

Analytical formulas for calculating extremal ranks and inertias of quadratic matrix-valued functions and their applications

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Abstract. A group of analytical formulas for calculating the global maximal and minimal ranks and inertias of the quadratic matrix-valued function

$$\phi(X) = (AXB + C)M(AXB + C)^* + D$$

are established and their consequences are presented, where A, B, C and D are given complex matrices with A and C Hermitian. As applications, necessary and sufficient conditions for the two general quadratic matrix-valued functions

$$\left(\sum_{i=1}^k A_i X_i B_i + C \right) M \left(\sum_{i=1}^k A_i X_i B_i + C \right)^* + D, \quad \sum_{i=1}^k (A_i X_i B_i + C_i) M_i (A_i X_i B_i + C_i)^* + D$$

to be semi-definite are derived, respectively, where A_i, B_i, C_i, C, D, M_i and M are given matrices with M_i, M and D Hermitian, $i = 1, \dots, k$. Löwner partial ordering optimizations of the two matrix-valued functions are studied and their solutions are characterized.

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1. Introduction

A matrix-valued function for complex matrices is a map between matrix spaces, which can generally be written as $Y = f(X)$ for $Y \in \mathbb{C}^{m \times n}$ and $X \in \mathbb{C}^{p \times q}$, or briefly, $f: \mathbb{C}^{m \times n} \rightarrow \mathbb{C}^{p \times q}$, where $\mathbb{C}^{m \times n}$ and $\mathbb{C}^{p \times q}$ are two complex matrix spaces. Linear and quadratic matrix-valued functions, as usual, as common representatives of various matrix-valued functions, are extensively studied from theoretical and applied points of view.

In this paper, we consider the following two types of quadratic matrix-valued functions

$$\phi(X_1, \dots, X_k) = \left(\sum_{i=1}^k A_i X_i B_i + C \right) M \left(\sum_{i=1}^k A_i X_i B_i + C \right)^* + D, \quad (1.1)$$

$$\psi(X_1, \dots, X_k) = \sum_{i=1}^k (A_i X_i B_i + C_i) M_i (A_i X_i B_i + C_i)^* + D, \quad (1.2)$$

where A_i, B_i, C_i, C, D, M_i and M are given matrices with M_i, M and D Hermitian, X_i is a variable matrix, $i = 1, \dots, k$. Eqs. (1.1) or (1.2) for $k = 1$ is

$$\phi(X) = (AXB + C)M(AXB + C)^* + D, \quad (1.3)$$

where $A \in \mathbb{C}^{n \times p}$, $B \in \mathbb{C}^{m \times q}$, $C \in \mathbb{C}^{n \times q}$, $D \in \mathbb{C}_{\mathbb{H}}^n$ and $M \in \mathbb{C}_{\mathbb{H}}^q$ are given, and $X \in \mathbb{C}^{p \times m}$ is a variable matrix. We treat it as a combination $\phi = \tau \circ \rho$ of the following two simple linear and quadratic Hermitian matrix-valued functions:

$$\rho: X \rightarrow AXB + C, \quad \tau: Y \rightarrow YMY^* + D. \quad (1.4)$$

For different choices of the given matrices, this quadratic function between matrix spaces includes many ordinary quadratic forms and quadratic matrix-valued functions as its special cases, such as, $\mathbf{x}^* A \mathbf{x}$, XAX^* , DXX^*D^* , $(X - C)M(X - C)^*$, etc.

It is well known that quadratic functions in elementary mathematics and ordinary quadratic forms in linear algebra have a fairly complete theory with a long history and numerous applications. Much of the beauty of these quadratic objects were highly appreciated by mathematicians in all times, and many of the fundamental ideas of quadratic functions and quadratic forms were developed in all branches of mathematics. While the mathematics of classic quadratic forms has been established for about one and a half century, various extensions

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of classic quadratic forms to some general settings were conducted from theoretical and applied point of view, in particular, quadratic matrix-valued functions and the corresponding quadratic matrix equations and quadratic matrix inequalities often appear briefly when needed to solve a variety of problems in mathematics and applications. These quadratic objects have many attractive features both from manipulative and computational point of view, and there is an intensive interest in studying behaviors of quadratic matrix-valued functions, quadratic matrix equations and quadratic matrix inequalities. In fact, any essential development on the researches of quadratic objects will lead to many progresses in both mathematics and applications. Compared with the theory of ordinary quadratic functions and forms, two distinctive features of quadratic matrix-valued functions are the freedom of entries in variable matrices and the non-commutativity of matrix algebra. So that there is no a general theory for describing behaviors of a given quadratic matrix-valued function with multiple terms. In particular, to solve an optimization problem on a quadratic matrix-valued function is believed to be NP hard in general, and thus there is a long way to go to establish a perfect theory on quadratic matrix-valued functions. In recent years, the present author conducted a seminal study on quadratic matrix-valued functions in [8, 10], which gave an initial quantitative understanding of the nature of the matrix rank and inertia optimization problems, in particular, a simple linearization method was introduced for studying quadratic or nonlinear matrix-valued functions, and many explicit formulas were established for calculating the extremum ranks and inertias of some simple quadratic matrix-valued functions. For applications of quadratic matrix-valued functions, quadratic matrix equations and quadratic matrix inequalities in optimization theory, system and control theory, see the references given in [8, 10].

Throughout this paper,

$\mathbb{C}^{m \times n}$ stands for the set of all $m \times n$ complex matrices;

$\mathbb{C}_{\mathbb{H}}^m$ stands for the set of all $m \times m$ complex Hermitian matrices;

A^* , $r(A)$ and $\mathcal{R}(A)$ stand for the conjugate transpose, rank and range (column space) of a matrix $A \in \mathbb{C}^{m \times n}$, respectively;

I_m denotes the identity matrix of order m ;

$[A, B]$ denotes a row block matrix consisting of A and B ;

the Moore–Penrose inverse of $A \in \mathbb{C}^{m \times n}$, denoted by A^\dagger , is defined to be the unique solution X satisfying the four matrix equations $AXA = A$, $XAX = X$, $(AX)^* = AX$ and $(XA)^* = XA$;

the symbols E_A and F_A stand for $E_A = I_m - AA^\dagger$ and $F_A = I_n - A^\dagger A$;

an $X \in \mathbb{C}^{n \times m}$ is called a g -inverse of $A \in \mathbb{C}^{m \times n}$, denoted by A^- , if it satisfies $AXA = A$;

an $X \in \mathbb{C}_{\mathbb{H}}^m$ is called a Hermitian g -inverse of $A \in \mathbb{C}_{\mathbb{H}}^m$, denoted by A^\sim , if it satisfies $AXA = A$; called a reflexive Hermitian g -inverse of $A \in \mathbb{C}_{\mathbb{H}}^m$, denoted by A_r^\sim , if it satisfies $AXA = A$ and $XAX = X$;

$i_+(A)$ and $i_-(A)$, called the partial inertia of $A \in \mathbb{C}_{\mathbb{H}}^m$, are defined to be the numbers of the positive and negative eigenvalues of A counted with multiplicities, respectively;

$A \succ 0$ ($A \succcurlyeq 0$, $\prec 0$, $\preccurlyeq 0$) means that A is Hermitian positive definite (positive semi-definite, negative definite, negative semi-definite);

two $A, B \in \mathbb{C}_{\mathbb{H}}^m$ are said to satisfy the inequality $A \succ B$ ($A \succcurlyeq B$) in the Löwner partial ordering if $A - B$ is positive definite (positive semi-definite).

2. Problem formulation

Matrix rank and inertia optimization problems are a class of discontinuous optimization problems, in which decision variables are matrices running over certain matrix sets, while the rank and inertia of the variable matrices are taken as integer-valued objective functions. Because rank and inertia of matrices are always integers, no approximation methods are allowed to use when finding the maximal and minimal possible ranks and inertias of a matrix-valued function. So that matrix rank and inertia optimization problems are not consistent with anyone of the ordinary continuous and discrete problems in optimization theory. Less people paid attention to this kind of optimization problems, and no complete theory was established. But, the present author has been working on this topic with great effort in the past 30 years, and contribute a huge amount of results on matrix rank and inertia optimization problems.

A major purpose of this paper is to develop a unified optimization theory on ranks, inertias and partial orderings of quadratic matrix-valued functions by using pure algebraic operations of matrices, which enables us to handle many mathematical and applied problems on behaviors of quadratic matrix-valued functions, quadratic matrix equations and quadratic matrix inequalities. The rank and inertia of a (Hermitian) matrix are two oldest basic concepts in linear algebra for describing the dimension of the row/column vector space and the sign distribution of the eigenvalues of the square matrix, which are well understood and are easy to compute by the well-known elementary or congruent matrix operations. These two quantities play an essential role in characterizing algebraic properties of (Hermitian) matrices. Because concepts of ranks and inertias are so

generic in linear algebra, it is doubtful that a primary work in linear algebra is to establish (expansion) formulas for calculating ranks and inertias of matrices as more as possible. However, this valuable work was really neglected in the development of linear algebra, and a great chance for discovering thousands of rank and inertia formulas, some of which are given in Lemmas 3.2, 3.3, 3.5 and 3.6 below, were lost in the earlier period of linear algebra. This paper tries to make some essential contributions on establishing formulas for ranks and inertias of some quadratic matrix-valued functions.

Taking the rank and inertia of (1.3) as inter-valued objective functions, we solve the following problems:

Problem 2.1. For the function in (1.3), establish explicit formulas for calculating the following global extremum ranks and inertias

$$\max_{X \in \mathbb{C}^{p \times m}} r[\phi(X)], \quad \min_{X \in \mathbb{C}^{p \times m}} r[\phi(X)], \quad \max_{X \in \mathbb{C}^{p \times m}} i_{\pm}[\phi(X)], \quad \min_{X \in \mathbb{C}^{p \times m}} i_{\pm}[\phi(X)]. \quad (2.1)$$

Problem 2.2. For the function in (1.3),

- (i) establish necessary and sufficient conditions for the existence of an $X \in \mathbb{C}^{p \times m}$ such that

$$\phi(X) = 0; \quad (2.2)$$

- (ii) establish necessary and sufficient conditions for the following inequalities

$$\phi(X) \succ 0, \quad \phi(X) \succcurlyeq 0, \quad \phi(X) \prec 0, \quad \phi(X) \preccurlyeq 0 \quad (2.3)$$

to hold for an $X \in \mathbb{C}^{p \times m}$, respectively;

- (iii) establish necessary and sufficient conditions for

$$\phi(X) \succ 0, \quad \phi(X) \succcurlyeq 0, \quad \phi(X) \prec 0, \quad \phi(X) \preccurlyeq 0 \quad \text{for all } X \in \mathbb{C}^{p \times m} \quad (2.4)$$

to hold, respectively, namely, to give identifying conditions for $\phi(X)$ to be a positive definite, positive semi-definite, negative definite, negative semi-definite function on complex matrices, respectively.

Problem 2.3. For the function in (1.3), establish necessary and sufficient conditions for the existence of $\widehat{X}, \widetilde{X} \in \mathbb{C}^{p \times m}$ such that

$$\phi(X) \succcurlyeq \phi(\widehat{X}), \quad \phi(X) \preccurlyeq \phi(\widetilde{X}) \quad (2.5)$$

hold for all $X \in \mathbb{C}^{p \times m}$, respectively, and derive analytical expressions of the two matrices \widehat{X} and \widetilde{X} .

3. Preliminary results

The ranks and inertias are two generic indices in finite dimensional algebras. The results related to these indices are unreplacable by any other quantitative tools in mathematics. A simple but striking fact about the indices is stated in following lemma.

Lemma 3.1. Let \mathcal{S} be a subset in $\mathbb{C}^{m \times n}$, and let \mathcal{H} be a subset in $\mathbb{C}_{\mathbb{H}}^m$.

