Convergence and Perturbation Resilience of Dynamic String-Averaging Projection Methods

Yair Censor[1] and Alexander J. Zaslavski[2]

[1]Department of Mathematics, University of Haifa
Mt. Carmel, Haifa 31905, Israel
(yair@math.haifa.ac.il)

[2]Department of Mathematics
The Technion – Israel Institute of Technology
Technion City, Haifa 32000, Israel
(ajzasl@techunix.technion.ac.il)

February 29, 2012. Revised: May 24, 2012.

Abstract

We consider the convex feasibility problem (CFP) in Hilbert space and concentrate on the study of string-averaging projection (SAP) methods for the CFP, analyzing their convergence and their perturbation resilience. In the past, SAP methods were formulated with a single predetermined set of strings and a single predetermined set of weights. Here we extend the scope of the family of SAP methods to allow iteration-index-dependent variable strings and weights and term such methods dynamic string-averaging projection (DSAP) methods. The bounded perturbation resilience of DSAP methods is relevant and important for their possible use in the framework of the recently developed superiorization heuristic methodology for constrained minimization problems.

Keywords and phrases: Dynamic string-averaging, projection methods, Perturbation resilience, fixed point, Hilbert space, metric projection, nonexpansive operator, superiorization method, variable strings, variable weights.

2010 Mathematics Subject Classification (MSC): 65K10, 90C25

1 Introduction

In this paper we consider the *convex feasibility problem* (CFP) in Hilbert space H. Let $C_1, C_2, \ldots, C_m$ be nonempty closed convex subsets of H, where m is a natural number, and define

$$C := \cap_{i=1}^{m} C_i. \tag{1}$$

Assuming consistency, i.e., that $C \neq \emptyset$, the CFP requires to find an element $x^* \in C$. We concentrate on the study of *string-averaging projection* (SAP) *methods* for the CFP and analyze their convergence and their perturbation resilience. SAP methods were first introduced in [11] and subsequently studied further in [4, 5, 12, 13, 14, 19], see also [3, Example 5.20]. They were also employed in applications [27, 29].

The class of *projection methods* is understood here as the class of methods that have the property that they can reach an aim related to the family of sets $\{C_1, C_2, \ldots, C_m\}$, such as, but not only, solving the CFP, or solving an optimization problem with these sets as constraints, by performing projections (orthogonal, i.e., least Euclidean distance, or others) onto the individual sets C_i. The advantage of such methods occurs in situations where projections onto the individual sets are computationally simple to perform. Such methods have been in recent decades extensively investigated mathematically and used experimentally with great success on some huge and sparse real-world applications, consult, e.g., [2, 9, 17, 18] and the books [3, 7, 8, 15, 16, 21, 22, 23].

Within the class of projection methods, SAP methods do not represent a single algorithm but rather, what might be called, an *algorithmic scheme,* which means that by making a specific choice of strings and weights in SAP, along with choices of other parameters in the scheme, a deterministic algorithm for the problem at hand can be obtained.

In all these works, SAP methods were formulated with a single predetermined set of strings and a single predetermined set of weights. Here we extend the scope of the family of SAP methods to allow iteration-index-dependent variable strings and weights. We term such SAP methods *dynamic string-averaging projection* (DSAP) *methods.* This is reminiscence of the analogous development of *block-iterative projection* (BIP) *methods* for the CFP wherein iteration-index-dependent variable blocks and weights are permitted [1]. For such DSAP methods we prove here convergence and bounded perturbation resilience.

The significance of DSAP in practice cannot be exaggerated. SAP methods, in their earlier non-dynamic versions, have been applied to the important real-world application of *proton Computerized Tomography* (pCT), see, e.g., [28, 27], which presents a computationally huge-size problem. The efforts to use parallel computing for the application of SAP to pCT is ongoing and will benefit from the DSAP. This is so because the flexibility of varying string lengths, string members and weights dynamically has direct bearing on load balancing between processors that run in parallel and should be loaded in a way that will minimize idle time of processors that await others to complete their jobs. Such experimental work will hopefully see light elsewhere.

