On Traveling Salesman Games with Asymmetric Costs

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Abstract

We consider cooperative traveling salesman games with non-negative asymmetric costs satisfying the triangle inequality. We construct a stable cost allocation with budget balance guarantee equal to the Held-Karp integrality gap for the asymmetric traveling salesman problem, using the parsimonious property and a previously unknown connection to linear production games. We also show that our techniques extend to larger classes of network design games. We then provide a simple example showing that our cost allocation does not necessarily achieve the best possible budget balance guarantee, even among cost allocations stable for the game defined by the Held-Karp relaxation, and discuss its implications on future work on traveling salesman games.

1 Introduction

Cooperative game theory provides a framework for determining "fair" ways of sharing the costs of joint ventures. The *traveling salesman game* is motivated by the following setting: A set of independent retailers $N = \{1, \dots, n\}$, or *players*, share a delivery vehicle stationed at a depot. For any subset of retailers S, $v(S)$ is the transportation cost the vehicle incurs to visit retailers in S, and therefore $v(N)$ is the total cost all the retailers would have to collectively pay in order to have the vehicle visit them. How should the retailers share this cost? One well-studied way of answering this question is the α-*core* (Faigle and Kern 1993). Let $\chi \in \mathbb{R}^N$ be a cost allocation: χ_i is the cost allocated to retailer $i \in N$. For any $\alpha \geq 1$, the α-core is the set of all cost allocations $\chi \in \mathbb{R}^N$ that satisfy the following constraints:

$$\sum_{i \in N} \chi_i \geq \frac{1}{\alpha} v(N), \tag{1a}$$

$$\sum_{i \in S} \chi_i \leq v(S) \qquad \text{for all } \emptyset \neq S \subsetneq N. \tag{1b}$$

Constraint (1a) requires that the cost allocation χ be α-*budget-balanced*: the total cost allocated, $\sum_{i \in N} \chi_i$, is at least a fraction $1/\alpha$ of the cost $v(N)$ collectively incurred by the players. We sometimes refer to α as the *budget balance guarantee*. Constraints (1b) require that the cost allocation χ be *stable*; that is, no subset of players would be better off by acting on its own. When $\alpha = 1$, the set of all cost allocations χ that satisfy (1) is simply called the *core* (Gillies 1959).

Traveling salesman games were first studied by Potters et al. (1991), who restricted their attention to the case when costs are symmetric, and showed that even in this case, traveling salesman games may have an empty core. Tamir (1988), Faigle and Kern (1993), and Faigle et al. (1998) studied the relationship between the α-core of symmetric traveling salesman games and various linear programming (LP) relaxations of the symmetric traveling salesman problem (TSP). In particular, Faigle et al. (1998) showed how to compute a cost allocation in the α_{HKS}-core of symmetric traveling salesman games, where α_{HKS} is the integrality gap of the Held-Karp bound for the symmetric TSP (Held and Karp 1970). This gap α_{HKS} has been shown to be at most $3/2$ (Wolsey 1980), and is widely conjectured to be $4/3$. On the other hand, traveling salesman games with asymmetric costs have not been studied until recently. Bläser and Shankar Ram (2008) showed how to use the approximation algorithms of Frieze et al. (1982) and Kaplan et al. (2005) for the asymmetric TSP to compute cost allocations in the $O(\log n)$-core of asymmetric traveling salesman games.

In this note, we establish a connection between the α-core of asymmetric traveling salesman games, the *parsimonious property* and the theory of *linear production games*. In particular, we use a reformulation of Nguyen's (2011) LP relaxation, which satisfies an extension of the parsimonious property of Goemans and Bertsimas (1993) to directed graphs. By replicating some of the constraints in this LP relaxation, we then show that the cooperative game associated with this LP relaxation is actually a linear production game (Owen 1975). As a result, the dual multipliers of this LP relaxation can be used to construct a cost allocation in the LP relaxation game's core, which in turn lies in the α_{HKA}-core of asymmetric traveling salesman games, where α_{HKA} is the integrality gap of the Held-Karp bound for the asymmetric TSP. Until recently, the best known bound on α_{HKA} was $O(\log n)$ (Frieze et al. 1982); Asadpour et al. (2010) improved this bound to $O(\log n / \log \log n)$. As a result, our cost allocation is in the α-core of asymmetric traveling salesman games with the asymptotically best bounds on α known to date. Our work extends the result of Faigle et al. (1998) to asymmetric traveling salesman games and simplifies their arguments by applying the parsimonious property and establishing a new and simple connection between traveling salesman games and linear production games. Furthermore, we discuss how our results extend to larger classes of network design games, namely *strongly k-connected Eulerian graph games* and undirected *survivable network design games*.

