A Newton-Fixed Point Homotopy Algorithm For Nonlinear Complementarity Problems With Generalized Monotonicity *

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Abstract

In this paper has been considered probability-one global convergence of NFPH (Newton-Fixed Point Homotopy) algorithm for system of nonlinear equations and has been proposed a probability-one homotopy algorithm to solve a regularized smoothing equation for NCP with generalized monotonicity. Our results provide a theoretical basis to develop a new computational method for nonlinear equation systems and complementarity problems. Some preliminary numerical experiments shows that our NFPH method is useful and promissing for difficult nonlinear problems.

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1 Introduction

NCP (Nonlinear Complementarity Problem) is to find $x \in R^n$ such that

$$x_i \geq 0, \quad f_i(x) \geq 0, \quad x_i f_i(x) = 0, \quad i = 1, ..., n. \tag{1.1}$$

Zhao and Li[1] studied several properties of a homotopy solution path associated with nonlinear quasi-monotone complementarity problems. They established a sufficient condition to

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assure the existence and boundedness of this homotopy solution path. Their results provide a theoretical basis to develop a new computational method for quasi-monotone complementarity problems. In [2], Billups and Watson considered probability-one global convergence of an interior FPH (Fixed Point Homotopy) algorithm for bounded MCP(Mixed Compementarity Problem). Their idea is to reformulate the MCP as a system of equations using FB(Fischer-Burmeister)- NCP function and then solve smooth approximations of this system with a homotopy method. Billups [3] has considered probability-one global convergence of FPH algorithm using smoothing of FB-NCP function for MCP satisfying a coercivity or generalized monotonicity and strict feasibility. Watson[4] has considered probability-one global convergence of FPH algorithm for monotone complementarity problem. His method involved reformulating the NCP as a system of smooth equations and applying a homotopy method to solve this system. In the context of Newton-based methods, such smooth reformulations of complementarity problems are inferior to nonsmooth reformulations due to slow local convergence for degenerate solutions. In contrast, nonsmooth reformulations allow much faster (superlinear or quadratic) convergence to degenerate solutions. In [5], Hotta and Yoshise considered global convergence of a non-interior homotopy algorithm using CHKS(Chen-Harker-Kanzow-Smale)-smoothing function in case that map f is P_0-mapping and has an interior feasible point, or f is monotone and has an interior feasible point. In this paper, we propose NFPH (Newton-Fixed Point Homotopy) algorithm for nonlinear system and consider its probability-one global convergence, and extend it to solve a regularized CHKS-smoothing of nonsmooth reformulation for NCP with generalized monotonicity.

2 Homotopy method for nonlinear system

Our probability-one homotopy algorithm is based on the following Theorem.

Theorem 2.1. ([2] and [3]) *Let $F : R^n \to R^n$ be a C^2-function and suppose there exists C^2-map $\rho : R^m \times [0,1) \times R^n \to R^n$ such that*

(i) the $n \times (m+1+n)$ Jacobian matrix $D\rho(a,\lambda,x)$ has rank n on the set

$$\rho^{-1}(0) = \{(a,\lambda,x) \in R^m \times [0,1) \times R^n | \rho(a,\lambda,x) = 0\}$$

(ii) for any fixed $a \in R^m$ and $\lambda = 0$, the equation $\rho_a(\lambda,x) \equiv \rho(a,\lambda,x) = 0$ has a unique solution $x^a \in R^n$,

(iii) $\rho(a,1,x) = F(x)$ for any fixed $a \in R^m$,

(iv) $\rho^{-1}(0)$ is bounded for any fixed $a \in R^m$.

Then for almost all $a \in R^m$ (in the sense of Lebesgue measure) there exists a zero curve γ_a of ρ_a , along which the Jacobian matrix $D\rho_a$ has rank n , emanating from $(0,x^a)$ and reaching a zero $\bar{x}$ of F at $\lambda = 1$. Moreover, γ_a does not intersect itself and is disjoint from any other zeros of ρ_a .

The expression "reaching a zero" means that there exists a sequence of points $\{(\lambda_k, x^k)\}$ in γ_a, accumulating at $(1, \bar{x})$. The popular homotopy used often in practice is the FPH defined by

$$\rho(a, \lambda, x) = \lambda F(x) + (1 - \lambda)(x - a).$$

In this paper, we consider NFPH defined by

$$\rho(a, \lambda, x) = \lambda F(x) + (1 - \lambda) G(x, a), \tag{2.1}$$

where $G(x, a) = F(x) - F(a) + A(x - a)$ and A is a symmetric and positive definite matrix. In what follows all of consideration will be made under the following assumption:

Assumption 1. F is a C^2-mapping and there exists a symmetric and positive definite matrix A such that $F'(x) + A$ is nonsingular for every $x \in R^n$.

Remark 2.1. Under the assumption 1, the map $\rho(a, \lambda, x)$ defined by (2.1) satisfies conditions (i)$\sim$(iii) of Theorem 2.1. □

The following Theorem provides a sufficient condition for condition (iv) of Theorem 2.1 to be satisfied.