The so extended DSAP algorithmic scheme is presented in Section 2 and the convergence analysis of it is done in Section 3. In Section 4 we quote the definition of bounded perturbation resilience and prove that the DSAP with iteration-index-dependent variable strings and weights is bounded perturbation resilient. There we also comment about the importance and relevance of this bounded perturbation resilience to the recently developed superiorization heuristic methodology.

2 The string-averaging projection method and the dynamic SAP with variable strings and weights

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $|| \cdot ||$. Originally, string-averaging is more general than SAP because it can employ operators other than projections and convex combinations. But, on the other hand, it is formulated for fixed strings as follows. Let the *string* I_t

be an ordered subset of $\{1, 2, \ldots, m\}$ of the form

$$I_t = (i_1^t, i_2^t, \ldots, i_{m(t)}^t), \tag{2}$$

with $m(t)$ the number of elements in I_t, for $t = 1, 2, \ldots, M$, where M is the number of strings. Suppose that there is a set $S \subseteq H$ such that there are operators $R_1, R_2, \ldots, R_m$ mapping S into S and an additional operator R which maps S^M into S.

Algorithm 1 ***The string-averaging algorithmic scheme of [11]***
Initialization: *$x^0 \in S$ is arbitrary.*
Iterative Step: *given the current iterate x^k,*
(i) calculate, for all $t = 1, 2, \ldots, M$,

$$T_t(x^k) = R_{i_{m(t)}^t} \cdots R_{i_2^t} R_{i_1^t}(x^k), \tag{3}$$

(ii) and then calculate

$$x^{k+1} = R(T_1(x^k), T_2(x^k), \ldots, T_M(x^k)). \tag{4}$$

For every $t = 1, 2, \ldots, M$, this algorithmic scheme applies to x^k successively the operators whose indices belong to the t-th string. This can be done in parallel for all strings and then the operator R maps all end-points onto the next iterate x^{k+1}. This is indeed an algorithm provided that the operators $\{R_i\}_{i=1}^m$ and R all have algorithmic implementations. In this framework we get a *sequential algorithm* by the choice $M = 1$ and $I_1 = (1, 2, \ldots, m)$ and a *simultaneous algorithm* by the choice $M = m$ and $I_t = (t),\ t = 1, 2, \ldots, M$.

Now we proceed to construct our proposed DSAP method with variable strings and weights. For each $x \in H$, nonempty set $E \subseteq H$ and $r > 0$ define the distance

$$d(x, E) = \inf\{||x - y|| \mid \ y \in E\} \tag{5}$$

and the closed ball

$$B(x, r) = \{y \in H \mid \ ||x - y|| \leq r\}. \tag{6}$$

The following proposition and corollary are well-known.

Proposition 2 *If D be a nonempty closed convex subset of H then for each $x \in H$ there is a unique point $P_D(x) \in D$, called* **`the projection of`** *x* **`onto`** *D, satisfying*

$$||x - P_D(x)|| = \inf\{||x - y|| \mid y \in D\}. \tag{7}$$

Moreover,

$$||P_D(x) - P_D(y)|| \leq ||x - y|| \text{ for all } x, y \in H, \tag{8}$$

and for each $x \in H$ and each $z \in D$,

$$\langle z - P_D(x), x - P_D(x) \rangle \leq 0. \tag{9}$$

Corollary 3 *If D be a nonempty closed convex subset of H then for each $x \in H$ and each $z \in D$,*

$$||z - P_D(x)||^2 + ||x - P_D(x)||^2 \leq ||z - x||^2. \tag{10}$$

For $i = 1, 2, \ldots, m$, we denote $P_i = P_{C_i}$. An *index vector* is a vector $t = (t_1, t_2, \ldots, t_p)$ such that $t_i \in \{1, 2, \ldots, m\}$ for all $i = 1, 2, \ldots, p$. For a given index vector $t = (t_1, t_2, \ldots, t_q)$ we denote its *length* by $p(t) = q$, and define the operator $P[t]$ as the product of the individual projections onto the sets whose indices appear in the index vector t, namely,