- Under $m = n$, \mathcal{S} has a nonsingular matrix if and only if $\max_{X \in \mathcal{S}} r(X) = m$.
- Under $m = n$, all $X \in \mathcal{S}$ are nonsingular if and only if $\min_{X \in \mathcal{S}} r(X) = m$.
- $0 \in \mathcal{S}$ if and only if $\min_{X \in \mathcal{S}} r(X) = 0$.
- $\mathcal{S} = \{0\}$ if and only if $\max_{X \in \mathcal{S}} r(X) = 0$.
- \mathcal{H} has a matrix $X \succ 0$ ($X \prec 0$) if and only if $\max_{X \in \mathcal{H}} i_+(X) = m$ ($\max_{X \in \mathcal{H}} i_-(X) = m$).
- All $X \in \mathcal{H}$ satisfy $X \succ 0$ ($X \prec 0$), namely, \mathcal{H} is a subset of the cone of positive definite matrices (negative definite matrices), if and only if $\min_{X \in \mathcal{H}} i_+(X) = m$ ($\min_{X \in \mathcal{H}} i_-(X) = m$).
- \mathcal{H} has a matrix $X \succcurlyeq 0$ ($X \preccurlyeq 0$) if and only if $\min_{X \in \mathcal{H}} i_-(X) = 0$ ($\min_{X \in \mathcal{H}} i_+(X) = 0$).
- All $X \in \mathcal{H}$ satisfy $X \succcurlyeq 0$ ($X \preccurlyeq 0$), namely, \mathcal{H} is a subset of the cone of positive semi-definite matrices (negative semi-definite matrices), if and only if $\max_{X \in \mathcal{H}} i_-(X) = 0$ ($\max_{X \in \mathcal{H}} i_+(X) = 0$).

The question of whether a given matrix-valued function is semi-definite everywhere is ubiquitous in matrix theory and applications. Lemma 3.1(e)–(h) assert that if certain explicit formulas for calculating the global maximal and minimal inertias of Hermitian matrix-valued functions are established, we can use them as a quantitative tool, as demonstrated in Sections 2–7 below, to derive necessary and sufficient conditions for the matrix-valued functions to be definite or semi-definite. In addition, we are able to use these inertia formulas to establish various matrix inequalities in the Löwner partial ordering, and to solve many matrix optimization problems in the Löwner partial ordering.

The following are obvious or well known (see [8]), which will be used in the latter part of this paper.

Lemma 3.2. Let $A \in \mathbb{C}_H^m$, $B \in \mathbb{C}_H^n$, $Q \in \mathbb{C}^{m \times n}$, and $P \in \mathbb{C}^{p \times m}$ with $r(P) = m$. Then,

$$i_{\pm}(PAP^*) = i_{\pm}(A), \quad (3.1)$$

$$i_{\pm}(\lambda A) = \begin{cases} i_{\pm}(A) & \text{if } \lambda > 0 \\ i_{\mp}(A) & \text{if } \lambda < 0 \end{cases}, \quad (3.2)$$

$$i_{\pm} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = i_{\pm}(A) + i_{\pm}(B), \quad (3.3)$$

$$i_+ \begin{bmatrix} 0 & Q \\ Q^* & 0 \end{bmatrix} = i_- \begin{bmatrix} 0 & Q \\ Q^* & 0 \end{bmatrix} = r(Q). \quad (3.4)$$

Lemma 3.3 ([7]). Let $A \in \mathbb{C}_H^m$, $B \in \mathbb{C}^{m \times n}$, $D \in \mathbb{C}_H^n$, and let

$$M_1 = \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} A & B \\ B^* & D \end{bmatrix}.$$

Then, the following expansion formulas hold

$$i_{\pm}(M_1) = r(B) + i_{\pm}(E_B A E_B), \quad r(M_1) = 2r(B) + r(E_B A E_B), \quad (3.5)$$

$$i_{\pm}(M_2) = i_{\pm}(A) + i_{\pm} \begin{bmatrix} 0 & E_A B \\ B^* E_A & D - B^* A^\dagger B \end{bmatrix}, \quad r(M_2) = r(A) + r \begin{bmatrix} 0 & E_A B \\ B^* E_A & D - B^* A^\dagger B \end{bmatrix}. \quad (3.6)$$

In particular, the following hold.

(a) If $A \succ 0$, then

$$i_+(M_1) = r[A, B], \quad i_-(M_1) = r(B), \quad r(M_1) = r[A, B] + r(B). \quad (3.7)$$

(b) If $A \preceq 0$, then

$$i_+(M_1) = r(B), \quad i_-(M_1) = r[A, B], \quad r(M_1) = r[A, B] + r(B). \quad (3.8)$$

(c) If $\mathcal{R}(B) \subseteq \mathcal{R}(A)$, then

$$i_{\pm}(M_2) = i_{\pm}(A) + i_{\pm}(D - B^* A^\dagger B), \quad r(M_2) = r(A) + r(D - B^* A^\dagger B). \quad (3.9)$$

(d) $r(M_2) = r(A) \Leftrightarrow \mathcal{R}(B) \subseteq \mathcal{R}(A)$ and $D = B^* A^\dagger B$.

(e) $M_2 \succ 0 \Leftrightarrow A \succ 0$, $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ and $D - B^* A^\dagger B \succ 0$.

Lemma 3.4 ([5]). Let $A \in \mathbb{C}^{m \times p}$, $B \in \mathbb{C}^{q \times n}$ and $C \in \mathbb{C}^{m \times n}$ be given. Then the matrix equation $AXB = C$ is consistent if and only if $\mathcal{R}(C) \subseteq \mathcal{R}(A)$ and $\mathcal{R}(C^*) \subseteq \mathcal{R}(B^*)$, or equivalently, $AA^\dagger CB^\dagger B = C$. In this case, the general solution can be written as

$$X = A^\dagger CB^\dagger + F_A V_1 + V_2 E_B, \quad (3.10)$$

where V_1 and V_2 are arbitrary matrices. In particular, $AXB = C$ has a unique solution if and only if

$$r(A) = p, \quad r(B) = q, \quad \mathcal{R}(C) \subseteq \mathcal{R}(A), \quad \mathcal{R}(C^*) \subseteq \mathcal{R}(B^*). \quad (3.11)$$

Lemma 3.5 ([3, 6, 11]). Let $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{m \times p}$ and $C \in \mathbb{C}^{q \times n}$ be given, and $X \in \mathbb{C}^{p \times q}$ be a variable matrix. Then, the global maximal and minimal ranks of $A - BXC$ are given by

$$\max_{X \in \mathbb{C}^{p \times q}} r(A - BXC) = \min \left\{ r[A, B], \quad r \begin{bmatrix} A \\ C \end{bmatrix} \right\}, \quad (3.12)$$

$$\min_{X \in \mathbb{C}^{p \times q}} r(A - BXC) = r[A, B] + r \begin{bmatrix} A \\ C \end{bmatrix} - r \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}. \quad (3.13)$$

Lemma 3.6 ([4]). Let $f(X) = A + BXC + (BXC)^*$ where $A \in \mathbb{C}_H^m$, $B \in \mathbb{C}^{m \times n}$ and $C \in \mathbb{C}^{p \times m}$ be given, $X \in \mathbb{C}^{n \times p}$ be a variable matrix. Then,

$$\max_{X \in \mathbb{C}^{n \times p}} r[f(X)] = \min \left\{ r[A, B, C^*], \quad r \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}, \quad r \begin{bmatrix} A & C^* \\ C & 0 \end{bmatrix} \right\}, \quad (3.14)$$

$$\min_{X \in \mathbb{C}^{n \times p}} r[f(X)] = 2r[A, B, C^*] + \max\{s_+ + s_-, t_+ + t_-, s_+ + t_-, s_- + t_+\}, \quad (3.15)$$

$$\max_{X \in \mathbb{C}^{n \times p}} i_{\pm}[f(X)] = \min \left\{ i_{\pm} \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}, \quad i_{\pm} \begin{bmatrix} A & C^* \\ C & 0 \end{bmatrix} \right\}, \quad (3.16)$$

$$\min_{X \in \mathbb{C}^{n \times p}} i_{\pm}[f(X)] = r[A, B, C^*] + \max\{s_{\pm}, t_{\pm}\}, \quad (3.17)$$

where

$$s_{\pm} = i_{\pm} \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} - r \begin{bmatrix} A & B & C^* \\ B^* & 0 & 0 \end{bmatrix}, \quad t_{\pm} = i_{\pm} \begin{bmatrix} A & C^* \\ C & 0 \end{bmatrix} - r \begin{bmatrix} A & B & C^* \\ C & 0 & 0 \end{bmatrix}.$$

4. Main results

We first solve Problem 1.1 through a linearization method and Theorem 1.8.

Theorem 4.1. *Let $\phi(X)$ be as given in (1.3), and define*

$$N_1 = \begin{bmatrix} D + CMC^* & A \\ A^* & 0 \end{bmatrix}, \quad N_2 = \begin{bmatrix} D + CMC^* & CMB^* & A \\ A^* & 0 & 0 \end{bmatrix}, \quad (4.1)$$

$$N_3 = \begin{bmatrix} D + CMC^* & CMB^* \\ BMC^* & BMB^* \end{bmatrix}, \quad N_4 = \begin{bmatrix} D + CMC^* & CMB^* & A \\ BMC^* & BMB^* & 0 \end{bmatrix}. \quad (4.2)$$

Then, the global maximum and minimum ranks and inertias of $\phi(X)$ are given by

$$\max_{X \in \mathbb{C}^{p \times m}} r[\phi(X)] = \min \{ r[D + CMC^*, CMB^*, A], r(N_1), r(N_3) \}, \quad (4.3)$$

$$\min_{X \in \mathbb{C}^{p \times m}} r[\phi(X)] = 2r[D + CMC^*, CMB^*, A] + \max\{s_1, s_2, s_3, s_4\}, \quad (4.4)$$

$$\max_{X \in \mathbb{C}^{p \times m}} i_{\pm}[\phi(X)] = \min \{ i_{\pm}(N_1), i_{\pm}(N_3) \}, \quad (4.5)$$

$$\min_{X \in \mathbb{C}^{p \times m}} i_{\pm}[\phi(X)] = r[D + CMC^*, CMB^*, A] + \max \{ i_{\pm}(N_1) - r(N_2), i_{\pm}(N_3) - r(N_4) \}, \quad (4.6)$$

where

$$s_1 = r(N_1) - 2r(N_2), \quad s_2 = r(N_3) - 2r(N_4), \\ s_3 = i_+(N_1) + i_-(N_3) - r(N_2) - r(N_4), \quad s_4 = i_-(N_1) + i_+(N_3) - r(N_2) - r(N_4).$$

Proof. It is easy to verify from (3.6) that

$$i_{\pm}[(AXB + C)M(AXB + C)^* + D] = i_{\pm} \begin{bmatrix} -M & M(AXB + C)^* \\ (AXB + C)M & D \end{bmatrix} - i_{\pm}(-M), \quad (4.7)$$

$$r[(AXB + C)M(AXB + C)^* + D] = r \begin{bmatrix} -M & M(AXB + C)^* \\ (AXB + C)M & D \end{bmatrix} - r(M). \quad (4.8)$$

that is, the rank and inertia of $\phi(X)$ in (1.3) can be calculated by those of the following linear matrix-valued function

$$\begin{aligned} \psi(X) &= \begin{bmatrix} -M & M(AXB + C)^* \\ (AXB + C)M & D \end{bmatrix} \\ &= \begin{bmatrix} -M & MC^* \\ CM & D \end{bmatrix} + \begin{bmatrix} 0 \\ A \end{bmatrix} X [BM, 0] + \begin{bmatrix} MB^* \\ 0 \end{bmatrix} X^* [0, A^*]. \end{aligned} \quad (4.9)$$

Note from (4.7) and (4.8) that

$$\max_{X \in \mathbb{C}^{p \times m}} r[\phi(X)] = \max_{X \in \mathbb{C}^{p \times m}} r[\psi(X)] - r(A), \quad (4.10)$$

$$\min_{X \in \mathbb{C}^{p \times m}} r[\phi(X)] = \min_{X \in \mathbb{C}^{p \times m}} r[\psi(X)] - r(A), \quad (4.11)$$

$$\max_{X \in \mathbb{C}^{p \times m}} i_{\pm}[\phi(X)] = \max_{X \in \mathbb{C}^{p \times m}} i_{\pm}[\psi(X)] - i_{\mp}(A), \quad (4.12)$$

$$\min_{X \in \mathbb{C}^{p \times m}} i_{\pm}[\phi(X)] = \min_{X \in \mathbb{C}^{p \times m}} i_{\pm}[\psi(X)] - i_{\mp}(A). \quad (4.13)$$

