# Some criteria for error bounds in set optimization

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**Abstract**

We obtain sufficient and/or necessary conditions for global/local error bounds for the distances to some sets appeared in set optimization studied with both the set approach and vector approach (sublevel sets, constraint sets, sets of *all* Pareto efficient/ Henig proper efficient/super efficient solutions, sets of solutions *corresponding to one* Pareto efficient/Henig proper efficient/super efficient value) and sufficient conditions for metric subregulality of a set-valued map at efficient solutions. All criteria except one are described in terms of the Mordukhovich coderivative and coderivative of convex analysis. Our techniques are based on scalarization by mean of the Hiriart-Urruty signed distance function, on exploiting criteria in terms of subdifferentials for error bounds of a lower semicontinuous function and estimates for subdifferentials of marginal functions. We also consider the single-valued case and provide illustrating examples.

**Key Words:** Error bound, metric subregularity, set-valued optimization problem, constraint set, solution set, subdifferential, coderivative, Pareto efficient solution, Henig efficient solution, super efficient solution.

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## 1 Introduction

During the last two decades, the problem of error bounds for a lower semicontinuous (l.s.c.) function has received a lot of attention of researchers. It can be stated as follows, see [3, 4, 10]. Let $X$ be a metric space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a l.s.c. function. Denote $\text{dom} f := \{x \in X \mid f(x) \neq \infty\}$. We say that $f$ has a *global error bound* at the level $\alpha$ if there exists a scalar $\tau > 0$ such that

$$\tau d(x; [f \leq \alpha]) \leq [f(x) - \alpha]^+ \text{ for all } x \in X$$

and that $f$ satisfies the *error bound property* at $\bar{x} \in \text{dom} f$ or the distance to the set $[f \leq f(\bar{x})]$ has a *local error bound* at $\bar{x}$ if there exists a scalar $\tau > 0$ such that

$$\tau d(x; [f \leq f(\bar{x})]) \leq [f(x) - f(\bar{x})]^+ \text{ for all } x \text{ near } \bar{x}.$$

Here, $[f \leq \alpha] := \{x \in X \mid f(x) \leq \alpha\}$, $[\alpha]^+ := \max\{0, \alpha\}$ and for a set $U \subset X$, $d(x; U)$ denotes the distance from $x$ to $U$. In the special case $\alpha = \min_X f$ with $\bar{x} \in \text{argmin} f := \{x \in X \mid f(x) = \min_X f\}$, one is concerned with the global/local error bounds of the distance to the set of minimizers of $f$

$$\tau d(x; \text{argmin } f) \leq f(x) - \min_X f.$$

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Error bound properties have important applications in subdifferential calculus, optimality conditions, sensitivity analysis and convergence of numerical methods. A huge literature deals with criteria for them in terms of various derivative-like objects defined either in the primal space (directional derivatives, slopes, etc.) or in the dual space (different kinds of subdifferentials), see [3, 4, 6, 10, 39] for an overview.

Closely related to error bounds is the notion of metric subregularity of a set-valued map $F$ acting between normed spaces $X$ and $Y$. Following Dontchev and Rockafellar [8], $F$ is said to be *metrically subregular* at $\bar{x}$ for $\bar{y} \in F(\bar{x})$ if there exists $\kappa \in ]0, \infty[$ such that

$$d(x; F^{-1}(\bar{y})) \leq \kappa d(\bar{y}; F(x)) \text{ for all } x \text{ naer } \bar{x}.$$

The metric subregularity property for the set-valued map $F$ was introduced by Ioffe [23, 24] using the terminology "regularity at a point" and has been widely considered especially in connection with feasibility problems (constraint sets, generalized equations...) and their associated maps, see [9, 13, 20, 24, 25, 41, 42].

*The aim of this paper* is to study error bounds in relation to set-valued optimization problems with data being a set-valued map $F$ from a metric (or normed) space $X$ into a normed space $Y$ partially ordered by a closed convex cone $K$. We obtain criteria for error bounds for the distances to sublevel sets, constraint sets, different solutions sets and for metric subregularity of $F$ at some efficient solutions.

Our *main results* are the following (for the definitions, see Section 2):
1. We establish a sufficient condition for error bounds for the distance to a sublevel set

$$\tau d(x; [F \preceq_l A]) \leq [\sup_{a \in A} \inf_{y \in F(x)} \Delta_{-K}(y - a)]^+ \text{ for all } x \in X,$$

where $\preceq_l$ is an order relation induced by $K$ on $2^Y$ (this order relation has been considered in the set approach to set optimization in the first time by Kuroiwa [29]), $A \subset Y$ is a nonempty set, $[F \preceq_l A] := \{x \in X \mid F(x) \preceq_l A\}$ and $\Delta_{-K}$ is the Hiriart-Urruty signed distance associated to the cone $-K$.

2. We obtain sufficient and/or necessary conditions in terms of the Mordukhovich coderivatives and coderivative of convex analysis for global/local error bounds for the distances to some sets appeared in set optimization

$$\tau d(x; S) \leq [\inf_{y \in F(x)} \Delta_{-K}(y - \bar{y})]^+ \text{ for all } x \in X \text{ or } x \text{ near } \bar{x},$$

where $S$ is one of the sets: (i) a special sublevel set $[F \preceq_l \bar{y}]$; (ii) the constrain set of the form $\{x \in X \mid G(x) \cap \mathcal{C} \neq \emptyset\}$ with $G$ being a set-valued map and $\mathcal{C}$ being a closed convex cone; (iii) the set of of solutions of a unconstrained set-valued optimization problem $(\mathcal{P})$ *corresponding to one* Pareto efficient/Henig proper efficient/super efficient value $\bar{y}$ or the set of *all* Pareto efficient/ Henig proper efficient/super efficient solutions of $(\mathcal{P})$.

3. We show, by the way, that some obtained sufficient conditions for local error bounds are in fact sufficient for the metric subregularity of the map $F$ at a Pareto efficient/ Henig proper efficient/super efficient solution.

Our *techniques* are based on scalarization by mean of the Hiriart-Urruty signed distance function [22], on exploiting criteria in terms of subdifferentials for error bounds of a l.s.c. function [3, 4, 10] and estimates of subdifferentials of marginal functions [15, 36, 37].

Note that, to the best of our knowledge, there are quite a few attempts in considering error bounds for maps with values in partially ordered spaces even in the single-valued case: Bernarczuk and Kruger [11] obtained some conditions in terms of various slopes and subdifferentials

of the distance functions to several level sets corresponding to Pareto or weak Pareto efficiency while Liu, Ng and Yang [34] considered a *linear* vector-valued map. The cases with Henig properly efficient solutions and super efficient solutions have not been considered yet even for vector-single-valued maps.

The paper is organized as follows. In Section 2 we present preliminaries from set optimization, variational analysis and some known results about error bounds of a l.s.c. function. In Section 3 we establish a sufficient condition for the error bound for distances to a sublevel set $[F \preceq_l A]$. In Section 4, we prove sufficient and/or necessary conditions for the error bound for distances to a special sublevel set $[F \preceq_l \bar{y}]$ and provide examples in case with constraint sets. Section 5 is devoted to error bounds for the distances to different efficient solution sets and for metric subregularity at different efficient solutions.

## 2  Preliminaries

### 2.1  Set-valued Optimization Problems

Let $Y$ be a topological vector space (t.v.s.) with the dual $Y^*$. For a nonempty set $U$, by int$U$, cl$U$, cone$U$ and conv$U$ we mean the interior, the closure, the conic hull and the convex hull of $U$, respectively. A closed unit ball in a normed space $Y$ is denoted by $\mathbb{B}_Y$. Let $K \subset Y$ be a nonempty closed pointed convex cone with apex at zero (pointedness means $K \cap (-K) = \{0\}$). The cone $K$ induces a partial order $\leq_K$ on $Y$: for $y_1, y_2 \in Y$, we write $y_1 \leq_K y_2$ if $y_2 - y_1 \in K$. Denote $K^+ = \{y^* \in Y^* \mid y^*(k) \geq 0 \text{ for all } k \in K\}$ and $K^{+i} = \{y^* \in Y^* \mid y^*(k) > 0 \text{ for all } k \in K \setminus \{0\}\}$. A convex set $\Theta \subset Y$ is called a base for $K$ if $0 \notin \text{cl}\Theta$ and $K = \{t\theta \mid t \in \mathbb{R}_+, \theta \in \Theta\}$. When $\Theta$ is bounded, we say that $K$ has a bounded base. It is known that $K^{+i} \neq \emptyset$ iff $K$ has a base, the nonnegative orthants in $\mathbb{R}^n$, $C_{[0,1]}$, $L^p_{[0,1]}$, $l^p$ $(1 \leq p < \infty)$ have bases and the nonnegative orthants in $\mathbb{R}^n$, $L^1_{[0,1]}$, $l^1$ have bounded bases [28]. Assuming that $Y$ is a normed space and $\Theta$ is a base of $K$, denote

$$\delta := \inf\{\|\theta\| \mid \theta \in \Theta\} > 0 \tag{1}$$

and let $K_\eta$ (with $\eta \in ]0, \delta[$) be a convex pointed closed cone defined by

$$K_\eta = \text{cl cone}(\Theta + \eta\mathbb{B}_Y). \tag{2}$$

We recall some concepts of efficienct points of a set [12, 19, 27].

**Definition 2.1.** *Let $A$ be a nonempty subset of $Y$ and $\bar{a} \in A$. We say that*

(i) *$\bar{a}$ is a Pareto efficient point of $A$ w.r.t. the cone $K$ if $a \not\leq_K \bar{a}$ for all $a \in A$, $a \neq \bar{a}$ or, equivalently, $(A - \bar{a}) \cap (-K \setminus \{0\}) = \emptyset$.*

(ii) *supposing that $Y$ is a normed space and $K$ has a base $\Theta$, $\bar{a}$ is a Henig properly efficient point of $A$ w.r.t. the cone $K$ if there exists $\eta \in ]0, \delta[$ such that $\text{clcone}(A - \bar{a}) \cap (-K_\eta) = \{0\}$.*

(iii) *supposing that $Y$ is a normed space, $\bar{a}$ is a super efficient point of $A$ if there exists $\rho > 0$ such that $\text{clcone}(A - \bar{a}) \cap (\mathbb{B}_Y - K) \subset \rho\mathbb{B}_Y$.*

Denote the sets of efficient points defined in Definition 2.1 by $Min(A, K)$, $He(A, K)$ and $SE(A, K)$, respectively. It is known that $SE(A, K) \subset He(A, K) \subset Min(A, K)$ and that if $K$ has a bounded base then $SE(A, K) = He(A, K)$. Moreover, $\bar{a} \in He(A, K)$ iff $\bar{a} \in Min(A, K_\eta)$ for some $\eta \in ]0, \delta[$ [17, Proposition 3.3].

Throughout the paper, $F$ is a set-valued map from a set $X$ into t.v.s. $Y$. In set optimization, one considers a unconstrained set-valued optimization problem $(\mathcal{P})$

$$\text{Minimize } F(x) \text{ subject to } x \in X$$

and a constrained set-valued optimization problem $(\mathcal{CP})$ with $X$ being replaced by a constraint set

$$\text{Minimize } F(x) \text{ subject to } G(x) \cap \mathcal{C} \neq \emptyset,$$

where $G$ is a set-valued map from a set $X$ into a t.v.s. $Z$, and $\mathcal{C} \subset Z$ is a closed convex cone ($\mathcal{C} \neq \{0\}$).

In this paper, we will be concerned with sets of solutions of $(\mathcal{P})$ and sublevel sets defined by the set-valued map $F$ as well as with the constraint set of $(\mathcal{CP})$ defined above. Let us note that there are two approaches, the vector approach and the set approach, to defining main concepts in set optimization (efficient solutions, sublevel sets). In the *vector approach*, efficient solutions of $(\mathcal{P})$ are defined through the corresponding concepts of efficient points for a set given in Definition 2.1.