$$P[t] := P_{t_q} P_{t_{q-1}} \cdots P_{t_1}, \tag{11}$$

and call it a *string operator*. A finite set Ω of index vectors is called *fit* if for each $i \in \{1, 2, \ldots, m\}$, there exists a vector $t = (t_1, t_2, \ldots, t_p) \in \Omega$ such that $t_s = i$ for some $s \in \{1, 2, \ldots, p\}$. For each index vector t the string operator is nonexpansive, since the individual projections are, i.e.,

$$||P[t](x) - P[t](y)|| \leq ||x - y|| \text{ for all } x, y \in H, \tag{12}$$

and also

$$P[t](x) = x \text{ for all } x \in C. \tag{13}$$

Denote by $\mathcal{M}$ the collection of all pairs (Ω, w), where Ω is a fit finite set of index vectors and

$$w : \Omega \to (0, \infty) \text{ is such that } \sum_{t \in \Omega} w(t) = 1. \tag{14}$$

A pair $(\Omega, w) \in \mathcal{M}$ and the function w were called in [5] an *amalgamator* and a *fit weight function*, respectively. For any $(\Omega, w) \in \mathcal{M}$ define the convex combination of the end-points of all strings defined by members of Ω by

$$P_{\Omega,w}(x) := \sum_{t \in \Omega} w(t) P[t](x), \ x \in H. \tag{15}$$

It is easy to see that

$$||P_{\Omega,w}(x) - P_{\Omega,w}(y)|| \le ||x - y|| \text{ for all } x, y \in H, \tag{16}$$

and that

$$P_{\Omega,w}(x) = x \text{ for all } x \in C. \tag{17}$$

We will make use of the following bounded regularity condition, see [2, Definition 5.1].

Condition 4 *For each $\varepsilon > 0$ and each $M > 0$ there exists $\delta = \delta(\varepsilon, M) > 0$ such that for each $x \in B(0, M)$ satisfying $d(x, C_i) \le \delta$, $i = 1, 2, \ldots, m$, the inequality $d(x, C) \le \varepsilon$ holds.*

The next proposition follows from [2, Proposition 5.4(iii)] but we present its proof here for the reader's convenience.

Proposition 5 *If the space H is finite-dimensional then Condition 4 holds.*

Proof. Assume to the contrary that there exist $\varepsilon > 0$, $M > 0$ and a sequence $\{x^k\}_{k=0}^{\infty} \subset B(0, M)$ such that

$$\text{for each integer } k \ge 1, \ d(x^k, C_i) \le 1/k, \ i = 1, 2, \ldots, m \text{ and } d(x^k, C) \ge \varepsilon. \tag{18}$$

We assume, without loss of generality, that there exists the limit

$$\lim_{k \to \infty} x^k = \tilde{x}. \tag{19}$$

Then the closedness of $B(0, M)$ and (18) imply that

$$\tilde{x} \in B(0, M) \cap C_i, \ i = 1, 2, \ldots, m. \tag{20}$$

Together with (1) and (19) this implies that $d(x^k, C) < \varepsilon/2$ for all sufficiently large natural numbers k, contradicting (18) and proving the proposition. ∎

In the sequel we will assume that Condition 4 holds. We fix a number $\Delta \in (0, 1/m)$ and an integer $\bar{q} \geq m$ and denote by $\mathcal{M}_* \equiv \mathcal{M}_*(\Delta, \bar{q})$ the set of all $(\Omega, w) \in \mathcal{M}$ such that the lengths of the strings are bounded and the weights are all bounded away from zero, namely,

$$\mathcal{M}_* := \{(\Omega, w) \in \mathcal{M} \mid p(t) \leq \bar{q} \text{ and } w(t) \geq \Delta \text{ for all } t \in \Omega\}. \tag{21}$$

The dynamic string-averaging projection (DSAP) method with variable strings and variable weights is described by the following algorithm.