For real-world instances, we expect the actual budget balance guarantee of our cost allocation to be considerably better than these bounds, since the Held-Karp bound has been empirically observed to be very close to the optimal cost. Furthermore, our cost allocation result combined with any future improvement on the integrality gap of the Held-Karp bound for the asymmetric TSP will immediately imply a better bound on α for the worst case α-core for asymmetric traveling salesman games. Notwithstanding this guarantee, one can ask if our cost allocation is in the best possible α-core for a given asymmetric traveling salesman game. It turns out that this is not the case, even for surprisingly simple game instances. We conclude by discussing such instances, and their implications for future work on cost allocations for traveling salesman games.

2 Formulations, relaxations and integrality gaps

Let G be a complete graph on vertices $N \cup \{0\}$, where $N = \{1, \dots, n\}$. Let A denote the set of arcs of G. The vertices in N represent *players* or *cities*, and the vertex 0 represents a *depot*. Each arc $(i, j) \in A$ has cost $c_{ij} \in \mathbb{R}_{\geq 0}$. We assume that the costs satisfy the triangle inequality; that is, for any $i, j, k \in N \cup \{0\}$ such that $i \neq j \neq k$, we have that $c_{ij} + c_{jk} \geq c_{ik}$. When $c_{ij} = c_{ji}$ for all $i, j \in N \cup \{0\}$, we say the costs are *symmetric*; otherwise, we say the costs are *asymmetric*. For any subset of players $S \subseteq N$, let G_S denote the (complete) subgraph of G induced by $S \cup \{0\}$, with set of arcs A_S (note that A_S contains arcs $(0, i)$ and $(i, 0)$ for all $i \in S$). A *traveling salesman game* is a cooperative game (N, v) where $v(S)$ is the optimal value of the TSP on G_S. Following standard terminology from cooperative game theory, we refer to a subset of players as a *coalition*, and the set of all players N as the *grand coalition*. To ease the notational burden in the remainder of the paper, for any vector x and set Y, we let $x(Y) = \sum_{i \in Y} x_i$.

For any coalition $\emptyset \neq S \subseteq N$, the following integer program finds a tour in $S \cup \{0\}$ of minimum length:

$$
\begin{aligned}
\text{IP}(S): \ \min_x \quad & \sum_{a \in A_S} c_a x_a && && \text{(2a)}\\
\text{s.t.} \quad & x(\delta_S^+(\{i\})) = 1 && \text{for all } i \in S \cup \{0\}, && \text{(2b)}\\
& x(\delta_S^-(\{i\})) = 1 && \text{for all } i \in S \cup \{0\}, && \text{(2c)}\\
& x(\delta_S^+(T)) \geq 1 && \text{for all } T \subseteq S \cup \{0\} : 1 \leq |T| \leq |S|, && \text{(2d)}\\
& x_a \in \{0, 1\} && \text{for all } a \in A_S, && \text{(2e)}
\end{aligned}
$$

where

$$
\begin{aligned}
\delta_S^+(T) &= \{(i, j) \in A_S : i \in T, j \in (S \cup \{0\}) \setminus T\},\\
\delta_S^-(T) &= \{(i, j) \in A_S : i \in (S \cup \{0\}) \setminus T, j \in T\}.
\end{aligned}
$$

In other words, $v(S)$ is equal to the optimal value of $\text{IP}(S)$. (We define $v(\emptyset) = 0$.) For any $\emptyset \neq S \subseteq N$, let $\text{LP}(S)$ denote the LP relaxation of $\text{IP}(S)$ obtained by replacing the binary constraints (2e) with the non-negativity constraints $x_a \geq 0$ for all $a \in A_S$. For all $\emptyset \neq S \subseteq N$, let $v_{\text{LP}}(S)$ denote the optimal value of $\text{LP}(S)$, also known as the *Held-Karp bound* for the asymmetric TSP on G_S.

