riemannian metric g on a domain $\Omega \subset \mathbb{R}^n$ is called *Hessian metric* if it can be locally expressed as the Hessian of a real-valued function, $g = \sum_{i,j=1}^n \frac{\partial^2 F}{\partial x^i \partial x^j} dx^i dx^j$. The function F is called the (local) *potential* of the Hessian metric. A Riemannian metric g on a domain $\Omega \subset \mathbb{C}^n$ is called *Kählerian metric* if it can be locally expressed as the complex Hessian of a real-valued function, $g = \sum_{i,j=1}^n \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j} dz^i d\bar{z}^j$. The function φ is called the (local) *potential* of the Kählerian metric. A Hessian metric g with potential F on a regular convex domain $\Omega \subset \mathbb{R}^n$ defines a Kählerian metric g^T with potential $\varphi(x + iy) = 4F(x)$ on the tube domain T_Ω. On the other hand, every Kählerian metric g^T on T_Ω with potential $\varphi(x + iy)$ depending only on the real part x of the argument defines a Hessian metric g with potential $F(x) = \frac{1}{4}\varphi(x)$ on Ω (cf. [19, Prop. 2.6, p.20]). It is not hard to check that with the metrics g, g^T on Ω, T_Ω, respectively, the embeddings $f_c : \Omega \to T_\Omega$ defined by $f_c : x \mapsto x + ic$ are isometries for every $c \in \mathbb{R}^n$. Moreover, if g is complete, then g^T is complete too [5, p.344].

On a Riemannian manifold one can define different *curvature tensors*. One of them is the *Ricci tensor* Ric, which is, similar to the metric tensor, a symmetric second order tensor. If the Ricci tensor is proportional to the metric tensor, $\mathrm{Ric} = \lambda g$, then the manifold is called *Einstein*. For a Kählerian metric with potential φ, the Ricci tensor is given by [15, p.90]

$$\mathrm{Ric} = -\sum_{i,j=1}^n \frac{\partial^2 \log\det\left(\frac{\partial^2 \varphi}{\partial z^k \partial \bar{z}^l}\right)}{\partial z^i \partial \bar{z}^j} dz^i d\bar{z}^j. \tag{5}$$

Note that multiplication of the potential, and hence the metric with a positive constant does not change the Ricci tensor. Hence the proportionality constant λ can be normalized to $-1, 0, +1$ by an appropriate scaling of the metric. A manifold with a Kählerian metric which is Einstein is called *Kähler-Einstein*. The next result shows that Kähler-Einstein manifolds are quite rigid objects.

Proposition 2.1. *[4, Prop. 5.5, p.528] Let (M, g), (M', g') be two complete Kähler-Einstein manifolds with the same (negative[3]) proportionality constant λ. Then any biholomorphic map between M, M' is an isometry.*

In particular, a complex manifold admits at most one Kähler-Einstein metric with a given negative proportionality constant, and this metric must be invariant with respect to all biholomorphisms of the manifold.

We shall now consider the Kählerian metric g^T on a tube domain T_Ω defined by a Hessian metric g on Ω. From (5) it follows that $\mathrm{Ric} = \lambda g^T$ if and only if the potential F of g can be chosen such that

$$\log\det\left(\frac{\partial^2 F}{\partial x^k \partial x^l}\right) + 4\lambda F = 0. \tag{6}$$

In this case the metric g is called *Einstein-Hessian* [19, Def. 3.3, p.41]. Next we give an existence result.

Proposition 2.2. *[5, Theorem 4.4, p.365] Let $\Omega \subset \mathbb{R}^n$ be a regular convex domain. Then there exists a locally strongly convex smooth solution $F : \Omega \to \mathbb{R}$ to (6) with $\lambda = -\frac{1}{2}$, such that $\lim_{x \to \partial\Omega} F(x) = +\infty$ and the Einstein-Hessian metric defined by the potential F on Ω is complete.*

Thus on every regular convex domain $\Omega \subset \mathbb{R}^n$ there exists a smooth complete Einstein-Hessian metric g generating a Kähler-Einstein metric g^T on T_Ω with proportionality constant $-\frac{1}{2}$. Note that an affine automorphism $x \mapsto Ax + b$ of Ω extends to a biholomorphism $z \mapsto Az + b$ of the tube domain

[3] The proof of this result is based on [24, Theorem 3], where it is required that the bounds on the Ricci curvature are negative. Hence one must include the condition that the proportionality constant is negative, which was omitted in the original formulation. We will use this result only in the case of a negative proportionality constant.

T_Ω. Hence from Proposition 2.1 it follows that the Einstein-Hessian metric is unique and invariant with respect to all affine automorphisms of Ω. This metric is called the *Cheng-Yau metric*.

The uniqueness of the Cheng-Yau metric implies that for $\lambda < 0$ the locally strongly convex solution of (6) with boundary condition $F|_{\partial\Omega} = +\infty$ is also unique. If $x \mapsto \tilde{x} = Ax + b$ is an affine coordinate transformation of $\mathbb{R}^n$, then the solution $\tilde{F}(\tilde{x})$ of (6) in the new coordinates is given by

$$\tilde{F}(\tilde{x}) = F(x) + \frac{1}{2\lambda} \log|\det A|. \tag{7}$$

In particular, F remains invariant if $\det A = \pm 1$. Let us summarize the obtained results.