Algorithm 6 ***The DSAP method with variable strings and variable weights***
Initialization: *select an arbitrary* $x^0 \in H$,
Iterative step: *given a current iteration vector* x^k *pick a pair* $(\Omega_k, w_k) \in \mathcal{M}_*$ *and calculate the next iteration vector by*

$$x^{k+1} = P_{\Omega_k, w_k}(x^k). \tag{22}$$

3 Convergence analysis

In this section we present our convergence analysis for the DSAP method with variable strings and variable weights, Algorithm 6. The main theorem is the following.

Theorem 7 *Let the following assumptions hold:*
(i) $M_0 > 0$ *is such that*

$$B(0, M_0) \cap C \neq \emptyset. \tag{23}$$

(ii) $\varepsilon > 0$, $M > 0$ *and* $\delta > 0$ *are such that*

$$\text{if } x \in B(0, 2M_0 + M) \text{ and } d(x, C_i) \leq \delta,\ i = 1, 2, \ldots, m, \text{ then } d(x, C) \leq \varepsilon/4. \tag{24}$$

(iii) γ *is a positive number that satisfies*

$$\bar{q}\gamma^{1/2} \leq \delta. \tag{25}$$

(iv) k_0 *is a natural number that satisfies*

$$k_0 > (\gamma\Delta)^{-1}(M + M_0)^2. \tag{26}$$

Under these assumptions, if Condition 4 holds then any sequence $\{x^k\}_{k=0}^{\infty} \subset H$*, generated by Algorithm 6 with* $||x^0|| \leq M$*, converges in the norm of* H*,* $\lim_{k\to\infty} x^k \in C$ *and*

$$||x^k - \lim_{s\to\infty} x^s|| \leq \varepsilon \text{ for all integers } k \geq k_0. \tag{27}$$

We use the notation and the definitions from Section 2 and prove first the next two lemmas as tools for the proof of Theorem 7.

Lemma 8 *Let* $t = (t_1, t_2, \ldots, t_p)$ *be an index vector,* $x \in H$ *and* $z \in C$*. Then*

$$||z - x||^2 \geq ||z - P[t](x)||^2 + ||x - P_{t_1}(x)||^2 \tag{28}$$

$$+ \sum_{\substack{i=1 \\ i<p}}^{p} \|P_{t_i+1}P_{t_i}\cdots P_{t_1}(x) - P_{t_i}\cdots P_{t_1}(x)\|^2. \tag{29}$$

Proof. By Corollary 3,

$$||z - x||^2 \geq ||z - P_{t_1}(x)||^2 + ||x - P_{t_1}(x)||^2 \tag{30}$$

and using the same corollary we also have, for each integer i satisfying $1 \leq i < p$,

$$||z - P_{t_i}\cdots P_{t_1}x||^2 \geq ||z - P_{t_i+1}P_{t_i}\cdots P_{t_1}(x)||^2 + ||P_{t_i+1}P_{t_i}\cdots P_{t_1}x - P_{t_i}\cdots P_{t_1}(x)||^2. \tag{31}$$

Combining (30) and (31) we obtain (28) and the lemma is proved. ■

For an index vector $t = (t_1, t_2, \ldots, t_{p(t)})$ and a vector $x \in H$ let us define the function

$$\phi[t](x) := ||x - P_{t_1}(x)||^2 + \sum_{\substack{i=1 \\ i<p}}^{p} \|P_{t_i+1}P_{t_i}\cdots P_{t_1}(x) - P_{t_i}\cdots P_{t_1}(x)\|^2. \tag{32}$$

Lemma 8 and (32) then imply that, for each $x \in H$ and each $z \in C$,

$$||z - x||^2 \geq ||z - P[t](x)||^2 + \phi[t](x). \tag{33}$$

Lemma 9 *Let $x \in H$, $z \in C$ and $(\Omega, w) \in \mathcal{M}_*$. Then*

$$||z-x||^2 \geq ||z - P_{\Omega,w}(x)||^2 + \Delta \sum_{t\in\Omega} \phi[t](x). \tag{34}$$