Theorem 2.2. *Suppose that $F : R^n \to R^n$ satisfies the assumption 1 and there exists $\tilde{x} \in R^n$ and $M > 0$ such that*

$$\|\tilde{x}\|_{A^{\frac{1}{2}}} < M, \tag{2.2}$$

where A is such as in the assumption 1 and $\|x\|_{A^{\frac{1}{2}}} = \sqrt{x^T A x}$. If it holds

$$(x - \tilde{x})^T F(x) \geq 0 \tag{2.3}$$

for every $x \in R^n$ such that $\|\tilde{x} - x\|_{A^{\frac{1}{2}}} \geq 2M$, then each zero curve γ_a of $\rho_a(\lambda, x) \equiv \rho(a, \lambda, x)$ defined by (2.1) is bounded for every $a \in R^n$ such that

$$a + A^{-1}F(a) \in B = \{x \in R^n | \|x\|_{A^{\frac{1}{2}}} < M\}$$

and there exists a zero curve γ_a , emanating from $(0, a)$ and reaching a zero $\bar{x}$ of F at $\lambda = 1$ for almost all $a \in R^n$ with $a + A^{-1}F(a) \in B$. In particular, if $F'(\bar{x})$ is nonsingular, γ_a has finite arc length.

Proof. Let $\|x\|_{A^{\frac{1}{2}}} \geq 3M$. Then we have

$$\|x - a'\|_{A^{\frac{1}{2}}} \geq \|x\|_{A^{\frac{1}{2}}} - \|a'\|_{A^{\frac{1}{2}}} \geq 3M - M = 2M$$

for every $a' \in B$, and since $\|\tilde{x} - a'\|_{A^{\frac{1}{2}}} \leq \|\tilde{x}\|_{A^{\frac{1}{2}}} + \|a'\|_{A^{\frac{1}{2}}} < 2M$ by (2.2) , we have

$$(x - \tilde{x})^T A(x - a') = (x - a' + a' - \tilde{x})^T A(x - a') =$$

$$= \|x - a'\|^2_{A^{\frac{1}{2}}} - (\tilde{x} - a')^T A(x - a') \geq \|x - a'\|^2_{A^{\frac{1}{2}}} - \|\tilde{x} - a'\|_{A^{\frac{1}{2}}} \|x - a'\|_{A^{\frac{1}{2}}} =$$

$$= \|x - a'\|_{A^{\frac{1}{2}}} \left(\|x - a'\|_{A^{\frac{1}{2}}} - \|\tilde{x} - a'\|_{A^{\frac{1}{2}}} \right) > 0$$

using generalized Schwartz inequality. Consequently, we have

$$(x - \tilde{x})^T A(x - a') > 0 \tag{2.4}$$

for every $a' \in B$ and $x \in R^n$ with $\|x\|_{A^{\frac{1}{2}}} \geq 3M$. Thus, if $a' = a + A^{-1}F(a) \in B$, we have $(x - \tilde{x})^T (\lambda F(x) + (1 - \lambda)G(x, a)) > 0$ for $x \in R^n$ with $\|x\|_{A^{\frac{1}{2}}} \geq 3M$, i.e.

$$(x - \tilde{x})^T \rho_a(\lambda, x) > 0 \tag{2.5}$$

for any $\lambda \in [0, 1]$ by (2.3) and (2.4) because $\|x - \tilde{x}\|_{A^{\frac{1}{2}}} \geq 2M$. The inequality (2.5) implies that $\rho_a(\lambda, x) \neq 0$ for $\lambda \in [0, 1]$ and $x \in \{x \in R^n | \|x\|_{A^{\frac{1}{2}}} \geq 3M\}$. Therefore, zero curve γ_a of ρ_a is contained in set $[0, 1] \times \{x \in R^n | \|x\|_{A^{\frac{1}{2}}} < 3M\}$ and is bounded, which proves the first proposition of Theorem together with Theorem 2.1. If $F'(\bar{x})$ is nonsingular, , which implies that γ_a has finite arc length. □

Definition 2.1. *A function $F : R^n \to R^n$ is said to be pseudomonotone at $\tilde{x}$ if F satisfies the following at $\tilde{x}$:*

$$(x - \tilde{x})^T F(\tilde{x}) \geq 0 \textit{ for every } x \in R^n \textit{ implies that } (x - \tilde{x})^T F(x) \geq 0 \textit{ for every } x \in R^n.$$

Corollary 2.1. *Suppose that the assumption 1 is satisfied and solution set of $F(x) = 0$ is bounded, i.e. $\|\tilde{x}\|_{A^{\frac{1}{2}}} < M$ for every solution $\tilde{x}$ of $F(x) = 0$. If $F(x)$ is pseudo-monotone at each solution $\tilde{x}$, then there exists a zero curve γ_a of ρ_a , emanating from $(0, a)$ and reaching a zero $\bar{x}$ of ρ_a at $\lambda = 1$ for almost every $a \in R^n$ with $a + A^{-1}F(a) \in B$, where A is such as in the assumption 1.*

Proof. For every solution $\tilde{x}$ of $F(x) = 0$ and for every $x \in R^n$ with $\|x - \tilde{x}\|_{A^{\frac{1}{2}}} \geq 2M$, we have $(x - \tilde{x})^T F(\tilde{x}) = 0$. Hence, by the pseudo-monotonicity of $F(x)$ at $\tilde{x}$, we have

$$(x - \tilde{x})^T F(x) \geq 0 \tag{2.6}$$

for every $x \in R^n$ with $\|x - \tilde{x}\|_{A^{\frac{1}{2}}} \geq 2M$, which implies (2.3). Thus, Theorem 2.2 completes the proof of the corollary. □

Remark 2.2. Suppose that $\tilde{x}$ is a solution of $F(x) = 0$. If F is monotone in $\tilde{x}$, F is pseudo-monotone at $\tilde{x}$. If F is pseudo-monotone at $\tilde{x}$, F satisfies (2.3). □

3 Homotopy algorithm for NCP

Definition 3.1. *If a function $\varphi : R^2 \to R$ is such that $\varphi(a, b) = 0 \Leftrightarrow a \geq 0, b \geq 0, ab = 0$, then φ is called NCP function.*