**Definition 2.2.** *We say that $\bar{x} \in X$ is an "N" efficient solution of $(\mathcal{P})$ if there exists $\bar{y} \in F(\bar{x})$ called an "N" efficient value of $(\mathcal{SP})$ such that $\bar{y}$ is an "N" efficient point of $F(X) := \cup_{x \in X} F(x)$, where "N" may be Pareto, Henig properly, super.*

The *set approach* proposed by Kuroiwa in [29] is based on an order relation $\preceq_l$ induced by the cone $K$ on the power set $2^Y$ of all nonempty subsets of $Y$: for two nonempty sets $A$ and $B$

$$A \preceq_l B \text{ if and only if } B \subseteq A + K \tag{3}$$

and $\bar{x} \in X$ is said to be a minimal solution of $(\mathcal{P})$ w.r.t. the order $\preceq_l$ if

$$F(x) \preceq_l F(\bar{x}) \text{ for some } x \in X \text{ implies } F(\bar{x}) \preceq_l F(x).$$

With this order relation in hand, one considers a sublevel set $[F \preceq_l A]$ of $F$ at the level $A \in 2^Y$ defined by

$$[F \preceq_l A] := \{x \in X \mid F(x) \preceq_l A\}.$$

Note that Kuroiwa's set approach has been used by many authors and we mention several works [1, 16, 18, 21, 30, 31, 32] devoted to this approach.

## 2.2 Subdifferentials and coderivatives

In this subsection, we recall the notions of subdifferentials and coderivatives [2, 36, 38], some estimates of subdifferentials of a marginal function [15, 36, 37], the Hiriart-Urruty signed distance function [22] and prove some properties of this function.

In this subsection, let $X$ and $Y$ be normed spaces with the duals $X^*$ and $Y^*$. Let $\Omega \subset X$ be a nonempty set. The *Fréchet normal cone* to $\Omega$ at $x$, denoted by $N_F(x; \Omega)$, is given by

$$N_F(x; \Omega) := \{x^* \in X^* \mid \limsup_{x' \xrightarrow{\Omega} x} \frac{\langle x^*, x' - x \rangle}{\|x' - x\|} \leq 0\}.$$

The *Mordukhovich normal cone* to $\Omega$ at $x$, denoted by $N_M(x; \Omega)$, is defined in Asplund space settings by

$$N_M(x; \Omega) := \limsup_{x' \xrightarrow{\Omega} x} N_F(x'; \Omega),$$

where the limit in the right-hand side means the sequential Kuratowski-Painlevé upper limit with respect to the norm topology in $X$ and the weak-star $\omega^*$ topology in $X^*$. Recall that a Banach space is *Asplund* if every continuous convex function defined on it is Fréchet differentiable on a dense set of points. Examples of Asplund spaces are $\mathbb{R}^n$, $L_{[0,1]}^p$ and $l^p$ ($1 < p < \infty$).

Note that when $\Omega$ is convex, the Mordukhovich normal cone reduce to the *normal cone of convex analysis*, denoted by $N(x; \Omega)$

$$N(x; \Omega) := \{x^* \in X^* \mid \langle x^*, v \rangle \leq 0 \text{ for all } v \in \text{cl}(\cup_{t>0} \frac{1}{t}(\Omega - x))\}.$$

Let $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a l.s.c. function. The *Mordukhovich subdifferential* $\partial_M f(x)$ of $f$ at $x \in \text{dom} f$ is defined by mean of the Mordukhovich normal cone to its epigraph as follows

$$\partial_M f(x) := \{x^* \in X^* \mid (x^*, -1) \in N_M((x, f(x)); \text{epi} f)\}.$$

When $f$ is convex, the Mordukhovich subdifferential coincides with the *subdifferential of convex analysis*, denoted by $\partial f$, which can be defined either through the normal cone of convex analysis or by

$$\partial f(x) := \{x^* \in X^* \mid \langle x^*, x' - x \rangle \leq f(x') - f(x) \text{ for all } x' \in \text{dom} f\}.$$

We recall a version of chain rules to be used in our study.

**Proposition 2.3.** *Assume that $X$ and $Y$ are Asplund spaces, $f : X \rightarrow Y$ is a strictly differentiable map and $\phi : Y \rightarrow \mathbb{R}$ is a convex Lipschitz function. Then*

$$\partial_M(\phi \circ f)(x) = \{[\nabla f(x)]^*(y^*) \mid y^* \in \partial \phi(f(x))\}.$$

Here, $\nabla f(x)$ and $[\nabla f(x)]^*$ are the strict derivative of $f$ at $x$ and its adjoint, and $(\phi \circ f)(x) := \phi(f(x))$. Proposition 2.3 is a consequence of [36, Theorem 3.41(iii)].

Let $F : X \rightrightarrows Y$ be as before a set-valued map. Denote by $\text{dom} F$ the domain of $F$, i.e. $\text{dom} F := \{x \in X \mid F(x) \neq \emptyset\}$ and by $\text{gr} F$ its graph, i.e., $\text{gr} F := \{(x, y) \mid y \in F(x)\}$. We say that $F$ is closed if its graph is closed and $F$ is convex if its graph is convex. We always assume that the set-valued map $F$ is closed. The *Mordukhovich coderivative* $D_M^* F(x, y)$ of $F$ at $(x, y) \in \text{gr} F$ is defined by mean of the Mordukhovich normal cone to its graph as follows: for $y^* \in Y^*$

$$D_M^* F(x, y)(y^*) := \{x^* \in X^* \mid (x^*, -y^*) \in N_M((x, y)); \text{gr} F)\}.$$

When $F$ is convex, the *coderivative of convex analysis* $D^* F(x, y)$ of $F$ at $(x, y) \in \text{gr} F$ is defined by mean of the normal cone of convex analysis to its graph in the similar way.

Note that in [35] Mordukhovich introduced the notion of coderivative of a set-valued map regardless of the normal cone used. After he suggested this approach to differentiability of maps, we may consider different specific coderivatives for set-valued maps generated by different normal cones to their graphs. We refer the reader to Mordukhovich's book [36] for the history of coderivatives.

Next, we recall some estimates of subdifferentials of a marginal function which play an essential role in our study. Let be given a function $\varphi : Y \rightarrow \mathbb{R}$. We associate with $F$ and $\varphi$ the *marginal function* $m : X \rightarrow \bar{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$

$$m(x) := \inf\{\varphi(y) \mid y \in F(x)\}$$

and the *minimum set*

$$V(x) := \{y \in F(x) \mid \varphi(y) = m(x)\}$$

with the convention that $\inf \emptyset = \infty$ and $V(x) = \emptyset$ when $x \notin \text{dom} F$. It is easy to see that if $F(x)$ is compact and $\varphi$ is l.s.c. at $x \in \text{dom} \varphi$, then $V(x) \neq \emptyset$. Moreover, it is well known that, under natural hypotheses, the function $m$ inherited continuity and convexity properties from $F$

and $\varphi$. In the remaining of this subsection, unless otherwise specified, $X$ and $Y$ are Banach spaces. Recall that $V$ is *lower semicompact* at $x$ [37] if there exists a neighborhood $U$ of $x$ such that for any sequence $x_n \to x$ there is a sequence $y_n \in V(x_n)$ that contains a subsequence convergent in the norm topology of $Y$. Lower semicompactness is useful for dealing with the infinite-dimensional case and is often satisfied in finite-dimensional spaces.

The first estimate is a special Lipschitz case of [37, Theorem 6.1].

**Proposition 2.4.** *Assume that $X$ and $Y$ are Asplund spaces, $F$ is u.s.c. compact-valued, $\varphi$ is Lipschitz and $V$ is lower semicompact at $x \in$ dom $m$. Then for any $y_x \in V(x)$, we have $\partial_M m(x) \subseteq \cup_{y^* \in \partial_M \varphi(y_x)} D_M^* F(x, y_x)(y^*)$.*

In the convex case, we have a stronger conclusion, see [15, Theorem 3.3].

**Proposition 2.5.** *Assume that $F$ is convex u.s.c. compact-valued, $\varphi$ is convex and continuous at a point in $F(X)$. Assume further that either $X$ is separable or $\varphi$ is l.s.c. on its domains. Then for any $y_x \in V(x)$, we have $\partial m(x) = \cup_{y^* \in \partial \varphi(y_x)} D^* F(x, y_x)(y^*)$.*

**Remark 2.6.** *The assertion of Proposition 2.5 remains true if we relax the convexity assumption on $F$ while retaining convexity of the function $m$.*

The reader is referred to [15] for examples and comments on the above estimates.

We conclude this subsection with some properties of the Hiriart-Urruty signed distance function [22]. To a nonempty set $U$ in a Banach space $Y$ we associate a function $\Delta_U$ given by

$$\Delta_U(y) := d(y; U) - d(y; Y \setminus U).$$

This function possesses nice properties, especially when $U$ has a nonempty interior, and has been used for scalarization in vector optimization in several works [7, 14, 15]. We list some known properties of this function [15, 22].

**Proposition 2.7.** *Let $U$ be a nonempty subset of $Y$.*

(a) *The function $\Delta_U$ is Lipschitz of rank 1 on $Y$.*

(b) *If $U$ is closed then for any $y \in Y$, $y \notin U$ iff $\Delta_U(y) > 0$.*

(c) *If $U$ is convex then $\Delta_U$ is convex, if $U$ is a cone then $\Delta_U$ is positively homogeneous and if $U$ is a closed convex cone then $\Delta_U$ satisfies the triangle inequality, i.e., for any $y_1$, $y_2 \in Y$, one has $\Delta_U(y_1 + y_2) \le \Delta_U(y_1) + \Delta_U(y_2)$.*

We establish some useful properties of the subdifferential of the function $\Delta_{-K}$.

**Proposition 2.8.** *Let $K$ be a closed convex cone in a Banach space $Y$. Then*

(a) *$\partial \Delta_{-K}(u) \subseteq K^+ \cap \mathbb{B}_{Y^*}$ for any $u \in Y$.*

(b) *$0 \notin \partial \Delta_{-K}(u)$ if $u \notin -K$ or $\operatorname{int} K \ne \emptyset$.*

(c) *$0 \in \partial \Delta_{-K}(u)$ for any $u \in -K$ iff $\operatorname{int} K = \emptyset$.*

*Proof.* (a) Let $y^* \in \partial \Delta_{-K}(u)$. The triangle property of the function $\Delta_{-K}$ (Proposition 2.7(c)) implies that for any $k \in K$ one has $\langle y^*, -k \rangle \le \Delta_{-K}(u-k) - \Delta_{-K}(u) \le \Delta_{-K}(-k) = d(-k; -K) - d(-k; Y \setminus (-K)) \le 0$. Thus, $\langle y^*, k \rangle \ge 0$ for all $k \in K$. Since the cone $K$ is closed, we get $y^* \in K^*$. Further, Proposition 2.7(a) implies that $\partial \Delta_{-K}(u) \subset \mathbb{B}_{Y^*}$.

(b) Let $u \in Y$. If $u \notin -K$ then $\Delta_{-K}(u) > 0 = \Delta_{-K}(0)$ and if $K$ has a nonempty interior with $k \in \mathrm{int} K$ then $\Delta_{-K}(-k) < 0$ and $\Delta_{-K}(-tk)$ tends to $-\infty$ as $t \to \infty$. Thus, in both cases, the convex function $\Delta_{-K}$ does not attain its minimum at $u$ and hence, $0 \notin \partial \Delta_{-K}(u)$.

(c) By (b), we need only to prove the " if" part. Since $\Delta_{-K}(u) > 0$ for $u \notin -K$ and $\Delta_{-K}(u) = 0$ for $u \in -K$, the convex function $\Delta_{-K}$ attains its minimum at $u$ and hence, $0 \in \partial \Delta_{-K}(u)$. $\qquad\square$

**Example 2.9.** *Let* $Y = \mathbb{R}^2$, $K = \mathbb{R}_+^2$. *Then*

$$\partial \Delta_{-K}(y,z) = \begin{cases} \{(u,v) \in \mathbb{R}_+^2 \mid u^2 + v^2 \le 1 \le u + v\} & \text{if } y = z = 0 \\ \{(y/\sqrt{y^2+z^2}, z/\sqrt{y^2+z^2}\} & \text{if } y > 0, z > 0 \\ \{(0,1)\} & \text{if } y \le 0, y - z < 0 \\ \{(1,0)\} & \text{if } z \le 0, y - z > 0 \\ \mathrm{conv}\{(0,1),(1,0))\} & \text{if } y = z < 0 \end{cases}$$

To characterize the subdifferential of the function $\Delta_{-K_\eta}$, we need the following result about relations between the dual cones of $K_\eta$ and $K$.