Applying Lemma 3.6 to (4.9), we first obtain

$$\max_{X \in \mathbb{C}^{p \times m}} r[\psi(X)] = \min \{ r(H), r(G_1), r(G_2) \}, \quad (4.14)$$

$$\min_{X \in \mathbb{C}^{p \times m}} r[\psi(X)] = 2r(H) + \max \{ s_+ + s_-, t_+ + t_-, s_+ + t_-, s_- + t_+ \}, \quad (4.15)$$

$$\max_{X \in \mathbb{C}^{p \times m}} i_{\pm}[\psi(X)] = \min \{ i_{\pm}(G_1), i_{\pm}(G_2) \}, \quad (4.16)$$

$$\min_{X \in \mathbb{C}^{p \times m}} i_{\pm}[\psi(X)] = r(H) + \max \{ s_{\pm}, t_{\pm} \}, \quad (4.17)$$

where

$$H = \begin{bmatrix} -M & MC^* & 0 & MB^* \\ CM & D & A & 0 \end{bmatrix}, G_1 = \begin{bmatrix} -M & MC^* & 0 \\ CM & D & A \\ 0 & A^* & 0 \end{bmatrix}, G_2 = \begin{bmatrix} -M & MC^* & MB^* \\ CM & D & 0 \\ BM & 0 & 0 \end{bmatrix},$$

$$H_1 = \begin{bmatrix} -M & MC^* & 0 & MB^* \\ CM & D & A & 0 \\ 0 & A^* & 0 & 0 \end{bmatrix}, H_2 = \begin{bmatrix} -M & MC^* & MB^* & 0 \\ CM & D & 0 & A \\ BM & 0 & 0 & 0 \end{bmatrix},$$

and

$$s_{\pm} = i_{\pm}(G_1) - r(H_1), \quad t_{\pm} = i_{\pm}(G_2) - r(H_2).$$

It is easy to derive from Lemmas 3.2 and 3.3, elementary matrix operations and congruence matrix operations that

$$r(H) = r(M) + r[D + CMC^*, CMB^*, A], \quad (4.18)$$

$$r(H_1) = r \begin{bmatrix} D + CMC^* & CMB^* & A \\ A^* & 0 & 0 \end{bmatrix}, \quad r(H_2) = r(M) + r \begin{bmatrix} D + CMC^* & CMB^* & A \\ BMC^* & BMB^* & 0 \end{bmatrix}, \quad (4.19)$$

$$i_{\pm}(G_1) = i_{\mp}(M) + i_{\pm} \begin{bmatrix} C + CMC^* & A \\ A^* & 0 \end{bmatrix}, \quad i_{\pm}(G_2) = i_{\mp}(M) + i_{\pm} \begin{bmatrix} D + CMC^* & CMB^* \\ BMC^* & BMB^* \end{bmatrix}. \quad (4.20)$$

Hence,

$$r(G_1) = r(M) + r(N_1), \quad r(G_2) = r(M) + r(N_3), \quad (4.21)$$

$$s_{\pm} = i_{\pm}(G_1) - r(H_1) = i_{\pm}(N_1) - r(N_2) - i_{\pm}(M), \quad (4.22)$$

$$t_{\pm} = i_{\pm}(G_2) - r(H_2) = i_{\pm}(N_3) - r(N_4) - i_{\pm}(M). \quad (4.23)$$

Substituting (4.18)–(4.23) into (4.15)–(4.17), and then (4.14)–(4.17) into (4.10)–(4.13), we obtain (4.4)–(4.7). \square

Without loss of generality, we assume in what follows that both $A \neq 0$ and $BMB^* \neq 0$ in (1.3). Applying Lemma 1.4 to (4.3)–(4.6), we obtain the following results.

Corollary 4.2. *Let $\phi(X)$ be as given in (1.3), and N_1 and N_3 be the matrices of (4.1) and (4.2). Then,*

- (a) *There exists an $X \in \mathbb{C}^{p \times m}$ such that $\phi(X)$ is nonsingular if and only if $r[D + CMC^*, CMB^*, A] = n$, $r(N_1) \geq n$ and $r(N_3) \geq n$.*
- (b) *$\phi(X)$ is nonsingular for all $X \in \mathbb{C}^{p \times m}$ if and only if $r(D + CMC^*) = n$, and one of the following four conditions holds*
 - (i) *$BMC^*(D + CMC^*)^{-1}A = 0$ and $A^*(D + CMC^*)^{-1}A = 0$.*
 - (ii) *$BMC^*(D + CMC^*)^{-1}A = 0$ and $BMC^*(D + CMC^*)^{-1}CMB^* = BMB^*$.*
 - (iii) *$A^*(D + CMC^*)^{-1}A \succcurlyeq 0$, $BMB^* - BMC^*(D + CMC^*)^{-1}CMB^* \succcurlyeq 0$, $\mathcal{R}[A^*(D + CMC^*)^{-1}CMB^*] \subseteq \mathcal{R}[A^*(D + CMC^*)^{-1}A]$ and $\mathcal{R}[BMC^*(D + CMC^*)^{-1}A] \subseteq \mathcal{R}[BMB^* - BMC^*(D + CMC^*)^{-1}CMB^*]$.*
 - (iv) *$A^*(D + CMC^*)^{-1}A \preccurlyeq 0$, $BMB^* - BMC^*(D + CMC^*)^{-1}CMB^* \preccurlyeq 0$, $\mathcal{R}[A^*(D + CMC^*)^{-1}CMB^*] \subseteq \mathcal{R}[A^*(D + CMC^*)^{-1}A]$, and $\mathcal{R}[BMC^*(D + CMC^*)^{-1}A] \subseteq \mathcal{R}[BMB^* - BMC^*(D + CMC^*)^{-1}CMB^*]$.*
- (c) *There exists an $X \in \mathbb{C}^{p \times m}$ such that $\phi(X) = 0$, namely, the matrix equation in (2.2) is consistent, if and only if*

$$\mathcal{R}(D + CMC^*) \subseteq \mathcal{R}[A, CMB^*], \quad r(M_1) = 2r(A), \quad 2r[A, CMB^*] + r(N_3) - 2r(N_4) \preccurlyeq 0,$$

$$r[A, CMB^*] + i_+(N_3) - r(N_4) \leq 0, \quad r[A, CMB^*] + i_-(N_3) - r(N_4) \leq 0.$$

- (d) *There exists an $X \in \mathbb{C}^{p \times m}$ such that $\phi(X) \succ 0$, namely, the matrix inequality is feasible, if and only if*

$$i_+(N_1) = n \text{ and } i_+(N_3) \geq n, \quad \text{or} \quad i_+(N_1) \geq n \text{ and } i_+(N_3) = n.$$

- (e) *There exists an $X \in \mathbb{C}^{p \times m}$ such that $\phi(X) \prec 0$, the matrix inequality is feasible, if and only if*

$$i_-(N_1) = n \text{ and } i_-(N_3) \geq n, \quad \text{or} \quad i_-(N_1) \geq n \text{ and } i_-(N_3) = n.$$

- (f) *$\phi(X) \succ 0$ for all $X \in \mathbb{C}^{p \times m}$, namely, $\phi(X)$ is a completely positive matrix-valued function, if and only if*

$$D + CMC^* \succ 0, \quad N_3 \succcurlyeq 0, \quad \mathcal{R} \begin{bmatrix} A \\ 0 \end{bmatrix} \subseteq \mathcal{R}(N_3).$$

(g) $\phi(X) \prec 0$ for all $X \in \mathbb{C}^{p \times m}$ namely, $\phi(X)$ is a completely negative matrix-valued function, if and only if

$$D + CMC^* \prec 0, N_3 \preceq 0, \mathcal{R} \begin{bmatrix} A \\ 0 \end{bmatrix} \subseteq \mathcal{R}(N_3).$$

(h) There exists an $X \in \mathbb{C}^{p \times m}$ such that $\phi(X) \succcurlyeq 0$, namely, the matrix inequality is feasible, if and only if

$$r[D + CMC^*, CMB^*, A] + i_-(N_1) \leq r(N_2) \text{ and } r[D + CMC^*, CMB^*, A] + i_-(N_3) \leq r(N_4).$$

(i) There exists an $X \in \mathbb{C}^{p \times m}$ such that $\phi(X) \preceq 0$, namely, the matrix inequality is feasible, if and only if

$$r[D + CMC^*, CMB^*, A] + i_+(N_1) \leq r(N_2) \text{ and } r[D + CMC^*, CMB^*, A] + i_+(N_3) \leq r(N_4).$$

(j) $\phi(X) \succcurlyeq 0$ for all $X \in \mathbb{C}^{p \times m}$, namely, $\phi(X)$ is a positive a positive matrix-valued function, if and only if $N_3 \succcurlyeq 0$.

(k) $\phi(X) \preceq 0$ for all $X \in \mathbb{C}^{p \times m}$, namely, $\phi(X)$ is a negative matrix-valued function, if and only if $N_3 \preceq 0$.

Proof. We only show (b). Under the condition $r(D + CMC^*) = n$, (4.4) reduces to

$$\min_{X \in \mathbb{C}^{p \times m}} r[\phi(X)] = 2n + \max\{s_1, s_2, s_3, s_4\}, \quad (4.24)$$

where

$$\begin{aligned} s_1 &= r[A^*(D + CMC^*)^{-1}A] - 2r[A^*(D + CMC^*)^{-1}CMB^*, A^*(D + CMC^*)^{-1}A] - n, \\ s_2 &= r[BMB^* - BMC^*(D + CMC^*)^{-1}CMB^*] \\ &\quad - 2r[BMB^* - BMC^*(D + CMC^*)^{-1}CMB^*, BMC^*(D + CMC^*)^{-1}A] - n, \\ s_3 &= i_-[A^*(D + CMC^*)^{-1}A] + i_-[BMB^* - BMC^*(D + CMC^*)^{-1}CMB^*] \\ &\quad - r[A^*(D + CMC^*)^{-1}CMB^*, A^*(D + CMC^*)^{-1}A] \\ &\quad - r[BMB^* - BMC^*(D + CMC^*)^{-1}CMB^*, BMC^*(D + CMC^*)^{-1}A] - n, \\ s_4 &= i_+[A^*(D + CMC^*)^{-1}A] + i_+[BMB^* - BMC^*(D + CMC^*)^{-1}CMB^*] \\ &\quad - r[A^*(D + CMC^*)^{-1}CMB^*, A^*(D + CMC^*)^{-1}A] \\ &\quad - r[BMB^* - BMC^*(D + CMC^*)^{-1}CMB^*, BMC^*(D + CMC^*)^{-1}A] - n. \end{aligned}$$

Setting (4.24) equal to n , we see that $\phi(X)$ is nonsingular for all $X \in \mathbb{C}^{p \times m}$ if and only if $r(D + CMC^*) = n$, and one of the following four rank equalities holds

- (i) $r[A^*(D + CMC^*)^{-1}A] = 2r[A^*(D + CMC^*)^{-1}CMB^*, A^*(D + CMC^*)^{-1}A]$;
- (ii) $r[BMB^* - BMC^*(D + CMC^*)^{-1}CMB^*] = 2r[BMB^* - BMC^*(D + CMC^*)^{-1}CMB^*, BMC^*(D + CMC^*)^{-1}A]$;
- (iii) $i_-[A^*(D + CMC^*)^{-1}A] + i_-[BMB^* - BMC^*(D + CMC^*)^{-1}CMB^*] = r[A^*(D + CMC^*)^{-1}CMB^*, A^*(D + CMC^*)^{-1}A] + r[BMB^* - BMC^*(D + CMC^*)^{-1}CMB^*, BMC^*(D + CMC^*)^{-1}A]$;
- (iv) $i_+[A^*(D + CMC^*)^{-1}A] + i_+[BMB^* - BMC^*(D + CMC^*)^{-1}CMB^*] = r[A^*(D + CMC^*)^{-1}CMB^*, A^*(D + CMC^*)^{-1}A] + r[BMB^* - BMC^*(D + CMC^*)^{-1}CMB^*, BMC^*(D + CMC^*)^{-1}A] - n$;

which are further equivalent to the result in (b) by comparing both sides of the four rank equalities. \square

Whether a given function is positive or nonnegative everywhere is a fundamental research subject in both elementary and advanced mathematics. It was realized in matrix theory that the complexity status of the definite and semi-definite feasibility problems of a general matrix-valued function is NP-hard. Corollary 4.2(d)–(k), however, show that we are really able to characterize the definiteness and semi-definiteness of (1.3) by using some ordinary and elementary methods. These results set up a criterion for characterizing definiteness and semi-definiteness of nonlinear matrix-valued functions, and will prompt more investigations on this challenging topic. In particular, definiteness and semi-definiteness of some nonlinear matrix-valued functions generated from (1.3) can be identified. We shall present them in another paper.

Recall that a Hermitian matrix A can uniquely be decomposed as the difference of two disjoint Hermitian positive semi-definite definite matrices

$$A = A_1 - A_2, \quad A_1 A_2 = A_2 A_1 = 0, \quad A_1 \succcurlyeq 0, \quad A_2 \succcurlyeq 0. \quad (4.25)$$

Applying this assertion to (1.3), we obtain the following result.