The function $\varphi(a,b) = a + b - \sqrt{(a-b)^2}$ is a popular NCP function called min-function. Then NCP (1.1) is equivalent to solve a system of nonlinear equations

$$\varphi\left(x_i, f_i(x)\right) = 0, \quad i = 1, ..., n$$

and the problem (1.1) is reduced to solve the following nonlinear system .

$$F(x,y) = \begin{pmatrix} f(x) - y \\ \Phi(x,y) \end{pmatrix}, \tag{3.1}$$

where $\Phi(x,y) = (\varphi(x_1, f_1(x)), ..., \varphi(x_n, f_n(x)))^T$. The min-function $\varphi(a,b)$ is nondifferentiable in case of $a = b$ and this function is approximated by the following smooth function:

$$\varphi_\mu(a,b) = a + b - \sqrt{(a-b)^2 + 4\mu^2} \tag{3.2}$$

Let's approximate (3.1) by the following smooth system using (3.2):

$$F^\mu(x,y) = \begin{pmatrix} f(x) - y + \mu x \\ \Phi_\mu(x,y) + \mu y \end{pmatrix}, \tag{3.3}$$

where $\Phi_\mu(x,y) = (\varphi_\mu(x_1, f_1(x)), ..., \varphi_\mu(x_n, f_n(x)))^T$. Then $F^0(x,y) = F(x,y)$. The smooth approximation often used for (3.1) is

$$\begin{pmatrix} f(x) - y \\ \Phi_\mu(x,y) \end{pmatrix}.$$

and (3.3) is a regularized smoothing for (3.1).

Lemma 3.1. *For every $\mu > 0, x$ and y, we have*

$$\|\Phi_\mu(x,y) - \Phi(x,y)\| \le 2\mu\sqrt{n}. \tag{3.4}$$

Proof. When $\xi > 0, \eta > 0$, we have $\sqrt{\xi + \eta} > \sqrt{\xi}$ and

$$(\sqrt{\xi + \eta} - \sqrt{\xi})^2 = \xi + \eta - 2\sqrt{\xi + \eta}\sqrt{\xi} + \xi \le 2\xi + \eta - 2\sqrt{\xi}\sqrt{\xi} = \eta.$$

Thus, letting $\xi_i = (x_i - y_i)^2, \quad \eta_i = (2\mu)^2, \quad i = 1, ..., n$, it holds

$$\|\Phi_\mu(x,y) - \Phi(x,y)\|^2 = \sum_{i=1}^{n} [\sqrt{(x_i - y_i)^2 + 4\mu^2} - \sqrt{(x_i - y_i)^2} \quad]^2 \le 4n\mu^2$$

which gives us the inequality (3.4).

□

Let's construct a homotopy to solve $F^{\mu}(x, y) = 0$ as follows.

$$\mu(\lambda) = \beta(1 - \lambda), \quad \lambda \in [0, 1], \quad \beta > 0, \tag{3.5}$$

$$\rho_a(\lambda, z) = \lambda F^{\mu(\lambda)}(z) + (1 - \lambda) G^{\mu(\lambda)}(z, a), \tag{3.6}$$

where

$$z = \begin{pmatrix} x \\ y \end{pmatrix}, \quad G^{\mu(\lambda)}(z, a) = F^{\mu(\lambda)}(z) - F^{\mu(\lambda)}(a) + A(z - a).$$

Let's denote zero set of ρ_a for fixed a by

$$\rho_a^{-1}(0) = \{(\lambda, z) | \rho_a(\lambda, z) = 0, \quad \lambda \in [0, 1)\}$$

Theorem 3.1. *Suppose that the assumption 1 is satisfied for some matrix $A = cI$ and $F^{\mu(\lambda)}(z)$ defined by (3.3), where I is an identity matrix and c is a positive constant. If $\|a\|_{A^{\frac{1}{2}}} < M, \quad \|\tilde{z}\|_{A^{\frac{1}{2}}} < M, \quad M \geq 2\sqrt{cn}$ and*

$$(z - \tilde{z})^T F(z) \geq 0 \tag{3.7}$$

for every $z \in R^{2n}$ with $\|z - \tilde{z}\|_{A^{\frac{1}{2}}} \geq 2M$, then there exists a zero curve γ_a of ρ_a defined by (3.6), emanating from $(0, a)$ and reaching a zero $\bar{z}$ of F at $\lambda = 1$ for almost all $a \in R^{2n}$ with $a + A^{-1}F^{\mu(\lambda)}(a) \in B = \{z \in R^{2n} | \|z\|_{A^{\frac{1}{2}}} < M\}$.

Proof. First, let's prove that

$$(z - \tilde{z})^T F^{\mu(\lambda)}(z) \geq 0 \tag{3.8}$$

for every $z \in R^{2n}$ with $\|z - \tilde{z}\|_{A^{\frac{1}{2}}} \geq 2M$. Let μ denote $\mu(\lambda)$ for simplicity. We have

