A family of polytopes in the 0/1-cube with Gomory-Chvátal rank at least ((1+1/6)n - 4)

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Abstract

We provide a family of polytopes $P \subseteq [0,1]^n$ whose Gomory-Chvátal rank is at least $((1+1/6)n-4)$.

1 Introduction

The Gomory-Chvátal procedure is used to find a set of half-spaces whose intersection identifies the integer hull P_I of a polyhedron $P = \{x \in R^n \| Ax \leq b\}$. The procedure was given by Chvátal [2] and Gomory [3],[4] and as one of today's standards, the Gomory-Chvátal procedure is widely used in solving integer programming problems. Research such as [6], [7], [8], [10] has given us good insight on the subject, however finding tight bounds for the rank of a polytope $P \in$ 0/1 cube is still an open question.

In [8], Pokutta and Schultz have shown that the rank of every polytope in 0/1 cube is bounded by $O(n^2 logn)$ and there exist a family of polytopes whose rank exceeds n. Although for the class of polytopes in [8], there exists an $\epsilon > 0$ such that their rank is greater or equal to $(1+\epsilon)n$ for infinitely many n, no example satisfying the inequality for any particular $\epsilon \in R^+$ has been given. In this paper we are defining a class of polytopes whose rank is greater or equal to $(1+1/6)n-4$ for all $n>0$, hence the rank exceeds $(1+\epsilon)n$ for every $\epsilon \in (0,1/6)$ as n gets large enough.

2 Preliminaries

Gomory-Chvátal closure of a polytope $P = \{x \in R^n \| Ax \leq b\}$ where $A \in Z^{mxn}$ and $b \in Z^m$ is given as:

$$P' := \bigcap_{\lambda \in R^m_+, \lambda A \in Z^n} \{x : \lambda A x \leq \lfloor \lambda b \rfloor\} \tag{1}$$

where P' is also a polytope. If we apply this closure over and over again, we reach a fixed point after finitely many number of closure operations. The smallest number of closures to be applied for reaching such a point(which is a polytope) is called the rank of the polytope and this point is equal to P_I which is the convex hull of $P \cap Z^n$. We will make use of the following lemma and definition in the next section.

The polytope B_n is defined in [13] by

$$B_n = \{x \in [0,1]^n \| \sum_{i \in S} x_i + \sum_{i \in [n] \ S} 1 - x_i \geq 1, \forall S \subseteq [n]\}. \tag{2}$$

B_n is important for analyzing the polytopes in 0/1 cube which have empty integer hull. If we take a polytope (with empty integer hull) in 0/1 cube which strictly contains B_n, then the closure of that polytope will be equal to B_n and the remaining closure operations can be worked out by just considering B_n to get the following result.

Lemma 1. *([13],Corollary 3.4)* $B_n^{(n-2)} = \{\frac{1}{2}e\}$

This residual point which is left after $n-2$ closures will give us a good estimate for $(P_n)^{n-2}$ where P_n is the polytope that we are going to define in the next section (in 3).

3 Polytopes $P \subseteq [0,1]^n$ with rank $(1+1/6)n-4$

3.1 Definition of P

We define the following family of polytopes.

For every positive and odd integer n, let $c \in R^n$ such that $c(i) = 1$ for $i \in \{1 \cdots n-1\}$ and $c(n) = (n-1)/2$. Let $S_c = \{z_y \| y \in \{0,1\}^n, c^t y \neq (n-1)/2\}$ where for every $y \in \{0,1\}^n$, $z_y \in \{-1,1\}^n$ such that $z_y(i) = -1$ if $y(i) = 0$ and $z_y(i) = 1$ if $y(i) = 1$. For this case we define P_n as:

$$P_n = [0,1]^n \bigcap_{z \in S_c} (\{x \| z^t x \leq max_{y \in \{1,0\}^n}\{z^t y\} - \frac{1}{2}\}). \tag{3}$$

For every positive and even integer n,

$$P_n = \{(x_1, \cdots, x_{n-1}, 0) \| (x_1, \cdots, x_{n-1}) \in P_{n-1}\}. \quad (4)$$

For odd valued n we can see that $(P_n)_I \cap Z^n = \{x \in \{0,1\}^n \| c^t x = (n-1)/2\}$. In other words, we cut out small simplices originated at the vertices of the 0/1 cube that do not belong to the hyperplane $c^t x = (n-1)/2$ to create P_n. For even valued n we have placed P_{n-1} on the facet $\{x \in [0,1]^n \| (0, \cdots, 0, 1)^t x = 0\}$ of the n-dimensional 0/1-cube to create P_n.