Proof. Since the function $u \to ||u-z||^2$, $u \in H$, is convex it follows from (14), (15), (32), (33), and the definition of $\mathcal{M}_*$ in (21) that

$$\begin{aligned} ||z - P_{\Omega,w}(x)||^2 &= ||z - \sum_{t\in\Omega} w(t)P[t](x)||^2 \leq \sum_{t\in\Omega} w(t)||z - P[t](x)||^2 \\ &\leq \sum_{t\in\Omega} w(t)\left(||z-x||^2 - \phi[t](x)\right) = ||z-x||^2 - \sum_{t\in\Omega} w(t)\phi[t](x) \\ &\leq ||z-x||^2 - \Delta \sum_{t\in\Omega} \phi[t](x). \end{aligned} \tag{35}$$

This implies (34) and completes the proof. ■

Now we are ready to prove Theorem 7.

Proof of Theorem 7. We first wish to show that there is a natural number $\ell \leq k_0$ such that

$$\sum_{t\in\Omega_\ell} \phi[t](x^{\ell-1}) \leq \gamma. \tag{36}$$

To this end we assume to the contrary, that

$$\text{for all } k = 1, 2, \ldots, k_0, \ \sum_{t\in\Omega_k} \phi[t](x^{k-1}) > \gamma, \tag{37}$$

and take some $\theta \in B(0, M_0) \cap C$. By Lemma 9, by the fact that $(\Omega_k, w_k) \in \mathcal{M}_*$ for all $k \geq 0$, by (22) and by (37), we have

$$||\theta - x^{k-1}||^2 \geq ||\theta - x^k||^2 + \Delta \sum_{t\in\Omega_k} \phi[t](x^{k-1}) > ||\theta - x^k||^2 + \Delta\gamma. \tag{38}$$

By the choice of θ and the fact that $||x^0|| \leq M$ we obtain

$$\begin{aligned} (M+M_0)^2 &\geq ||\theta - x^0||^2 - ||\theta - x^{k_0}||^2 \\ &= \sum_{k=1}^{k_0} (||\theta - x^{k-1}||^2 - ||\theta - x^k||^2) \geq k_0\Delta\gamma \end{aligned} \tag{39}$$

which implies

$$k_0 \leq (\Delta\gamma)^{-1}(M+M_0)^2. \tag{40}$$

This contradicts Assumption (iv) of the theorem thus showing that there exists a natural number $\ell \leq k_0$ such that (36) holds.

From Lemma 9, the choice of θ, the fact that $(\Omega_k, w_k) \in \mathcal{M}_*$ for all $k \geq 0$, the iterative step (22) and $||x^0|| \leq M$, we conclude that

$$||\theta - x^{\ell-1}||^2 \leq ||\theta - x^0||^2 \leq (M_0+M)^2, \tag{41}$$

which yields by the triangle inequality

$$||x^{\ell-1}|| \leq 2M_0 + M. \tag{42}$$

In order to use the last inequality to invoke Assumption (ii) of the theorem we show next that

$$d(x^{\ell-1}, C_i) \leq \delta,\ i = 1, 2, \ldots, m. \tag{43}$$

Assume that $s \in \{1, 2, \ldots, m\}$. Since the set Ω_ℓ is fit there is a $t = (t_1, t_2, \ldots, t_{p(t)}) \in \Omega_\ell$ such that

$$s = t_q \text{ for some } q \in \{1, 2, \ldots, p(t)\}. \tag{44}$$

From (36) and the fact that $t \in \Omega_\ell$, we know that $\phi[t](x^{\ell-1}) \leq \gamma$. Together with (32) this implies that

$$||x^{\ell-1} - P_{t_1}(x^{\ell-1})|| \leq \gamma^{1/2}, \tag{45}$$

therefore, for any index $1 \leq i \leq p(t)$ satisfying $i < p(t)$,

$$||P_{t_{i+1}} P_{t_i} \cdots P_{t_1}(x^{\ell-1}) - P_{t_i} \cdots P_{t_1}(x^{\ell-1})|| \leq \gamma^{1/2}. \tag{46}$$