$$\begin{aligned} (z - \tilde{z})^T F^{\mu}(z) &= (z - \tilde{z})^T F(z) + (z - \tilde{z})^T (F^{\mu}(z) - F(z)) = \\ &= (z - \tilde{z})^T F(z) + (x - \tilde{x})^T \mu x + (y - \tilde{y})^T (\Phi_{\mu}(z) - \Phi(z)) + (y - \tilde{y})^T \mu y \\ &= (z - \tilde{z})^T F(z) + \mu (z - \tilde{z})^T z + (y - \tilde{y})^T (\Phi_{\mu}(z) - \Phi(z)) \\ &= (z - \tilde{z})^T F(z) + \mu \|z - \tilde{z}\|^2 + \mu (z - \tilde{z})^T \tilde{z} + (y - \tilde{y})^T (\Phi_{\mu}(z) - \Phi(z)) \\ &\geq (z - \tilde{z})^T F(z) + \mu \|z - \tilde{z}\|^2 - \mu \|z - \tilde{z}\| \|\tilde{z}\| - \|y - \tilde{y}\| \|\Phi_{\mu}(z) - \Phi(z)\| \\ &\geq (z - \tilde{z})^T F(z) + \mu \|z - \tilde{z}\|^2 - \mu \|z - \tilde{z}\| \|\tilde{z}\| - \|z - \tilde{z}\| \|\Phi_{\mu}(z) - \Phi(z)\|, \end{aligned}$$

which gives us

$$\begin{aligned} (z - \tilde{z})^T F^{\mu}(z) &\geq (z - \tilde{z})^T F(z) + \mu \|z - \tilde{z}\|^2 - \mu \|z - \tilde{z}\| \|\tilde{z}\| - 2\mu\sqrt{n} \|z - \tilde{z}\| \\ &= (z - \tilde{z})^T F(z) + \mu \|z - \tilde{z}\| (\|z - \tilde{z}\| - \|\tilde{z}\| - 2\sqrt{n}) \end{aligned} \tag{3.9}$$

by (3.4). It follows from $A = cI$ that $\|x\|_{A^{\frac{1}{2}}} = \sqrt{c}\|x\|$, which implies (3.8) by (3.7) and (3.9) because $\|z - \tilde{z}\| \geq \frac{2M}{\sqrt{c}} \geq \|\tilde{z}\| + \frac{M}{\sqrt{c}} \geq \|\tilde{z}\| + 2\sqrt{n}$. Therefore, a zero curve γ_a of ρ_a is bounded for each $a \in R^{2n}$ with $a + A^{-1}F^{\mu(\lambda)}(a) \in B = \{z \in R^{2n} | \|z\|_{A^{\frac{1}{2}}} < M\}$ by Theorem 2.2, and there exists a γ_a of ρ_a , emanating from $(0, a)$ and reaching a zero $\bar{z}$ of $F^{\mu(\lambda)}$ at $\lambda = 1$ for almost every $a \in R^{2n}$ with $a + A^{-1}F^{\mu(\lambda)}(a) \in B$. Then, $\bar{z}$ is a zero of F because $F^{\mu(1)}(z) = F^0(z) = f(z)$ by (3.3) and (3.5). □

Lemma 3.2. ([4]). *For every* μ, $\varphi_\mu(a,b) = a + b - \sqrt{(a-b)^2 + 4\mu^2} = d$ *if and only if*

$$\left(a - \frac{d}{2}, b - \frac{d}{2}\right) \geq 0, \quad \left(a - \frac{d}{2}\right)\left(b - \frac{d}{2}\right) = \mu^2.$$

Let $\lambda \in (0,1)$ and $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$. Then, it follows from (3.6) that for $a = \begin{pmatrix} a' \\ a'' \end{pmatrix}$, $\rho_a(\lambda, z) = 0$ if and only if

$$\Phi_\mu(z) = v, \tag{3.10}$$

$$y = f(x) + r, \quad r = \mu x - (1-\lambda)[f(a') - a'' + \mu a' - A_1(x - a')], \tag{3.11}$$

$$v = -\mu y + (1-\lambda)[\Phi_\mu(a) + \mu a'' - A_2(y - a'')] \tag{3.12}$$

By Lemma 3.2, it follows from (3.10) that $\rho_a(\lambda, z) = 0$ if and only if

$$\left(x_i - \frac{v_i}{2}, y_i - \frac{v_i}{2}\right) \geq 0, \quad \left(x_i - \frac{v_i}{2})(y_i - \frac{v_i}{2}\right) = \mu^2. \tag{3.13}$$

Lemma 3.3. *Let* $a = \begin{pmatrix} a' \\ a'' \end{pmatrix} \geq \begin{pmatrix} \beta \\ \beta \end{pmatrix}$, *where* β *is the constant given in (3.5). Then, we have* $y > 0$ *for every* $(\lambda, z) \in \rho^{-1}(0)$ *with* $z = (x, y)$.

Proof. Let's prove that $\Phi_\mu(a) \geq 0$. If otherwise, there exists an i such that $\varphi_\mu(a'_i, a''_i) = a'_i + a''_i - \sqrt{(a'_i - a''_i)^2 + 4\mu^2} < 0$. Then, $a'_i + a''_i < \sqrt{(a'_i - a''_i)^2 + 4\mu^2}$ and it follows $a'_i a''_i < \mu^2$. Thus, $\beta^2 < \beta^2$ by (3.5) and the condition of Lemma, which is a contradiction. Therefore, $\Phi_\mu(a) \geq 0$. Assume that there exists an index i with $y_i \leq 0$. Then, it follows that $v_i > 0$ by (3.12) and $\Phi_\mu(a) \geq 0$. But, by (3.10),

$$v_i = \varphi_\mu(x_i, y_i) \leq \varphi(x_i, y_i) = 2\min\{x_i, y_i\} \leq 2y_i \leq 0,$$

which is a contradiction. Thus, we have $y > 0$. □

Lemma 3.4. $a = \begin{pmatrix} a' \\ a'' \end{pmatrix} \geq \begin{pmatrix} \beta \\ \beta \end{pmatrix}$, *where* β *is the constant given in (3.5) and* $(\bar{\lambda}, \bar{z})$ *be any limit of a smooth zero curve* $\gamma_a \subset \rho^{-1}(0)$. *Let*

$$P_1 = \{i | \bar{x}_i = \infty\}, \quad P_2 = \{i | \bar{y}_i = \infty\}, and \quad \{i | \bar{x}_i = -\infty\}.$$