3.2 Analysis

In this section we are going to derive the rank of the polytope defined in 3.

Lemma 2. *Let H , G be two polytopes in the 0/1 cube such that $H \subseteq G$ and $H_I = G_I$, then for every $k \in Z^+$ we have $H^k \subseteq G^k$.*

Proof.For every $c \in R^n$ we have $max_{x \in H}(c^t x) \leq max_{x \in G}(c^t x)$ hence $\{x \| c^t x \leq \lfloor max_{x \in H}(c^t x) \rfloor\} \subseteq \{x \| c^t x \leq \lfloor max_{x \in G}(c^t x) \rfloor\} \Rightarrow H^1 \subseteq G^1$. By induction the proof follows. □

Since $P_n \supseteq B_n$, from lemma 1 we have $(P_n)^{n-2} \supseteq \{e\frac{1}{2}\}$. Moreover $(P_n)^{n-2} \supseteq (P_n)_I$ hence $(P_n)^{n-2} \supseteq conv((P_n)_I \cup \{e\frac{1}{2}\}) \supset (P_n)_I$ which means we still need more subsequent closures to reach $(P_n)_I$.

Lemma 3. $(P_n)^{n-2} \supset \{ce \| \frac{1}{2} \geq c \geq \frac{1}{3}\}$.

Proof. First observe that $e\frac{1}{3} \in P_I$. Since $conv((P_n)_I \cup \{e\frac{1}{2}\}) \subseteq (P_n)^{n-2}$ then $(P_n)^{n-2} \supset conv(\{e/3, e/2\})$ which concludes the proof. □

Now let's focus on the cutting planes for the polytope $(P_n)^k$ where $k \geq n-2$. Take any $v \in Z^n$ and let $\mu = max_{x \in (P_n)_I}(v^t x)$. Observe that such μ has to be an integer because all extreme points of $(P_n)_I$ are integer valued by definition.

Lemma 4. *For every positive odd integer n and for every $v \in Z^n$, if $\mu = max_{x \in (P_n)_I}(v^t x)$ then $v^t e \leq 3\mu$ moreover for $\mu \in [0, (n-1)/2)$ we have $v^t e \leq 2\mu$.*

Proof. To see this we have to look at the set of extreme points of $(P_n)_I$ which is equal to $\{(0 \cdots 001)\} \cup \{y_S \| |S| = (n-1)/2, S \subseteq \{1, \cdots, n-1\}\}$ where $y_S(i) = 1$ if $i \in S$ and $y_S(i) = 0$ otherwise. For every $S \subseteq \{1, \cdots, n-1\}\}$, if $|S| = (n-1)/2$ then $\bar{S} = (\{1, \cdots, n-1\} - S) \subseteq \{1, \cdots, n-1\}\}$ also has $(n-1)/2$ elements hence $y_{\bar{S}}$ is also an extreme point of $(P_n)_I$. Now we can write $v^t e = v^t (1 \cdots 1)' = v^t (0 \cdots 001) + v^t y_S + v^t y_{\bar{S}}$, since each term must be smaller or equal to μ we conclude that $v^t e \leq 3\mu$.