Proposition 2.3. *Let $\Omega \subset \mathbb{R}^n$ be a regular convex domain. Then the equation $\log\det\left(\frac{\partial^2 F}{\partial x^2}\right) = 2F$ on Ω with boundary condition $F|_{\partial\Omega} = +\infty$ has a unique smooth locally strongly convex solution. This solution is invariant with respect to all unimodular affine automorphisms of Ω and the Cheng-Yau metric $\sum_{i,j=1}^n \frac{\partial^2 F}{\partial x^i \partial x^j} dx^i dx^j$ defined on Ω by the potential F is complete.*

For domains Ω which can be represented as a product of two domains, we have the following evident result.

Lemma 2.4. *Let $\Omega_n \subset \mathbb{R}^n$, $\Omega_m \subset \mathbb{R}^m$ be regular convex domains, and let $F_n : \Omega_n \to \mathbb{R}$, $F_m : \Omega_m \to \mathbb{R}$ be the solutions of the boundary value problem in Proposition 2.3 for the domains Ω_n, Ω_m. Then the solution of this boundary value problem on the product domain $\Omega = \Omega_n \times \Omega_m$ is given by $F(x,y) = F_n(x) + F_m(y)$.* □

The solution F thus shares with the universal barrier of Nesterov and Nemirovski its existence and uniqueness as well as the invariance properties with respect to unimodular automorphisms and the behaviour under the operation of taking products of sets. In the next sections we shall show that it is indeed a barrier if Ω is a regular convex cone.

3 A splitting theorem

In this section we give an exposition of a relation between the potential F from Proposition 2.3 and a class of hypersurfaces in $\mathbb{R}^n$, the *affine hyperspheres*. We shall use this opportunity to present a general splitting theorem on the metrics generated by logarithmically homogeneous functions on regular convex cones, which can be applied to arbitrary barriers. The splitting theorem is formalized in Proposition 3.1. The other results shown in this section, Propositions 3.2 and 3.3, will be used in the subsequent section for showing invariance of the potential of the Cheng-Yau metric with respect to duality and its self-concordance, respectively.

Assume the conditions of Proposition 2.3 and suppose that the linear map $x \mapsto \tilde{x} = \alpha x$ is an automorphism of Ω for all $\alpha > 0$, i.e., Ω is the interior of a regular convex cone. For $\lambda = -\frac{1}{2}$ we obtain from (7) that $F(\alpha x) = F(x) - n \log\det\alpha$ for all $\alpha > 0$, i.e., F satisfies (3) with $\nu = n$. That the potential of the Cheng-Yau metric on regular convex cones is logarithmically homogeneous was already mentioned in [14, pp.426–427] without a detailed proof. In order to prove in the next section that F satisfies condition (1), we shall need a result of Loftin [14] on the Hessian metrics generated by general smooth locally strongly convex logarithmically homogeneous functions on the interior of regular convex cones.

To this end, we will need some basics of centro-affine geometry. Let $H \subset \mathbb{R}^n$ be a smooth locally strongly convex hypersurface such that the convex hull of H is separated from the origin, and such that every line through the origin intersects H at most once. Assume that if a line intersects H, then it intersects H transversally. Note that the level surfaces of a barrier on a regular convex cone $K \subset \mathbb{R}^n$ are of this type. On H there exists a metric h that is invariant with respect to linear automorphisms of $\mathbb{R}^n$, the *centro-affine metric*. The exact definition can be found in [17, p.15], but we will need only the following description. Choose a basis of $\mathbb{R}^n$ such that in the corresponding coordinate system $x^0, \dots, x^{n-1}$ the x^0-coordinate is strictly positive on H. To each point $x = (x^0, \dots, x^{n-1})^T \in H$ we can then assign the coordinate vector $y = (y^1, \dots, y^{n-1})^T \in \mathbb{R}^{n-1}$ with $y^k = \frac{x^k}{x^0}$, $k = 1, \dots, n-1$. The value of the coordinate x^0 on H then becomes a function f of y. We shall say that H is the *radial graph* of the function f. The centro-affine metric on H is then given by the expression $h = -u^{-1}\sum_{i,j=1}^{n-1}\frac{\partial^2 u}{\partial y^i \partial y^j}dy^i dy^j$ with $u = -\frac{1}{f}$.

We shall now return to logarithmically homogeneous functions. Let $F : K^o \to \mathbb{R}^n$ be a smooth locally strongly convex function on the interior of a regular convex cone K satisfying (3) with some $\nu > 0$. Let $F_\alpha = \{x \in K^o \,|\, F(x) = \alpha\}$ be the level surfaces of F. For every $\alpha \in \mathbb{R}$, there exists a diffeomorphism $I_\alpha : F_\alpha \times \mathbb{R}_+ \to K^o$ given by $I_\alpha : (x, s) \mapsto sx$. The following result states that the Hessian metric defined by F on K^o splits into a direct product under the inverse of I_α.

Proposition 3.1. *[14, Theorem 1, p.428] Assume the notations and conditions of the previous paragraph. Then the Riemannian manifold (K^o, F''), consisting of the interior of K equipped with the Hessian metric generated by F, is isometric under I_α^{-1} to the product $(F_\alpha, \nu h) \times (\mathbb{R}_+, \nu s^{-2}ds^2)$, where h is the centro-affine metric on F_α.*

Since all 1-dimensional Riemannian manifolds are locally isometric to the Euclidean space $\mathbb{R}^1$, all the information contained in the metric F'' is contained in the non-trivial $(n-1)$-dimensional factor and can thus be encoded by the centro-affine metric h of the level surfaces of F.