Then, $\bar{\lambda} < 1$ *implies that (i) for every* $j \in P_1$, j *does not belong to* P_2 *and* $f_j(\bar{x}) = -\infty$, *and that (ii)* $P_2 = N$. *And* $\bar{\lambda} = 1$ *implies that (i)* $\bar{x}_i \leq 0$ *and* $f_i(\bar{x}) = \infty$ *for every* $i \in P_2$, *(ii)* $f_i(\bar{x}) = \infty$ *for every* $i \in N$ *and (iii)* $\bar{y}_i = 0$ *and* $f_i(\bar{x}) \leq 0$ *for every* $i \in P_1$.

Proof. 1) Consider the case of $\bar{\lambda} < 1$. If $j \in P_1$, then $\bar{x}_j - \frac{\bar{v}_j}{2} = \infty$ where $\bar{v} = \Phi_{\bar{\mu}}(\bar{z})$ and $\bar{\mu} = \mu(\bar{\lambda})$. Suppose the contrary. Then, there is a finite $C > 0$ such that $\bar{x}_j - \frac{\bar{v}_j}{2} = C$, and $\bar{v}_j = \infty$ because $\bar{x}_j = \frac{\bar{v}_j}{2} + C = \infty$. Thus, $\bar{y}_j = -\infty$ by (3.12). But, by Lemma 3.3, $\bar{y}_j \geq 0$ which is a contradiction. By (3.13), $\bar{x}_j - \frac{\bar{v}_j}{2} = \infty$ implies that $\bar{y}_j - \frac{\bar{v}_j}{2} = 0$. Suppose that $j \in P_2$. Then $\bar{y}_j = \infty$ and $\bar{v}_j = -\infty$ by (3.12), which implies $\bar{y}_j - \frac{\bar{v}_j}{2} = \infty$, contradicting $\bar{y}_j - \frac{\bar{v}_j}{2} = 0$. Therefore, j does not belong to P_2 , i.e. $\bar{y}_j$ is a finite positive, from which it follows that $\bar{x}_j = \infty$ implies $\bar{r}_j = \infty$, and $f_j(\bar{x}) = -\infty$ by (3.11).

Now, let's prove $P_2 = N$. Let $i \in P_2$. Then, we have $\bar{v}_i = -\infty$ by (3.12), and $\bar{y}_i - \frac{\bar{v}_i}{2} = \infty$, which implies $\bar{x}_i - \frac{\bar{v}_i}{2} = 0$ and $\bar{x}_i = -\infty$ by (3.13). Hence, $i \in N$ and $P_2 \subset N$. Let $i \in N$. Then, $\bar{x}_i = -\infty$ and $\bar{r}_i = -\infty$ by (3.11). Because $\bar{x}_i \geq \frac{\bar{v}_i}{2}$ by (3.13), we have $\bar{v}_i = -\infty$. Therefore, $\bar{y}_i = \infty$, i.e. $i \in P_2$ by (3.12), which implies $N \subset P_2$. Thus, $P_2 = N$.

2) Let's consider the case of $\bar{\lambda} = 1$. In this case, by (3.5) , we have $\bar{\mu} = 0$, and by (3.13)

$$\left(\bar{x}_i - \frac{\bar{v}_i}{2}, \bar{y}_i - \frac{\bar{v}_i}{2}\right) \geq 0, \quad \left(\bar{x}_i - \frac{\bar{v}_i}{2}\right)\left(\bar{y}_i - \frac{\bar{v}_i}{2}\right) = 0. \tag{3.14}$$

First, let's prove the proposition (i). If $i \in P_2$, then $\bar{v}_i \leq 0$ and $\bar{y}_i - \frac{\bar{v}_i}{2} = \infty$ by (3.12), and

$$\bar{x}_i - \frac{\bar{v}_i}{2} = 0 \tag{3.15}$$

by (3.14). Consider the case of $\bar{v}_i = 0$. Then, $\bar{x}_i = 0$ by (3.15), and $\bar{r}_i = 0$ and $\bar{y}_i = f_i(\bar{x}) = \infty$ by (3.11). Consider the case of $\bar{v}_i < 0$. Then, $\bar{x}_i < 0$ by (3.15), and $\bar{r}_i \leq 0$ and $\infty = \bar{y}_i \leq f_i(\bar{x})$ by (3.11).

Second, let's prove the proposition (ii). If $i \in N$, then $\bar{v}_i = -\infty$ by (3.14) and $\bar{y}_i = \infty$ by (3.12), which implies that $f_i(\bar{x}) = \infty$ by the proposition (i).