**Proposition 2.10.** *Let* $\delta$ *be the scalar defined by (1) and* $K_\eta$ ($\eta \in {]}0, \delta[$) *be the cone defined by (2). Then for the cone* $K_\eta^+ := (K_\eta)^+$ *one has*

$$K_\eta^+ \setminus \{0\} \subset \{y^* \in K^{+i} \mid \inf_{\theta \in \Theta} y^*(\theta) \ge \eta \|y^*\|\} \subset K^{+si}$$

*where* $K^{+si} := \{y^* \in K^+ \mid \inf_{\theta \in \Theta} y^*(\theta) > 0\}$ *and if* $K$ *has a bounded base then*

$$K_\eta^+ \setminus \{0\} \subset \mathrm{int} K^+.$$

*Proof.* Let $y^* \in K_\eta^+ \setminus \{0\}$ and $\theta \in \Theta$. By the definition of the cone $K_\eta$, we have $\theta - \eta \mathbb{B}_Y \subset K_\eta$. For all $e \in \mathbb{B}_Y$ we have $y^*(\theta - \eta e) \ge 0$ or $y^*(\theta) \ge \eta y^*(e)$ and hence, $y^*(\theta) \ge \eta \|y^*\|$. It follows that $\inf_{\theta \in \Theta} y^*(\theta) \ge \eta \|y^*\| > 0$. Now assume that $\Theta$ is bounded. Then $\sup_{\theta \in \Theta} \|\theta\| = \rho > 0$. Let $y^* \in K_\eta^+ \setminus \{0\}$. We have to show that $y^* + r \mathbb{B}_{Y^*} \subset K^+$, where $r = \eta \|y^*\|/(2\rho)$. Let $e^* \in \mathbb{B}_{Y^*}$ be an arbitrary vector. For any $\theta \in \Theta$, we have

$$\begin{aligned}(y^* + re^*)(\theta) &\ge y^*(\theta) - \eta\|y^*\|/(2\rho)e^*(\theta) \ge \eta\|y^*\| - \eta\|y^*\|/(2\rho)\|e^*\|\|\theta\| \\ &\ge \eta\|y^*\| - \eta\|y^*\|/(2\rho)\rho = \eta\|y^*\|/2 > 0.\end{aligned}$$

Since $\Theta$ is a base of $K$, it follows that $y^* + re^* \in K^+$ and hence $y^* + r\mathbb{B}_{Y^*} \subset K^+$. $\qquad\square$

We are ready now to characterize the subdifferential of the function $\Delta_{-K_\eta}$.

**Corollary 2.11.** *Let* $\delta$ *and* $K_\eta$ ($\eta \in {]}0, \delta[$) *be the scalar and the cone defined by (1) and (2), respectively. Then* $\partial \Delta_{-K_\eta}(u) \subseteq K^{+si} \cap \mathbb{B}_{Y^*}$ *and if* $K$ *has a bounded base then* $\partial \Delta_{-K_\eta}(u) \subset \mathrm{int} K^+ \cap \mathbb{B}_{Y^*}$ *for any* $u \in Y$.

*Proof.* Observe that the pointed convex cone $K_\eta$ is closed and has a nonempty interior. The assertion follows from Propositions 2.10 and 2.8. $\qquad\square$

## 2.3 Existence of error bounds for a l.s.c. function

In this subsection, we recall some known criteria ensuring error bounds for a l.s.c. function that will be used to obtain similar ones for the case with a closed set-valued map.

Let $X$ be a complete metric space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a l.s.c. function.

**Definition 2.12.** *[4, Definition 2.2] (case $\beta = \infty$) For $\alpha \in \mathbb{R}$, we let $\sigma_\alpha(f)$ denote the supremum of the $\tau$'s in $[0, +\infty[$ such that*

$$\tau d(x; [f \le \alpha]) \le f(x) - \alpha \quad \text{for all } x \in [\alpha < f],$$

*with the convention $\sigma_\alpha(f) = 0$ if $[f \le \alpha] = \emptyset$ and $\sigma_\alpha(f) = \infty$ if $[f \le \alpha] = X$.*

**Definition 2.13.** *[10, Definition 1] For $\bar{x} \in domf$, we let $\mathrm{Er}f(\bar{x})$ denote the supremum of the $\tau$'s in $[0, +\infty[$ such that*

$$\tau d(x; [f \le f(\bar{x})]) \le [f(x) - f(\bar{x})]^+ \quad \text{for all } x \text{ near } \bar{x}.$$

In case of a l.s.c. function, the following conditions are known to provide sufficient and/or necessary criteria for the error bound property.

**Theorem 2.14.** *Let $X$ be a complete metric space, $f : X \to \mathbb{R} \cup \{\infty\}$ be a proper l.s.c. function. Assume that for all $x \in [\alpha < f]$ there exists $x' \in [\alpha \le f]$, $x' \ne x$ such that $f(x') + d(x; x') \le f(x)$. Then $[f \le \alpha] \ne \emptyset$, and $d(x; [f \le \alpha]) \le [f(x) - \alpha]^+$ for all $x \in domf$.*

**Theorem 2.15.** *Let $X$ be a Banach space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper l.s.c. function.*

*(a) Suppose that $X$ is an Asplund space. Then we have $\sigma_\alpha(f) \ge \inf_{x \in [\alpha < f]} d_M(0; \partial f(x))$.*

*(b) (convex case) Suppose that $f$ is convex. Then we have $\sigma_\alpha(f) = \inf_{x \in [\alpha < f]} d(0; \partial f(x))$.*

**Theorem 2.16.** *Let $X$ be a Banach space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper l.s.c. function. Let $\bar{x} \in domf$.*

*(a) Suppose that $X$ is an Asplund space. Then function $f$ has an error bound at $\bar{x}$ if $\liminf_{(x,f(x)) \to (\bar{x}, f(\bar{x}))} d_M(0; \partial f(x)) > 0$.*

*(b) (finite-dimensional case) Suppose that $X$ is a finite-dimensional space. Then the function $f$ has an error bound at $\bar{x}$ if $0 \notin \partial_M f(\bar{x})$,*

*(c) (convex case) Suppose that, in addition, $f$ is convex. The function $f$ has an error bound at $\bar{x}$ if $0 \notin \partial f(\bar{x})$. Moreover, we have*

$$\liminf_{(x,f(x)) \to (\bar{x}, f(\bar{x}))} d(0; \partial f(x)) > 0 \iff \mathrm{Er}f(\bar{x}) > 0.$$

Note that Theorem 2.14 is Theorem 1.3 of [3] , the assertions (a) of Theorem 2.15 is a special case of [4, Proposition 4.1, Corollary 4.1 and Remark 4.1 (b)] ($\beta = \infty$, $\gamma = \alpha$) and the assertion (b) of Theorem 2.15 is immediate from [4, Theorems 3.1 and 3.2] while conclusions of Theorem 2.16 are the conditions C6, C9, C13 and C14 of [10].

# 3 A sufficient condition for a global error bound of the distance to a sublevel set

From now on, unless otherwise specified, $Y$ is a normed space and $K \subset Y$ is a closed convex cone inducing the order relation $\preceq_l$ by (3). In this section, we assume that $X$ is a complete metric space and $A \subset Y$ is a nonempty set. A sufficient condition for the existence of a global error bound for the distance to the sublevel set $[F \preceq_l A]$ reads as follows.

**Theorem 3.1.** *Suppose that*

(i) $K_0 \subset K$ is a set such that $\inf_{k_0 \in K_0} d(k_0; Y \setminus K) \geq 1$.

(ii) The set $A$ is bounded.

(iii) The set-valued map $F$ is $K$-upper semicontinuous and has compact values.

(iv) For any $x \notin [F \preceq_l A]$ there exists $x' \notin [F \preceq_l A]$, $x' \neq x$ such that

$$F(x') + d(x, x')K_0 \preceq_l F(x).$$

Then $[F \preceq_l A] \neq \emptyset$ and

$$d(x; [F \preceq_l A]) \leq [\sup_{a \in A} \inf_{y \in F(x)} \Delta_{-K}(y - a)]^+ \quad \text{for all } x \in X. \tag{4}$$

Recall that $F$ is $K$-upper semicontinuous ($K$-u.s.c.) at $\bar{x} \in X$ [33] if for any open set $V$ such that $F(\bar{x}) \subset V$, there exists an open neighborhood $U$ of $\bar{x}$ such that $F(x) \subset V + K$ for any $x \in U$. It is clear that if $F$ is upper semicontinuous (u.s.c.) then it is $K$-u.s.c.

Note that (4) reduces to the classical scalar form

$$d(x; [f \leq a]) \leq [f(x) - a]^+ \quad \text{for all } x \in X$$

when $Y = \mathbb{R}$, $K = \mathbb{R}^1_+$, $F$ is a (single-valued) function $f : X \to \mathbb{R}$ and $A = \{a\}$.

To prove Theorem 3.1, we will apply a scalarization technique and Theorem 2.14. We associate to the map $F$ and the set $A$ a function $g : X \to \mathbb{R}$ defined by

$$g(x) := \sup_{a \in A} \inf_{y \in F(x)} \Delta_{-K}(y - a). \tag{5}$$

Some properties of the function $g$ is formulated in the following.

**Proposition 3.2.** (a) Suppose that $F(x)$ and $A$ are bounded sets. Then $-\infty < g(x) < +\infty$.

(b) If $F$ is $K$-u.s.c. at $\bar{x}$ then $g$ is l.s.c. at $\bar{x}$.

*Proof.* As the function $\Delta_{-K}$ is Lipschitz of rank 1, the assertion (a) follows from the boundedness of the sets $F(x)$ and $A$. To prove (b), we show that $g_a(x) := \inf_{y \in F(x)} \Delta_{-K}(y - a)$ is l.s.c. at $\bar{x}$ for every $a \in A$. Since $F$ is $K$-u.s.c. at $\bar{x}$, for $\epsilon > 0$ there exists $\delta > 0$ such that

$$F(x) \subset F(\bar{x}) + \epsilon \mathbb{B}_Y + K \quad \text{for any } x \in X \text{ such that } d(x; \bar{x}) \leq \delta.$$

Fix $x \in X$ such that $d(x; \bar{x}) \leq \delta$. For any $y \in F(x)$ there exist $e \in \mathbb{B}_Y$, $\bar{y} \in F(\bar{x})$ and $k \in K$ such that $y = \bar{y} - \epsilon e + k$. Properties of the signed distance function (see Proposition 2.7) imply

$$
\begin{aligned}
\Delta_{-K}(y - a) &= \Delta_{-K}(\bar{y} - \epsilon e + k - a) \geq \Delta_{-K}(\bar{y} - a) - \Delta_{-K}(\epsilon e - k) \geq \Delta_{-K}(\bar{y} - a) - \\
&\quad -\Delta_{-K}(\epsilon e) - \Delta_{-K}(-k) \geq \Delta_{-K}(\bar{y} - a) - \Delta_{-K}(\epsilon e) \geq \Delta_{-K}(\bar{y} - a) - \epsilon e \\
&\geq \inf_{y' \in F(\bar{x})} \Delta_{-K}(y' - a) - \epsilon = g_a(\bar{x}) - \epsilon,
\end{aligned}
$$

which gives $g_a(x) - g_a(\bar{x}) \geq -\epsilon$ for any $x \in X$ such that $d(x; \bar{x}) \leq \delta$. Therefore, $g_a$ is l.s.c. at $\bar{x}$ for every $a \in A$. Finally, since $g(x) = \sup_{a \in A} g_a(x)$, the lower semicontinuity of $g$ at $\bar{x}$ is immediate from the one of the functions $g_a$ at this point. $\square$

**Proposition 3.3.** (a) $F(x) \preceq_l A$ implies $g(x) \leq 0$.

(b) Suppose that $F(x)$ is compact. Then $g(x) \leq 0$ implies $F(x) \preceq_l A$.

*Proof.* (a) Fix $a \in A$. Since $F(x) \preceq_l A$, there exist $y \in F(x)$ and $k \in K$ such that $a = y + k$. Then we have $\Delta_{-K}(y - a) = \Delta_{-K}(-k) = -d(-k; Y \setminus (-K)) \leq 0$. It follows that $\inf_{y \in F(x)} \Delta_{-K}(y - a) \leq 0$ and hence, $g(x) = \sup_{a \in A} \inf_{y \in F(x)} \Delta_{-K}(y - a) \leq 0$.