Corollary 4.3. Let $\phi(X)$ be as given in (1.3). Then, $\phi(X)$ can always be decomposed as

$$\phi(X) = \phi_1(X) - \phi_2(X), \quad (4.26)$$

where

$$\phi_1(X) = (AXB + C)M_1(AXB + C)^* + D_1, \quad \phi_2(X) = (AXB + C)M_2(AXB + C)^* + D_2$$

satisfy

$$\phi_1(X) \succcurlyeq 0, \quad \phi_2(X) \succcurlyeq 0 \quad (4.27)$$

for all $X \in \mathbb{C}^{p \times m}$.

Proof. Note from (4.25) that the two Hermitian matrices D and M in (1.3) can uniquely be decomposed as

$$\begin{aligned} D &= D_1 - D_2, \quad D_1 D_2 = D_2 D_1 = 0, \quad D_1 \succcurlyeq 0, \quad D_2 \succcurlyeq 0, \\ M &= M_1 - M_2, \quad M_1 M_2 = M_2 M_1 = 0, \quad M_1 \succcurlyeq 0, \quad M_2 \succcurlyeq 0, \end{aligned}$$

So that $\phi_1(X)$ and $\phi_2(X)$ in (4.26) are positive matrix-valued functions. \square

Corollary 4.4. Let $\phi(X)$ be as given in (1.3), and suppose that $AXB + C = 0$ has a solution, i.e., $\mathcal{R}(C) \subseteq \mathcal{R}(A)$ and $\mathcal{R}(C^*) \subseteq \mathcal{R}(B^*)$, and let $N = \begin{bmatrix} D & A \\ A^* & 0 \end{bmatrix}$. Then,

$$\max_{X \in \mathbb{C}^{p \times m}} r[\phi(X)] = \min \{ r[A, D], \quad r(BMB^*) + r(D) \}, \quad (4.28)$$

$$\begin{aligned} \min_{X \in \mathbb{C}^{p \times m}} r[\phi(X)] &= \max \{ 2r[A, D] - r(M), \quad r(D) - r(BMB^*), \quad r[A, D] + i_-(D) - i_+(BMB^*) - i_-(N), \\ &\quad r[A, D] + i_+(D) - i_-(BMB^*) - i_+(N) \}, \end{aligned} \quad (4.29)$$

$$\max_{X \in \mathbb{C}^{p \times m}} i_{\pm}[\phi(X)] = \min \{ i_{\pm}(M), \quad i_{\pm}(BMB^*) + i_{\pm}(D) \}, \quad (4.30)$$

$$\min_{X \in \mathbb{C}^{p \times m}} i_{\pm}[\phi(X)] = \max \{ r[A, D] - i_{\mp}(N), \quad i_{\pm}(D) - i_{\mp}(BMB^*) \}. \quad (4.31)$$

Corollary 4.5. Let

$$\phi(X) = (AX + C)M(AX + C)^* + D, \quad (4.32)$$

where $A \in \mathbb{C}^{n \times p}$, $C \in \mathbb{C}^{n \times m}$, $D \in \mathbb{C}_{\mathbb{H}}^n$ and $M \in \mathbb{C}_{\mathbb{H}}^m$ are given, and $X \in \mathbb{C}^{p \times m}$ is a variable matrix, and define

$$N_1 = \begin{bmatrix} D + CMC^* & A \\ A^* & 0 \end{bmatrix}, \quad N_2 = \begin{bmatrix} D & CM & A \\ A^* & 0 & 0 \end{bmatrix}. \quad (4.33)$$

Then,

$$\max_{X \in \mathbb{C}^{p \times m}} r[\phi(X)] = \min \{ r[A, CM, D], \quad r(N_1), \quad r(M) + r(D) \}, \quad (4.34)$$

$$\min_{X \in \mathbb{C}^{p \times m}} r[\phi(X)] = 2r[A, CM, D] + \max \{ s_1, \quad s_2, \quad s_3, \quad s_4 \}, \quad (4.35)$$

$$\max_{X \in \mathbb{C}^{p \times m}} i_{\pm}[\phi(X)] = \min \{ i_{\pm}(N_1), \quad i_{\pm}(M) + i_{\pm}(D) \}, \quad (4.36)$$

$$\min_{X \in \mathbb{C}^{p \times m}} i_{\pm}[\phi(X)] = r[D, CM, A] + \max \{ i_{\pm}(N_1) - r(N_2), \quad i_{\pm}(M) + i_{\pm}(D) - r(N_4) \}, \quad (4.37)$$

where

$$\begin{aligned} s_1 &= r(N_1) - 2r(N_2), \quad s_2 = r(D) - 2r[A, D] - r(M), \\ s_3 &= i_+(N_1) - r(N_2) - i_-(D) - r[A, D] - i_+(M), \quad s_4 = i_-(N_1) - r(N_2) - i_+(D) - r[A, D] - i_-(M). \end{aligned}$$

Corollary 4.6. Let

$$\phi(X) = (XB + C)M(XB + C)^* + D, \quad (4.38)$$

where $B \in \mathbb{C}^{p \times m}$, $C \in \mathbb{C}^{n \times m}$, $D \in \mathbb{C}_H^n$ and $M \in \mathbb{C}_H^m$ are given, and $X \in \mathbb{C}^{n \times p}$ is a variable matrix, and define

$$N_1 = \begin{bmatrix} D + CMC^* & CMB^* \\ BMC^* & BMB^* \end{bmatrix}, \quad N_2 = [BMB^*, BMC^*]. \quad (4.39)$$

Then,

$$\max_{X \in \mathbb{C}^{p \times m}} r[\phi(X)] = n, \quad (4.40)$$

$$\min_{X \in \mathbb{C}^{p \times m}} r[\phi(X)] = \max\{0, r(N_1) - 2(N_2), i_+(N_1) - r(N_2), i_-(N_1) - r(N_2)\}, \quad (4.41)$$

$$\max_{X \in \mathbb{C}^{p \times m}} i_{\pm}[\phi(X)] = \min\{n, i_{\pm}(N_1)\}, \quad (4.42)$$

$$\min_{X \in \mathbb{C}^{p \times m}} i_{\pm}[\phi(X)] = n + \max\{0, i_{\pm}(N_1) - r(N_2)\}. \quad (4.43)$$

We next solve the two quadratic optimization problems in (2.5), where the two matrices $\phi(\widehat{X})$ and $\phi(\widetilde{X})$, when they exist, are called the global maximal and minimal matrices of $\phi(X)$ in (1.3) in the Löwner partial ordering, respectively.

Corollary 4.7. *Let $\phi(X)$ be as given in (1.3), and let $N = \begin{bmatrix} D + CMC^* & CMB \\ B^*MC & B^*MB \end{bmatrix}$. Then, there exists an $\widehat{X} \in \mathbb{C}^{p \times m}$ such that*

$$\phi(X) \succcurlyeq \phi(\widehat{X}) \quad (4.44)$$

holds for all $X \in \mathbb{C}^{p \times m}$ if and only if

$$BMB^* \succcurlyeq 0, \quad \mathcal{R}(CMB^*) \subseteq \mathcal{R}(A), \quad \mathcal{R}(BMC^*) \subseteq \mathcal{R}(BMB^*). \quad (4.45)$$

In this case,

(a) *The matrix $\widehat{X} \in \mathbb{C}^{p \times m}$ satisfying (4.44) is the solution of the linear matrix equation*

$$A\widehat{X}BMB^* + CMB^* = 0. \quad (4.46)$$

Correspondingly,

$$\widehat{X} = -A^\dagger CMB^*(BMB^*)^\dagger + F_A V_1 + V_2 E_{BMB^*}, \quad (4.47)$$

$$\phi(\widehat{X}) = D + CMC^* - CMB^*(BMB^*)^\dagger BMC^*, \quad (4.48)$$

$$\phi(X) - \phi(\widehat{X}) = (AXB + C)MB^*(BMB^*)^\dagger BM(AXB + C)^*, \quad (4.49)$$

where V_1 and V_2 are arbitrary matrices.

(b) *The inertias and ranks of $\phi(\widehat{X})$ and $\phi(X) - \phi(\widehat{X})$ are given by*

$$i_+[\phi(\widehat{X})] = i_+(N) - r(BMB^*), \quad i_-[\phi(\widehat{X})] = i_-(N), \quad r[\phi(\widehat{X})] = r(N) - r(BMB^*), \quad (4.50)$$

$$i_+[\phi(X) - \phi(\widehat{X})] = r[\phi(X) - \phi(\widehat{X})] = r(AXBMB^* + CMB^*). \quad (4.51)$$

(c) *The matrix $\widehat{X} \in \mathbb{C}^{p \times m}$ satisfying (4.44) is unique if and only if*

$$r(A) = p, \quad \mathcal{R}(CMB^*) \subseteq \mathcal{R}(A), \quad BMB^* \succ 0; \quad (4.52)$$

under this condition,

$$\widehat{X} = -A^\dagger CMB^*(BMB^*)^{-1}, \quad (4.53)$$

$$\phi(\widehat{X}) = D + CMC^* - CMB^*(BMB^*)^{-1} BMC^*, \quad (4.54)$$

$$\phi(X) - \phi(\widehat{X}) = (AXB + C)MB^*(BMB^*)^{-1} BM(AXB + C)^*. \quad (4.55)$$

(d) $\widehat{X} = 0$ *is a solution of (4.44) if and only if $BMB^* \succcurlyeq 0$ and $CMB^* = 0$. In this case, $\phi(0) = D + CMC^*$.*

(e) $\widehat{X} = 0$ *is the unique solution (4.44) if and only if $r(A) = p$, $CMB^* = 0$ and $BMB^* \succ 0$. In this case, $\phi(0) = D + CMC^*$.*

(f) *There exists an $\widehat{X} \in \mathbb{C}^{p \times m}$ such that*

$$\phi(X) \succ \phi(\widehat{X}) \succ 0 \quad (4.56)$$

holds for all $X \in \mathbb{C}^{p \times m}$ if and only if

$$\mathcal{R}(CMB^*) \subseteq \mathcal{R}(A) \text{ and } N \succ 0. \quad (4.57)$$

In this case, the matrix $\widehat{X} \in \mathbb{C}^{p \times m}$ satisfying (4.56) is unique if and only if

$$r(A) = p, \quad \mathcal{R}(CMB^*) \subseteq \mathcal{R}(A), \quad BMB^* \succ 0, \quad N \succ 0. \quad (4.58)$$

Proof. Let

$$\psi(X) = \phi(X) - \phi(\widehat{X}) = (AXB + C)M(AXB + C)^* - (A\widehat{X}B + C)M(A\widehat{X}B + C)^*.$$

Then, $\phi(X) \succ \phi(\widehat{X})$ is equivalent to $\psi(X) \succ 0$. Under $A \neq 0$, we see from Corollary 2.2(j) that $\psi(X) \succ 0$ holds for all $X \in \mathbb{C}^{p \times m}$ if and only if

$$\begin{bmatrix} CMC^* - (A\widehat{X}B + C)M(A\widehat{X}B + C)^* & CMB^* \\ BMC^* & BMB^* \end{bmatrix} \succ 0, \quad (4.59)$$

which, by Lemma 3.3(e), is further equivalent to

$$BMB^* \succ 0, \quad \mathcal{R}(BMC^*) \subseteq \mathcal{R}(BMB^*), \quad (4.60)$$

$$CMC^* - (A\widehat{X}B + C)M(A\widehat{X}B + C)^* - CMB^*(BMB^*)^\dagger BMC^* \succ 0. \quad (4.61)$$

In this case, it is easy to verify

$$\begin{aligned} & CMC^* - (A\widehat{X}B + C)M(A\widehat{X}B + C)^* - CMB^*(BMB^*)^\dagger BMC^* \\ &= -(A\widehat{X}BMB^* + CMB^*)(BMB^*)^\dagger (A\widehat{X}BMB^* + CMB^*)^*, \end{aligned} \quad (4.62)$$

and therefore, the inequality in (4.61) is equivalent to $CMB^* + A\widehat{X}BMB^* = 0$. By Lemma 3.4, this matrix equation is solvable if and only if $\mathcal{R}(CMB^*) \subseteq \mathcal{R}(A)$ and $\mathcal{R}(BMC^*) \subseteq \mathcal{R}(BMB^*)$. In this case, the general solution of the equation is (4.47) by Lemma 3.4, and (4.61) becomes

$$CMC^* - (A\widehat{X}B + C)M(A\widehat{X}B + C)^* - CMB^*(BMB^*)^\dagger BMC^* = 0.$$

Thus (4.48) and (4.49) follow. The results in (b)–(f) follow from (a). \square

The following corollary can be shown similarly.