Finally, let's prove the proposition (iii). If $i \in P_1$, we have $\bar{x}_i - \frac{\bar{v}_i}{2} = \infty$ by (3.12) and

$$\bar{y}_i - \frac{\bar{v}_i}{2} = 0 \tag{3.16}$$

by (3.14). Suppose that $i \in P_2$. Then, from the proposition (i), it follows that $\bar{x}_i \leq 0$, contradicting $i \in P_1$. Hence, i does not belong to P_2 and $\bar{v}_i = 0$ by (3.12) because $\bar{\mu} = 0$ and $\bar{\lambda} = 1$. Thus $\bar{y}_i = 0$ by (3.16), and we have $f_i(\bar{x}) + \bar{r}_i = 0$ and $\bar{r}_i \geq 0$ by (3.11). Consider the case of $\bar{r}_i = 0$. Then $\bar{y}_i = f_i(\bar{x}) = 0$. Consider the case of $\bar{r}_i > 0$. Then $f_i(\bar{x}) + \bar{r}_i = 0 \geq f_i(\bar{x})$. □

Theorem 3.2. *Let* $a = \begin{pmatrix} a' \\ a'' \end{pmatrix} \geq \begin{pmatrix} \beta \\ \beta \end{pmatrix}$, *where* β *is the constant given in (3.5) and suppose that a map* $f : R^n \to R^n$ *satisfies* $f(a') > 0$ *and*

$$(x - y)^T (f(x) - f(y)) \geq 0 \tag{3.17}$$

for some $\delta > 0$ *and for every* x *and* y *with* $\|x - y\| \geq \delta$, *and the assumption 1 is satisfied for* $F^{\mu(\lambda)}(z)$ *defined by (3.3). Then, there is a smooth zero curve* $\gamma_a \subset \rho^{-1}(0)$, *emanating from* $(0, a)$ *and reaching a NCP solution* $\bar{z} = (\bar{x}, \bar{y})$ *with* $\bar{y} = f(\bar{x})$ *at* $\lambda = 1$.

Proof. If conditions of the Theorem are satisfied, conditions $(i) \sim (iii)$ of Theorem 2.1 are satisfied by Remark 2.1. Then, there exists a smooth zero curve $\gamma_a \subset \rho^{-1}(0)$ of $\rho_a(\lambda, z)$, along which the Jacobian matrix $D\rho_a(\lambda, z)$ has rank n , emanating from $(0, a)$ and not intersecting itself by Lemma 2.2 of [4]. Suppose that γ_a is unbounded. Then there is an unbounded sequence $\{(\lambda_k, z^k)\} \subset \gamma_a$ such that $\lambda_k \to \tilde{\lambda} \in [0,1], z^k \to \tilde{z}$ and $\|\tilde{z}\| = \infty$ as $k \to \infty$, where $\tilde{z} = (\tilde{x}, \tilde{y})$.

Let $\tilde{\lambda} < 1$. Then $\tilde{\lambda} = 0$ is impossible because the zero curve γ_a can not return the starting point $(0, a)$ and the $\rho_a(0, z)$ has a unique simple zero $z = a$.

Let i be not in $F = \{i | \tilde{z}_i \text{ is finite}\}$. First, consider the case of $i \in P_1$. Then, $f_i(\tilde{x}) = -\infty$ by Lemma 3.4. Thus, we have

$$(\tilde{x}_i - a'_i)(f_i(\tilde{x}) - f_i(a')) = -\infty. \tag{3.18}$$

Second, consider the case of $i \in N$. Then, $\tilde{x}_i = -\infty$ and $\tilde{y}_i = \infty$ by Lemma 3.4, and it follows from (3.11) that $f_i(\tilde{x}) = \infty$. Therefore, we obtain the (3.18) again. Third, consider the case when $\tilde{x}_i$ is finite. Then $\tilde{y}_i$ should be infinite because i does not belong to F , and $\tilde{y}_i = \infty$. But, $P_2 = N$ by Lemma 3.4 and $\tilde{x}_i = -\infty$, contradicting the finiteness of $\tilde{x}_i$. Hence, we have (3.18) for all i that do not belong to F .

Suppose that $\|\tilde{x} - a'\| < \delta$ for every $\delta > 0$. Then $\tilde{x} = a'$ because δ is arbitrary. Hence, $\tilde{x}$ is bounded, which implies that $\tilde{y}$ should be unbounded. Since $P_2 = N$, if $\tilde{y}_i = \infty$ for some i , then $\tilde{x}_i = -\infty$, contradicting $\tilde{x}_i = a'_i$. Therefore, there is $\delta > 0$ such that $\|\tilde{x} - a'\| \geq \delta$. Then, by (3.17), we have $(\tilde{x} - a')^T(f(\tilde{x}) - f(a')) \geq 0$. Thus, taking account of (3.18), there is $j \in F$ such that

$$(\tilde{x}_i - a'_i)(f_i(\tilde{x}) - f_i(a')) = \infty.$$

The following two cases are possible.

$$(i) \qquad \tilde{x}_j > a'_j, \quad f_j(\tilde{x}) = \infty, \tag{3.19}$$

$$(ii) \qquad \tilde{x}_j < a'_j, \quad f_j(\tilde{x}) = -\infty. \tag{3.20}$$

Consider the case (i). Because of $j \in F$, $\tilde{x}_j$ is bounded and so is $\tilde{r}_j$ by (3.11). Then, we have $\tilde{y}_j = \infty$ by (3.19) because $\tilde{y}_j = f_j(\tilde{x}) + \tilde{r}_j$, contradicting $j \in F$. Consider the case (ii). Taking account of (3.20), we have $\tilde{y}_j = -\infty$ similarly to the proof of the case (i), while $\tilde{y}_j \geq 0$ by Lemma 3.3. We have a contradiction again.