(b) Suppose to the contrary that $g(x) \leq 0$ but $F(x) \npreceq_l A$ or $A \nsubseteq F(x) + K$. Then there exists $a \in A$ such that $a \notin F(x) + K$. For any $y \in F(x)$ one has $y - a \in Y \setminus (-K)$ and since $K$ is closed, one gets that $\Delta_{-K}(y - a) = d(y - a; -K) > 0$. Since $F(x)$ is compact, $\inf_{y \in F(x)} \Delta_{-K}(y - a) > 0$. Thus, we have $g_a(x) > 0$ for some $a \in A$, and hence we obtain $g(x) > 0$, a contradiction. $\square$

**Corollary 3.4.** *If $F$ has compact values then $[F \preceq_l A] = [g \leq 0]$.*

**Proposition 3.5.** *Let $K_0 \subset K$ be a set such that $\inf_{k_0 \in K_0} d(k_0; Y \setminus K) \geq 1$. Then*

$$F(x') + d(x; x')K_0 \preceq_l F(x) \quad \text{implies} \quad g(x') + d(x; x') \leq g(x).$$

*Proof.* Observe that for any $k_0 \in K_0$, we have $\Delta_{-K}(-k_0) = -d(-k_0; Y \setminus (-K)) = -d(k_0; Y \setminus K) \leq -1$. Now, suppose that $F(x') + d(x; x')K_0 \preceq_l F(x)$. By the definition, we have $F(x) \subseteq F(x') + d(x; x')K_0 + K$. Fix $a \in A$. For any $y \in F(x)$, there exist $y' \in F(x')$, $k_0 \in K_0$ and $k \in K$ such that $y = y' + d(x; x')k_0 + k$. The properties of the signed distance function (see Proposition 2.7) imply

$$\Delta_{-K}(y - a) = \Delta_{-K}(y' + d(x; x')k_0 + k - a) \geq \Delta_{-K}(y' - a) - \Delta_{-K}(-d(x; x')k_0) - \\ -\Delta_{-K}(-k) \geq \Delta_{-K}(y' - a) + d(x; x') \geq \inf_{u \in F(x')} \Delta_{-K}(u - a) + d(x; x').$$

Since $y \in F(x)$ and $a \in A$ are arbitrarily chosen, we obtain

$$\sup_{a \in A} \inf_{y \in F(x)} \Delta_{-K}(y - a) \geq \sup_{a \in A} \inf_{u \in F(x')} \Delta_{-K}(u - a) + d(x; x'),$$

which means that $g(x') + d(x; x') \leq g(x)$. $\square$

Now we are ready to prove Theorem 3.1.

*Proof.* Observe that by Proposition 3.2, the function $g$ is l.s.c. and by Proposition 3.5, the assumption (d) becomes

For any $x \in [g > 0]$ there exists $x' \in [g > 0]$, $x' \neq x$ such that $g(x') + d(x; x') \leq g(x)$.

Theorem 2.14 applied to the function $g$ implies that $[g \leq 0] \neq \emptyset$ and $d(x; [g \leq 0]) \leq [g(x)]^+$ for all $x \in X$. The assertion follows from Corollary 3.4 and (5). $\square$

# 4 Error bounds for the distances to a special sublevel set and to a constraint set

Throughout the section, unless otherwise stated, we assume that $X$ and $Y$ are Banach spaces. The sublevel set under consideration is of the form (here, $\preceq_l \bar{y}$ stands for $\preceq_l \{\bar{y}\}$)

$$[F \preceq_l \bar{y}] := \{x \in X \mid F(x) \preceq_l \bar{y}\}. \tag{6}$$

The class of sublevel sets of the form (6) covers the constraint set

$$\{x \in X \mid G(x) \cap (-\mathcal{C}) \neq \emptyset\}$$

and, as it will be shown in Section 5, some efficient solution sets of $(\mathcal{P})$. In this section, we formulate criteria for error bounds for the distance to the sublevel set and illustrate them by

examples about the constraint set and postfone the cases with efficient solution sets to Section 5.

In what follows, by $g$ we mean the scalarizing function

$$g(x) := \inf_{y \in F(x)} \Delta_{-K}(y - \bar{y})$$

and by $V(x)$ we mean the minimum set

$$V(x) := \{y \in F(x) \mid \Delta_{-K}(y - \bar{y}) = g(x)\}.$$

## 4.1 Global error bounds

Following Definition 2.12, the global error bound $\sigma(g)$ of the function $g$ at the level $\alpha = 0$ can be defined as follows

$$\sigma(g) := \sup\{\tau > 0 \mid \tau d(x; [g \leq 0]) \leq [g(x)]^+ \quad \text{for all } x \in X\}.$$

We define a global error bound for the set-valued map $F$ as follows.

**Definition 4.1.** *The global error bound $\sigma(F)$ for the set-valued map $F$ at the level $\bar{y}$ is the supremum of the $\tau$'s in $[0, +\infty[$ such that*

$$\tau d(x; [F \preceq_l \bar{y}]) \leq [\inf_{y \in F(x)} \Delta_{-K}(y - \bar{y})]^+ \quad \text{for all } x \in X,$$

*with the convention $\sigma(F) = 0$ if $[F \preceq_l \bar{y}] = \emptyset$ and $\sigma(F) = \infty$ if $[F \preceq_l \bar{y}] = X$.*

**Remark 4.2.** *Assume that $F$ is compact-valued. If $x \in [F \preceq_l \bar{y}]$ then Corollary 3.4 implies that $g(x) \leq 0$. If $x \notin [F \preceq_l \bar{y}]$ then it is easy to see that $g(y) = \inf_{y \in F(x)} d_{-K}(y - \bar{y}) > 0$. Therefore, $\sigma(F)$ can be equivalently defined as the supremum of the $\tau$'s in $[0, +\infty[$ such that*

$$\tau d(x; [F \preceq_l \bar{y}]) \leq \inf_{y \in F(x)} d_{-K}(y - \bar{y}) \quad \text{for all } x \notin [F \preceq_l \bar{y}].$$

Below is a simple but useful relation between the error bound of $F$ and that of $g$.

**Proposition 4.3.** *Suppose that $F$ has compact values. Then*

$$\sigma(F) = \sigma(g). \tag{7}$$

*Proof.* Corollary 3.4 implies that $[F \preceq_l \bar{y}] = [g \leq 0]$. The assertion follows from this equality and the definitions of $g$, $\sigma(g)$ and $\sigma(F)$. $\square$

The first estimate for $\sigma(F)$ in terms of the Mordukhovich coderivative reads as follows.

**Theorem 4.4.** *Suppose that $X$, $Y$ are Asplund spaces, $F$ is closed $K$-u.s.c. compact-valued and $V$ is lower semicompact. Then*

$$\sigma(F) \geq \inf_{x \notin [F \preceq_l \bar{y}]} d(0; \cap_{y_x \in V(x)} \cup_{y^* \in \partial_M \Delta_{-K}(y_x - \bar{y})} D^*_M F(x, y_x)(y^*)). \tag{8}$$

*Proof.* Note that under the assumptions of the theorem, the function $g$ is l.s.c. by Proposition 3.2. Applying Theorem 2.15 (a) to the function $g$, we obtain

$$\sigma(g) \geq \inf_{x \in [g > 0]} d(0; \partial_M g(x)).$$

On the other hand, by Proposition 2.4, we have

$$\partial_M g(x) \subseteq \cap_{y_x \in V(x)} \cup_{y^* \in \partial \Delta_{-K}(y_x - \bar{y})} D_M^* F(x, y_x)(y^*).$$

Therefore, by Corollary 3.4 we have

$$\begin{aligned}
\sigma(g) &\geq \inf_{x \in [g>0]} d(0; \cap_{y_x \in V(x)} \cup_{y^* \in \partial \Delta_{-K}(y_x - \bar{y})} D_M^* F(x, y_x)(y^*)) \\
&= \inf_{x \notin [F \preceq_l \bar{y}]} d(0; \cap_{y_x \in V(x)} \cup_{y^* \in \partial \Delta_{-K}(y_x - \bar{y})} D_M^* F(x, y_x)(y^*)),
\end{aligned}$$

which together with (7) yield (8).  □

We will illustrate the results obtained in this section by examples with the constraint set $\{x \in X \mid G(x) \cap (-\mathcal{C}) \neq \emptyset\} = [G \preceq_l 0]$, where $\preceq_l$ is induced by the cone $\mathcal{C}$ in $2^Z$ in the way similar to (3). For the reader's convenience, we do not change either $F$ by $G$ or $K$ by $\mathcal{C}$.

**Example 4.5.** *Let* $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $K = \mathbb{R}_+^2$. *Consider a set-valued map*

$$F(x) = \begin{cases} \{(u, v) \mid (u - x - 1)^2 + v^2 \leq 1\} & \text{if } x \geq 0 \\ \{(u, v) \mid (u - 1)^2 + v^2 \leq 1\} & \text{otherwise} \end{cases}$$

*One can check that $F$ is u.s.c. and compact-valued. The graph of $F$ is a cylinder "broken" at the "height" $x = 0$, i.e. it is not convex but is locally convex around $(x, y) \in \mathrm{gr} F$ for any $x > 0$ and $y = (x, 0) \in F(x)$. Consider the constraint set*

$$\mathcal{A} := \{x \in X \mid F(x) \cap (-\mathbb{R}_+^2) \neq \emptyset\} = \{x \in X \mid F(x) \preceq_l (0, 0)\}$$

*(here, $\bar{y} = (0, 0)$). Observe that $\mathcal{A} = ]-\infty, 0]$. Assume that $x \notin \mathcal{A}$. One can check that $V(x) = \{(x, 0)\}$, $y_x = (x, 0)$, $\partial \Delta_{-\mathbb{R}_+^2}(y_x) = \{(1, 0)\}$ (see Example 2.9) and the normal cone of convex analysis to the graph of $F$ at $(x, y_x)$ is given by $N((x, y_x); \mathrm{gr} F) = \{t(1, -1, 0) \mid t \in \mathbb{R}_+\}$. Recall that the Mordukhovich coderivative $D_M^* F(x, y_x)(y^*)$ coincides with the coderivative $D^* F(x, y_x)(y^*)$ of convex analysis. Hence,*

$$\cup_{y^* \in \partial \Delta_{-\mathbb{R}_+^2}(y_x - \bar{y})} D^* F(x, y_x)(y^*) = \{1\}$$

*and Theorem 4.4 implies that $\sigma(F) \geq 1$.*

Let us go to the convex case. Recall that the set-valued map $F$ is convex if its graph is convex and is $K$-convex if for any $x_1, x_2 \in X$ and $\lambda \in [0, 1]$ one has

$$\lambda F(x_1) + (1 - \lambda) F(x_2) \subseteq F(\lambda x_1 + (1 - \lambda) x_2) + K.$$

$(F(\lambda x_1 + (1 - \lambda) x_2) \leq_K \lambda F(x_1) + (1 - \lambda) F(x_2)$ when $F$ is single-valued).

The following proposition shows that $g$ inherits the convexity property of $F$.