The fact that $(\Omega_k, w_k) \in \mathcal{M}_*$ for all $k \geq 0$ guarantees that $p(t) \leq \bar{q}$, and the δ whose existence is guaranteed by Condition 4, Assumption (iii) of the theorem, along with (45) and (46) imply that for any $i \in \{1, \ldots, p(t)\}$,

$$||P_{t_i} \cdots P_{t_1}(x^{\ell-1}) - x^{\ell-1}|| \leq i\gamma^{1/2} \leq p(t)\gamma^{1/2} \leq \bar{q}\gamma^{1/2} \leq \delta \tag{47}$$

and thus that

$$d(x^{\ell-1}, C_{t_i}) \leq \delta. \tag{48}$$

Together with (44) this implies that

$$d(x^{\ell-1}, C_s) \leq \delta. \tag{49}$$

Since (49) holds for any $s \in \{1, 2, \ldots, m\}$, we use (42) and Assumption (ii) of the theorem to state that

$$d(x^{\ell-1}, C) \leq \varepsilon/4 \tag{50}$$

and that there is a $z \in H$ such that

$$z \in C \text{ and } ||x^{\ell-1} - z|| < \varepsilon/3. \tag{51}$$

By (16), (17), (51), the fact that $(\Omega_k, w_k) \in \mathcal{M}_*$ for all $k \geq 0$ and the iterative step (22) we have

$$||x^k - z|| \leq ||x^{\ell-1} - z|| < \varepsilon/3 \text{ for all integers } k \geq \ell - 1. \tag{52}$$

This implies that for all integers $k_1, k_2 \geq \ell - 1$ it is true that $||x^{k_1} - x^{k_2}|| < \varepsilon$. Since $\varepsilon > 0$ is arbitrary it follows that $\{x^k\}_{k=0}^{\infty}$ is a Cauchy sequence and that the limit $\lim_{k\to\infty} x^k$ in the norm exists. By (52)

$$||z - \lim_{k\to\infty} x^k|| \leq \varepsilon/3. \tag{53}$$

Since $\varepsilon > 0$ is arbitrary it follows from (51) that $\lim_{k\to\infty} x^k \in C$. By (52), since $\ell \leq k_0$, and using (37) for all integers $k \geq k_0$ we may write

$$||x^k - \lim_{s\to\infty} x^s|| \leq ||x^k - z|| + ||z - \lim_{s\to\infty} x^s|| \leq \varepsilon/3 + \varepsilon/3 \tag{54}$$

which completes the proof of Theorem 7. ■

4 Perturbation resilience of dynamic string-averaging with variable strings and weights

In this section we prove the bounded perturbation resilience of the DSAP method with variable strings and weights. We use the notations and the definitions from the previous sections. The next definition was originally given with a finite-dimensional Euclidean space R^J instead of the Hilbert space H that we inserted into it below.

Definition 10 *[10, Definition 1] Given a problem T, an algorithmic operator $\mathcal{A} : H \to H$ is said to be* **`bounded perturbations resilient`** *if the following is true: if the sequence $\{x^k\}_{k=0}^{\infty}$, generated by $x^{k+1} = \mathcal{A}(x^k)$, for all $k \geq 0$, converges to a solution of T for all $x^0 \in H$, then any sequence $\{y^k\}_{k=0}^{\infty}$ of points in H generated by $y^{k+1} = \mathcal{A}(y^k + \beta_k v^k)$, for all $k \geq 0$, also converges to a solution of T provided that, for all $k \geq 0$, $\beta_k v^k$ are* **`bounded perturbations`**, *meaning that $\beta_k \geq 0$ for all $k \geq 0$ such that $\sum_{k=0}^{\infty} \beta_k < \infty$ and the sequence $\{v^k\}_{k=0}^{\infty}$ is bounded.*

We will make use of the following theorem that was proved in [6, Theorem 3.2].