Corollary 4.8. *Let $\phi(X)$ be as given in (1.3), and let $N = \begin{bmatrix} D + CMC^* & CMB \\ B^*MC & B^*MB \end{bmatrix}$. Then, there exists an $\widetilde{X} \in \mathbb{C}^{p \times m}$ such that*

$$\phi(X) \preceq \phi(\widetilde{X}) \quad (4.63)$$

holds for all $X \in \mathbb{C}^{p \times m}$ if and only if

$$BMB^* \preceq 0, \quad \mathcal{R}(CMB^*) \subseteq \mathcal{R}(A), \quad \mathcal{R}(BMC^*) \subseteq \mathcal{R}(BMB^*). \quad (4.64)$$

In this case,

(a) *The matrix \widetilde{X} satisfying (4.63) is the solution of the linear matrix equation*

$$A\widetilde{X}BMB^* + CMB^* = 0. \quad (4.65)$$

Correspondingly,

$$\widetilde{X} = -A^\dagger CMB^*(BMB^*)^\dagger + F_A V_1 + V_2 E_{BMB^*}, \quad (4.66)$$

$$\phi(\widetilde{X}) = D + CMC^* - CMB^*(BMB^*)^\dagger BMC^*, \quad (4.67)$$

$$\phi(X) - \phi(\widetilde{X}) = (AXB + C)MB^*(BMB^*)^\dagger BM(AXB + C)^*, \quad (4.68)$$

where V_1 and V_2 are arbitrary matrices.

(b) The inertias and ranks of $\phi(\tilde{X})$ and $\phi(X) - \phi(\tilde{X})$ are given by

$$i_+[\phi(\tilde{X})] = i_+(N), \quad i_-[\phi(\tilde{X})] = i_-(N) - r(BMB^*), \quad r[\phi(\tilde{X})] = r(N) - r(BMB^*), \quad (4.69)$$

$$i_-[\phi(X) - \phi(\tilde{X})] = r[\phi(X) - \phi(\tilde{X})] = r(AXBMB^* + CMB^*). \quad (4.70)$$

(c) The matrix $\tilde{X} \in \mathbb{C}^{p \times m}$ satisfying (4.63) is unique if and only if

$$r(A) = p, \quad \mathcal{R}(CMB^*) \subseteq \mathcal{R}(A), \quad BMB^* \prec 0. \quad (4.71)$$

In this case,

$$\tilde{X} = -A^\dagger CMB^*(BMB^*)^{-1}, \quad (4.72)$$

$$\phi(\tilde{X}) = D + CMC^* - CMB^*(BMB^*)^{-1}BMC^*, \quad (4.73)$$

$$\phi(X) - \phi(\tilde{X}) = (AXB + C)MB^*(BMB^*)^{-1}BM(AXB + C)^*. \quad (4.74)$$

(d) $\tilde{X} = 0$ is a solution of (4.63) if and only if $BMB^* \preceq 0$ and $CMB^* = 0$. In this case, $\phi(0) = D + CMC^*$.

(e) $\tilde{X} = 0$ is the unique solution of (4.63) if and only if $r(A) = p$, $CMB^* = 0$ and $BMB^* \prec 0$. In this case, $\phi(0) = D + CMC^*$.

(f) There exists an $\tilde{X} \in \mathbb{C}^{p \times m}$ such that

$$\phi(X) \preceq \phi(\tilde{X}) \preceq 0 \quad (4.75)$$

holds for all $X \in \mathbb{C}^{p \times m}$ if and only if

$$\mathcal{R}(CMB^*) \subseteq \mathcal{R}(A) \quad \text{and} \quad N \preceq 0. \quad (4.76)$$

In this case, the matrix $\tilde{X} \in \mathbb{C}^{p \times m}$ satisfying (4.75) is unique if and only if

$$r(A) = p, \quad \mathcal{R}(CMB^*) \subseteq \mathcal{R}(A), \quad BMB^* \prec 0, \quad N \preceq 0. \quad (4.77)$$

5. The convexity and concavity of $\phi(X)$ in (1.3)

As usual, the matrix-valued function $\phi(X)$ in (1.3) is said to be convex if and only if

$$\phi\left(\frac{1}{2}X_1 + \frac{1}{2}X_2\right) \preceq \frac{1}{2}\phi(X_1) + \frac{1}{2}\phi(X_2) \quad (5.1)$$

holds for all $X_1, X_2 \in \mathbb{C}^{p \times m}$; said to be concave if and only if

$$\phi\left(\frac{1}{2}X_1 + \frac{1}{2}X_2\right) \succcurlyeq \frac{1}{2}\phi(X_1) + \frac{1}{2}\phi(X_2) \quad (5.2)$$

holds for all $X_1, X_2 \in \mathbb{C}^{p \times m}$. It is easy to verify that

$$\phi\left(\frac{1}{2}X_1 + \frac{1}{2}X_2\right) - \frac{1}{2}\phi(X_1) - \frac{1}{2}\phi(X_2) = -\frac{1}{4}A(X_1 - X_2)BMB^*(X_1 - X_2)^*A^*, \quad (5.3)$$

which is a special case of (1.3). Applying Theorem 4.1 to (5.3), we obtain the following result.

Theorem 5.1. Let $\phi(X)$ be as given in (1.3) with $A \neq 0$ and $BMB^* \neq 0$. Then,

$$\max_{X_1 \neq X_2, X_1, X_2 \in \mathbb{C}^{p \times m}} r\left[\phi\left(\frac{1}{2}X_1 + \frac{1}{2}X_2\right) - \frac{1}{2}\phi(X_1) - \frac{1}{2}\phi(X_2)\right] = \min\{r(A), r(BMB^*)\}, \quad (5.4)$$

$$\min_{X_1 \neq X_2, X_1, X_2 \in \mathbb{C}^{p \times m}} r\left[\phi\left(\frac{1}{2}X_1 + \frac{1}{2}X_2\right) - \frac{1}{2}\phi(X_1) - \frac{1}{2}\phi(X_2)\right] = \begin{cases} 1 & BMB^* \succ 0 \text{ and } r(A) = p \\ 1 & BMB^* \prec 0 \text{ and } r(A) = p \\ 0 & \text{otherwise} \end{cases}, \quad (5.5)$$

$$\max_{X_1 \neq X_2, X_1, X_2 \in \mathbb{C}^{p \times m}} i_+\left[\phi\left(\frac{1}{2}X_1 + \frac{1}{2}X_2\right) - \frac{1}{2}\phi(X_1) - \frac{1}{2}\phi(X_2)\right] = \min\{r(A), i_-(BMB^*)\}, \quad (5.6)$$

$$\max_{X_1 \neq X_2, X_1, X_2 \in \mathbb{C}^{p \times m}} i_-\left[\phi\left(\frac{1}{2}X_1 + \frac{1}{2}X_2\right) - \frac{1}{2}\phi(X_1) - \frac{1}{2}\phi(X_2)\right] = \min\{r(A), i_+(BMB^*)\}, \quad (5.7)$$

$$\min_{X_1 \neq X_2, X_1, X_2 \in \mathbb{C}^{p \times m}} i_+\left[\phi\left(\frac{1}{2}X_1 + \frac{1}{2}X_2\right) - \frac{1}{2}\phi(X_1) - \frac{1}{2}\phi(X_2)\right] = \begin{cases} 1 & BMB^* \prec 0 \text{ and } r(A) = p \\ 0 & BMB^* \not\prec 0 \text{ or } r(A) < p \end{cases}, \quad (5.8)$$

$$\min_{X_1 \neq X_2, X_1, X_2 \in \mathbb{C}^{p \times m}} i_-\left[\phi\left(\frac{1}{2}X_1 + \frac{1}{2}X_2\right) - \frac{1}{2}\phi(X_1) - \frac{1}{2}\phi(X_2)\right] = \begin{cases} 1 & BMB^* \succ 0 \text{ and } r(A) = p \\ 0 & BMB^* \not\succ 0 \text{ or } r(A) < p \end{cases}. \quad (5.9)$$

In consequence,

- (a) *There exist $X_1, X_2 \in \mathbb{C}^{p \times m}$ with $X_1 \neq X_2$ such that $\phi(\frac{1}{2}X_1 + \frac{1}{2}X_2) - \frac{1}{2}\phi(X_1) - \frac{1}{2}\phi(X_2)$ is nonsingular if and only if both $r(A) = n$ and $r(BMB^*) \succ n$.*
- (b) *There exist $X_1, X_2 \in \mathbb{C}^{p \times m}$ with $X_1 \neq X_2$ such that $\phi(\frac{1}{2}X_1 + \frac{1}{2}X_2) = \frac{1}{2}\phi(X_1) + \frac{1}{2}\phi(X_2)$ if and only if $BMB^* \not\prec 0$ and $BMB^* \not\prec 0$, or $r(A) < p$.*
- (c) *There exist $X_1, X_2 \in \mathbb{C}^{p \times m}$ with $X_1 \neq X_2$ such that $\phi(\frac{1}{2}X_1 + \frac{1}{2}X_2) \succ \frac{1}{2}\phi(X_1) + \frac{1}{2}\phi(X_2)$ if and only if both $BMB^* \prec 0$ and $r(A) = n$.*
- (d) *There exist $X_1, X_2 \in \mathbb{C}^{p \times m}$ with $X_1 \neq X_2$ such that $\phi(\frac{1}{2}X_1 + \frac{1}{2}X_2) \prec \frac{1}{2}\phi(X_1) + \frac{1}{2}\phi(X_2)$ if and only if both $BMB^* \succ 0$ and $r(A) = n$.*
- (e) *There exist $X_1, X_2 \in \mathbb{C}^{p \times m}$ with $X_1 \neq X_2$ such that $\phi(\frac{1}{2}X_1 + \frac{1}{2}X_2) \succcurlyeq \frac{1}{2}\phi(X_1) + \frac{1}{2}\phi(X_2)$ if and only if either $BMB^* \not\prec 0$ or $r(A) < p$.*
- (f) *There exist $X_1, X_2 \in \mathbb{C}^{p \times m}$ with $X_1 \neq X_2$ such that $\phi(\frac{1}{2}X_1 + \frac{1}{2}X_2) \preccurlyeq \frac{1}{2}\phi(X_1) + \frac{1}{2}\phi(X_2)$ if and only if either $BMB^* \not\prec 0$ or $r(A) < p$.*
- (g) *$\phi(\frac{1}{2}X_1 + \frac{1}{2}X_2) \succcurlyeq \frac{1}{2}\phi(X_1) + \frac{1}{2}\phi(X_2)$ for all $X_1, X_2 \in \mathbb{C}^{p \times m}$ with $X_1 \neq X_2$ if and only if $BMB^* \preccurlyeq 0$.*
- (h) *$\phi(\frac{1}{2}X_1 + \frac{1}{2}X_2) \preccurlyeq \frac{1}{2}\phi(X_1) + \frac{1}{2}\phi(X_2)$ for all $X_1, X_2 \in \mathbb{C}^{p \times m}$ with $X_1 \neq X_2$ if and only if $BMB^* \succcurlyeq 0$.*
- (i) *If $\phi(X)$ is a positive semi-definite matrix-valued function, then $\phi(X)$ is convex.*
- (j) *If $\phi(X)$ is a negative semi-definite matrix-valued function, then $\phi(X)$ is concave.*

6. Semi-definiteness of general Hermitian quadratic matrix-valued functions and solutions of the corresponding partial ordering optimization problems

As an extension of (1.3), we consider the general matrix-valued quadratic function

$$\phi(X_1, \dots, X_k) = \left(\sum_{i=1}^k A_i X_i B_i + C \right) M \left(\sum_{i=1}^k A_i X_i B_i + C \right)^* + D, \quad (6.1)$$

where $0 \neq A_i \in \mathbb{C}^{n \times p_i}$, $B_i \in \mathbb{C}^{m_i \times q}$, $C \in \mathbb{C}^{n \times q}$, $D \in \mathbb{C}_{\mathbb{H}}^n$ and $M \in \mathbb{C}_{\mathbb{H}}^q$ are given, and $X_i \in \mathbb{C}^{p_i \times m_i}$ is a variable matrix, $i = 1, \dots, k$. We treat it as a combined non-homogeneous linear and quadratic Hermitian matrix-valued function $\phi = \tau \circ \psi$:

$$\psi: \mathbb{C}^{p_1 \times m_1} \oplus \dots \oplus \mathbb{C}^{p_k \times m_k} \rightarrow \mathbb{C}^{n \times q}, \quad \tau: \mathbb{C}^{n \times q} \rightarrow \mathbb{C}_{\mathbb{H}}^n.$$

This general quadratic function between matrix space includes many ordinary Hermitian quadratic matrix-valued functions as its special cases. Because more than one variable matrices occur in (6.1), we do not know at current time how to establish analytical formulas for the extremum ranks and inertias of (6.1). In this section, we only consider the following problems:

- (i) establish necessary and sufficient conditions for $\phi(X_1, \dots, X_k) \succcurlyeq 0$ ($\phi(X_1, \dots, X_k) \preccurlyeq 0$) to hold for all X_1, \dots, X_k ;
- (ii) establish necessary and sufficient conditions for the existence of $\widehat{X}_1, \dots, \widehat{X}_k$ and $\widetilde{X}_1, \dots, \widetilde{X}_k$ such that

$$\phi(X_1, \dots, X_k) \succcurlyeq \phi(\widehat{X}_1, \dots, \widehat{X}_k), \quad \phi(X_1, \dots, X_k) \preccurlyeq \phi(\widetilde{X}_1, \dots, \widetilde{X}_k) \quad (6.2)$$

hold for all X_1, \dots, X_k in the Löwner partial ordering, respectively, and give analytical expressions of $\widehat{X}_1, \dots, \widehat{X}_k$ and $\widetilde{X}_1, \dots, \widetilde{X}_k$.