We shall now describe a link between the potential of the Cheng-Yau metric on the interior of a regular convex cone K and *affine hyperspheres*. Affine hyperspheres are particular hypersurfaces in $\mathbb{R}^n$ equipped with an intrinsic metric and known in affine differential geometry already for a century [22], [1]. Of interest for conic optimisation is the class of complete hyperbolic affine hyperspheres, which is related to regular convex cones by the *Calabi conjecture* [2, p.22]. The conjecture has been proven by the efforts of many authors, a synthesis of the proof is given in [13, Section 2]. The conjecture associates to every regular convex cone K a unique one-parametric family $(S_\alpha)_{\alpha>0}$ of complete hyperbolic affine hyperspheres. The hypersurfaces S_α are closed, smooth, locally strongly convex, and related by homothety, $S_\alpha = \alpha S_1$. Moreover, each of the hypersurfaces S_α is asymptotic to the boundary ∂K of K, and the union $\bigcup_{\alpha>0} S_\alpha$ equals the interior K^o of K. Choose a basis of $\mathbb{R}^n$ such that in the corresponding coordinates $x^0, \dots, x^{n-1}$ the affine section $\Omega = \{x \in K^o \,|\, x^0 = 1\}$ of the regular convex cone K is bounded, and let $y^k = x^k$, $k = 1, \dots, n-1$ be coordinates on Ω. The hypersurface S_1 can be defined as the radial graph of the function $f(y) = -\frac{1}{u(y)}$ [7, Prop. 1, p.391], where $u : \Omega \to \mathbb{R}$ is the unique locally strongly convex solution of the boundary value problem

$$\det u'' = (-u)^{-n-1}, \qquad u|_{\partial\Omega} = 0. \tag{8}$$

The parameter α is related to quantity called *mean curvature*, which measures in some sense the radius of the hypersphere. If $\mathbb{R}^n$ is equipped with the volume element emanating from the chosen basis, then the surface S_1 is an affine hypersphere with mean curvature $H = -1$ [7, Prop. 1, p.391], and its metric coincides with the centro-affine metric h [17, p.43].

Sasaki [18, pp.73–74] has shown that the level surfaces of the potential F from Proposition 2.3 are exactly the complete hyperbolic affine hyperspheres which are asymptotic to the cone K. More precisely, we have[4]

$$F(x^0, x^0y^1, \dots, x^0y^{n-1}) = -n\log(-x^0u(y)) + \frac{n}{2}\log n,$$

where u is the convex solution of (8), and the hypersurface S_1 is the level surface $F_{\frac{n}{2}\log n}$. This relation has been later rediscovered as the main result of [14].

Affine hyperspheres behave well under duality. In particular, we have the following result.

Proposition 3.2. *[7, Prop. 1, p.391] Let $M \subset \mathbb{R}^n$ be an affine hypersphere. Define a map $N : M \to \mathbb{R}_n$ into the dual space as follows. For $x \in M$, let $N(x)$ be the element $p \in \mathbb{R}_n$ such that the kernel of p is parallel to the tangent space to M at x and such that $\langle x, p\rangle = 1$. Then the image $N[M] \subset \mathbb{R}_n$ is also an affine hypersphere.*

The map defined in Proposition 3.2 is called the *conormal map* [17, p.57]. Note that the map $x \mapsto p = -F'(x)$ takes the level surfaces of F to the level surfaces of its Legendre transform $F^* : (K^*)^o \to \mathbb{R}$ [16, Section 2.4.3], and that $\langle p, x\rangle \equiv n$. From Proposition 3.2 it then follows that the level surfaces of F^* are also affine hyperspheres, and hence F^* is a potential of the Cheng-Yau metric on the dual cone

[4] In [18] the dimension of the cone is $n+1$ and the constant in the relation $\mathrm{Ric} = \lambda g$ is given by $\lambda = n+1$, but it is straightforward to derive the equation for the normalization considered here.

K^*. This is in contrast with the behaviour of the universal barrier, whose Legendre transform on the dual cone is in general not the universal barrier of the dual cone.

Of importance for our proof will be the following inequality on the Ricci tensor of the metric h on the level surfaces of F.

$$-(n-2)h \preceq \mathrm{Ric} \preceq 0. \tag{9}$$

The first inequality here comes from [2, eq. (2.7), p.24], the second one from [2, Theorem 5.1, p.31].

Combining (9) and Proposition 3.1 and taking into account that the curvature of a 1-dimensional Riemannian manifold vanishes identically and the Cheng-Yau metric is logarithmically homogeneous with $\nu = n$, we obtain the following result.

Proposition 3.3. *Let $K \subset \mathbb{R}^n$ be a regular convex cone and let $F : K^o \to \mathbb{R}$ be the potential of the Cheng-Yau metric $g = F''$ from Proposition 2.3. Then the Ricci tensor of this metric obeys the inequalities*

$$-\frac{n-2}{n} g \preceq \mathrm{Ric} \preceq 0,$$

and the radial direction is in the kernel of the Ricci tensor. □

4 Self-concordance

Let $K \subset \mathbb{R}^n$ be a regular convex cone and let g be the Cheng-Yau metric on K^o with potential F. We are now in a position to prove the self-concordance of F.