**Proposition 4.6.** *Suppose that $F$ is either convex or $K$-convex. Then the function $g$ is convex.*

*Proof.* Let $x_1, x_2 \in X$ and $\lambda \in [0, 1]$ be given. Denote $x_\lambda = \lambda x_1 + (1 - \lambda) x_2$. Suppose first that $F$ is convex. Let $\epsilon > 0$ be an arbitrary scalar. Let $y_1 \in F(x_1)$ and $y_2 \in F(x_2)$ be such that $g(x_1) + \epsilon/2 \geq \Delta_{-K}(y_1 - \bar{y})$ and $g(x_2) + \epsilon/2 \geq \Delta_{-K}(y_2 - \bar{y})$. Denote $y_\lambda = \lambda y_1 + (1 - \lambda) y_2$. As $F$ is convex, we have $y_\lambda \in F(x_\lambda)$. Sine the function $\Delta_{-K}$ is convex (see Proposition 2.7(c)), we obtain

$$\begin{aligned}
g(x_\lambda) &= \inf_{y' \in F(x_\lambda)} \Delta_{-K}(y' - \bar{y}) \leq \Delta_{-K}(y_\lambda - \bar{y}) = \Delta_{-K}(\lambda y_1 + (1 - \lambda) y_2 - \bar{y}) \\
&\leq \lambda \Delta_{-K}(y_1 - \bar{y}) + (1 - \lambda) \Delta_{-K}(y_2 - \bar{y}) \leq \lambda [g(x_1) + \epsilon/2] + (1 - \lambda)[g(x_2) + \epsilon/2] \\
&\leq \lambda g(x_1) + (1 - \lambda) g(x_2) + \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrarily chosen, we deduce that $g(\lambda x_1 + (1 - \lambda)x_2) = g(x_\lambda) \le \lambda g(x_1) + (1 - \lambda)g(x_2)$, which means that the function $g$ is convex. Now, suppose that $F$ is $K$-convex. By the definition, for any $y_1 \in F(x_1)$ and $y_2 \in F(x_2)$ there exist $y_\lambda \in F(x_\lambda)$ and $k \in K$ such that $\lambda y_1 + (1 - \lambda)y_2 = y_\lambda + k$. The convexity and the triangle inequality of the function $\Delta_{-K}$ (see Proposition 2.7(c)) imply

$$
\begin{aligned}
g(x_\lambda) &= \inf_{y' \in F(x_\lambda)} \Delta_{-K}(y' - \bar{y}) \le \Delta_{-K}(y_\lambda - \bar{y}) \\
&\le \Delta_{-K}(y_\lambda - \bar{y}) + d_{Y \setminus (-K)}(-k) = \Delta_{-K}(y_\lambda - \bar{y}) - \Delta_{-K}(-k) \\
&\le \Delta_{-K}(y_\lambda - \bar{y} + k) = \Delta_{-K}(\lambda y_1 + (1 - \lambda)y_2 - \bar{y}) \\
&\le \lambda \Delta_{-K}(y_1 - \bar{y}) + (1 - \lambda)\Delta_{-K}(y_2 - \bar{y}).
\end{aligned}
$$

Since $y_1 \in F(x_1)$ and $y_2 \in F(x_2)$ are arbitrarily chosen, we deduce that $g(\lambda x_1 + (1 - \lambda)x_2) = g(x_\lambda) \le \lambda g(x_1) + (1 - \lambda)g(x_2)$, which means that the function $g$ is convex. $\qquad\square$

Let us formulate an estimate in terms of coderivative of convex analysis for the global error bound $\sigma(F)$ in the convex case.

**Theorem 4.7.** *Suppose that one of the following conditions is satisfied.*

*(i) The set-valued map $F$ is u.s.c. convex/$K$-convex and compact-valued.*

*(ii) $X$ is reflexive and the set-valued map $F$ is $K$-u.s.c. closed convex/$K$-convex and compact-valued.*

*Then for any $y_x \in V(x)$ we have*

$$\sigma(F) = \inf_{x \notin [F \preceq_l \bar{y}]} d(0; \cup_{y^* \in \partial \Delta_{-K}(y_x - \bar{y})} D^* F(x, y_x)(y^*)). \tag{9}$$

*Proof.* Note that under the assumptions of the theorem, the function $g$ is l.s.c. by Proposition 3.2 and is convex by Proposition 4.6. Applying Theorem 2.15(b) to the function $g$, we obtain $\sigma(g) = \inf_{x \in [g > 0]} d(0; \partial g(x))$. On the other hand, by Proposition 2.5, for any $y_x \in V(x)$ we have $\partial g(x) = \cup_{y^* \in \partial \Delta_{-K}(y_x - \bar{y})} D^* F(x, y_x)(y^*)$. Corollary 3.4 implies

$$
\begin{aligned}
\sigma(g) &= \inf_{x \in [g > 0]} d(0; \cup_{y^* \in \partial \Delta_{-K}(y_x - \bar{y})} D^* F(x, y_x)(y^*)) \\
&= \inf_{x \notin [F \preceq_l \bar{y}]} d(0; \cup_{y^* \in \partial \Delta_{-K}(y_x - \bar{y})} D^* F(x, y_x)(y^*)),
\end{aligned}
$$

which together with (7) yield (9). $\qquad\square$

**Remark 4.8.** *Note that for any $y^*$ in the right-hand side of (8) and (9) one has*

$$y^* \in (K^* \setminus \{0\}) \cap \mathbb{B}_{Y^*}. \tag{10}$$

*Indeed, when $x \notin [F \preceq_l \bar{y}]$, Corollary 3.4 implies that $g(x) > 0$ and, therefore, $\Delta_{-K}(y_x - \bar{y}) = g(x) > 0$ as $y_x \in V(x)$. Proposition 2.7(b) yields that $y_x - \bar{y} \notin -K$ and the desired relation (10) follows from Proposition 2.8.*

**Example 4.9.** *Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $K = \mathbb{R}_+^2$. Consider a set-valued map*

$$F(x) = \{(u, v) \mid (u - x - 1)^2 + v^2 \le 1\}.$$

*One can check that $F$ is u.s.c. and compact-valued. The graph of $F$ is a "leaning" cylinder, i.e. $F$ is convex. Consider the constraint set*

$$\mathcal{A} := \{x \in X \mid F(x) \cap (-\mathbb{R}_+^2) \ne \emptyset\} = \{x \in X \mid F(x) \preceq_l (0, 0)\}$$

*(here, $\bar{y} = (0, 0)$). Observe that $\mathcal{A} = ]-\infty, 0]$. For $x \notin \mathcal{A}$ one has $V(x) = \{(x, 0)\}$, $y_x = (x, 0)$, $\Delta_{-\mathbb{R}_+^2}(y_x) = \{(1, 0)\}$ (see Example 2.9) and $N((x, y_x); \mathrm{gr} F) = \{t(1, -1, 0) \mid t \in \mathbb{R}_+\}$. Hence, $\cup_{y^* \in \partial \Delta_{-\mathbb{R}_+^2}(y_x - \bar{y})} D^* F(x, y_x)(y^*) = \{1\}$. Theorem 4.7 implies that $\sigma(F) = 1$.*

Next, we consider the case with a single- vector-valued map $f : X \to Y$. Denote

$$[f \leq_K \bar{y}] := \{x \in X \mid f(x) \leq_K \bar{y}\}.$$

**Definition 4.10.** *For $\bar{y} \in Y$, we let $\sigma_{\bar{y}}(f)$ denote the supremum of the $\tau$'s in $[0, +\infty[$ such that*

$$\tau d(x; [f \leq_K \bar{y}]) \leq [\Delta_{-K}(f(x) - \bar{y})]^+ \quad \text{for all } x \in X$$

*with the convention $\sigma_{\bar{y}}(f) = 0$ if $[f \leq_K \bar{y}] = \emptyset$ and $\sigma_{\bar{y}}(f) = \infty$ if $[f \leq_K \bar{y}] = X$.*

In such a case, the scalarizing function $g$ has the form $g(x) = \Delta_{-K}(f(x) - \bar{y})$. Applying the chain rule recalled in Theorem 2.3 and using the same techniques as above, one can easily derive from Theorems 2.15 the following result.

**Theorem 4.11.** *Assume that $f$ is strictly differentiable. If $X$, $Y$ are Asplund spaces, then*

$$\sigma_{\bar{y}}(f) \geq \inf_{x \notin [f \leq_K \bar{y}]} d_M(0; \{[\nabla f(x)]^*(y^*) \mid y^* \in \partial\Delta_{-K}(f(x) - \bar{y})\})$$

*and if $f$ is $K$-convex, then*

$$\sigma_{\bar{y}}(f) = \inf_{x \notin [f \leq_K \bar{y}]} d(0; \{[\nabla f(x)]^*(y^*) \mid y^* \in \partial\Delta_{-K}(f(x) - \bar{y})\}).$$

**Example 4.12.** *Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$.*
*(a) Consider a function $f(x) = (x^3, x)$. It is easy to see that $f$ is strictly differentiable (but it is not $\mathbb{R}^2_+$-convex), $\nabla f(x) = (3x^2, 1)$ and $\mathcal{A} := [f \leq_{\mathbb{R}^2_+} (0, 0)] =] - \infty, 0]$. Assume that $x \notin \mathcal{A}$, i.e. $x > 0$. Then $f(x) - (0, 0) = (x^3, x)$ and*

$$\partial\Delta_{-\mathbb{R}^2_+}(f(x)) = \{(x^3/\sqrt{x^6 + x^2}, x/\sqrt{x^6 + x^2})\} = \{(x^2/\sqrt{x^4 + 1}, 1/\sqrt{x^4 + 1})\}$$

*(see Example 2.9). One can check that $U := \{[\nabla f(x)]^*(y^*) \mid y^* \in \partial\Delta_{-\mathbb{R}^2_+}(f(x))\} = \{(3x^4 + 1)/\sqrt{x^4 + 1}\}$ and $d(0; U) = (3x^4 + 1)/\sqrt{x^4 + 1} \geq 1$ for all $x \notin \mathcal{A}$ . Theorem 4.11 (a) implies the existence of $\tau \in ]0, 1[$ such that*

$$\tau d(x; [f \leq_K 0]) \leq [\Delta_{-\mathbb{R}^2_+}(f(x))]^+ \quad \text{for all } x \in \mathbb{R}.$$

*(b) Consider a function*

$$f(x) = \begin{cases} (e^x - 5, -\ln x) & \text{if } x \geq 1 \\ (e^x - 5, 1 - x) & \text{otherwise} \end{cases}$$

*It is easy to see that $f$ is $\mathbb{R}^2_+$-convex strictly differentiable and $\mathcal{A} := [f \leq_{\mathbb{R}^2_+} (0, 0)] = [1, \ln 5]$; moreover, $\nabla f(x) = (e^x, -1/x)$ if $x \geq 1$ and $\nabla f(x) = (e^x, -1)$, otherwise. Assume that $x \notin \mathcal{A}$. If $x < 1$, then $\partial\Delta_{-\mathbb{R}^2_+}(f(x)) = \{(0, 1)\}$ (see Example 2.9) and $U := \{[\nabla f(x)]^*(y^*) \mid y^* \in \partial\Delta_{-\mathbb{R}^2_+}(f(x))\} = \{\langle (e^x, -1), (0, 1)\rangle\} = \{-1\}$. If $x > \ln 5$, then $\partial\Delta_{-\mathbb{R}^2_+}(f(x))\} = \{(1, 0)\}$ (see Example 2.9) and $U = \{\langle (e^x, -1/x), (1, 0)\rangle\} = \{e^x\}$. It follows from Theorem 4.11 (b) that $\sigma_{(0,0)}(f) = \inf_{x \notin [f \leq_{\mathbb{R}^2_+} (0,0)]} d(0; U) = 1$.*

## 4.2 Local error bounds

We introduce a concept of a local error bound for the set-valued map $F$ as follows.

**Definition 4.13.** *Let $\bar{y} \in F(\bar{x})$. We say that $F$ satisfies the (local) error bound property at $\bar{x} \in domF$ w.r.t. $\bar{y}$ if there exists a positive real number $\tau$ such that*

$$\tau d(x; [F \preceq_l \bar{y}]) \leq [\inf_{y \in F(x)} \Delta_{-K}(y - \bar{y})]^+ \quad \text{for all } x \text{ near } \bar{x}$$

*and the local error bound $\mathrm{Er}F(\bar{x}, \bar{y})$ of $F$ near $\bar{x}$ w.r.t. $\bar{y}$ is the supremum of such scalars $\tau$.*

Recall that the function $g$ with $g(\bar{x}) = 0$ is said to satisfy the (local) error bound property at $\bar{x}$ if there exists a positive real number $\tau$ such that

$$\tau d(x; [g \leq 0]) \leq [g(x)]^+ \quad \text{for all } x \text{ near } \bar{x}$$

and its local error bound $\mathrm{Er}g(\bar{x})$ near $\bar{x}$ is the supremum of such scalars $\tau$, see Definition 2.13. Let us formulate a simple but important fact.

**Proposition 4.14.** *Suppose that $F$ has compact values and $\bar{y} \in V(\bar{x})$. Then*

$$\mathrm{Er}F(\bar{x}, \bar{y}) = \mathrm{Er}g(\bar{x}). \tag{11}$$

*Proof.* Observe that $\bar{y} \in V(\bar{x})$ implies $g(\bar{x}) = 0$. Corollary 3.4 yields that $[F \preceq_l \bar{y}] = [g \leq 0] = [g \leq g(\bar{x})]$. The assertion is immediate from the definitions of $g$, $\mathrm{Er}F(\bar{x}, \bar{y})$ and $\mathrm{Er}g(\bar{x})$. $\square$

First criterion in terms of the Mordukhovich coderivative for the local error bound property reads as follows.