Theorem 6.1. *Let $\phi(X_1, \dots, X_k)$ be as given in (6.1), and define $B^* = [B_1^*, \dots, B_k^*]$. Also let*

$$N = \begin{bmatrix} D + CMC^* & CMB^* \\ BMC^* & BMB^* \end{bmatrix}. \quad (6.3)$$

Then,

- (a) $\phi(X_1, \dots, X_k) \succcurlyeq 0$ for all $X_1 \in \mathbb{C}^{p_1 \times m_1}, \dots, X_k \in \mathbb{C}^{p_k \times m_k}$ if and only if $N \succcurlyeq 0$.
- (b) $\phi(X_1, \dots, X_k) \preccurlyeq 0$ for all $X_1 \in \mathbb{C}^{p_1 \times m_1}, \dots, X_k \in \mathbb{C}^{p_k \times m_k}$ if and only if $N \preccurlyeq 0$.
- (c) *There exist $\widehat{X}_1, \dots, \widehat{X}_k$ such that*

$$\phi(X_1, \dots, X_k) \succcurlyeq \phi(\widehat{X}_1, \dots, \widehat{X}_k) \quad (6.4)$$

holds for all $X_1 \in \mathbb{C}^{p_1 \times m_1}, \dots, X_k \in \mathbb{C}^{p_k \times m_k}$ if and only if

$$BMB^* \succcurlyeq 0, \quad \mathcal{R}(BMC^*) \subseteq \mathcal{R}(BMB^*). \quad (6.5)$$

In this case, the matrices $\widehat{X}_1, \dots, \widehat{X}_k$ are the solutions of the linear matrix equation

$$\sum_{i=1}^k A_i \widehat{X}_i B_i M B^* = -C M B^*. \quad (6.6)$$

Correspondingly,

$$\phi(\widehat{X}_1, \dots, \widehat{X}_k) = D + C M C^* - C M B^* (B M B^*)^\dagger B M C^*, \quad (6.7)$$

$$\phi(X_1, \dots, X_k) - \phi(\widehat{X}_1, \dots, \widehat{X}_k) = \left(\sum_{i=1}^k A_i X_i B_i + C \right) M B^* (B M B^*)^\dagger B M \left(\sum_{i=1}^k A_i X_i B_i + C \right)^*. \quad (6.8)$$

(d) There exist $\widetilde{X}_1, \dots, \widetilde{X}_k$ such that

$$\phi(X_1, \dots, X_k) \preceq \phi(\widetilde{X}_1, \dots, \widetilde{X}_k) \quad (6.9)$$

holds for all $X_1 \in \mathbb{C}^{p_1 \times m_1}, \dots, X_k \in \mathbb{C}^{p_k \times m_k}$ if and only if

$$B M B^* \preceq 0, \quad \mathcal{R}(B M C^*) \subseteq \mathcal{R}(B M B^*). \quad (6.10)$$

In this case, the matrices $\widetilde{X}_1, \dots, \widetilde{X}_k$ are the solutions of the linear matrix equation

$$\sum_{i=1}^k A_i \widetilde{X}_i B_i M B^* = -C M B^*. \quad (6.11)$$

Correspondingly,

$$\phi(\widetilde{X}_1, \dots, \widetilde{X}_k) = D + C M C^* - C M B^* (B M B^*)^\dagger B M C^*, \quad (6.12)$$

$$\phi(X_1, \dots, X_k) - \phi(\widetilde{X}_1, \dots, \widetilde{X}_k) = \left(\sum_{i=1}^k A_i X_i B_i + C \right) M B^* (B M B^*)^\dagger B M \left(\sum_{i=1}^k A_i X_i B_i + C \right)^*. \quad (6.13)$$

Proof. Rewrite (6.1) as

$$\begin{aligned} \phi(X_1, \dots, X_k) &= \left(A_1 X_1 B_1 + \sum_{i=2}^k A_i X_i B_i + C \right) M \left(A_1 X_1 B_1 + \sum_{i=2}^k A_i X_i B_i + C \right)^* \\ &\quad + D, \end{aligned} \quad (6.14)$$

and applying Corollary 4.7 to it, we see that $\phi(X_1, \dots, X_k) \succeq 0$ for all $X_1 \in \mathbb{C}^{p_1 \times m_1}$ if and only if

$$\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \sum_{i=2}^k A_i X_i B_i + C \\ B_1 \end{bmatrix} M \begin{bmatrix} \sum_{i=2}^k A_i X_i B_i + C \\ B_1 \end{bmatrix}^* \succeq 0 \quad (6.15)$$

for all $X_2 \in \mathbb{C}^{p_2 \times m_2}, \dots, X_k \in \mathbb{C}^{p_k \times m_k}$. Note that

$$\begin{aligned} &\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \sum_{i=2}^k A_i X_i B_i + C \\ B_1 \end{bmatrix} M \begin{bmatrix} \sum_{i=2}^k A_i X_i B_i + C \\ B_1 \end{bmatrix}^* \\ &= \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} + \left(\sum_{i=2}^k \begin{bmatrix} A_i \\ 0 \end{bmatrix} X_i B_i + \begin{bmatrix} C \\ B_1 \end{bmatrix} \right) M \left(\sum_{i=2}^k \begin{bmatrix} A_i \\ 0 \end{bmatrix} X_i B_i + \begin{bmatrix} C \\ B_1 \end{bmatrix} \right)^*. \end{aligned} \quad (6.16)$$

Applying Corollary 4.7, we see that this matrix is positive semi-definite for all $X_2 \in \mathbb{C}^{p_2 \times m_2}$ if and only if

$$\begin{bmatrix} D & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \sum_{i=3}^k \begin{bmatrix} A_i \\ 0 \end{bmatrix} X_i B_i + \begin{bmatrix} C \\ B_1 \end{bmatrix} \\ B_2 \end{bmatrix} M \begin{bmatrix} \sum_{i=3}^k \begin{bmatrix} A_i \\ 0 \end{bmatrix} X_i B_i + \begin{bmatrix} C \\ B_1 \end{bmatrix} \\ B_2 \end{bmatrix}^* \succeq 0 \quad (6.17)$$

for all $X_3 \in \mathbb{C}^{p_3 \times m_3}, \dots, X_k \in \mathbb{C}^{p_k \times m_k}$. Thus, we obtain by induction that $\phi(X_1, \dots, X_k) \succeq 0$ for all $X_1 \in \mathbb{C}^{p_1 \times m_1}, \dots, X_k \in \mathbb{C}^{p_k \times m_k}$ if and only if

$$\begin{bmatrix} D + C M C^* & C M B_1^* & \dots & C M B_k^* \\ B_1 M C^* & B_1 M B_1^* & \dots & B_1 M B_k^* \\ \vdots & \vdots & \ddots & \vdots \\ B_k M C^* & B_k M B_1^* & \dots & B_k M B_k^* \end{bmatrix} = \begin{bmatrix} D + C M C^* & C M B^* \\ B M C^* & B M B^* \end{bmatrix} \succeq 0,$$

establishing (a).

Let

$$\begin{aligned}
& \rho(X_1, \dots, X_k) \\
&= \phi(X_1, \dots, X_k) - \phi(\widehat{X}_1, \dots, \widehat{X}_k) \\
&= \left(\sum_{i=1}^k A_i X_i B_i + C \right) M \left(\sum_{i=1}^k A_i X_i B_i + C \right)^* - \left(\sum_{i=1}^k A_i \widehat{X}_i B_i + C \right) M \left(\sum_{i=1}^k A_i \widehat{X}_i B_i + C \right)^*. \tag{6.18}
\end{aligned}$$

Then, $\phi(X_1, \dots, X_k) \succcurlyeq \phi(\widehat{X}_1, \dots, \widehat{X}_k)$ for all $X_1 \in \mathbb{C}^{p_1 \times m_1}, \dots, X_k \in \mathbb{C}^{p_k \times m_k}$ is equivalent to

$$\rho(X_1, \dots, X_k) \succcurlyeq 0 \text{ for all } X_1 \in \mathbb{C}^{p_1 \times m_1}, \dots, X_k \in \mathbb{C}^{p_k \times m_k}. \tag{6.19}$$

From (a), (6.19) holds if and only if

$$\left[- \left(\sum_{i=1}^k A_i \widehat{X}_i B_i + C \right) M \left(\sum_{i=1}^k A_i \widehat{X}_i B_i + C \right)^* + CMC^* \quad \begin{array}{c} CMB^* \\ BMB^* \end{array} \right] \succcurlyeq 0, \tag{6.20}$$

which, by (4.59)–(4.61), is further equivalent to

$$BMB^* \succcurlyeq 0, \quad \mathcal{R}(BMC^*) \subseteq \mathcal{R}(BMB^*), \tag{6.21}$$

$$CMC^* - \left(\sum_{i=1}^k A_i \widehat{X}_i B_i + C \right) M \left(\sum_{i=1}^k A_i \widehat{X}_i B_i + C \right)^* - CMB^*(BMB^*)^\dagger BMC^* \succcurlyeq 0. \tag{6.22}$$

In this case,

$$\begin{aligned}
& CMC^* - \left(\sum_{i=1}^k A_i \widehat{X}_i B_i + C \right) M \left(\sum_{i=1}^k A_i \widehat{X}_i B_i + C \right)^* - CMB^*(BMB^*)^\dagger BMC^* \\
&= - \left(\sum_{i=1}^k A_i \widehat{X}_i B_i + C \right) MB^*(BMB^*)^\dagger BM \left(\sum_{i=1}^k A_i \widehat{X}_i B_i + C \right)^* \tag{6.23}
\end{aligned}$$

holds, and therefore, (6.22) is equivalent to $CMB^* + \sum_{i=1}^k A_i \widehat{X}_i B_i MB^* = 0$. This is a general two-sided linear matrix equation involving k unknown matrices. The existence of solutions of this equation and its general solution can be derived from the Kronecker product of matrices. The details are omitted here. Result (d) can be shown similarly. \square

Two consequences of Theorem 6.1 are given below.

