Approximation of Matrix Rank Function and Its Application to Matrix Rank Minimization

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Abstract

Matrix rank minimization problem is widely applicable in many fields such as control, signal processing and system identification. However, the problem is in general NP-hard, and it is computationally hard to solve directly in practice. In this paper, we provide a new kind of approximation functions for the rank of matrix, and the corresponding approximation problems can be used to approximate the matrix rank minimization problem within any level of accuracy. Furthermore, the monotonicity and the Fréchet derivative of the approximation functions are discussed. On this basis, we design a new method, which is called as successive projected gradient method, for solving the matrix rank minimization problem by using the projected gradient method to find the stationary point of a series of approximation problems. Finally, the convergence analysis and the preliminary numerical results of the successive projected gradient method for the rank minimization problem are given.

Mathematics Subject Classifications: 65K05, 90C30
Key words. Matrix rank minimization, approximation function, projected gradient method, successive projected gradient method

1. Introduction

Let $\mathcal{R}^n$ be the n-dimensional Euclidean space, and $\mathcal{R}^{m\times n}, \mathcal{S}^n$ be the $m \times n$ real matrix space and the set of $n \times n$ real symmetric matrices, respectively. Given $X, Y \in \mathcal{R}^{m\times n}, b \in \mathcal{R}^n$, we use $\langle X, Y\rangle = tr(X^\top Y)$ to denote the inner product of X and Y. $\| X \|_F$, $\| X \|_*$, $\| b \|$, $\sigma_i(X)$ and $\sigma_i^2(X)$ denote the Frobenius norm, nuclear norm for matrix, the Euclidean norm for vector, the

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i-th largest singular value of X and $(\sigma_i(X))^2$, respectively. $X \succ 0 (\succeq 0)$ means that $X \in \mathcal{S}^n$ is positive (semi)definite. At last, I stands for the identical matrix with the proper dimension.

In this paper, we consider the following matrix rank minimization problem

$$\min \ \ rank(X) \qquad \text{s.t.} \ \ X \in \mathcal{C}, \tag{1.1}$$

where the feasible area $\mathcal{C} \subseteq \mathcal{R}^{m\times n}$.

The problem (1.1) has found many applications in system control, matrix completion, machine learning, image reconstruction and distance geometry, etc. For details, one can refer to the literature [18] and its references.

Matrix rank minimization problem is NP-hard since it includes the cardinality minimization as a special case [13]. A lot of algorithms constructed for (1.1) are based on the idea of the nuclear norm heuristic, i.e., to replace the objective of (1.1) by the nuclear norm $\| X \|_*$, and to solve the following optimization problem

$$\min \ \ \| X \|_* \qquad \text{s.t.} \quad X \in \mathcal{C}.$$

Under some conditions, the solution to the nuclear norm heuristic coincides with the minimum rank solution. For details, see [4, 6, 11, 12, 13] and their references.

In reference [18], Zhao approximated the matrix rank function in (1.1) by the generic approximation function

$$\phi_\varepsilon(X) = tr[X^\top (XX^\top + \varepsilon I)^{-1} X] = \sum_{i=1}^{\iota} \frac{\sigma_i^2(X)}{\sigma_i^2(X) + \varepsilon},$$

where $\iota = \min\{m, n\}$ and $\varepsilon > 0$. Some properties and applications about $\phi_\varepsilon(\cdot)$ are introduced in references [1, 18]. There are two advantages of the generic approximation function $\phi_\varepsilon(\cdot)$ compared with nuclear norm $\| \cdot \|_*$. Firstly, the function $\phi_\varepsilon(\cdot)$ is differentiable; secondly, for any $X \in \mathcal{R}^{m\times n}$, $\phi_\varepsilon(X)$ approaches to $rank(X)$ as ε approaches to zero. Zhao reformulated the generic approximation problem into nonlinear semidefinite programming (SDP) problem or bilevel SDP problem in [18]. However, the generic approximation problem, nonlinear SDP problem and bilevel SDP problem are all hard to be solved directly.

Taking the construction of $\phi_\varepsilon(\cdot)$ into consideration, we design a new kind of approximation functions to approximate the function $rank(\cdot)$ in this paper. Among all these approximation functions, some can be regarded as the bridge connecting the nuclear norm with the rank of matrix, and some can be used to design the new approximation problem, which can be solved by the classical projected gradient method.