For the case where $\mu \in [0, (n-1)/2)$ first let $T = \{S \| S \subseteq \{1, \cdots, n-1\}, |S| = (n-1)/2\}$ and choose $Q \in T$ such that $v^t y_Q = max_{S \in T}(v^t y_S)$ (observe that $max_{S \in T}(v^t y_S) \leq \mu$). If $v^t y_Q \geq 0$ then $v^t y_{\bar{Q}} \leq 0$ where $\bar{Q} = (\{1, \cdots, n-1\} - Q)$. To prove this first observe that $\exists i \in Q$ such that $v(i) \leq 0$ otherwise $v^t y_Q \geq (n-1)/2 > \mu \geq max_{S \in T}(v^t y_S)$ which is a contradiction. Moreover if $v^t y_{\bar{Q}} > 0$ then $\exists j \in \bar{Q}$ such that $v(j) > 0$. But then for $S' = (Q - \{i\}) \cup \{j\}$ we have $v^t y_{S'} > v^t y_Q = max_{S \in T}(v^t y_S)$ hence a contradiction. We conclude that for such μ, $v^t e = v^t (1 \cdots 1)' = v^t (0 \cdots 001) + v^t y_Q + v^t y_{\bar{Q}} \leq 2\mu$ which concludes our proof □

Lemma 5. *For every positive odd integer n and for every $v \in Z^n$, if $max_{x \in (P_n)_I}(v^t x) = \mu < 0$ then $c(e^t v) \leq \mu$ for every $c \geq 1/3$.*

Proof. From lemma (4) we have $e^t v \leq 3\mu$ hence $c(e^t v) \leq c(3\mu)$. For $c \geq 1/3$ we have $3c \geq 1$ and since $\mu < 0$ we have $c(3\mu) \leq \mu$ which completes the proof □

Lemma 6. *For every positive odd integer n and for every $v \in Z^n$, if $0 \leq max_{x \in (P_n)_I}(v^t x) = \mu < (n-1)/2$ then $c(e^t v) \leq \mu$ for every $c \in [1/3, 1/2]$.*

Proof. From lemma (4) for such μ we have $v^t e \leq 2\mu$ and $\mu \geq 0$ so the result follows □

Lemma 7. *For every positive odd integer n, for every $c \in [1/3, 1/2]$ if $ce \in (P_n)^k$ then $max\{c - 1/(n-1), 1/3\}e \in (P_n)^{k+1}$.*

Proof. Once again let $v \in Z^n$ and $\mu = max_{x \in (P_n)_I}(v^t x)$ (remember that such μ is an integer). We are going to make the analysis by dividing the problem into three cases.

Let $\mu < 0$. From lemma 5 we know that such a cutting plane will not cut out the point ce since $\lfloor max_{x \in (P_n)^k}(v^t x) \rfloor \geq \mu \geq c(e^t v)$.

Let $\mu \geq 0$ and $v^t e \leq 2\mu$. $c(v^t e) \leq \mu \leq \lfloor max_{x \in (P_n)^k}(v^t x) \rfloor$ and again such a cutting plane will not cut out ce.

Let $\mu \geq 0$ and $v^t e > 2\mu$. From lemma 4 we have to have $\mu \geq (n-1)/2$. But then

$v^te(1/(n-1)) > 2\mu(1/(n-1)) \geq (n-1)(1/(n-1)) \geq 1 \Rightarrow \lfloor max_{x\in(P_n)^k}(v^tx)\rfloor \geq \lfloor c(v^te)\rfloor \geq (c-1/(n-1))v^te$. Since $e/3 \in (P_n)_I$ no cutting plane can cut it out and together we conclude that no cutting plane can cut out $max\{c-1/(n-1), 1/3\}e$.
These three cases complete the proof. □

Lemma 8. *For every positive odd integer n, rank of (P_n) is at least $n(1+1/6)-3$.*

Proof. From lemma 3 we have $(P_n)^{n-2} \supseteq conv((P_n)_I \cup \{e\frac{1}{2}\}) \supset (P_n)_I$ then from lemma 7 and by simple induction we get $(P_n)^{n-2+l} \supset conv((P_n)_I \cup \{e(1/2-l/(n-1))\})$ for $\{l \in Z^+ \| (1/2-l/(n-1)) \geq 1/3\}$. Since $conv((P_n)_I \cup \{e(1/2-(l-1)/(n-1))\}) \supset (P_n)_I$ (the first expression strictly contains the integer hull of polytope P_n) we can see that $rank(P_n) \geq n-2+l$. Let l_m be the maximum l satisfying $l \in Z^+$, $(1/2-l/(n-1)) \geq 1/3$ then $l_m = \lfloor(1/2-1/3)/(1/(n-1))\rfloor \geq (n/6-1)$. We conclude that $rank(P_n) \geq n-2+l_m \geq n(1+1/6)-3$. □