Corollary 6.2. *Let*

$$\psi(X_1, \dots, X_k) = \sum_{i=1}^k (A_i X_i B_i + C_i) M_i (A_i X_i B_i + C_i)^* + D, \tag{6.24}$$

where $0 \neq A_i \in \mathbb{C}^{n \times p_i}$, $B_i \in \mathbb{C}^{m_i \times q_i}$, $C \in \mathbb{C}^{n \times q_i}$, $D \in \mathbb{C}_{\mathbb{H}}^n$ and $M_i \in \mathbb{C}_{\mathbb{H}}^{q_i}$ are given, and $X_i \in \mathbb{C}^{p_i \times m_i}$ is a variable matrix, $i = 1, \dots, k$. Also define

$$B = \text{diag}(B_1, \dots, B_k), \quad C = [C_1, \dots, C_k], \quad M = \text{diag}(M_1, \dots, M_k), \quad N = \begin{bmatrix} D + CMC^* & CMB^* \\ BMC^* & BMB^* \end{bmatrix}.$$

Then,

- (a) $\psi(X_1, \dots, X_k) \succcurlyeq 0$ for all $X_1 \in \mathbb{C}^{p_1 \times m_1}, \dots, X_k \in \mathbb{C}^{p_k \times m_k}$ if and only if $N \succcurlyeq 0$.
- (b) $\psi(X_1, \dots, X_k) \preccurlyeq 0$ for all $X_1 \in \mathbb{C}^{p_1 \times m_1}, \dots, X_k \in \mathbb{C}^{p_k \times m_k}$ if and only if $N \preccurlyeq 0$.
- (c) There exist $\widehat{X}_1, \dots, \widehat{X}_k$ such that

$$\psi(X_1, \dots, X_k) \succcurlyeq \psi(\widehat{X}_1, \dots, \widehat{X}_k) \tag{6.25}$$

holds for all $X_1 \in \mathbb{C}^{p_1 \times m_1}, \dots, X_k \in \mathbb{C}^{p_k \times m_k}$ if and only if

$$B_i M_i B_i^* \succcurlyeq 0, \quad \mathcal{R}(B_i M_i C_i^*) \subseteq \mathcal{R}(B_i M_i B_i^*), \quad i = 1, \dots, k. \tag{6.26}$$

In this case, the matrices $\widehat{X}_1, \dots, \widehat{X}_k$ satisfying (6.25) are the solutions of the k linear matrix equations

$$A_i \widehat{X}_i B_i M_i B_i^* = -C_i M_i B_i^*, \quad i = 1, \dots, k. \quad (6.27)$$

Correspondingly,

$$\psi(\widehat{X}_1, \dots, \widehat{X}_k) = D + \sum_{i=1}^k C_i M_i C_i^* - \sum_{i=1}^k C_i M_i B^*(B_i M_i B_i^*)^\dagger B_i M_i C_i^*, \quad (6.28)$$

$$\psi(X_1, \dots, X_k) - \psi(\widehat{X}_1, \dots, \widehat{X}_k) = \sum_{i=1}^k (A_i X_i B_i + C_i) M_i B_i^* (B_i M_i B_i^*)^\dagger B_i M_i (A_i X_i B_i + C_i)^*. \quad (6.29)$$

(d) There exist $\widetilde{X}_1, \dots, \widetilde{X}_k$ such that

$$\psi(X_1, \dots, X_k) \preceq \psi(\widetilde{X}_1, \dots, \widetilde{X}_k) \quad (6.30)$$

holds for all $X_1 \in \mathbb{C}^{p_1 \times m_1}, \dots, X_k \in \mathbb{C}^{p_k \times m_k}$ if and only if

$$B_i M_i B_i^* \preceq 0, \quad \mathcal{R}(B_i M_i C_i^*) \subseteq \mathcal{R}(B_i M_i B_i^*), \quad i = 1, \dots, k. \quad (6.31)$$

In this case, the matrices $\widetilde{X}_1, \dots, \widetilde{X}_k$ satisfying (6.30) are the solutions of the k linear matrix equations

$$A_i \widetilde{X}_i B_i M_i B_i^* = -C_i M_i B_i^*, \quad i = 1, \dots, k. \quad (6.32)$$

Correspondingly,

$$\psi(\widetilde{X}_1, \dots, \widetilde{X}_k) = D + \sum_{i=1}^k C_i M_i C_i^* - \sum_{i=1}^k C_i M_i B^*(B_i M_i B_i^*)^\dagger B_i M_i C_i^*, \quad (6.33)$$

$$\psi(X_1, \dots, X_k) - \psi(\widetilde{X}_1, \dots, \widetilde{X}_k) = \sum_{i=1}^k (A_i X_i B_i + C_i) M_i B_i^* (B_i M_i B_i^*)^\dagger B_i M_i (A_i X_i B_i + C_i)^*. \quad (6.34)$$

Proof. Rewrite (6.24) as

$$\begin{aligned} \psi(X_1, \dots, X_k) &= [A_1 X_1 B_1 + C_1, \dots, A_k X_k B_k + C_k] M [A_1 X_1 B_1 + C_1, \dots, A_k X_k B_k + C_k]^* + D \\ &= [A_1 X_1 [B_1, \dots, 0] + \dots + A_k X_k [0, \dots, B_k] + [C_1, \dots, C_k]] M \\ &\quad \times [A_1 X_1 [B_1, \dots, 0] + \dots + A_k X_k [0, \dots, B_k] + [C_1, \dots, C_k]]^* + D, \end{aligned} \quad (6.35)$$

which is a special case of (6.1). Applying Theorem 6.1 to it, we obtain the result desired. \square

Corollary 6.3. Let

$$\psi(X_1, \dots, X_k) = [A_1 X_1 B_1 + C_1, \dots, A_k X_k B_k + C_k] M [A_1 X_1 B_1 + C_1, \dots, A_k X_k B_k + C_k]^* + D, \quad (6.36)$$

where $0 \neq A_i \in \mathbb{C}^{n \times p_i}$, $B_i \in \mathbb{C}^{m_i \times q_i}$, $C_i \in \mathbb{C}^{n \times q_i}$, $D \in \mathbb{C}_H^n$ and $M \in \mathbb{C}_H^{q_1 + \dots + q_k}$ are given, and $X_i \in \mathbb{C}^{p_i \times m_i}$ is variable matrix, $i = 1, \dots, k$. Also define

$$B = \text{diag}(B_1, \dots, B_k) \quad \text{and} \quad C = [C_1, \dots, C_k], \quad N = \begin{bmatrix} D + CMC^* & CMB^* \\ BMC^* & BMB^* \end{bmatrix}.$$

Then,

- (a) $\psi(X_1, \dots, X_k) \succeq 0$ for all $X_1 \in \mathbb{C}^{p_1 \times m_1}, \dots, X_k \in \mathbb{C}^{p_k \times m_k}$ if and only if $N \succeq 0$.
- (b) $\psi(X_1, \dots, X_k) \preceq 0$ for all $X_1 \in \mathbb{C}^{p_1 \times m_1}, \dots, X_k \in \mathbb{C}^{p_k \times m_k}$ if and only if $N \preceq 0$.
- (c) There exist $\widehat{X}_1, \dots, \widehat{X}_k$ such that

$$\psi(X_1, \dots, X_k) \succeq \psi(\widehat{X}_1, \dots, \widehat{X}_k) \quad (6.37)$$

holds for all $X_1 \in \mathbb{C}^{p_1 \times m_1}, \dots, X_k \in \mathbb{C}^{p_k \times m_k}$ if and only if

$$BMB^* \succeq 0, \quad \mathcal{R}(BMC^*) \subseteq \mathcal{R}(BMB^*). \quad (6.38)$$

In this case, the matrices $\widehat{X}_1, \dots, \widehat{X}_k$ satisfying (6.37) are the solutions of the linear matrix equation

$$\sum_{i=1}^k A_i \widehat{X}_i B_i M B^* = -C M B^*. \quad (6.39)$$

Correspondingly,

$$\psi(\widehat{X}_1, \dots, \widehat{X}_k) = D + C M C^* - C M B^* (B M B^*)^\dagger B M C^*, \quad (6.40)$$

$$\begin{aligned} & \psi(X_1, \dots, X_k) - \psi(\widehat{X}_1, \dots, \widehat{X}_k) \\ &= [A_1 X_1 B_1 + C_1, \dots, A_k X_k B_k + C_k] M B^* (B M B^*)^\dagger B M [A_1 X_1 B_1 + C_1, \dots, A_k X_k B_k + C_k]^*. \end{aligned} \quad (6.41)$$

(d) There exist $\widetilde{X}_1, \dots, \widetilde{X}_k$ such that

$$\psi(X_1, \dots, X_k) \preceq \psi(\widetilde{X}_1, \dots, \widetilde{X}_k) \quad (6.42)$$

holds for all $X_1 \in \mathbb{C}^{p_1 \times m_1}, \dots, X_k \in \mathbb{C}^{p_k \times m_k}$ if and only if

$$B M B^* \preceq 0, \quad \mathcal{R}(B M C^*) \subseteq \mathcal{R}(B M B^*). \quad (6.43)$$

In this case, the matrices $\widetilde{X}_1, \dots, \widetilde{X}_k$ satisfying (6.42) are the solutions of the linear matrix equation

$$\sum_{i=1}^k A_i \widetilde{X}_i B_i M B^* = -C M B^*. \quad (6.44)$$

Correspondingly,

$$\psi(\widetilde{X}_1, \dots, \widetilde{X}_k) = D + C M C^* - C M B^* (B M B^*)^\dagger B M C^*, \quad (6.45)$$

$$\begin{aligned} & \psi(X_1, \dots, X_k) - \psi(\widetilde{X}_1, \dots, \widetilde{X}_k) \\ &= [A_1 X_1 B_1 + C_1, \dots, A_k X_k B_k + C_k] M B^* (B M B^*)^\dagger B M [A_1 X_1 B_1 + C_1, \dots, A_k X_k B_k + C_k]^*. \end{aligned} \quad (6.46)$$

Proof. Rewrite (6.36) as

$$\begin{aligned} \psi(X_1, \dots, X_k) &= [A_1 X_1 B_1 + C_1, \dots, A_k X_k B_k + C_k] M [A_1 X_1 B_1 + C_1, \dots, A_k X_k B_k + C_k]^* + D \\ &= [A_1 X_1 [B_1, \dots, 0] + \dots + A_k X_k [0, \dots, B_k] + [C_1, \dots, C_k]] M \\ &\quad \times [A_1 X_1 [B_1, \dots, 0] + \dots + A_k X_k [0, \dots, B_k] + [C_1, \dots, C_k]]^* + D, \end{aligned} \quad (6.47)$$

which is a special case of (6.1). Applying Theorem 6.1 to it, we obtain the result desired. \square

Many consequences can be derived from the results in this section. For instance,

(i) the semi-definiteness and the global extremum matrices in the Löwner partial ordering of the following constrained QHMF

$$\phi(X) = (A X B + C) M (A X B + C)^* + D \quad \text{s.t.} \quad P X Q = R$$

can be derived;

(ii) the semi-definiteness and the global extremum matrices in the Löwner partial ordering of the following matrix expressions that involve partially specified matrices

$$\begin{bmatrix} A & B \\ C & ? \end{bmatrix} M \begin{bmatrix} A & B \\ C & ? \end{bmatrix}^* + N, \quad \begin{bmatrix} ? & B \\ C & ? \end{bmatrix} M \begin{bmatrix} ? & B \\ C & ? \end{bmatrix}^* + N, \quad \begin{bmatrix} A & ? \\ ? & ? \end{bmatrix} M \begin{bmatrix} A & ? \\ ? & ? \end{bmatrix}^* + N$$

can be derived. In particular, necessary and sufficient conditions can be derived for the following inequalities

$$\begin{bmatrix} A & B \\ C & ? \end{bmatrix} \begin{bmatrix} A & B \\ C & ? \end{bmatrix}^* \preceq I, \quad \begin{bmatrix} ? & B \\ C & ? \end{bmatrix} \begin{bmatrix} ? & B \\ C & ? \end{bmatrix}^* \preceq I, \quad \begin{bmatrix} A & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} A & ? \\ ? & ? \end{bmatrix}^* \preceq I$$

to always hold in the Löwner partial ordering.

7. Some optimization problems on the matrix equation $AXB = C$

Consider the following linear matrix equation

$$AXB = C, \quad (7.1)$$

where $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{p \times q}$ and $C \in \mathbb{C}^{m \times q}$ are given, and $X \in \mathbb{C}^{n \times p}$ is an unknown matrix. Eq. (7.1) is one of the best known matrix equations in matrix theory. Many papers on this equation and its applications can be found in the literature. In the Penrose's seminal paper [5], the consistency conditions and the general solution of (7.1) were completely derived by using generalized inverse of matrices. If (7.1) is not consistent, people often need to find its approximation solutions under various optimal criteria, in particular, the least-squares criterion is ubiquitously used in optimization problems which almost always admits an explicit global solution. For (7.1), the least-squares solution is defined to be a matrix $X \in \mathbb{C}^{n \times p}$ that minimizes the quadratic objective function

$$\text{trace}[(C - AXB)(C - AXB)^*] = \text{trace}[(C - AXB)^*(C - AXB)]. \quad (7.2)$$

The normal equation corresponding to (7.2) is given by

$$A^*AXBB^* = A^*CB^*, \quad (7.3)$$

which is always consistent, and the following result is well known.

Lemma 7.1. *The general least-squares solution of (7.1) can be written as*

$$X = A^\dagger CB^\dagger + F_A V_1 + V_2 E_B, \quad (7.4)$$

where $V_1, V_2 \in \mathbb{C}^{n \times p}$ are arbitrary.

Define the two QHMFs in (7.2) as

$$\phi_1(X) := (C - AXB)(C - AXB)^*, \quad \phi_2(X) := (C - AXB)^*(C - AXB). \quad (7.5)$$

Note that

$$r[\phi_1(X)] = r[\phi_2(X)] = r(C - AXB). \quad (7.6)$$

Hence, we first obtain the following result from Lemma 3.5.