The paper is organized as follows. In Section 2, a kind of approximation functions of $rank(\cdot)$ is introduced, and some properties of these functions and the corresponding approximation problems are deduced. The successive projected gradient method for solving the original problem (1.1) by using the projected gradient method to find the stationary point of a series of approximation problems is shown in Section 3, and the convergence theory about the successive projected gradient method is also given in this section. At last, the numerical result of the successive projected gradient method for solving the rank minimization problems with certain special feasible areas and the final discussion of this paper are given in Section 4 and Section 5, respectively.

2. Approximation functions and approximation problems

In this section, we will construct a new class of approximation functions on the basis of the construction of the generic approximation function. Then some properties of these approximation functions and the corresponding approximation problems will be given.

It is clear that, for any fixed X, $\phi_\varepsilon(X) \to rank(X)$ as $\varepsilon \to 0$, but the generic approximation function $\phi_\varepsilon(\cdot)$ is always majored by the rank function $rank(\cdot)$ for all $X \in \mathcal{R}^{m\times n}$. So, we want to design a new kind of approximation functions of the matrix rank valued around $rank(\cdot)$ over $\mathcal{R}^{m\times n}$, and ε is introduced as another variable in this kind of functions, i.e.,

$$\Phi(X,\varepsilon) = (1+\varepsilon^\varrho)tr\Big\{[X^\top(XX^\top+\varepsilon I)^{-1}X]^\beta\Big\} = \sum_{i=1}^{\iota}\frac{(1+\varepsilon^\varrho)\sigma_i^{2\beta}(X)}{(\sigma_i^2(X)+\varepsilon)^\beta}, \quad (2.1)$$

where $\varrho \in (0,1)$ is a rational number and $\beta \in \{2^{-k}|k=0,1,2,\cdots\}$. By using the new approximation functions, we can deduce the corresponding approximation problems as

$$\min \ \ \Phi(X,\varepsilon) \quad \text{s.t.} \ \ X\in\mathcal{C}. \quad (2.2)$$

We will show that the matrix rank minimization problem (1.1) can be solved by finding the stationary point of a series of approximation problems (2.2) in Section 3.

For X fixed, we now discuss the relationship between $\Phi(X,\varepsilon)$ and $rank(X)$ in two different cases depending on $\beta = 2^{-k}$ $(k=1,2,\cdots)$ and $\beta=1$.

Firstly, if $\beta = 2^{-k}, (k=1,2,\cdots)$, we have $\lim_{\varepsilon\to 0}\Phi(X,\varepsilon) = rank(X)$. In particular, if $\beta=\varrho=1/2$, then

$$\Phi(X,\varepsilon) = \sum_{i=1}^{\iota}\frac{(1+\varepsilon^{\frac{1}{2}})\sigma_i(X)}{(\sigma_i^2(X)+\varepsilon)^{\frac{1}{2}}},$$

which follows that $\lim_{\varepsilon\to 0}\Phi(X,\varepsilon) = rank(X)$ and $\lim_{\varepsilon\to\infty}\Phi(X,\varepsilon) = \| X \|_*$. So, $rank(X)$ and $\| X \|_*$ can be regarded as the two different extreme cases of $\Phi(X,\varepsilon)$ if $\beta=\varrho=1/2$.

Secondly, suppose $\beta = 1$. It is clear that $r := rank(X) = \sum_{i=1}^{\iota} \delta(\sigma_i(X))$, where

$$\delta(x) = \begin{cases} 1, & \text{if } x \in \mathcal{R} \setminus \{0\}, \\ 0, & \text{if } x = 0, \end{cases}$$

and there are three cases should be discussed when we compare the difference between $\frac{(1+\varepsilon^{\varrho})\sigma_i^2(X)}{\sigma_i^2(X)+\varepsilon}$ and $\delta(\sigma_i(X))$ for $i = 1, 2 \cdots, \iota$.

Case (I), when $\sigma_i(X) = 0$, the single term $\frac{(1+\varepsilon^{\varrho})\sigma_i^2(X)}{\sigma_i^2(X)+\varepsilon} = 0$, which is equal to $\delta(\sigma_i(X))$.

Case (II), when $0 < \sigma_i(X) < \varepsilon^{\frac{1-\varrho}{2}}$, we have $\varepsilon^{\varrho}\sigma_i^2(X) < \varepsilon \Longrightarrow \frac{(1+\varepsilon^{\varrho})\sigma_i^2(X)}{\sigma_i^2(X)+\varepsilon} < 1 = \delta(\sigma_i(X))$ and $\delta(\sigma_i(X)) - \frac{(1+\varepsilon^{\varrho})\sigma_i^2(X)}{\sigma_i^2(X)+\varepsilon} = \frac{\varepsilon-\varepsilon^{\varrho}\sigma_i^2(X)}{\sigma_i^2(X)+\varepsilon} < \varepsilon^{\varrho}(\frac{\varepsilon^{1-\varrho}}{\sigma_i^2(X)} - 1)$.