Lemma 9. *For every positive even integer n, rank of P_n is at least $n(1+1/6)-4$.*

Proof. First let's show that for P_n defined in 4, $P_n^k = \{(x_1,\cdots,x_{n-1},0) \| (x_1,\cdots,x_{n-1}) \in P_{n-1}^k\}$. For $v \in \{(0,\cdots,0,1),(0,\cdots,0,-1)\}$ we have $\lfloor max_{y\in P}(v^ty)\rfloor = 0$ (so the closure of P_n resides on the hyperplane $\{x \in R^n \| (0,\cdots,0,1)^tx = 0\}$). Since $\forall v \in Z^n$ we have $\{(x_1,\cdots,x_{n-1},0) \in R^n \| v^tx \leq \lfloor max_{y\in P_n}(v^ty)\rfloor\} = \{(x_1,\cdots,x_{n-1},0) \in R^n \| (v_1,\cdots,v_{n-1})x \leq \lfloor max_{y\in P_{n-1}}((v_1,\cdots,v_{n-1})y)\rfloor\} \Rightarrow P_n' = \{(x_1,\cdots,x_{n-1},0) \| (x_1,\cdots,x_{n-1}) \in P_{n-1}'\}$ and the result follows by induction.
Now let k be the smallest integer such that $P_n^k = (P_n)_I$. By the previous result we have $(P_n)_I = \{(x_1,\cdots,x_{n-1},0) \| (x_1,\cdots,x_{n-1}) \in P_{n-1}^k\} \Rightarrow P_{n-1}^k = (P_{n-1})_I$ and by lemma 8 we must have $k \geq (n-1)(1+1/6)-3$. Since k is an integer, we have $k \geq n(1+1/6)-4$ and we conclude that $rank(P_n) \geq n(1+1/6)-4$. Now we can complete our derivation of the rank of P_n, for all n as follows.

Theorem 1. *For every positive integer n, rank of P_n is at least $n(1+1/6)-4$.*

Proof. Lemma 8 and lemma 9 together gives the result. □

Since the latter term in theorem 1 is a constant, for every $\epsilon \in [0,1/6)$ there exists an n_ϵ such that for $n \geq n_\epsilon$ we have $rank(P_n) \geq (1+\epsilon)n$.

3.3 Discussion

Now that we have defined a family of polytopes and derived its rank, it is time for a quick discussion. The approach that we have used is purely geometrical. We started with the convex hull of a subset of vertices $\{0,1\}^n$ which lie on the same hyperplane, then "expanded" this polytope to a full dimensional polytope. Integer hull this expanded polytope is equal to the convex hull of the chosen vetrices and from lemma 2 we can see that this expansion will give us a maximum ranked polytope in the 0/1 cube among all the polytopes that have the same integer hull. In the analysis we derived the lowerbound by observing the line segment connecting $e/2$ and $e/3$. For odd n since $(P_n)_I$ lies on a hyperplane which contains only $e/3$ from that line segment it is clear that the whole line segment has to be cut out before we reach $(P_n)_I$. We have shown that we can divide this line segment into equal length subsegments such that at most one subsegment can be cut at each closure operation which led to our lower bound on the rank. For the even valued n we considered $n-1$ dimensional polytope living on a facet of the n-dimensional 0/1-cube where the bound for the $n-1$ dimensional case ($n-1$ is odd in this case) immediately applied.

4 Conclusion

We have defined a family of polytopes that realizes a rank greater or equal to $(1+1/6)n-4$. We believe this lower bound can be improved by considering expansions of convex hulls of subsets of $\{0,1\}^n$ which reside on a hyperplane. No full dimensional P_I will not lead to a higher rank since each of its facets must be reached at some closure and the facet that is reached last will define the rank of such polytope. This approach may be used to derive tight upper bounds for the same kind of polytopes. Searching for a maximum rank polytope formed by the expansion of $n-1$ dimensional integer hulls will give an upper bound for the polytopes in 0/1-cube.

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