Nguyen (2011) extended the work of Goemans and Bertsimas (1993) to show that $\text{LP}(S)$ satisfies the *parsimonious property*: If degree balance is enforced at each node $i \in S \cup \{0\}$,

$$
x(\delta_S^+(\{i\})) = x(\delta_S^-(\{i\})) \qquad \text{for } i \in S \cup \{0\}, \tag{3}
$$

then the degree constraints (2b)–(2c) can be removed from $\text{LP}(S)$ without affecting its optimal value. In other words, the optimal value of $\text{LP}(S)$ is equal to the optimal value of the following linear program:

$$
\begin{aligned}
\text{LP}_4(S): \ \min_x \quad & \sum_{a \in A_S} c_a x_a && && \text{(4a)}\\
\text{s.t.} \quad & x(\delta_S^+(\{i\})) = x(\delta_S^-(\{i\})) && \text{for all } i \in S \cup \{0\}, && \text{(4b)}\\
& x(\delta_S^+(T)) \geq 1 && \text{for all } T \subseteq S \cup \{0\} : 1 \leq |T| \leq |S|, && \text{(4c)}\\
& x_a \geq 0 && \text{for all } a \in A_S. && \text{(4d)}
\end{aligned}
$$

Now consider the following variant of LP(S) for any $\emptyset \neq S \subseteq N$, in which we only have constraints *corresponding to subsets of players* (i.e., *without* the depot vertex 0):

$$
\begin{aligned}
\mathrm{LP}_5(S): \quad \min_x \quad & \sum_{a \in A_S} c_a x_a && && \text{(5a)}\\
& x(\delta_S^+(\{i\})) = 1 && \text{for all } i \in S, && \text{(5b)}\\
& x(\delta_S^-(\{i\})) = 1 && \text{for all } i \in S, && \text{(5c)}\\
& x(\delta_S^+(T)) \geq 1 && \text{for all } T \subseteq S : 2 \leq |T| \leq |S|, && \text{(5d)}\\
& x_a \geq 0 && \text{for all } a \in A_S. && \text{(5e)}
\end{aligned}
$$

Let $v_k(S)$ denote the optimal value of $\mathrm{LP}_k(S)$. We use the parsimonious property to show that the optimal values of LP(S) and $\mathrm{LP}_5(S)$ are in fact equal.

Lemma 1. *$v_{\mathrm{LP}}(S) = v_4(S) = v_5(S)$ for all $S \subseteq N$.*

Note that the first equality in the above lemma is due to Nguyen (2011). See also the work by Nguyen (2012) and Toriello (2013) for other TSP relaxations related to the Held-Karp bound that differentiate the depot from other cities.

Proof. Fix $\emptyset \neq S \subseteq N$. First, we show $v_4(S) \leq v_5(S)$. Suppose x is an optimal solution to $\mathrm{LP}_5(S)$. We prove that x is a feasible solution to $\mathrm{LP}_4(S)$. Before doing so, we make the following observation. Take any $T \subseteq S$. By (5b) and (5c), we have

$$
\begin{aligned}
|T| &= \sum_{i \in T} x(\delta_S^+(\{i\})) = x(A_S(T)) + x(\delta_S^+(T)),\\
|T| &= \sum_{i \in T} x(\delta_S^-(\{i\})) = x(A_S(T)) + x(\delta_S^-(T)),
\end{aligned}
$$

where $A_S(T) = \{(i,j) \in A_S : i \in T, j \in T\}$. Therefore,

$$
x(\delta_S^+(T)) = x(\delta_S^-(T)) \qquad \text{for all } T \subseteq S. \tag{6}
$$

Now for any $i \in S$, by (5b) and (5c), we have $x(\delta_S^+(\{i\})) = 1 = x(\delta_S^-(\{i\}))$. By (6), we have that $x(\delta_S^+(\{0\})) = x(\delta_S^-(S)) = x(\delta_S^+(S)) = x(\delta_S^-(\{0\}))$. So, x satisfies (4b). For any $T \subseteq S$, we have $x(\delta_S^+(T)) \geq 1$ by (5b)-(5d). For any $T \subseteq S \cup \{0\}$ such that $0 \in T$, we have

$$
x(\delta_S^+(T)) = x(\delta_S^-(S \cup \{0\} \setminus T)) \overset{\text{(i)}}{=} x(\delta_S^+(S \cup \{0\} \setminus T)) \overset{\text{(ii)}}{\geq} 1.
$$

Equality (i) holds by (6) and inequality (ii) holds by (5b)-(5d), since $0 \in T$ and therefore $S \cup \{0\} \setminus T \subseteq S$. So, x satisfies (4c). In addition, x clearly satisfies (4d). Therefore, x is a feasible solution to $\mathrm{LP}_4(S)$, and $v_4(S) \leq v_5(S)$.