Theorem 7.2. *Let $\phi_1(X)$ and $\phi_2(X)$ be as given in (7.5). Then,*

$$\max_{X \in \mathbb{C}^{n \times p}} r[\phi_1(X)] = \max_{X \in \mathbb{C}^{n \times p}} r[\phi_2(X)] = \max_{X \in \mathbb{C}^{n \times p}} r(C - AXB) = \min \left\{ r[A, C], r \begin{bmatrix} B \\ C \end{bmatrix} \right\}, \quad (7.7)$$

$$\min_{X \in \mathbb{C}^{n \times p}} r[\phi_1(X)] = \min_{X \in \mathbb{C}^{n \times p}} r[\phi_2(X)] = \min_{X \in \mathbb{C}^{n \times p}} r(C - AXB) = r[A, C] + r \begin{bmatrix} B \\ C \end{bmatrix} - r \begin{bmatrix} C & A \\ B & 0 \end{bmatrix}. \quad (7.8)$$

Applying Theorem 4.1 to (7.5), we obtain the following result.

Theorem 7.3. *Let $\phi_1(X)$ and $\phi_2(X)$ be as given in (7.5). Then, the following hold.*

(a) *There exists an $\widehat{X} \in \mathbb{C}^{n \times p}$ such that $\phi_1(X) \succcurlyeq \phi_1(\widehat{X})$ holds for all $X \in \mathbb{C}^{n \times p}$ if and only if*

$$\mathcal{R}(CB^*) \subseteq \mathcal{R}(A). \quad (7.9)$$

In this case,

$$\widehat{X} = A^\dagger CB^\dagger + F_A V_1 + V_2 E_B, \quad \phi_1(\widehat{X}) = CC^* - CB^\dagger BC^*, \quad (7.10)$$

where $V_1, V_2 \in \mathbb{C}^{n \times p}$ are arbitrary.

(b) *There exists an $\widehat{X} \in \mathbb{C}^{n \times p}$ such that $\phi_2(X) \succcurlyeq \phi_2(\widehat{X})$ holds for all $X \in \mathbb{C}^{n \times p}$ if and only if*

$$\mathcal{R}(C^*A) \subseteq \mathcal{R}(B^*). \quad (7.11)$$

In this case,

$$\widehat{X} = A^\dagger CB^\dagger + F_A V_1 + V_2 E_B, \quad \phi_2(\widehat{X}) = C^*C - C^*AA^\dagger C, \quad (7.12)$$

where $V_1, V_2 \in \mathbb{C}^{n \times p}$ are arbitrary.

Theorem 7.3 also motivates us to obtain the following consequence.

Theorem 7.4. *Let $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{p \times q}$ and $C \in \mathbb{C}^{m \times q}$ be given. Then, there always exist an $X \in \mathbb{C}^{n \times p}$ that satisfies*

$$\min_{\succcurlyeq} \{ A^*(C - AXB)(C - AXB)^*A : X \in \mathbb{C}^{n \times p} \}, \quad (7.13)$$

$$\min_{\succcurlyeq} \{ B(C - AXB)^*(C - AXB)B^* : X \in \mathbb{C}^{n \times p} \}, \quad (7.14)$$

and the general solution is given by

$$\begin{aligned} & \operatorname{argmin}_{\succcurlyeq} \{ A^*(C - AXB)(C - AXB)^*A : X \in \mathbb{C}^{n \times p} \} \\ &= \operatorname{argmin}_{\succcurlyeq} \{ B(C - AXB)^*(C - AXB)B^* : X \in \mathbb{C}^{n \times p} \} \\ &= \operatorname{argmin}_{X \in \mathbb{C}^{n \times p}} \operatorname{tr} [(C - AXB)(C - AXB)^*] \\ &= A^\dagger C B^\dagger + F_A V_1 + V_2 E_B, \end{aligned} \quad (7.15)$$

where V_1 and V_2 are arbitrary matrices, namely, the solutions of the three minimization problems in (7.15) are the same.

For (7.1), the weighted least-squares solutions with respect to positive semi-definite matrices M and N are defined to be matrices $X \in \mathbb{C}^{n \times p}$ that satisfy

$$\operatorname{trace} [(C - AXB)M(C - AXB)^*] = \min, \quad \operatorname{trace} [(C - AXB)^*N(C - AXB)] = \min, \quad (7.16)$$

respectively. In this case, the two QHMFs in (7.16) are

$$\phi_1(X) = (C - AXB)M(C - AXB)^*, \quad \phi_2(X) = (C - AXB)^*N(C - AXB), \quad (7.17)$$

so that the theory on the ranks and inertias of $\phi_1(X)$ and $\phi_2(X)$ can be established routinely.

Recall that the least-squares solution of a linear matrix equation is defined by minimizing the trace of certain QHMF. For example, the least-squares solution of the well-known linear matrix equation

$$AXB + CYD = E, \quad (7.18)$$

where A, B, C, D are given, are two matrices X and Y such that

$$\begin{aligned} & \operatorname{trace} [(E - AXB - CYD)(E - AXB - CYD)^*] \\ &= \operatorname{trace} [(E - AXB - CYD)^*(E - AXB - CYD)] = \min. \end{aligned}$$

Correspondingly, solutions to the Löwner partial ordering minimization problems of the two QHMFs

$$(E - AXB - CYD)(E - AXB - CYD)^*, \quad (E - AXB - CYD)^*(E - AXB - CYD)$$

can be derived from Theorem 6.1.

8. Concluding remarks

We established in this paper a group of explicit formulas for calculating the global maximal and minimal ranks and inertias of (1.3) when X runs over the whole matrix space. By taking these rank and inertia formulas as quantitative tools, we characterized many algebraic properties of (1.3), including solvability conditions for some nonlinear matrix equations and inequalities generated from (1.3), and analytical solutions to the two well-known classic optimization problems on the $\phi(X)$ in the Löwner partial ordering. The results obtained and the techniques adopted for solving the matrix rank and inertia optimization problems enable us to make new extensions of some classic results on quadratic forms, quadratic matrix equations and quadratic matrix inequalities, and to derive many new algebraic properties of nonlinear matrix functions that can hardly be handled before. As a continuation of this work, we mention some research problems on QHMFs for further consideration.

- (i) Characterize algebraic and topological properties of generalized Stiefel manifolds composed by the collections of all matrices satisfying (4.3)–(4.6).

- (ii) The difference of (1.3) at two given matrices X , $X + \Delta X \in \mathbb{C}^{p \times m}$

$$\phi(X + \Delta X) - \phi(X)$$

is homogenous with respect to ΔX , so that we can add a restriction on its norm, for instance, $\|\Delta X\| = \sqrt{\text{tr}[(\Delta X)(\Delta X)^*]} < \delta$. In this case, establish formulas for calculating the maximal and minimal ranks and inertias of the difference with respect to $\Delta X \neq 0$, and use them to analyze the behaviors of $\phi(X)$ nearby X . Also note that any matrix $X = (x_{ij})_{p \times m}$ can be decomposed as $X = \sum_{i=1}^p \sum_{j=1}^m x_{ij} e_{ij}$. A precise analysis on the difference is to take $\Delta X = \lambda e_{ij}$ and to characterize the behaviors of the difference by using the corresponding rank and inertia formulas.

- (iii) Denote the real and complex parts of (1.3) as $\phi(X) = \phi_0(X) + i\phi_1(X)$, where $\phi_0(X)$ and $\phi_1(X)$ two real quadratic matrix-valued functions satisfying $\phi_0^T(X) = \phi_0(X)$ and $\phi_1^T(X) = -\phi_1(X)$. In this case, establish formulas for calculating the maximal and minimal ranks and inertias of $\phi_0(X)$ and $\phi_1(X)$, and use them to characterize behaviors of $\phi_0(X)$ and $\phi_1(X)$.
- (iv) Partition $\phi(X)$ in (1.3) as

$$\phi(X) = \begin{bmatrix} \phi_{11}(X) & \phi_{12}(X) \\ \phi_{12}^*(X) & \phi_{22}(X) \end{bmatrix}.$$

In this case, establish formulas for calculating the maximal and minimal ranks and inertias of the submatrices $\phi_{11}(X)$ and $\phi_{22}(X)$ with respect to X , and utilize them to characterize behaviors of these submatrices.

- (v) Most criteria related to vector and matrix optimizations are constructed via traces of matrices. An optimization theory for (1.3) can also be established by taking the trace of (1.3) as an objective function. In such a case, it would be of interest to characterize relations between the two optimization theories for (1.3) derived from trace and Löwner partial ordering.
- (vi) Establish formulas for calculating the extremum ranks and inertias of

$$(AXB + C)M(AXB + C)^* + D \quad \text{s.t.} \quad r(X) \leq k,$$

where $k \leq \min\{p, m\}$. This rank-constrained matrix-valued function is equivalent to the following biquadratic matrix-valued function

$$(AYZB + C)M(AYZB + C)^* + D, \quad Y \in \mathbb{C}^{p \times k}, \quad Z \in \mathbb{C}^{k \times m}.$$

Some previous results on positive semi-definiteness of biquadratic forms can be found in [1, 2].

- (vii) Establish formulas for calculating the maximal and minimal ranks and inertias of

$$(AXB + C)M(AXB + C)^* + D \quad \text{s.t.} \quad PX = Q \text{ or } XR = S.$$

This task could be regarded as extensions of classic equality-constrained quadratic programming problems.

- (viii) For two given QHMFs

$$\phi_i(X) = (A_i X B_i + C_i)M(A_i X B_i + C_i)^* + D_i, \quad i = 1, 2$$

of the same size, establish necessary and sufficient conditions for $\phi_1(X) \equiv \phi_2(X)$ to hold.

- (ix) Note that the QHMF in (1.3) is embed into the congruence transformation for a block Hermitian matrix consisting of the given matrices. This fact prompts us to construct some general nonlinear matrix-valued functions that can be embed in congruence transformations for block Hermitian matrices, for instance,

$$\begin{bmatrix} I_{m_1} & 0 & 0 \\ B_1 X_1 & I_{m_2} & 0 \\ B_2 X_2 B_1 X_1 & B_2 X_2 & I_{m_3} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{12}^* & A_{22} & A_{23} \\ A_{13}^* & A_{23}^* & A_{33} \end{bmatrix} \begin{bmatrix} I_{m_1} & X_1^* B_1^* & X_1^* B_1^* X_2^* B_2^* \\ 0 & I_{m_2} & X_2^* B_2^* \\ 0 & 0 & I_{m_3} \end{bmatrix} \\ = \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & \phi(X_1, X_2) \end{bmatrix},$$

where

$$\phi(X_1, X_2) = [B_2 X_2 B_1 X_1, B_2 X_2, I_{m_3}] \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{12}^* & A_{22} & A_{23} \\ A_{13}^* & A_{23}^* & A_{33} \end{bmatrix} \begin{bmatrix} X_1^* B_1^* X_2^* B_2^* \\ X_2^* B_2^* \\ I_{m_3} \end{bmatrix},$$

which is a special case of the following nonlinear matrix-valued function

$$\phi(X_1, X_2) = (A_1 X_1 B_1 + C_1)(A_2 X_2 B_2 + C_2)M(A_2 X_2 B_2 + C_2)^*(A_1 X_1 B_1 + C_1)^* + D.$$

In these cases, it would be of interest to establish possible formulas for calculating the extremum ranks and inertias of these nonlinear matrix-valued functions (biquadratic matrix-valued functions), in particular, to find criteria of identifying semi-definiteness of these nonlinear matrix-valued functions, and to solve the Löwner partial ordering optimization problems.

- (x) Two special forms of (6.1) and (6.24) by setting $X_1 = \cdots = X_k = X$ are

$$\left(\sum_{i=1}^k A_i X B_i + C \right) M \left(\sum_{i=1}^k A_i X B_i + C \right)^* + D, \quad \sum_{i=1}^k (A_i X B_i + C_i) M_i (A_i X B_i + C_i)^* + D.$$

In this case, find criteria for the QHMF to be semi-definite, and solve for its global extremum matrices in the Löwner partial ordering.

- (xi) Many expressions that involve matrices and their generalized inverses can be represented as quadratic matrix-valued functions, for instance,

$$D - B^* A_r^- B, \quad A - B B^- A (B B^-)^*, \quad A - B B^- A - A (B B^-)^* + B B^- A (B B^-)^*.$$

In these cases, it would be of interest to establish formulas for calculating the maximal and minimal ranks and inertias of these matrix expressions with respect to the reflexive Hermitian g -inverse A_r^- of a Hermitian matrix A , and g -inverse B^- of B . Some recent work on the ranks and inertias of the Hermitian Schur complement $D - B^* A_r^- B$ and their applications was given in [4, 9].

Another type of subsequent work is to reasonably extend the results in the precious sections to the corresponding operator-valued functions, for which less quantitative methods are allowed to use.

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