Case (III), when $\sigma_i(X) \geq \varepsilon^{\frac{1-\varrho}{2}}$, we have $\varepsilon^{\varrho}\sigma_i^2(X) \geq \varepsilon \Longrightarrow \frac{(1+\varepsilon^{\varrho})\sigma_i^2(X)}{\sigma_i^2(X)+\varepsilon} \geq 1 = \delta(\sigma_i(X))$ and $\frac{(1+\varepsilon^{\varrho})\sigma_i^2(X)}{\sigma_i^2(X)+\varepsilon} - \delta(\sigma_i(X)) = \frac{\varepsilon^{\varrho}\sigma_i^2(X)-\varepsilon}{\sigma_i^2(X)+\varepsilon} < \varepsilon^{\varrho}\frac{\sigma_i^2(X)-\varepsilon^{1-\varrho}}{\sigma_i^2(X)} < \varepsilon^{\varrho}$.

Hence, in the worst case, the difference between $rank(X)$ and $\Phi_{\varepsilon}(X)$ is $\max\{r\varepsilon^{\varrho}, \varepsilon^{\varrho}\sum_{i=1}^{r}(\frac{\varepsilon^{1-\varrho}}{\sigma_i^2(X)} - 1)\}$, which is a number in control.

For sufficiently small ε and any convergent sequence $\{X_k\}$ approaching to X, we have $\sigma_r(X_k) \geq \varepsilon^{\frac{1-\varrho}{2}}$ and $\sigma_{r+1}(X_k) < \varepsilon^{\frac{1-\varrho}{2}}$ for all k large enough. From the case (II) and (III), we know $\frac{(1+\varepsilon^{\varrho})\sigma_i^2(X_k)}{\sigma_i^2(X_k)+\varepsilon} \geq \delta(\sigma_i(X_k))$ for $i = 1, \cdots, r$ and $\frac{(1+\varepsilon^{\varrho})\sigma_i^2(X_k)}{\sigma_i^2(X_k)+\varepsilon} < \delta(\sigma_i(X_k))$ for $i \in \{r+1, \cdots, \iota\} \bigcap \{j | \sigma_j(X_k) \neq 0\}$. Instead of $\phi_{\varepsilon}(X_k)$ being always smaller than $rank(X_k)$, $\Phi(X_k, \varepsilon)$ takes value around $rank(X_k)$. Hence, it would be reasonable to use $\Phi(X_k, \varepsilon)$ to approximate $rank(X_k)$ when X_k close to X.

According to the above discussion, we have

Lemma 2.1 *For each $X \in \mathcal{R}^{m\times n}$, there exists accordingly a number $\varepsilon_\star > 0$ such that $rank(X) = \lceil \Phi(X, \varepsilon) \rceil$ for all $\varepsilon \in (0, \varepsilon_\star]$, where '$\lceil \cdot \rceil$' is rounding operator. Furthermore, if $\beta = 1$, the abstract error of $rank(X)$ and $\Phi(X, \varepsilon)$ satisfies $|rank(X) - \Phi(X, \varepsilon)| \leq \varepsilon^{\varrho} \max\{rank(X), \sum_{i=1}^{rank(X)} |\frac{\varepsilon^{1-\varrho}}{\sigma_i^2(X)} - 1|\}$.*

Let $\Phi(X, 0) = rank(X)$. According to Lemma 2.1, we can obtain the following results.

Proposition 2.2 *(a) The approximation functions $\Phi(\cdot, \cdot)$ are continuous everywhere in the region $\mathcal{R}^{m\times n} \times (0, \infty)$, and lower semicontinuous at $(X, 0)$, i.e.,*

$$\liminf_{(Y,\varepsilon)\to(X,0)} \Phi(Y, \varepsilon) \geq \Phi(X, 0) = rank(X).$$

(b) Let r^* be the minimum rank of (1.1). For given $\varepsilon > 0$, let Φ_{ε}^* and $X(\varepsilon)$ be the optimal value and an optimal solution of (2.2), respectively. Then

$$\lim_{\varepsilon\to 0} \Phi_{\varepsilon}^* = r^*$$

and any accumulation point of $X(\varepsilon)$, as ε approaching to zero, is a minimum rank solution of (1.1).
(c) If the optimal solution set of (1.1), denoted by $\mathcal{C}^*$, is bounded and $\beta = 1$, then there exists a constant $\delta > 0$ such that for any given $\varepsilon > 0$ the inequality

$$\Phi(X^*, \varepsilon) - \iota\varepsilon^{\varrho} \leq rank(X^*) \leq \Phi(X^*, \varepsilon) + (\frac{\iota\varepsilon^{1-\varrho}}{\delta^2} - \iota)\varepsilon^{\varrho}$$

holds for all $X^* \in \mathcal{C}^*$.