Theorem 11 *Let (Y, ρ) be a complete metric space, let $F \subset Y$ be a nonempty closed set, and let $T_i : Y \to Y$, $i = 1, 2, \ldots$, satisfy*

$$\rho(T_i(x), T_i(y)) \leq \rho(x, y) \text{ for all } x, y \in Y \text{ and all integers } i \geq 1, \tag{55}$$

and

$$T_i(z) = z \text{ for each } z \in F \text{ and each integer } i \geq 1. \tag{56}$$

Assume that for each $x \in Y$ and integer $q \geq 1$, the sequence $\{T_n \cdots T_q(x)\}_{n=q}^{\infty}$ converges to an element of F. Let $x^0 \in Y$, $\{\gamma_n\}_{n=1}^{\infty} \subset (0, \infty)$, $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\{x^n\}_{n=0}^{\infty} \subset Y$, and suppose that for all integers $n \geq 0$,

$$\rho(x^{n+1}, T_{n+1}(x^n)) \leq \gamma_{n+1}. \tag{57}$$

Then the sequence $\{x^n\}_{n=0}^{\infty}$ converges to an element of F.

The next theorem establishes the bounded perturbations resilience of DSAP.

Theorem 12 *Let $C_1, C_2, \ldots, C_m$ be nonempty closed convex subsets of H, where m is a natural number, $C := \cap_{i=1}^{m} C_i \neq \emptyset$, let $\{\beta_k\}_{k=0}^{\infty}$ be a sequence of nonnegative numbers such that $\sum_{k=0}^{\infty} \beta_k < \infty$, let $\{v^k\}_{k=0}^{\infty} \subset H$ be a norm bounded sequence, let $\{(\Omega_k, w_k)\}_{k=1}^{\infty} \subset \mathcal{M}_*$, for all $k \geq 0$, and let $x^0 \in H$. Then any sequence $\{x_k\}_{k=0}^{\infty}$, generated by Algorithm 6 in which (22) is replaced by*

$$x^{k+1} = P_{\Omega_k, w_k}(x^k + \beta_k v^k), \tag{58}$$

converges in the norm of H and its limit belongs to C.

Proof. The proof follows from Theorem 7 and from Theorem 11. ■

We conclude with a comment about the importance and relevance of this bounded perturbation resilience to the recently developed superiorization methodology. The superiorization methodology was first proposed (although without using this term) in [5] and subsequently investigated and developed further in [10, 20, 24, 25, 26, 28]. For the case of a minimization problem of an objective function ϕ over a family of constraints $\{C_i\}_{i=1}^{m}$, where each set C_i is a nonempty closed convex subset of R^n it works as follows. It applies to $C = \cap_{i=1}^{m} C_i$ a feasibility-seeking *projection method* capable of using projections onto the individual sets C_i in order to generate a sequence $\{x^k\}_{k=0}^{\infty}$ that converges to a point $x^* \in C$. It applies to C only such a feasibility-seeking projection method which is bounded perturbation resilient. Doing so, the superiorization method exploits this perturbation resilience to perform objective function reduction steps by doing negative subgradient moves with certain step sizes. Thus, in superiorization the feasibility-seeking algorithm leads the overall process and uses permissible perturbations, that do not spoil the feasibility-seeking, to periodically jump out from the overall process to do the subgradient objective function reduction step.

This has a great potential computational advantage and poses also interesting mathematical questions. It has been shown to be advantageous in some real-world problems in image reconstruction from projections, see the above mentioned references. Theorem 12 here, which established the bounded perturbations resilience of DSAP methods, makes it now possible to use, when the need arises, also DSAP methods within the framework of the superiorization heuristic methodology.

Acknowledgments. We gratefully acknowledge enlightening discussions with Gabor Herman on the topics discussed in this paper. The work of the first author was supported by the United States-Israel Binational Science Foundation (BSF) Grant number 200912 and US Department of Army Award number W81XWH-10-1-0170.

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