Next, we have $v_5(S) \leq v_{\mathrm{LP}}(S)$, since it is straightforward to see that any optimal solution to LP(S) is a feasible solution to $\mathrm{LP}_5(S)$. Finally, we have $v_{\mathrm{LP}}(S) = v_4(S)$ by the result of Nguyen (2011) mentioned above. Putting this all together, we have

$$
v_{\mathrm{LP}}(S) = v_4(S) \leq v_5(S) \leq v_{\mathrm{LP}}(S). \qquad \square
$$

Frieze et al. (1982) implicitly showed that $v(S) \leq \log(|S|+1)v_{\text{LP}}(S)$ for any $\emptyset \neq S \subseteq N$. In addition, Asadpour et al. (2010) showed that for any $\emptyset \neq S \subseteq N$, $v(S) \leq (2 + \frac{8\log(|S|+1)}{\log\log(|S|+1)})v_{\text{LP}}(S)$. In other words,

$$v(S) \leq \min\left\{\log(|S|+1), 2 + \frac{8\log(|S|+1)}{\log\log(|S|+1)}\right\}v_{\text{LP}}(S) \quad \text{for all } \emptyset \neq S \subseteq N. \tag{7}$$

3 Cost allocation

In order to find a cost allocation for the asymmetric traveling salesman game (N, v), we first consider the auxiliary cooperative game (N, v_{LP}) and establish a connection to linear production games (Owen 1975). A *linear production game* is a cooperative game (N, w), defined by an objective function vector c, constraint matrices B and D, and constraint vectors b^j and d^j for each player $j \in N$. For every coalition $S \subseteq N$, we define the cost $w(S)$ as the optimal value of a particular linear program:

$$w(S) = \min_x \quad cx \tag{8a}$$

$$\text{s.t.} \quad Bx = \sum_{j \in S} b^j, \tag{8b}$$

$$Dx \geq \sum_{j \in S} d^j, \tag{8c}$$

$$x \geq 0. \tag{8d}$$

We assume that c, B, D, b^j and d^j for $j \in N$ are given so that (8) has a finite optimal value for every coalition $\emptyset \neq S \subseteq N$. Owen (1975) showed that the following cost allocation is always in the core of a linear production game (N, w):

$$\chi_i = b^j\lambda + d^j\mu \qquad \text{for all } j \in N, \tag{9}$$

where λ and μ are respectively the optimal dual multipliers for constraints (8b) and (8c) in $w(N)$.

In order to connect (N, v_{LP}) with linear production games, we reformulate $\text{LP}_5(S)$ for $\emptyset \neq S \subseteq N$ by replicating the subtour elimination constraints (5d). For the constraint associated with $T \subseteq N$ we include a copy for each $i \in T$, but only "activate" the copies for players in the coalition S:

$$\text{LP}_{10}(S): \quad \min_x \quad \sum_{a \in A} c_a x_a \tag{10a}$$

$$x(\delta_N^+(\{i\})) = \sum_{j \in S} b_i^j \qquad \text{for all } i \in N, \tag{10b}$$

$$x(\delta_N^-(\{i\})) = \sum_{j \in S} b_i^j \qquad \text{for all } i \in N, \tag{10c}$$

$$x(\delta_N^+(T)) \geq \sum_{j \in S} d_{i,T}^j \qquad \text{for all } T \subseteq N : 2 \leq |T| \leq |N|, i \in T \tag{10d}$$

$$x_a \geq 0 \qquad \text{for all } a \in A, \tag{10e}$$

where

$$b_i^j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j; \end{cases} \qquad\qquad d_{i,T}^j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

As above, the superscripts for these right-hand side values indicate the player, whereas the subscripts indicate the constraint. This construction is reminiscent of prize-collecting TSP formulations (e.g. Balas 1989; Bienstock et al. 1993).