Therefore, we have $\tilde{\lambda} = 1$ and the equality (3.18) for $i \in P_2 \cup N$ by Lemma 3.4. For $i \in P_1$, we have the equality (3.18) again because $f(a') > 0$. Hence, it holds (3.18) for all i that do not belong to F . Suppose that $\|\tilde{x} - a'\| < \delta$ for every $\delta > 0$. Then $\tilde{x} = a'$, which implies that $\tilde{y}$ is unbounded. Thus, $P_2 \neq \varnothing$ and then $\tilde{x}_i \leq 0$ for $i \in P_2$ by Lemma 3.4, contradicting $\tilde{x}_i = a'_i > 0$. Therefore, there is $\delta > 0$ such that $\|\tilde{x} - a'\| \geq \delta$, and we have a contradiction again like the case of $\tilde{\lambda} < 1$. Therefore, the zero curve γ_a is bounded and it's accumulation point has form of $(1, \bar{z})$ by Lemma 2.3 of [4]. Then, $\bar{x}$ is a solution of (1.1) and $\bar{y} = f(\bar{x})$, where $\bar{z} = (\bar{x}, \bar{y})$. The proof is completed. □

The condition (3.17) for map $f : R^n \to R^n$ is called generalized monotonicity.

4 Preliminary numerical experiments

We can parameterize zero curve γ_a of ρ_a by its arclength s , that is, there exist continuously differentiable functions $x(s)$ and $\lambda(s)$ such that

$$\varrho_a(x(s), \lambda(s)) = 0, x(0) = x^0, \lambda(0) = 0. \tag{4.1}$$

By differentiating the first equation of (4.1), we obtain the following result: the homotopy path is determined by the following initial value problem of the ordinary differential equations.

$$\begin{gathered} D\varrho_a(x(s), \lambda(s)) \begin{pmatrix} \dot{x} \\ \dot{\lambda} \end{pmatrix} = 0, \\ x(0) = x^0, \lambda(0) = 0, \\ \| \left(\dot{x}(s), \dot{\lambda}(s) \right) \| = 1, \dot{\lambda}(0) > 0, , \end{gathered} \tag{4.2}$$

where $D\varrho_a(x(s), \lambda(s)) = \left(\frac{\partial \rho_a}{\partial x}, \frac{\partial \varrho_a}{\partial \lambda} \right)$. Then the x component of the solution $(x(s^*), \lambda(s^*))$ of (4.2), which satisfies $\lambda(s^*) = 1$, is the solution which we need to find. We carried out numerical experiments for nonlinear equations to show the performance of our NFPH method by PC Pentium IV, 3.19GHz, 1.00GB of RAM.

Our homotopy zero-curve trace was made by using ODE (ordinary differential equation) toolbox of MATLAB. In our experiment, we took $A = \alpha I, \alpha > 0$ and made use of 'ode45', ODE solvers of MATLAB, to solve (4.2).

Let S_f be the upper bound used in the ODE part for arc lengths$s$ and C_n be the number of intermediate checks $S_1, S_2, ...$ between 0 and S_f. Then interval $[0, S_f]$ is divided in $(C_n + 1)$ intervals of equal length. The zero-curve trace is finished as soon as one candidate solution has been found in any of the intervals defined by the checkpoints $[0, S_1, S_2, ..., S_f]$. If path-following is successful, we obtain the more refined solution 'nsol' using 'fsolve', nonlinear equation solver of MATLAB, with initial guess 'hsol' obtained by homotopy method. In the following tables, a is the starting point x^0, sol is a solution of the given problem, N_c is the number of checked intervals, $fhom = F(hsol)/\,(1 + \|hsol\|)$ and $fnew = F(nsol)/\,(1 + \|nsol\|)$.

Example 1.

$$F(x) = 2x - 4 + sin(2\pi x), x \in [-100, 100], a = 0, sol = 2.$$

Example 2.

$$F(x_1, x_2) = \begin{pmatrix} x_1^2 + x_2^2 - 1 \\ sin(x_1) - x_2) \end{pmatrix}, -100 \leq x_1, x_2 \leq 100, a = [0, 0]$$

Example 3.

$$F(x_1, x_2, x_3) = \begin{pmatrix} x_1 + 0.5x_2 + 0.3x_3 - 5 \\ 0.6x_1 + x_2 + 0.1x_3 - 7 \\ 0.2x_1 + 0.4x_2 + x_3 - 4 \end{pmatrix}$$

$$-100 \leq x_i \leq 100, i = 1, 2, 3, a = [0, 0, 0]$$

Example 4.

$$F(x) = arctan(100x)/\pi + sin(5x/(x^2 + 0.2))/2 + 0.1x, x \in [-2, 2], a = 0.2, sol = 0.$$

The results of numerical experiments are shown in the following tables.

$Table1(Example1, S_f = 2.5, C_n = 70)$

method	N_c	hsol	nsol	time(s)
$\alpha = 0.001$	14	2.0005	2.00000	1.0809
$\alpha = 50$	2	1.9783	2.00000	0.3327
FPH	20	2.0008	2.00000	1.5547
NH	14	2.0000	2.00000	0.9153

$Table2(Example1, S_f = 5, C_n = 70)$

method	N_c	hsol	nsol	time(s)
$\alpha = 0.001$	7	2.0005	2.00000	0.5911
$\alpha = 50$	1	2.0032	2.00000	0.1862
FPH	10	2.0008	2.00000	0.7333
NH	7	2.0000	2.00000	0.5098

$Table3(Example2, S_f = 20, C_n = 70)$

method	N_c	hsol	nsol	fhom	fnew	time(s)
$\alpha = 0.001$	3	-7.4234e-001 -6.7601e-001	-7.3908e-001 -6.7361e-001	4.0222e-003 -2.7211e-006	3.5192e-011 3.6061e-012	0.5689
$\alpha = 50$	1	-7.2803e-001 -7.3465e-001	-7.3908e-001 -6.7361e-001	3.4288e-003 3.4037e-003	1.6008e-012 1.7203e-013	1.4577
FPH	6	-7.3885e-001 -7.3465e-001	-7.3908e-001 -6.7361e-001	1.2271e-003 1.1222e-003	2.3637e-012 5.1031e-013	1.2431
NH	3	-7.3571e-001 -6.7112e-001	-7.3908e-001 -6.7361e-001	-4.1720e-003 -4.5865e-003	4.1332e-011 4.2909e-012	0.3311