First we will need an explicit expression of the Ricci tensor. Since the Ricci tensor is the trace of the Riemann curvature tensor over the first and third index, the Ricci tensor of a general Hessian metric with potential $F(x)$ is given by [19, Prop. 2.3, p.15]

$$\mathrm{Ric} = \frac{1}{4} \sum_{i,j,l,r,t,s=1}^{n} \left(\frac{\partial^2 F}{\partial x^2}\right)^{-1}_{it} \left(\frac{\partial^2 F}{\partial x^2}\right)^{-1}_{rs} \left(\frac{\partial^3 F}{\partial x^t \partial x^l \partial x^r} \frac{\partial^3 F}{\partial x^s \partial x^j \partial x^i} - \frac{\partial^3 F}{\partial x^t \partial x^i \partial x^r} \frac{\partial^3 F}{\partial x^s \partial x^l \partial x^j} \right) dx^j dx^l.$$

Now note that the potential of the Cheng-Yau metric satisfies the relation $\log\det\left(\frac{\partial^2 F}{\partial x^2}\right) = 2F$. Differentiating this with respect to x, we obtain

$$\sum_{j,k=1}^{n} \left(\frac{\partial^2 F}{\partial x^2}\right)^{-1}_{jk} \frac{\partial^3 F}{\partial x^i \partial x^j \partial x^k} = 2\frac{\partial F}{\partial x^i}. \tag{10}$$

Together with the relations $\sum_{j=1}^n \left(\frac{\partial^2 F}{\partial x^2}\right)^{-1}_{ij} \frac{\partial F}{\partial x^j} = -x^i$ [16, eq. (2.3.12), p.41] and

$$\sum_{k=1}^{n} \frac{\partial^3 F}{\partial x^i \partial x^j \partial x^k} x^k = -2\frac{\partial^2 F}{\partial x^i \partial x^j}, \tag{11}$$

which follows from [16, eq. (2.3.12), p.41] by differentiation, we obtain

$$\mathrm{Ric} = \sum_{j,l=1}^{n} \left(\frac{1}{4} \sum_{i,r,t,s=1}^{n} \left(\frac{\partial^2 F}{\partial x^2}\right)^{-1}_{it} \left(\frac{\partial^2 F}{\partial x^2}\right)^{-1}_{rs} \frac{\partial^3 F}{\partial x^t \partial x^l \partial x^r} \frac{\partial^3 F}{\partial x^s \partial x^j \partial x^i} - \frac{\partial^2 F}{\partial x^j \partial x^l} \right) dx^j dx^l.$$

From Proposition 3.3 we then get by virtue of $g = \sum_{j,l=1}^n \frac{\partial^2 F}{\partial x^j \partial x^l} dx^j dx^l$ the matrix inequalities

$$\frac{8}{n}\frac{\partial^2 F}{\partial x^2} \preceq \left(\sum_{i,r,t,s=1}^{n} \left(\frac{\partial^2 F}{\partial x^2}\right)^{-1}_{it} \left(\frac{\partial^2 F}{\partial x^2}\right)^{-1}_{rs} \frac{\partial^3 F}{\partial x^t \partial x^l \partial x^r} \frac{\partial^3 F}{\partial x^s \partial x^j \partial x^i} \right)_{jl} \preceq 4\frac{\partial^2 F}{\partial x^2}.$$

Multiplying this with a unit length vector ξ (as measured in the metric $g = \frac{\partial^2 F}{\partial x^2}$) from the left and from the right, we obtain in the middle of the inequality the squared Frobenius norm $||\cdot||_F$ of the matrix product $(F'')^{-1/2} F'''[\cdot,\cdot,\xi] (F'')^{-1/2}$. Let us state this result separately.

Lemma 4.1. *Let $K \subset \mathbb{R}^n$ be a regular convex cone and let F be a potential of the Cheng-Yau metric on K^o. Then at every point $x \in K^o$ and for every unit length tangent vector ξ at x we have*

$$2\sqrt{\frac{2}{n}} \le ||(F'')^{-1/2} F'''[\cdot,\cdot,\xi](F'')^{-1/2}||_F \le 2.$$

For an arbitrary point $x \in K^o$, we shall now derive an upper bound on the quantity

$$\max_{\xi: F''(x)[\xi,\xi]=1} F'''(x)[\xi,\xi,\xi]. \tag{12}$$

Let us choose a basis of $\mathbb{R}^n$ such that in the corresponding coordinate system the matrix $F''(x)$ is the identity matrix I, the vector x is proportional to the basis vector e_0, and the maximizing tangent vector ξ in (12) is located in the plane spanned by e_0, e_1. Then from $F''(x)[x,x] = n$ [16, eq. (2.3.14), p.41] we have $x = (\sqrt{n}, 0, \dots, 0)^T$, from $F'(x) = -F''(x)x$ [16, eq. (2.3.12), p.41] we have $F'(x) = (-\sqrt{n}, 0, \dots, 0)^T$, from (11) we have $F'''(x)[\cdot,\cdot,e_0] = -\frac{2}{\sqrt{n}} I$, and from (10) we obtain $\operatorname{trace} F'''(x)[\cdot,\cdot,e_1] = 0$. Since $F'''(x)$ is symmetric in all three indices, we have that

$$F'''(x)[\cdot,\cdot,e_1] = \begin{pmatrix} 0 & -\frac{2}{\sqrt{n}} & 0 & \cdots & 0 \\ -\frac{2}{\sqrt{n}} & & & & \\ 0 & & & & \\ \vdots & & A & & \\ 0 & & & & \end{pmatrix}$$

for some traceless symmetric $(n-1)\times(n-1)$ matrix A. By Lemma 4.1 we then get $||F'''(x)[\cdot,\cdot,e_1]||_F^2 = \frac{8}{n} + ||A||_F^2 \le 4$. If $\lambda_1, \dots, \lambda_{n-1}$ are the eigenvalues of A, we hence have

$$\sum_{i=1}^{n-1} \lambda_i = 0, \qquad \sum_{i=1}^{n-1} \lambda_i^2 \le 4 - \frac{8}{n}.$$

Let without restriction of generality $|\lambda_{n-1}| = \max_i |\lambda_i|$. Then $|\sum_{i=1}^{n-2} \lambda_i| = |\lambda_{n-1}|$ and hence $\sum_{i=1}^{n-2} \lambda_i^2 \ge \frac{\lambda_{n-1}^2}{n-2}$. It follows that $\frac{(n-1)\lambda_{n-1}^2}{n-2} \le 4 - \frac{8}{n}$, which yields $\max_i |\lambda_i| \le \frac{2(n-2)}{\sqrt{n(n-1)}}$, and hence $|F'''(x)[e_1,e_1,e_1]| \le \frac{2(n-2)}{\sqrt{n(n-1)}}$.