**Theorem 4.15.** *Suppose that $X$ and $Y$ are Asplund spaces, the set-valued map $F$ is closed $K$-u.s.c. compact-valued and $V$ is lower semicompact. Suppose further that $\bar{y} \in V(\bar{x})$. Then*

*(a) The set-valued map $F$ satisfies the local error bound property at $\bar{x}$ w.r.t. $\bar{y}$ if*

$$\liminf_{x \to \bar{x}, y_x \in V(x), \Delta_{-K}(y_x - \bar{y}) \to 0} d(0; \cup_{y^* \in \partial \Delta_{-K}(y_x - \bar{y})} D_M^* F(x, y_x)(y^*)) > 0. \tag{12}$$

*(b) If in addition, $X$ is a finite-dimensional space, then the set-valued map $F$ satisfies the local error bound property at $\bar{x}$ w.r.t. $\bar{y}$ if*

$$0 \notin \cap_{y_{\bar{x}} \in V(\bar{x})} \cup_{y^* \in \partial \Delta_{-K}(y_{\bar{x}} - \bar{y})} D_M^* F(\bar{x}, y_{\bar{x}})(y^*). \tag{13}$$

*Proof.* Note that under the assumptions of the theorem, the map $F$ is closed and the function $g$ is l.s.c. by Proposition 3.2. Proposition 2.4 implies that

$$\partial_M g(x) \subseteq \cup_{y^* \in \partial \Delta_{-K}(y_x - \bar{y})} D_M^* F(x, y_x)(y^*)$$

for any $y_x \in V(x)$ in the case (a) and

$$\partial_M g(\bar{x}) \subseteq \cap_{y_{\bar{x}} \in V(\bar{x})} \cup_{y^* \in \partial \Delta_{-K}(y_{\bar{x}} - \bar{y})} D_M^* F(x, y_{\bar{x}})(y^*)$$

in the case (b). The relation (12) and (13) yield $\liminf_{x \to \bar{x}, g(x) \to 0} d(0; \partial_M g(x)) > 0$. in the first case and $0 \notin \partial_M g(\bar{x})$) in the second case. In both cases, $g$ satisfies the local error bound property at $\bar{x}$ by Theorem 2.16 and hence, the relation (11) implies that $F$ satisfies the local error bound property at $\bar{x}$ w.r.t. $\bar{y}$. $\square$

**Remark 4.16.** *It should be noted that the relation (13) holds only when the cone $K$ has a nonempty interior (for instance, when $K$ is the nonnegative orthant in $\mathbb{R}^n$). Indeed, suppose that $\mathrm{int}\,K = \emptyset$. For any $y_{\bar{x}} \in V(\bar{x})$, one has $\Delta_{-K}(y_{\bar{x}} - \bar{y}) = g(\bar{x}) = 0$ and Proposition 2.7 (b) yields that $y_{\bar{x}} - \bar{y} \in -K$. Proposition 2.8 (c) then implies that $0 \in \partial \Delta_{-K}(y_{\bar{x}} - \bar{y})$. Since $0 \in D_M^* F(\bar{x}, y_{\bar{x}})(0)$, we have $0 \in \cup_{y^* \in \partial \Delta_{-K}(y_{\bar{x}} - \bar{y})} D_M^* F(\bar{x}, y_{\bar{x}})(y^*)$ and (13) cannot hold.*

Below is an example illustrating Theorem 4.15 in case of the constraint set.

**Example 4.17.** *(a) Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $K = \mathbb{R}_+^2$ and $F$ be the map*

$$F(x) = \begin{cases} \{(u,v) \mid (u - x - 1)^2 + v^2 \le 1\} & \text{if } x \in [0, \infty[ \\ \{(u,v) \mid (u + x - 1)^2 + v^2 \le 1\} & \text{if } x \in [-1, 0[ \\ \{(u,v) \mid (u - x - 3)^2 + v^2 \le 1\} & \text{if } x \in [-2, -1[ \\ \{(u,v) \mid (u - 1)^2 + v^2 \le 1\} & \text{if } x \in ]-\infty, -2[ \end{cases}$$

*One can check that $F$ is u.s.c. and compact-valued. Consider the constraint set*

$$\mathcal{A} := \{x \in X \mid F(x) \cap (-\mathbb{R}_+^2) \ne \emptyset\} = \{x \in X \mid F(x) \preceq_l (0,0)\}$$

*(here, $\bar{y} = (0,0)$). Observe that $\bar{x} = 0 \in \mathcal{A} = \{0\} \cup [-2, \infty]$.*
*We show first that (12) holds. Observe that for $x > 0$, one has $g(x) = x$, $V(x) = \{(x,0)\}$, $y_x = (x,0)$, $\partial \Delta_{-\mathbb{R}_+^2}(y_x - \bar{y}) = \partial \Delta_{-\mathbb{R}_+^2}(x,0) = \{(1,0)\}$ (see Example 2.9), the graph of $F$ is locally convex near $(x, y_x)$ and $D^* F(\bar{x}, \bar{y})(1,0) = \{1\}$. Further, for $x \in ]0, -1[$, one has $g(x) = -x$, $V(x) = \{(-x, 0)\}$, $y_x = (-x, 0)$, $\partial \Delta_{-\mathbb{R}_+^2}(y_x - \bar{y}) = \partial \Delta_{-\mathbb{R}_+^2}(-x, 0) = \{(1,0)\}$ (see Example 2.9), the graph of $F$ is locally convex near $(x, y_x)$ and $D^* F(\bar{x}, \bar{y})(1,0) = \{-1\}$. Hence,*

$$\liminf_{x \to \bar{x}, y_x \in V(x), \Delta_{-K}(y_x - \bar{y}) \to 0} d(0; \cup_{y^* \in \partial \Delta_{-K}(y_x - \bar{y})} D_M^* F(x, y_x)(y^*)) = 1.$$

*Theorem 4.15 yields that $F$ satisfies the local error bound property at $\bar{x} = 0$ w.r.t. $\bar{y} = (0,0)$.*
*Next, we show that (13) does not hold. Indeed, one can check that $V(\bar{x}) = \{(0,0)\}$, $y_{\bar{x}} = (0,0)$ and $N((\bar{x}, \bar{y}); \mathrm{gr} F) = \{t_1(1, -1, 0) - t_2(1, 1, 0) \mid t_1, t_2 \in \mathbb{R}_+\}$. Recall that $D_M^* F(\bar{x}, \bar{y})(y^*) = D^* F(\bar{x}, \bar{y})(y^*)$. Since $(0, -1, 0) \in N((\bar{x}, \bar{y}); \mathrm{gr} F)$ and $(1, 0) \in \partial \Delta_{-\mathbb{R}_+^2}(y_x - \bar{y}) = \partial \Delta_{-\mathbb{R}_+^2}(0,0)$ (see Example 2.9), we have $0 \in D^* F(\bar{x}, \bar{y})(1, 0)$.*
*(b) Let $F$ be the set-valued map of Example 4.5. We are interested in the local error bound property of $F$ at $\bar{x} = 0$ w.r.t. $\bar{y} = (0,0)$ or, in other words, in the existence of a local error bound for the distance to the constraint set*

$$\mathcal{A} := \{x \in X \mid F(x) \cap (-\mathbb{R}_+^2) \ne \emptyset\} = [F \preceq_l (0,0)]$$

*at $\bar{x} = 0$. We have $\bar{y} = (0,0) \in V(\bar{x}) = \{(0,0)\}$ and $\Delta_{-\mathbb{R}_+^2}(\bar{y} - \bar{y}) = \{(u,v) \mid u^2 + v^2 \le 1 \le u + v\}$ (see Example 2.9). Observe that the graph of $F$ is locally convex near $(\bar{x}, \bar{y}) = (0, 0, 0)$ and the Mordukhovich normal cone at that point coincides with the normal cone of convex analysis $N((0,0,0); \mathrm{gr} F) = \{t(1, -1, 0) \mid t \in \mathbb{R}^+\}$. Hence, $0 \notin \cup_{y^* \in \partial \Delta_{-\mathbb{R}_+^2}(0,0)} D^* F(0,0,0)(y^*)$. Thus, the relation (13) holds and Theorem 4.15 implies that $F$ satisfies the local error bound property at $\bar{x} = 0$ w.r.t. $\bar{y} = (0,0)$.*

Next, we consider the convex case.

**Theorem 4.18.** *Suppose that one of the following conditions is satisfied.*

*(i) The set-valued map $F$ is u.s.c., convex/K-convex and has compact values.*

*(ii) $X$ is reflexive and the set-valued map $F$ is closed K-u.s.c. convex/K-convex and compact-valued.*

*Suppose further that* $\bar{y} \in V(\bar{x})$. *Then the set-valued map* $F$ *satisfies the local error bound property at* $\bar{x}$ *w.r.t.* $\bar{y}$ *if*

$$0 \notin \cup_{y^* \in \partial \Delta_{-K}(y_{\bar{x}} - \bar{y})} D^* F(\bar{x}, y_{\bar{x}})(y^*), \tag{14}$$

*for some* $y_{\bar{x}} \in V(\bar{x})$. *Moreover, we have*

$$\liminf_{y_x \in V(x), x \to \bar{x}, \Delta_{-K}(y_x - \bar{y}) \to 0} d(0, \cup_{y^* \in \partial \Delta_{-K}(y_x - \bar{y})} D^* F(x, y_x)(y^*)) > 0 \iff \operatorname{Er} F(\bar{x}, \bar{y}) > 0. \tag{15}$$

*Proof.* Note that under the assumptions of the theorem, the function $g$ is l.s.c. by Proposition 3.2 and is convex by Proposition 4.6. Observe that by Proposition 2.5, we have

$$\partial g(\bar{x}) = \cup_{y^* \in \partial \Delta_{-K}(y_x - \bar{y})} D^* F(\bar{x}, y_x)(y^*),$$

for any $y_{\bar{x}} \in V(\bar{x})$. Therefore, the relation (14) implies $0 \notin \partial g(\bar{x})$. We also have

$$\liminf_{x \to \bar{x}, g(x) \to 0} d(0; \partial g(x)) =$$
$$= \liminf_{x \to \bar{x}, y_x \in V(x), \Delta_{-K}(y_x - \bar{y}) \to 0} d(0; \cup_{y^* \in \partial \Delta_{-K}(y_x - \bar{y})} D^* F(x, y_x)(y^*)).$$

The assertions follow from Theorem 2.16 (c) applied to the function $g$ and the relation (11). $\square$

**Remark 4.19.** *By the reasons mentioned in Remark 4.16, the relation (14) holds only when the interior of $K$ is nonempty, for instance, when $K$ is the nonnegative orthant in $\mathbb{R}^n$ or $C_{[0,1]}$ [28]. Another examples of cones with a nonempty interior are Bishop-Phelps cones in Banach spaces, which are representable in the form $K = \{y \in Y \mid \phi(y) \geq t\|y\|\}$ for some functional $\phi \in Y^*$ with $\|\phi\| > 1$ and some scalar $t > 0$ [26].*

Let us illustrate Theorem 4.18 by an example.

**Example 4.20.** *Let us return to the set-valued map $F$ in Example 4.5. We are interested in the local error bound property of $F$ at $\bar{x} = 0$ w.r.t. $\bar{y} = (0,0)$ or, in other words, in the existence of a local error bound for the distance to the constraint set*

$$\mathcal{A} := \{x \in X \mid F(x) \cap (-\mathbb{R}_+^2) \neq \emptyset\} = [F \preceq_l (0,0)]$$

*at $\bar{x} = 0$ (the order relation $\preceq_l$ is induced by the cone $\mathbb{R}_+^2$). In this case we have $\bar{y} = (0,0) \in V(\bar{x}) = \{(0,0)\}$ and $\Delta_{-\mathbb{R}_+^2}(\bar{y} - \bar{y}) = \{(u,v) \mid u^2 + v^2 \leq 1 \leq u + v\}$ (see Example 2.9). One can check that $N((0,0,0); \operatorname{gr} F) = \{t(1,-1,0) \mid t \in \mathbb{R}^+\}$. Hence, $0 \notin \cup_{y^* \in \partial \Delta_{-\mathbb{R}_+^2}(0,0)} D^* F(0,0,0)(y^*)$. Theorem 4.18 implies that $F$ satisfies the local error bound property at $\bar{x}$ w.r.t. $\bar{y}$.*

We consider now the case with a single- vector-valued map $f : X \to Y$.