$Table4(Example2, S_f = 5, C_n = 70)$

method	N_c	hsol	nsol	fhom	fnew	time(s)
$\alpha = 0.001$	10	-0.71376658 -0.65472611	-0.73908522 -0.67361213	-3.14294e -02 2.059234e-05	1.3983767e- 07 1.4943515e-08	1.4112
$\alpha = 50$	1	-0.72803942 -0.73465009	-0.73908513 -0.67361202	3.428826e-02 3.403779e-02	1.6008305e-012 1.7202905e-013	1.4576
FPH	21	-0.73915876 -0.67293244	-0.739085321 -0.673612169	-4.03203e-04 -3.67077e-04	2.337165e-07 9.0814323e-010	3.5987
NH	10	-0.7142857 -0.6550778	-0.739085223 -0.673612122	-3.080903e-02 -4.510372e-016	1.288977e-07 1.378482e-08	0.8984

$Table5(Example3, S_f = 30, C_n = 50)$

method	N_c	hsol	nsol	fhom	fnew	time(s)
$\alpha = 0.001$	3	1.67897430 5.89119023 1.32550592	1.67155427 5.86510265 1.31964807	3.0576e-003 4.2828e-003 2.4461e-003	4.3102e-012 9.4505e-012 1.5067e-011	0.9735
$\alpha = 50$	1	1.66456368 5.92393350 1.31394188	1.67155425 5.86510263 1.31964809	2.8405e-003 7.4144e-003 2.2530e-003	-1.5842e-011 -3.3679e-011 -1.3984e-012	0.9152
FPH	2	1.6731148 5.8574770 1.3209060	1.6715542 5.8651026 1.3196482	1.7504e-003 2.4505e-003 1.4003e-003	-1.3416e-011 2.0760e-012 1.0126e-011	0.6623
NH	3	1.6758000 5.8800000 1.3230000	1.6715542 5.8651026 1.3196482	1.7504e-003 2.4505e-003 1.4003e-003	-1.3416e-011 2.0760e-012 1.0126e-011	0.3325

$Table6(Example3, S_f = 2, C_n = 50)$

method	N_c	hsol	nsol	fhom	fnew	time(s)
$\alpha = 0.001$	33	1.50807356 5.29045112 1.19058642	1.67155426 5.86510262 1.31964809	-0.07385090 -0.10343821 -0.05908049	1.680643e-011 -1.5938702e-009 -6.454418e-010	6.7216
$\alpha = 50$	1	1.69024906 5.79197983 1.33466574	1.671554252 5.865102639 1.319648094	-0.001861054 -0.008413507 -0.001461469	6.8498442e-012 2.679456e-012 5.0761474e-011	0.7997
FPH	26	1.7520000 5.3961542 1.3848495	1.67155424 5.86510263 1.31964809	-0.01965900 -0.06054967 -0.01553925	-4.948133e-010 -4.022569e-010 -3.449198e-010	4.6629
NH	33	1.5047999 5.2800000 1.1880000	1.67155425 5.86510264 1.31964809	-0.075378066 -0.105529292 -0.060302452	6.0140810e-010 4.7703930e-010 6.6359281e-010	1.6579

$Table7(Example4, S_f = 5, C_n = 70)$

method	N_c	hsol	nsol	time(s)
$\alpha = 0.001$	37	-1.8652e-005	3.1351e-011	4.8317
$\alpha = 1$	18	1.9012e-004	7.6041e-020	2.3264
$\alpha = 75$	1	-2.3974e-004	5.3332-022	0.3118
FPH	7	-6.0612e-006	1.0732e-012	0.8831
NH	37	5.0773e-005	2.9810e-022	3.3849

As shown in Table 1 and Table 2, when $\alpha = 50$, our $NFPH$ is much better than FPH and NH for Exampl 1. For Example 2, our $NFPH$ with $\alpha = 50$ is better than others in the accuracy of obtained solution (Table 3 and Table 4). Our $NFPH \quad (C_f = 2, \quad \alpha = 50)$ is much better than others in both of time and accuracy(Table 6) for Example 3. Especially, the $NFPH$ with $\alpha = 75$ shows good performance for Example 4 (Table 7). From the above numerical results, we can see that choice of proper A can remarkably improve the performance of the $NFPH$.

5 Conclusion

This paper describes a probability-one homotopy algorithm for solving nonlinear systems of equations and complementarity problems. They are attractive because they are able to solve a qualitatively different class of problems than methods relying on merit functions. This claim is justified both theoretically and computationally. While the common homotopy used to solve nonlinear system $F(x) = 0$ is FPH defined by $\rho(a, \lambda, x) = \lambda F(x) + (1-\lambda)(x-a)$, in this paper we considered probability-one global convergence of the algorithm based on NFPH defined by

$$\rho(a, \lambda, x) = \lambda F(x) + (1 - \lambda)(F(x) - F(a) + A(x - a))$$

and extended the results to NCP with generalized monotonicity. The preliminary numerical experiments for some difficult nonlinear equations showed the robustness and fast convergence of the NFPH method. We expect the NFPH method would have better performance than NH(Newton Homotopy) or FPH method because NFPH combines the advantages of both NH and FPH.

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