Let us now estimate (12). Since the maximizer in (12) has the form $(\cos\varphi, \sin\varphi, 0, \dots, 0)^T$, we have

$$\begin{aligned} \max_{\xi: F''(x)[\xi,\xi]=1} F'''(x)[\xi,\xi,\xi] &= \max_{\varphi} \left(-\frac{2}{\sqrt{n}} \cos^3\varphi - \frac{6}{\sqrt{n}} \cos\varphi \sin^2\varphi + F'''(x)[e_1,e_1,e_1] \sin^3\varphi \right) \\ &\le \frac{2}{\sqrt{n}} \max_{\varphi} \left(-\cos^3\varphi - 3\cos\varphi\sin^2\varphi + \frac{n-2}{\sqrt{n-1}} \sin^3\varphi \right). \end{aligned}$$

It is easily checked that the stationary points of the function $f(\varphi) = -\cos^3\varphi - 3\cos\varphi\sin^2\varphi + \beta\sin^3\varphi$ in the interval $[0, 2\pi)$ are given by the roots $\varphi_1 = 0$, $\varphi_2 = \pi$ of the equation $\sin\varphi = 0$ and the roots $\varphi_3, \dots, \varphi_6$ of the equation $\cot 2\varphi = \frac{\beta}{2}$. The roots of the first equation yield $f(\varphi_{1,2}) = \pm 1$. The root φ_3 is given by $\sin\varphi_3 = \sqrt{\frac{1}{2} - \frac{\beta}{2\sqrt{4+\beta^2}}}$, $\cos\varphi_3 = \sqrt{\frac{1}{2} + \frac{\beta}{2\sqrt{4+\beta^2}}}$, the other roots $\varphi_4, \varphi_5, \varphi_6$ are obtained by adding multiples of $\frac{\pi}{2}$ to φ_3. From $\beta = 2\cot 2\varphi$, we obtain $f(\varphi) = -\cos\varphi + \sin^2\varphi(-2\cos\varphi + \beta\sin\varphi) = -\frac{1}{\cos\varphi}$, giving the values $f(\varphi_{3,\dots,6}) = \pm(4+\beta^2)^{1/4}\sqrt{\frac{\sqrt{4+\beta^2}}{2} \pm \frac{\beta}{2}}$. Finally, inserting $\beta = \frac{n-2}{\sqrt{n-1}}$, we obtain $f(\varphi_{3,\dots,6}) = \pm\sqrt{\frac{n}{n-1}}, \pm\sqrt{n}$, giving $\max_\varphi f(\varphi) = \sqrt{n}$. Thus $\max_{\xi: F''(x)[\xi,\xi]=1} F'''(x)[\xi,\xi,\xi] \le 2$ and we have proven the following result.

Lemma 4.2. *Let $K \subset \mathbb{R}^n$ be a regular convex cone and let F be the potential of the Cheng-Yau metric on K^o from Proposition 2.3. Then F satisfies (1) and is hence a self-concordant barrier on K with barrier parameter n.* □

Definition 4.3. We call the barrier from Lemma 4.2 the *Einstein-Hessian barrier*.

Corollary 4.4. *Let $K \subset \mathbb{R}^n$ be a regular convex cone. Then the barrier parameter of the optimal barrier on K does not exceed n.* $\square$

We shall summarize the obtained results in the following theorem.

Theorem 4.5. *Let $K \subset \mathbb{R}^n$ be a regular convex cone. Fix a basis of $\mathbb{R}^n$ and consider in the corresponding coordinate system the boundary value problem*

$$\log\det\left(\frac{\partial^2 F(x)}{\partial x^2}\right) = 2F(x),\ x \in K^o; \qquad F|_{\partial K} = +\infty. \tag{13}$$

This problem has a unique locally strictly convex solution $F : K^o \to \mathbb{R}$. This solution is a smooth logarithmically homogeneous self-concordant barrier, the Einstein-Hessian barrier, on K with barrier parameter n, and gives rise to an Einstein-Hessian metric F'' on K^o. It is invariant under unimodular basis changes of $\mathbb{R}^n$ and is determined up to an additive constant under arbitrary basis changes. In particular, it is invariant under the group of unimodular automorphisms of the cone K. The dual barrier $F^* : (K^*)^o \to \mathbb{R}$ differs from the solution of the above boundary value problem on the dual cone K^* by an additive constant, and hence its Hessian is an Einstein-Hessian metric on $(K^*)^o$. If $K = K_1 \times K_2$ is a product of regular convex cones, then the Einstein-Hessian barrier F on K is the sum of the Einstein-Hessian barriers on the individual factor cones K_1, K_2.

From the invariance properties it follows that the Einstein-Hessian barrier has the same level surfaces as the universal barrier on the class of homogeneous cones. In particular, all classical barriers used in the interior-point methods for solving conic programs over symmetric cones have the same level surfaces as the Einstein-Hessian barrier. This shows at the same time that the barrier parameter of the Einstein-Hessian barrier may be effectively smaller than the dimension n of the cone, i.e., that we might divide the barrier by a constant greater than 1 without violating the self-concordance property (1). Below in Section 6 we shall demonstrate this on the example of a non-homogeneous cone.