**Definition 4.21.** *$f$ is said to satisfy the local error bound property at $\bar{x}$ if there exists a positive scalar $\tau$ such that*

$$\tau d(x; [f \leq_K f(\bar{x})]) \leq [\Delta_{-K}(f(x) - f(\bar{x}))]^+ \quad \text{for all } x \text{ near } \bar{x}$$

*and its local error bound $\operatorname{Er} f(\bar{x})$ near $\bar{x}$ is the supremum of such scalars $\tau$.*

In this case, the scalarizing function $g$ has the form $g(x) := \Delta_{-K}(f(x) - f(\bar{x}))$. Applying the chain rule recalled in Theorem 2.3 and the arguments used above, one can easily derive from Theorem 2.16 the following result.

**Theorem 4.22.** *Assume that $f$ is strictly differentiable.*

(a) *Assume that X,Y are Asplund spaces. Then f satisfies the local error bound property at $\bar{x}$ if*

$$\liminf_{x \to \bar{x}} d_M(0; \{[\nabla f(x)]^*(y^*) \mid y^* \in \partial \Delta_{-K}(f(x) - f(\bar{x}))\}) > 0.$$

(b) *Assuming that K has a nonempty interior, if X is a finite-dimensional space or if f is K-convex, then f satisfies the local error bound property at $\bar{x}$ if*

$$0 \notin \{[\nabla f(\bar{x})]^*(y^*) \mid y^* \in \partial \Delta_{-K}(0)\}.$$

(c) *If f is K-convex, we have*

$$\liminf_{x \to \bar{x}} d(0; \{[\nabla f(x)]^*(y^*) \mid y^* \in \partial \Delta_{-K}(f(x) - f(\bar{x}))\}) > 0 \iff \mathrm{Er} f(\bar{x}) > 0.$$

We assuming that $K$ has a nonempty interior in the assertion (b) of Theorem 4.22 in order to ensure that $0 \notin \partial \Delta_{-K}(0)$. Below we present an example illustrating this theorem.

**Example 4.23.** *Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $K = \mathbb{R}_+^2$.*
*(a) Consider a function*

$$f(x) = \begin{cases} (2\sqrt{x+1} - 2, x) & \text{if } x \geq 0 \\ (x, x) & \text{otherwise} \end{cases}$$

*We are interested in the local error bound property of f at $\bar{x} = 0$. It is easy to see that f is strictly differentiable (but it is not $\mathbb{R}_+^2$-convex) and $0 \notin \{[\nabla f(0)]^*(y^*) \mid y^* \in \partial \Delta_{-\mathbb{R}_+^2}(0,0)\}$. Theorem 4.22 (b) implies that f satisfies the error bound property of f at $\bar{x} = 0$.*
*(b) Let us return to the function f in Example 4.12(b). We are interested in the error bound property of f at $\bar{x} = 1$. One can check that*

$$\{[\nabla f(1)]^*(y^*) \mid y^* \in \partial \Delta_{-\mathbb{R}_+^2}(0,0)\} = \{\langle (e, -1), (u, v) \rangle \mid u^2 + v^2 \leq 1 \leq u + v\}.$$

*Since $0 = \langle (e, -1), (u, v) \rangle$ with $u = 1/2[1/(1+e) + 1/\sqrt{1+e^2}]$, $v = e/2[1/(1+e) + 1/\sqrt{1+e^2}]$ and $(u, v) \in \partial \Delta_{-\mathbb{R}_+^2}(0,0)$, one has $0 \in \{[\nabla f(1)]^*(y^*) \mid y^* \in \partial \Delta_{-\mathbb{R}_+^2}(0,0)\}$. Therefore, the assertion (b) of Theorem 4.22 can not be applied. However, we have*

$$\liminf_{x \to \bar{x}} d(0; \{[\nabla f(x)]^*(y^*) \mid y^* \in \partial \Delta_{-K}(f(x) - f(\bar{x}))\}) = 1 > 0.$$

*Indeed, for $x > 1$ we have $f(x) - f(1) = (e^x - e, -\ln x)$, $\nabla f(x) = (e^x, -1/x)$, $\partial \Delta_{-\mathbb{R}_+^2}(f(x) - f(1)) = \{(1, 0)\}$, $\{[\nabla f(x)]^*(y^*) \mid y^* \in \partial \Delta_{-\mathbb{R}_+^2}(f(x) - f(1))\} = \{e^x\}$ and for $x < 1$ we have $f(x) - f(1) = (e^x - e, 1 - x)$, $\nabla f(x) = (e^x, -1)$, $\partial \Delta_{-\mathbb{R}_+^2}(f(x) - f(1)) = \{(0, 1)\}$, $\{[\nabla f(x)]^*(y^*) \mid y^* \in \partial \Delta_{-\mathbb{R}_+^2}(f(x) - f(1))\} = \{-1\}$. The assertion (c) of Theorem 4.22 implies that f satisfies the error bound property of f at $\bar{x} = 1$*

# 5 Error bounds for the distances to sets of efficient solutions and metric subregularity at efficient solutions

In this section, sufficient and/or necessary conditions ensuring error bounds for the distance to different solution sets of $(\mathcal{P})$ are formulated from the ones in Section 4 or are obtained by similar arguments. Some sufficient conditions are shown to be sufficient for metric subregularity at an efficient solution. In the remaining of the paper, we will use the same notations as in Section 4.

## 5.1 Error bounds for the distance to the set of Pareto efficient/ Henig properly efficient/superefficient solutions

Firstly, we consider the case $\bar{y}$ is one known Pareto efficient value of $(\mathcal{P})$, i.e. $\bar{y} \in Min(F(X), K)$. Denote by $S_{\bar{y}}^{\text{Pareto}}$ the set of Pareto efficient solutions corresponding to the Pareto efficient value $\bar{y}$, i.e.,

$$S_{\bar{y}}^{\text{Pareto}} := \{x \in X \mid \bar{y} \in F(x)\}$$

and by $S^{\text{Pareto}}$ the set of all Pareto efficient solutions of $(\mathcal{P})$, i.e.,

$$S^{\text{Pareto}} := \{x \in X \mid \text{there exists } y \in Min(F(X), K) \text{ such that } y \in F(x)\}.$$

Let us introduce the notions of error bounds. In the definitions below, $S$ stands for $S_{\bar{y}}^{\text{Pareto}}$ or $S^{\text{Pareto}}$.

**Definition 5.1.** *We define the global error bound $\sigma(S)$ for the distance $d(x; S)$ by*

$$\sigma(S) := \sup\{\tau > 0 \mid \tau d(x; S) \leq [\inf_{y \in F(x)} \Delta_{-K}(y - \bar{y})]^+ \text{ for all } x \in X\}$$

*with the convention that $\sigma(S) = \infty$ if $S = X$.*

**Definition 5.2.** *Assume that $\bar{x} \in S$ . We say that the distance $d(x; S)$ has the (local) error bound property at $\bar{x}$ if there exists a positive scalar $\tau$ such that*

$$\tau d(x; S) \leq [\inf_{y \in F(x)} \Delta_{-K}(y - \bar{y})]^+ \text{ for all } x \text{ near } \bar{x}$$

*and denote by $\mathrm{Er}S(\bar{x})$ the supremum of these $\tau$.*

It turns out that $S_{\bar{y}}^{\text{Pareto}}$ can be represented as a sublevel set considered in Section 4.

**Proposition 5.3.** *We have*
$$S_{\bar{y}}^{\text{Pareto}} = [F \preceq_l \bar{y}].$$

*Moreover, for any $x \in S_{\bar{y}}^{\text{Pareto}}$ we have $V(x) = \{\bar{y}\}$ and $g(x) = 0$; in particular, we have*

$$V(\bar{x}) = \{\bar{y}\}. \tag{16}$$

*Proof.* If $x \in S_{\bar{y}}$ then $\bar{y} \in F(x) \subset F(x) + K$ and $x \in [F \preceq_l \bar{y}]$. Suppose that $x \in [F \preceq_l \bar{y}]$. Then $\bar{y} \in F(x) + K$ and one can find $y \in F(x)$ and $k \in K$ such that $\bar{y} = y + k$. As $\bar{y} \in Min(F(X), K)$, we deduce that $k = 0$. Therefore, $\bar{y} \in F(x)$ and $x \in S_{\bar{y}}$. Further, if $x \in S_{\bar{y}}$, then $\bar{y} \in \mathrm{Min}(F(x), K)$. Hence, $y - \bar{y} \notin -K$, $\Delta_{-K}(y - \bar{y}) > 0 = \Delta_{-K}(\bar{y} - \bar{y})$ for all $y \in F(x)$ and the desired relations follow. $\square$

Proposition 5.3 and (16) imply that the results of Section 4 can be applied for obtaining sufficient and/or necessary conditions ensuring error bounds for the distance $d(x; S_{\bar{y}}^{\text{Pareto}})$. It suffices to replace $[F \preceq_l \bar{y}]$ by $S_{\bar{y}}^{\text{Pareto}}$, $\sigma(F)$ by $\sigma(S_{\bar{y}}^{\text{Pareto}})$ and $\mathrm{Er}F(\bar{x})$ by $\mathrm{Er}S_{\bar{y}}(\bar{x})$ everywhere in Theorems 4.4, 4.7, 4.15 and 4.18. In case $F$ is a strictly differentiable single-valued map, criteria for error bounds of the distance $d(x; S_{\bar{y}}^{\text{Pareto}})$ can be derived from Theorems 4.11 and 4.22. Note that by (16), the relation (13) of Theorem 4.15 becomes

$$0 \notin \cup_{y^* \in \partial \Delta_{-K}(0)} D_M^* F(\bar{x}, \bar{y})(y^*)$$

and the relation (14) of Theorem 4.18 becomes

$$0 \notin \cup_{y^* \in \partial \Delta_{-K}(0)} D^* F(\bar{x}, \bar{y})(y^*).$$

It should be noted, however, that these relations may not be satisfied due to the fact that the Fermat rule for Pareto efficient solutions holds (under some assumptions), i.e.,

$$0 \in D_M^* F(\bar{x}, \bar{y})(y^*) \quad \text{or} \quad 0 \in D^* F(\bar{x}, \bar{y})(y^*)$$

for some $y^* \in K^+$ when $\bar{y} \in Min(F(X), K)$ and $\bar{y} \in F(\bar{x})$ [5, 40]. In such a situation, criteria involving information around the point $\bar{x}$ such as (12) and (15) are more useful.

Now let us return to the set of all Pareto efficient solutions $S^{\text{Pareto}}$. Since $S_{\bar{y}}^{\text{Pareto}} \subset S^{\text{Pareto}}$, we have

$$\sigma(S_{\bar{y}}^{\text{Pareto}}) \leq \sigma(S^{\text{Pareto}})$$

and

$$\text{Er} S_{\bar{y}}^{\text{Pareto}}(\bar{x}) \leq \text{Er} S^{\text{Pareto}}(\bar{x}).$$

Therefore, any sufficient conditions for error bounds for the distance to the set $S_{\bar{y}}^{\text{Pareto}}$ is sufficient for the error bounds for the distance to the set $S^{\text{Pareto}}$ as well. Below, we formulate two consequences of Theorems 4.4, 4.15 and illustrate them by examples.