Lemma 2. (N, v_{LP}) *is a linear production game.*

Proof. (N, v_{10}) is a linear production game. In $\text{LP}_{10}(S)$, $x_a = 0$ for all $a \in A \setminus A_S$; using this observation and Lemma 1, it is straightforward to show that $v_{10}(S) = v_5(S) = v_{\text{LP}}(S)$ for all $\emptyset \neq S \subseteq N$. □

Lemma 2 allows us to construct a stable allocation for (N, v_{LP}) using dual multipliers. Let λ_i^+, λ_i^-, and μ_T respectively be optimal dual multipliers of $\text{LP}_5(N)$ for constraints (5b), (5c) and (5d). For each $T \subseteq N$ define weights $\gamma_{i,T} \geq 0$ for $i \in T$ satisfying $\sum_{i \in T} \gamma_{i,T} = 1$.

Corollary 3. *The cost allocation*

$$\chi_i = \lambda_i^+ + \lambda_i^- + \sum_{\substack{T \subseteq N : i \in T \\ |T| \geq 2}} \gamma_{i,T} \mu_T \qquad \textit{for all } i \in N \tag{11}$$

is in the core of (N, v_{LP}).

Proof. Because $(N, v_{\text{LP}}) = (N, v_{10})$ is a linear production game, by (9), the cost allocation

$$\chi_i = b^i \bar{\lambda} + d^i \bar{\mu} = \bar{\lambda}_i^+ + \bar{\lambda}_i^- + \sum_{\substack{T \subseteq N : i \in T \\ |T| \geq 2}} \bar{\mu}_{i,T} \qquad \text{for all } i \in N$$

is in the core of (N, v_{LP}), where $\bar{\lambda}_i^+$, $\bar{\lambda}_i^-$ and $\bar{\mu}_{i,T}$ are optimal dual multipliers of $\text{LP}_{10}(N)$ for constraints (10b), (10c), and (10d), respectively. Since $\text{LP}_{10}(N)$ has every right-hand side equal to 1, it is an exact replica of $\text{LP}_5(N)$ with $|T|$ copies of every subtour elimination constraint (5d) for each $T \subseteq N$. Therefore, $\bar{\lambda}_i^+ = \lambda_i^+$, $\bar{\lambda}_i^- = \lambda_i^-$ and $\bar{\mu}_{i,T} = \gamma_{i,T} \mu_T$ are optimal dual multipliers of $\text{LP}_{10}(N)$. □

The cost allocation in (11) allows a significant amount of freedom to determine how much each player is allocated. For example, we can assign uniform weights to all players of each subset, i.e. $\gamma_{i,S} = 1/|S|$. At the other extreme, we can fix an ordering of N and set $\gamma_{i,S} = 1$ if i is the minimal element of S, and 0 otherwise.

Since $v_{\text{LP}}(S) \leq v(S)$ for all $S \subseteq N$, Lemma 2 and Corollary 3 imply that the cost allocation χ defined in (11) is stable and α_{HKA}-budget-balanced for (N, v), where $\alpha_{\text{HKA}} = v(N)/v_{\text{LP}}(N)$, the integrality gap of $\text{LP}(N)$. These observations, combined with (7), give us our main result.

Theorem 4. *The cost allocation* χ *defined in* (11) *is in the* α*-core of the asymmetric traveling salesman game* (N, v)*, where*

$$\alpha = \min\left\{ \log(n+1), 2 + \frac{8 \log(n+1)}{\log\log(n+1)} \right\}. \tag{12}$$

Note that $\mathrm{LP}_5(N)$ can be solved in polynomial time using the ellipsoid method, with an algorithm for the minimum directed cut problem as a separation oracle. We can therefore find optimal dual multipliers for $\mathrm{LP}_5(N)$ in polynomial time (Grötschel et al. 1993). It follows that the cost allocation χ in (11) can be computed in polynomial time.

4 Extension to Network Design Games

In addition to the usual transportation perspective, the asymmetric TSP can be interpreted as the problem of finding a minimum-cost strongly connected Eulerian graph. As such, it falls into the larger class of network design and connectivity problems, which have many applications in their own right. As with the TSP, these problems naturally define cooperative games; we next discuss two important classes of these games.