If the cone K is 3-dimensional, then the corresponding affine hyperspheres are 2-dimensional surfaces. The theory of 2-dimensional affine hyperspheres is quite well developed [20],[23]. The special structure of the cubic form of 2-dimensional affine hyperspheres suggests the following conjecture.

Conjecture 4.6. *Let $K \subset \mathbb{R}^3$ be a regular convex cone. Then the Einstein-Hessian barrier on K can be scaled to an optimal barrier by multiplication with a constant.*

The conjecture is not true for higher-dimensional cones. A counterexample is the symmetric cone $L_3 \times \mathbb{R}_+ = \left\{x \in \mathbb{R}^4 \,|\, x_1^2 \geq \sqrt{x_2^2 + x_3^2},\ x_4 \geq 0\right\}$. The optimal barrier $-\log(x_1^2 - x_2^2 - x_3^2) - \log x_4$ on this cone has barrier parameter $\nu = 3$, while the Einstein-Hessian barrier $-\frac{3}{2}\log\frac{x_1^2 - x_2^2 - x_3^2}{3} - \log x_4$ cannot be multiplied by a constant strictly smaller than 1 without violating (1).

5 The Einstein-Hessian barrier as a minimal submanifold

We shall now provide a transparent geometric interpretation of the Einstein-Hessian barrier. To this end we need a construction introduced in [9, Section 4.3].

Let $K \subset \mathbb{R}^n$ be a regular convex cone and $K^* \subset \mathbb{R}_n$ its dual. Let $F : K^o \to \mathbb{R}$ be a logarithmically homogeneous self-concordant barrier on K and $F^* : (K^*)^o \to \mathbb{R}$ the dual barrier. Let further $\Pi : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}P^{n-1}$, $\Pi^* : \mathbb{R}_n \setminus \{0\} \to \mathbb{R}P_{n-1}$ be the projections of the ambient space $\mathbb{R}^n$ and its dual to the projective space and its dual, respectively. Consider the map $\mathcal{I} : K^o \to \mathbb{R}P^{n-1} \times \mathbb{R}P_{n-1}$ defined by $\mathcal{I} : x \mapsto (\Pi(x), \Pi^*(F'(x)))$. Since the map $x \mapsto -F'(x)$ takes rays in K^o to rays in $(K^*)^o$, the map $\mathcal{I}$ is constant on every ray in K^o. The image $M = \mathcal{I}[K^o]$ is hence a smooth $(n-1)$-dimensional submanifold of the $2(n-1)$-dimensional manifold

$$\mathcal{M} = \{(\tilde{x}, \tilde{p}) \in \mathbb{R}P^{n-1} \times \mathbb{R}P_{n-1} \,|\, \tilde{x} \not\perp \tilde{p}\}. \tag{14}$$

In Section 3 we have seen that the Hessian metric F'' on K^o splits in a direct product of Riemannian metrics, namely a 1-dimensional trivial radial factor, and a $(n-1)$-dimensional angular factor, whose metric is proportional to the centro-affine metric h on the level surfaces of F. Clearly the map $\mathcal{I}$ defines a diffeomorphism between this angular factor and the submanifold M, which equips M in a natural way with the centro-affine metric h. It is now the main result of [10, Section 4] that the metric h also arises on M as a consequence of its being a submanifold of $\mathcal{M}$. Namely, in [10, Section 3] we have shown that $\mathcal{M}$ carries a natural pseudo-Riemannian metric emanating from the projective cross-ratio, and that the restriction of this metric to the submanifold M coincides (up to a sign) with the centro-affine metric h. In this way the Hessian metric F'' on K^o can be recovered without knowledge of the barrier F if only the barrier parameter ν of F and the submanifold $M \subset \mathcal{M}$ are given.

Let us now characterize the Einstein-Hessian barrier in terms of the submanifold M. It turns out that the level surfaces of a logarithmically homogeneous function $F : K^o \to \mathbb{R}$ are affine hyperspheres if and only if the submanifold $M \subset \mathcal{M}$ generated by F is *minimal* [10, Cor. 4.1, p.51], i.e., a stationary point of the volume functional under variations of the submanifold with compact support. But the Einstein-Hessian barrier is exactly the barrier whose level surfaces are affine hyperspheres. Thus it can be characterized in a purely geometric way by the minimality of the induced submanifold M.

6 Example: the power cone

In this section we consider the Einstein-Hessian barrier on a family of non-homogeneous cones. Let $n \geq 3$, let $p, q \in (1, +\infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$, and consider the cone

$$K = \left\{ (x, y, z^T)^T \in \mathbb{R}^n \mid x \geq 0,\ y \geq 0,\ ||z||_2 \leq (\sqrt{p}x)^{1/p} (\sqrt{q}y)^{1/q} \right\}. \tag{15}$$

For $n = 3$, i.e., z scalar, we obtain the well-known self-dual *power cone*, which for the first time probably appeared in the incomplete classification of 3-dimensional positivity domains in [11, p.596][5]. For general n these cones have been considered in [3, p.94] and are also self-dual.