**Theorem 5.4.** *Suppose that $X$, $Y$ are Asplund spaces, $F$ is closed $K$-u.s.c. compact-valued and $V$ is lower semicompact. Then*

$$\sigma(S^{\text{Pareto}}) \geq \sigma(S_{\bar{y}}^{\text{Pareto}}) \geq \inf_{x \notin S_{\bar{y}}} d(0; \cap_{y_x \in V(x)} \cup_{y^* \in \partial_M \Delta_{-K}(y_x - \bar{y})} D_M^* F(x, y_x)(y^*)).$$

**Example 5.5.** *Let $F$ be the set-valued map of Example 4.5. It is easy to see that $\bar{y} = (0, 0)$ is a Pareto efficient point of $F(X)$ and that $S_{\bar{y}}^{\text{Pareto}} = S^{\text{Pareto}} = ]-\infty, 0]$. Observe that $S_{\bar{y}}^{\text{Pareto}}$ and $S^{\text{Pareto}}$ coincide with the set $\mathcal{A}$ considered in Example 4.5. Results obtained in this example yield that*

$$\sigma(S^{\text{Pareto}}) \geq \sigma(S_{\bar{y}}^{\text{Pareto}}) \geq \inf_{x \notin S_{\bar{y}}} d(0; \cap_{y_x \in V(x)} \cup_{y^* \in \partial_M \Delta_{-K}(y_x - \bar{y})} D_M^* F(x, y_x)(y^*)) \geq 1.$$

**Theorem 5.6.** *Suppose that $X$ and $Y$ are Asplund spaces, the set-valued map $F$ is closed $K$-u.s.c. compact-valued and $V$ is lower semicompact. Then the distances $d(x; S_{\bar{y}}^{\text{Pareto}})$ and $d(x; S^{\text{Pareto}})$ have the error bound property at $\bar{x}$ if*

$$\liminf_{x \to \bar{x}, y_x \in V(x), \Delta_{-K}(y_x - \bar{y}) \to 0} d(0; \cup_{y^* \in \partial \Delta_{-K}(y_x - \bar{y})} D_M^* F(x, y_x)(y^*)) > 0.$$

**Example 5.7.** *Let $F$ be the set-valued map of Example 4.17(a). It is easy to see that $\bar{y} = (0, 0)$ is a Pareto efficient point of $F(X)$ and that $S_{\bar{y}}^{\text{Pareto}} = S^{\text{Pareto}} = \{0\} \cup ]-\infty, -2]$. Observe that $S_{\bar{y}}^{\text{Pareto}}$ and $S^{\text{Pareto}}$ coincide with the set $\mathcal{A}$ considered in Example 4.17(a). Results obtained in this example yield that the distances $d(x; S_{\bar{y}}^{\text{Pareto}})$ and $d(x; S^{\text{Pareto}})$ have the error bound property at $\bar{x} = 0$.*

Now let us go to the cases $\bar{y}$ is one known Henig properly efficient value or a super efficient value of $(\mathcal{P})$. In the first case, one can define the set $S_{\bar{y}}^{\text{He}}$ of Henig properly efficient solutions of $(\mathcal{P})$ corresponding to the Henig proper efficient value $\bar{y}$, i.e.,

$$S_{\bar{y}}^{\text{He}} := \{x \in X \mid \bar{y} \in F(x)\}$$

and the set $S^{\text{He}}$ of all Henig properly efficient solutions of $(\mathcal{P})$. Since $\bar{y} \in He(F(X), K)$ iff $\bar{y} \in Min(F(X), K_\eta)$ [17, Proposition 3.3] for some scalar $\delta$ defined by (1) and some cone $K_\eta$ ($\eta \in ]0, \delta[$) defined by (2), the global/local error bounds for the distance $d(x; S_{\bar{y}}^{\text{He}})$ and $d(x; S^{\text{He}})$ as well as criteria ensuring these bounds can be defined and formulated in a similar way as for

the distance $d(x; S_{\bar{y}}^{\text{Pareto}})$, $d(x; S^{\text{Pareto}})$. In the second case, under the assumption that the base of $K$ is bounded, Henig efficiency and super efficiency coincide, and we can also consider error bounds for the set of super efficient solutions. The formulation of the corresponding results in these cases is left to the reader.

**Remark 5.8.** *In case of Henig properly efficient/super efficient solutions, we can simply assume $F$ to be $K$-u.s.c. or $K$-convex instead of assuming $F$ to be $K_\eta$-u.s.c. or $K_\eta$-convex. Moreover, in contrast to Remarks 4.8 and 4.16, in this case by Corollary 2.11 we have estimates $y^* \in K^{+si} \cap \mathbb{B}_{Y^*}$ and $y^* \in \text{int} K^+ \cap \mathbb{B}_{Y^*}$ which are stronger than the relation (10); moreover, the cone $K_\eta$ has a nonempty interior.*

### 5.2 Metric subregularity at efficient solutions

Observe that the set $S_{\bar{y}}^{\text{Pareto}}$ coincides with set $F^{-1}(\bar{y})$ appeared in the definition of metric subregularity. The simple relation formulated below allows us to connect the error bound property of the distance to this set at $\bar{x}$ and the metric subregularity of the map $F$ at $\bar{x}$ for $\bar{y}$.

**Proposition 5.9.** *The following inequality holds:*

$$[\inf_{y \in F(x)} \Delta_{-K}(y - \bar{y})]^+ \leq d(\bar{y}; F(x)) \quad \text{for all } x \in X.$$

*Proof.* The inequality follows from the fact that the function $\Delta_{-K}$ is Lipschizt of rank 1. □

It follows from the definition of the error bound property, Definition 5.2, and the definition of metric subregularity, recalled in Introduction, and Proposition 5.9 that sufficient conditions in terms of coderivatives of $F$ for the error bound property of the distance to the set $S_{\bar{y}}^{\text{Pareto}}$ at the Pareto solution $\bar{x}$ also are sufficient for the metric subregularity of the set-valued map $F$ near $\bar{x}$ to $\bar{y}$. This conclusion holds true also for case with Henig proper efficiency and, under the assumption that $K$ has a bounded base, for case with superefficiency.

As noted in [9], a major drawback of metric subregularity is the lack of a norm characterization in terms of coderivatives of the considered map except some sufficient conditions in terms of derivative/coderivative of the inverse map characterizing the calmness of the latter ( metrict subregularity has been shown to be equivalent to the calmness of the inverse map $F^{-1}$ [8]), see [9, 20, 24, 25, 41, 42]. Recently, some first- and second-order characterization for the metrict subregularity of of $F$ in terms of its derivatives and coderivatives have been obtained in [13].

## References

[1] M. ALONSO and L. RODRIGUEZ-MARIN, *Set-relations and optimality conditions in set-valued maps*, Nonlinear Anal., 63 (2005), pp. 1167-1179.

[2] J.-P. AUBIN and I. EKELAND, *Applied Nonlinear Analysis*, John Wiley & Sons, New York,1983.

[3] D. AZÉ, *A survey on error bounds for lower semicontinuous functions*, ESAIM: PROCEEDINGS, 13 (2003), pp. 1-17.

[4] D. AZÉ and J.-N. CORVELLEC, *Characterizations of error bounds for lower semicontinuous functions on metric spaces*, ESAIM Control Optim. Calc. Var., 10 (2004), pp. 409-425.

[5] T. Q. BAO and B. S. MORDUKHOVICH, *Relative Pareto minimizers for multiobjective problems: existence and optimality conditions*, Math. Program. , Ser. A, 122 (2010), pp. 301-347.

[6] P. BOSCH, A. JOURANI and R. HENRION, *Sufficient conditions for error bounds and applications* , Appl. Math. Optim., 50 (2004), pp. 161-181.

[7] M. CILIGOR-TRAVAIN, *On Lagrange-Kuhn-Tucker multiplier for Pareto optimization problems*, Numer. Func. Anal. and Optim., 15 (1994), pp. 689-693.

[8] A. L. DONTCHEV and R. T. ROCKAFELLAR, *Regularity and conditioning of solution mappings in variational analysis*, Set-Valued Anal., 12 (2004), pp. 79-109.

[9] A. L. DONTCHEV and R. T. ROCKAFELLAR, *Implicit Functions and Solution Mappings*, Springer, Dordrecht, 2009.

[10] M. J. FABIAN, R. HENRION, A. Y. KRUGER and J. V. OUTRATA, *Error bounds: necessary and sufficient conditions*, Set-valued Var. Anal., 18 (2010), pp. 121-149.

[11] E. M. BERNARCZUK and A. Y. KRUGER, *Error bounds for vector-valued functions: necessary and sufficient conditions*, Nonlinear Anal., 75 (2012), pp. 1124-1140.

[12] J. M. BORWEIN and D. ZHUANG, *Super efficiency in vector optimization*, Trans. Amer. Math. Soc., 338 (1993), pp. 105-122.

[13] H. GFRERER, *First order and second order characterizations of metric subregularity and calmness of constraint set mappings*, SIAM J. Optim., 21 (2011), pp. 1439-1447.

[14] V. V. GOROKHOVICH, *Convex and Nonsmooth Problems of Vector Optimization*, Nauka i Tekhnika, Minsk, 1990.

[15] T. X. D. HA, *The Ekeland variational principle for set-valued maps involving coderivatives*, J. Math. Anal.Appl., 286 (2003), pp. 509-523.

[16] T. X. D. HA, *Some variants of the Ekeland variational principle for a set-valued map*, J. Optim. Theory Appl. , 124 (2005), pp. 187-206.

[17] T. X. D. HA, *The Ekeland variational principle for Henig proper minimizers and super minimizers*, J. Math. Anal.Appl., 364 (2010), pp. 156-170.

[18] A. HAMEL and A. LÖHNE, *Minimal element theorems and Ekeland's principle with set relations*, J. Nonlinear Convex Anal., 7 (2006), pp. 19-37.

[19] M. I. HENIG, *Proper efficiency with respect to the cones*, J. Optim. Theory Appl. 36 (1982) pp. 387-407.

[20] R. HENRION and, J. OUTRATA, *A subdifferential condition for calmness of multifunctions*, J. Math. Anal. Appl. 258 (2001), pp. 110-130.

[21] E. HERNANDEZ and L. RODRIGUEZ-MARIN, *Existence theorems for set optimization problems*, Nonlinear Anal., 67 (2007), pp. 1726-1736.

[22] J. B. HIRIART-URRUTY, *New concepts in nondifferentiable programming*, Bull. Soc. Math. France, 60 (1979), pp. 57-85.

[23] A. D. IOFFE, *Regular points of Lipschitz functions*, Trans. Amer. Math. Soc., 251 (1979), pp. 61-69.

[24] A. D. IOFFE, *Metric regularity and subdifferential calculus*, Russian Math. Surv. , 55 (2000), pp. 501-558.

[25] A. D. IOFFE, *On regularity concepts in variational analysis*, J. Fixed Point Theory Appl., 8 (2010), pp. 339-363.

[26] J. JAHN, *Bishop-Phelps cones in optimization*, Int. J. Optim. Theory Methods Appl., 1 (2009), pp. 123-139.

[27] J. JAHN, *Vector Optimization - Theory, Applications, and Extensions*, Second Edition, Springer, Berlin, 2011.

[28] M. A. KRASNOSELSKI, *Positive Solutions of Operator Equations*, Nordhoff, Groningen, 1964.

[29] D. KUROIWA, *The natural criteria in set-valued optimization*, RIMS Kokyuroku, 1031 (1998), pp. 85-90.

[30] D. KUROIWA, *Some duality theorems of set-valued optimization with natural criteria*, Nonlinear anal. Convex Anal. (Niigata, 1998), pp. 221-228, World Sci. Publ., River Edge, NJ, 1999.

[31] D. KUROIWA, *Existence theorems of set optimization with set-valued maps*, J. Inf. Optim. Sci. , 24 (2003), pp. 73-84.

[32] A. LÖHNE, *Optimization with set relations: conjugate duality*, Optimization, 54 (2005), pp. 265-282.

[33] D. T. LUC, *Theory of Vector Optimization*, Springer-Verlag, Berlin, 1989.

[34] C. G. LIU, K. F. NG and W. H. YANG, *Merit functions in vector optimization*, Math.Programming, Ser. A, 119 (2009), pp. 215-237.

[35] B. S. MORDUKHOVICH, *Maximum principle in problems of time optimal control with nonsmooth contraints* , J. Appl. Math. Mech. , 40 (1976), pp. 960-969.

[36] B. S. MORDUKHOVICH, *Variational Analysis and Generalized differentiation, I: Basic Theory*, Grundlehren Series (Fundamental Principles of Mathematical Sciences) 330, Springer, Berlin, 2006.

[37] B. S. MORDUKHOVICH and Y. SHAO, *Nonsmooth sequential analysis in Asplund spaces*, Trans. Amer. Math. Soc. , 348 (1996), pp. 1235-1280.

[38] R. T. ROCKAFELLAR, *Convex Analysis*, Princeton University Press, Princeton, 1970.

[39] X. Y. ZHENG and K. F. NG, *Error bounds for lower semicontinuous functions in normed spaces*, SIAM J. Optim., 12 (2001), pp. 1-17.

[40] X. Y. ZHENG and K. F. NG, *Fermat rule for multifunctions in Banach spaces*, Math. Program., 104 (2005), pp. 69-90.

[41] X. Y. ZHENG and K. F. NG, *Metric subregularity and constraint qualifications for convex generalized equations in Banach spaces*, SIAM J. Optim., 18 (2007), pp. 437-460.

[42] X. Y. ZHENG and K. F. NG, *Metric subregularity for proximal generalized equations in Hilbert spaces*, Nonlinear Anal., 75 (2012), pp. 1686-1699.