First, consider the *strongly k-connected Eulerian graph game*. This game is set up similarly to the TSP game, except that the cost for a coalition $S \subseteq N$ is the minimum cost of a strongly k-connected Eulerian subgraph of G_S, i.e. a graph with $k \in \mathbb{Z}_{>0}$ arc-disjoint directed paths between any pair of nodes (including the depot 0), where every node's out-degree equals its in-degree. The cost for coalition $\emptyset \neq S \subseteq N$ is given by the integer program

$$\begin{aligned}
\mathrm{IP}_k(S): \min_x \quad & \sum_{a \in A_S} c_a x_a && && \text{(13a)}\\
\text{s.t.} \quad & x(\delta_S^+(\{i\})) = k && \text{for all } i \in S \cup \{0\}, && \text{(13b)}\\
& x(\delta_S^-(\{i\})) = k && \text{for all } i \in S \cup \{0\}, && \text{(13c)}\\
& x(\delta_S^+(T)) \geq k && \text{for all } T \subseteq S \cup \{0\} : 1 \leq |T| \leq |S|, && \text{(13d)}\\
& x_a \in \mathbb{Z}_{\geq 0} && \text{for all } a \in A_S. && \text{(13e)}
\end{aligned}$$

As with the asymmetric TSP, Nguyen (2011) showed that the LP relaxation of $\mathrm{IP}_k(S)$ satisfies the *parsimonious property*: if degree balance (3) is enforced, then the degree constraints (13b)–(13c) can be removed without affecting the optimal value.

Next we consider *survivable network design games*. In these games, costs are symmetric, we assume G is an undirected graph with node set $N \cup \{0\}$ and edge set E, and all other graph concepts are defined in an analogous fashion. Every player $i \in N$ has a *connectivity type* $r_i \in \mathbb{Z}_{\geq 0}$, indicating the number of edge-disjoint paths it requires to other nodes: There must be r_i edge-disjoint paths between node i and the depot node 0, and $r_{ij} = \min\{r_i, r_j\}$ edge-disjoint paths between nodes i and j. There may also be a set $D \subseteq N \cup \{0\}$ of nodes that must be *parsimonious*, i.e. whose degree must equal the minimum required to satisfy all connectivity requirements. For a coalition $\emptyset \neq S \subseteq N$, its cost is equal to the minimum cost of a subgraph satisfying these connectivity and degree requirements, and is given by the integer program

$$\begin{aligned}
\mathrm{IP}_r(S): \min_x \quad & \sum_{e \in E_S} c_e x_e && && \text{(14a)}\\
\text{s.t.} \quad & x(\delta_S(\{i\})) = r_i && \text{for all } i \in D \cap (S \cup \{0\}), && \text{(14b)}\\
& x(\delta_S(T)) \geq \max_{e \in \delta(T)} r_e && \text{for all } T \subseteq S \cup \{0\} : 1 \leq |T| \leq |S|, && \text{(14c)}\\
& x_e \in \mathbb{Z}_{\geq 0} && \text{for all } e \in E_S, && \text{(14d)}
\end{aligned}$$

where $r_0 = \max_{i \in S} r_i$, $r_{0i} = r_i$. In addition to symmetric TSP games, this class of games includes many other fundamental games, such as minimum spanning tree games, Steiner tree games, undirected edge-connectivity games, and others (see Goemans and Bertsimas 1993). The LP relaxation of $\mathrm{IP}_r(S)$ satisfies the parsimonious property: the degree constraints (14b) can be deleted without loss in optimal value (Goemans and Bertsimas 1993).

In a similar fashion to asymmetric TSP games, we can exploit these formulations' parsimonious properties and the constraint replication technique described in the previous section to connect these network design games to linear production games and obtain approximately budget balanced and stable allocations.

Theorem 5. *For both strongly k-connected Eulerian graph games and survivable network design games as defined above, the games defined by their LP relaxations can be formulated as linear production games. Consequently, these games have allocations in their α-core, where the budget balance guarantee α is the integrality gap of formulations* $\mathrm{IP}_k(N)$ *and* $\mathrm{IP}_r(N)$ *respectively.*

5 Discussion

Theorem 4 and the earlier result by Faigle et al. (1998) rely on an LP relaxation of the TSP to construct stable cost allocations, and the budget balance guarantees of these cost allocations match the integrality gaps of the LP relaxations. This motivates the natural question of whether the TSP integrality gap imposes a limit on the best possible budget balance guarantee achievable by an allocation that is stable for (N, v_{LP}). Surprisingly, this is not the case, even for very simple games. Consider a three-player instance with arc costs

$$\begin{aligned} c_{01} &= 1 & c_{02} &= 2 & c_{03} &= 1 \\ c_{10} &= 1 & c_{12} &= 1 & c_{13} &= 2 \\ c_{20} &= 2 & c_{21} &= 2 & c_{23} &= 1 \\ c_{30} &= 2 & c_{31} &= 1 & c_{32} &= 1. \end{aligned} \tag{15}$$