The cone K is invariant under unimodular automorphisms of the form

$$x \mapsto \alpha^{-1-\frac{n-2}{q}} x, \quad y \mapsto \alpha^{1+\frac{n-2}{p}} y, \quad z \mapsto \alpha^{\frac{1}{q}-\frac{1}{p}} U z,$$

where $\alpha > 0$ and U is an orthogonal matrix of size $n - 2$. Hence the Einstein-Hessian barrier on K, which must be invariant under these automorphisms and satisfies (3) with $\nu = n$, can be written in the form

$$F(x, y, z) = -\left(1 + \frac{n-2}{p}\right) \log(\sqrt{p}x) - \left(1 + \frac{n-2}{q}\right) \log(\sqrt{q}y) + \phi\left((\sqrt{p}x)^{-1/p} (\sqrt{q}y)^{-1/q} ||z|| \right), \tag{16}$$

where $\phi : [0, 1) \to \mathbb{R}$ is a function of a scalar variable such that $\lim_{t=1} \phi(t) = +\infty$.

We shall now determine this function. With $t = (\sqrt{p}x)^{-1/p} (\sqrt{q}y)^{-1/q} ||z||$ and $\rho = t\frac{d\phi}{dt}$ we have after some calculations

$$F'' = \begin{pmatrix} \frac{p(\rho+p+n-2)+t\frac{d\rho}{dt}}{p^2x^2} & \frac{t\frac{d\rho}{dt}}{px\,qy} & -\frac{t\frac{d\rho}{dt}z}{px\,||z||^2} \\ \frac{t\frac{d\rho}{dt}}{px\,qy} & \frac{q(\rho+q+n-2)+t\frac{d\rho}{dt}}{q^2y^2} & -\frac{t\frac{d\rho}{dt}z}{qy\,||z||^2} \\ -\frac{t\frac{d\rho}{dt}z}{px\,||z||^2} & -\frac{t\frac{d\rho}{dt}z}{qy\,||z||^2} & \frac{(t\frac{d\rho}{dt}-\rho)zz^T}{||z||^4} + \frac{\rho(||z||^2I-zz^T)}{||z||^4} \end{pmatrix}.$$

Using the relation $p + q = pq$, it follows that

$$\det F'' = \frac{\rho^{n-3}\left(t\frac{d\rho}{dt}(n\rho + (p+n-2)(q+n-2)) - \rho(\rho+p+n-2)(\rho+q+n-2)\right)}{px^2\,qy^2\,||z||^{2(n-2)}}.$$

The partial differential equation (13) then simplifies to the ordinary differential equation

$$\rho^{n-3}\left(t\frac{d\rho}{dt}(n\rho + (p+n-2)(q+n-2)) - \rho(\rho+p+n-2)(\rho+q+n-2)\right) = t^{2(n-2)}e^{2\phi},$$

[5] The cones obtained as homogenizations of the interiors of regular convex polygons are positivity domains which are not mentioned in the classification.

which using $p+q=pq$ again can be integrated to

$$\rho^{n-2}(\rho+p+n-2)(\rho+q+n-2)=t^{n-2}e^{\phi}(c+t^{n-2}e^{\phi}),$$

with c an integration constant. Since F is smooth on the interior of K, we must have $\frac{d\phi}{dt}|_{t=0}=0$, and the quantity $\rho(t)$ is of order $O(t^2)$ at $t=0$. Hence the value of the constant c must be zero, and we get

$$\rho^{n-2}(\rho+p+n-2)(\rho+q+n-2)=t^{2(n-2)}e^{2\phi}. \tag{17}$$

Differentiating (17) with respect to t and using (17) again to eliminate the exponent, we obtain with use of the relation $pq=p+q$ the equation

$$t\frac{d\rho}{dt}=\frac{2\rho(\rho+p+n-2)(\rho+q+n-2)}{n\rho+(p+n-2)(q+n-2)},$$

which again can be integrated to

$$\log t^2=c_1-\frac{1}{p}\log\left(1+\frac{p+n-2}{\rho}\right)-\frac{1}{q}\log\left(1+\frac{q+n-2}{\rho}\right),$$

with c_1 an integration constant. The relation $\lim_{t\to 1}\rho(t)=+\infty$ then yields $c_1=0$, and we have

$$\log t=-\frac{1}{2p}\log\left(1+\frac{p+n-2}{\rho}\right)-\frac{1}{2q}\log\left(1+\frac{q+n-2}{\rho}\right). \tag{18}$$

Inserting (18) into the logarithm of (17) yields

$$\phi=\frac{1}{2}\left(1+\frac{n-2}{p}\right)\log(\rho+p+n-2)+\frac{1}{2}\left(1+\frac{n-2}{q}\right)\log(\rho+q+n-2). \tag{19}$$

Relations (18),(19) give a parametric representation of the solution curve $(t,\phi(t))$, with the parameter ρ ranging from 0 to $+\infty$. It is also seen that for $p\neq 2$ there exists no closed-form expression for $\phi(t)$. Let us formalize this result.

Lemma 6.1. *The Einstein-Hessian barrier on the cone K defined by* (15) *is given by* (16)*, where the scalar function ϕ is defined implicitly by* (18),(19).

Numerical calculations indicate the following conjecture.

Conjecture 6.2. *The barrier parameter of the Einstein-Hessian barrier F on the cone K defined by* (15) *can be lowered to $\nu=\frac{n\max(p,q)}{\max(p,q)+n-2}$ by multiplying F by the factor $\frac{\max(p,q)}{\max(p,q)+n-2}$, i.e., the function $\frac{\max(p,q)}{\max(p,q)+n-2}F$ still satisfies* (1).