These costs give rise to an asymmetric traveling salesman game with $v(N) = 5$ and $v_{\mathrm{LP}}(N) = 9/2$, and so the cost allocation (11) has a budget balance guarantee of 10/9. In particular, an optimal set of dual multipliers of $\mathrm{LP}_5(N)$ is

$$\begin{gathered} \lambda_1^+ = -1/2 \qquad \lambda_2^+ = 0 \qquad \lambda_3^+ = -1/2 \\ \lambda_1^- = 1 \qquad \lambda_2^- = 3/2 \qquad \lambda_3^+ = 1 \\ \mu_{12} = 0 \qquad \mu_{13} = 0 \qquad \mu_{23} = 1/2 \qquad \mu_{123} = 3/2, \end{gathered}$$

and the cost allocation (11) with $\gamma_{i,T} = 1/|T|$ is

$$\chi_1 = 1 \qquad \chi_2 = 9/4 \qquad \chi_3 = 5/4.$$

However, the core of this game is nonempty: it is simple to check that the cost allocation $\hat{\chi}_1 = 1$, $\hat{\chi}_2 = \hat{\chi}_3 = 2$ is stable and budget-balanced. Even more surprisingly, since $v(S) = v_{\mathrm{LP}}(S)$ for all coalitions $\emptyset \neq S \subsetneq N$, this cost allocation $\hat{\chi}$ is in fact stable *with respect to* (N, v_{LP}): $\hat{\chi}$ satisfies the stability constraints (1b) with the characteristic function v_{LP} replacing v, just as the cost allocation (11) does.

This simple example motivates the following approach: Find a cost allocation stable with respect to (N, v_{LP}) that has the best possible budget balance. This can be formulated as the LP

$$\max_{\chi,\beta} \quad \beta \tag{16a}$$
$$\text{s.t.} \quad \beta v(N) \leq \chi(N) \leq v(N) \tag{16b}$$
$$\chi(S) \leq v_{\text{LP}}(S) \qquad \text{for all } \emptyset \neq S \subsetneq N, \tag{16c}$$

where a feasible solution (χ, β) corresponds to a cost allocation χ with a budget balance guarantee of $1/\beta$. Clearly, an optimal solution to (16) yields a stable cost allocation whose budget balance guarantee is at least as good as α_{HKA}, the budget balance guarantee of the cost allocation constructed in Section 3.

The separation problem for constraints (16c),

$$\min_{S \subsetneq N} \{v_{\text{LP}}(S) - \chi(S)\}, \tag{17}$$

is similar to the prize-collecting TSP, but substitutes the actual cost $v(S)$ of an optimal tour over the visited cities S with the corresponding LP relaxation value $v_{\text{LP}}(S)$. Of course, one would do better by requiring stability with respect to (N, v) instead of stability with respect to (N, v_{LP}). However, the separation problem for those constraints is the prize-collecting TSP (Balas 1989; Bienstock et al. 1993), which is known to be NP-hard. The hope is that this relaxed separation problem (17) is easier. If this "prize-collecting fractional TSP" could be solved in polynomial time, it could lead not only to efficient methods for finding better cost allocations for asymmetric traveling salesman games, but also to better approximation algorithms and integer programming approaches for the prize-collecting TSP.

Unfortunately, the problem (17) still has complex structure: for instance, the objective function of (17) is not necessarily submodular nor supermodular in S. In fact, the instance given in (15) already violates submodularity. We have thus far been unable to determine the complexity of this interesting separation problem, so we leave it as an open question.

One might also wonder if there are asymmetric traveling salesman games whose best possible budget balance guarantee matches (12). Faigle et al. (1998) proposed a family of symmetric traveling salesman games with Euclidean costs whose best possible budget balance guarantee is $4/3$. Of course, this lower bound also holds for asymmetric traveling salesman games. While this lower bound on the budget balance guarantee matches the best known lower bound on the integrality gap of the Held-Karp bound for the symmetric TSP, it does not match the best known bound for the asymmetric TSP, which is 2 (Charikar et al. 2006). One natural open question is whether one can construct an asymmetric traveling salesman game whose best possible budget balance guarantee is 2, perhaps by building upon the work of Charikar et al. (2006).

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