For $n=3$, i.e., when K is the usual 3-dimensional power cone, we have obtained a rigorous proof of this conjecture which will be reported separately. The corresponding value $\nu=\frac{3\max(p,q)}{\max(p,q)+1}$ of the barrier parameter is smaller than the best-known values $\nu=3$ (analytically) and $\nu=3-\frac{2}{\max(p,q)}$ (numerically) reported in [3, Section 3.1].

7 Conclusions

In this contribution we introduced a new self-concordant barrier on regular convex cones, the Einstein-Hessian barrier, which is universal in the sense that it exists and is unique on an arbitrary cone. It has better theoretical properties than the universal barrier introduced by Nesterov and Nemirovski, because in addition to the symmetries of the latter, it is also invariant with respect to duality. It has also a lower barrier parameter, which does not exceed the dimension of the cone. On homogeneous cones, both the universal barrier and the Einstein-Hessian barrier have the same level surfaces and hence essentially coincide.

The Einstein-Hessian barrier on a regular convex cone $K\subset\mathbb{R}^n$ can be characterized in many equivalent ways. It is the unique convex solution to the boundary value problem (13). Another characterization is that its level surfaces are the complete hyperbolic affine hyperspheres which are asymptotic to the boundary of the cone K. If Ω is a compact affine section of K with codimension 1, then a particular level surface of the Einstein-Hessian barrier can be described as the radial graph of $-u^{-1}$, where u is the solution of the boundary value problem (8). Still another characterization is that it corresponds to a minimal submanifold of the manifold (14).

References

[1] Wilhelm Blaschke and Kurt Reidemeister. *Affine Differentialgeometrie*. Springer, 1923.

[2] Eugenio Calabi. Complete affine hyperspheres I. In *Symposia Mathematica Vol. 10*, pages 19–38. Istituto Nazionale di Alta Matematica, Acad. Press, 1972.

[3] Robert Chares. *Cones and Interior-Point Algorithms for Structured Convex Optimization involving Powers and Exponentials*. PhD thesis, Université Catholique de Louvain, Louvain-la-Neuve, 2008.

[4] Shiu Yuen Cheng and Shing-Tung Yau. On the existence of a complete Kähler metric on non-compact complex manifolds and the regularity of Fefferman's equation. *Comm. Pure Appl. Math.*, 33(4):507–544, 1980.

[5] Shiu Yuen Cheng and Shing-Tung Yau. The real Monge-Ampère equation and affine flat structures. In *Proc. 1980 Beijing Symp. on Diff. Geom. and Diff. Eq., Vol. 1*, pages 339–370, 1982.

[6] Salvador Gigena. Integral invariants of convex cones. *J. Diff. Geom.*, 13:191–222, 1978.

[7] Salvador Gigena. On a conjecture by E.Calabi. *Geom. Dedicata*, 11:387–396, 1981.

[8] Osman Güler. Barrier functions in interior point methods. *Math. Oper. Res.*, 21(4):860–885, 1996.

[9] Roland Hildebrand. Barriers on projective convex sets. In *AIMS Proceedings*, pages 672–683, 2011.

[10] Roland Hildebrand. The cross-ratio manifold: a model of centro-affine geometry. *Int. Electron. J. Geom.*, 4(2):32–62, 2011.

[11] Max Koecher. Positivitätsbereiche im $\mathbf{R}^n$. *Amer. J. Math.*, 79:575–596, 1957.

[12] Adam Koranyi. The Bergman kernel function for tubes over convex cones. *Pac. Math. J.*, 12:1355–1359, 1962.

[13] An-Min Li, Udo Simon, and Guosong Zhao. *Global affine differential geometry of hypersurfaces*, volume 11 of *De Gruyter expositions in mathematics*. Walter de Gruyter, 1993.

[14] John C. Loftin. Affine spheres and Kähler-Einstein metrics. *Math. Res. Lett.*, 9(4):425–432, 2002.

[15] Andrei Moroianu. *Lectures on Kähler geometry*, volume 69 of *London Mathematical Society Student Texts*. Cambridge University Press, 2007.

[16] Yurii Nesterov and Arkadii Nemirovskii. *Interior-point polynomial algorithms in convex programming*, volume 13 of *SIAM Stud. Appl. Math.* SIAM, Philadelphia, 1994.

[17] Katsumi Nomizu and Takeshi Sasaki. *Affine differential geometry: geometry of affine immersions*, volume 111 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1994.

[18] Takeshi Sasaki. A note on characteristic functions and projectively invariant metrics on a bounded convex domain. *Tokyo J. Math.*, 8(1):49–79, 1985.

[19] Hirohiko Shima. *Geometry of Hessian Structures*. World Scientific, Hackensack, New Jersey, 2007.

[20] Udo Simon and Chang-Ping Wang. Local theory of affine 2-spheres. In *Differential Geometry: Riemannian geometry*, volume 54-3 of *Proceedings of Symposia in Pure Mathematics*, pages 585–598. American Mathematical Society, Los Angeles, 1993.

[21] Levent Tunçel and Song Xu. On homogeneous convex cones, the Carathéodory number, and the duality mapping. *Math. Oper. Res.*, 26(2):234–247, 2001.

[22] Gheorghe Tzitzeica. Sur une nouvelle classe de surfaces. *Rend. Circ. Mat. Palermo*, 25:180–187, 1908.

[23] Chang-Ping Wang. Some examples of complete hyperbolic affine 2-spheres in $\mathbf{R}^3$. In *Global Differential Geometry and Global Analysis*, volume 1481 of *Lecture Notes in Mathematics*, pages 272–280. Springer, 1991.

[24] Shing-Tung Yau. A general Schwarz lemma for Kähler manifolds. *Amer. J. Math.*, 100:197–203, 1978.