The proof of Proposition 2.2 is similar to that of the corresponding conclusions in [18], so we omit here. It is easy to see that the first conclusion in Lemma 2.1 and $(a), (b)$ in Proposition 2.2 still hold if $\beta = 1/k$ for $k = 1, 2, 3, \cdots$. From Lemma 2.1 and Proposition 2.2, we know it is reasonable to approximate the matrix rank problem (1.1) by using the approximation problem (2.2).

For further discussion, we suppose $\varrho = \frac{\xi_1}{\xi_2}$, where $0 < \xi_1 < \xi_2$ are all integers. Let $\{\varepsilon_k | k = 0, 1, 2, \cdots\}$ be a sequence of positive real numbers, and $\psi_k^j = \varepsilon_k^{j\varrho}\varepsilon_{k+1}^{j\varrho}\frac{\varepsilon_k^{1-j\varrho} - \varepsilon_{k+1}^{1-j\varrho}}{\varepsilon_k^{\varrho} - \varepsilon_{k+1}^{\varrho}}$, $j = 0, 1, 2, \cdots$. By ψ_{low}^j and ψ_{up}^j, we denote the positive constants $\min\left\{\frac{|j\xi_1 - \xi_2|}{\xi_1}\delta^{\frac{\xi_2 - 1}{\xi_2}}, \frac{|j\xi_1 - \xi_2|}{\xi_1}\delta^{\frac{j\xi_1 - 1}{\xi_2}}\right\}$ and $\max\left\{\frac{|j\xi_1 - \xi_2|}{\xi_1}, \frac{\xi_2}{\xi_1}\delta^{\frac{1-\xi_1}{\xi_2}}\right\}$, respectively, where the parameter $\delta \in (0, 1)$. Then we can deduce the following lemma.

Lemma 2.3 *Suppose $\varepsilon_k > \varepsilon_{k+1} \geq \delta\varepsilon_k$ holds for all k. Then, for all k, $\psi_k^j = 0$ if $j \geq 2$ and $\varrho = 1/j$; otherwise, $\psi_{low}^j \varepsilon_k^{1+(j-1)\varrho} <| \psi_k^j |< \psi_{up}^j \varepsilon_k^{1+(j-1)\varrho}$,*

Proof. It is easy to see that $\psi_k^j = 0$ if $j \geq 2$ and $\varrho = 1/j$.

For the rest conclusion, there are two cases should be discussed. Firstly, if $j \geq 2$ and $j\xi_1 > \xi_2$, then

$$\begin{aligned}
& \left|\varepsilon_k^{j\varrho}\varepsilon_{k+1}^{j\varrho}\frac{\varepsilon_k^{1-j\varrho} - \varepsilon_{k+1}^{1-j\varrho}}{\varepsilon_k^{\varrho} - \varepsilon_{k+1}^{\varrho}}\right| \\
= \; & \varepsilon_k\varepsilon_{k+1}\frac{\varepsilon_k^{\frac{j\xi_1 - \xi_2}{\xi_2}} - \varepsilon_{k+1}^{\frac{j\xi_1 - \xi_2}{\xi_2}}}{(\varepsilon_k^{\frac{1}{\xi_2}} - \varepsilon_{k+1}^{\frac{1}{\xi_2}})\left(\varepsilon_k^{\frac{\xi_1 - 1}{\xi_2}} + \varepsilon_k^{\frac{\xi_1 - 2}{\xi_2}}\varepsilon_{k+1}^{\frac{1}{\xi_2}} + \cdots + \varepsilon_{k+1}^{\frac{\xi_1 - 1}{\xi_2}}\right)} \\
= \; & \varepsilon_k\varepsilon_{k+1}\frac{\varepsilon_k^{\frac{j\xi_1 - \xi_2 - 1}{\xi_2}} + \varepsilon_k^{\frac{j\xi_1 - \xi_2 - 2}{\xi_2}}\varepsilon_{k+1}^{\frac{1}{\xi_2}} + \cdots + \varepsilon_{k+1}^{\frac{j\xi_1 - \xi_2 - 1}{\xi_2}}}{\varepsilon_k^{\frac{\xi_1 - 1}{\xi_2}} + \varepsilon_k^{\frac{\xi_1 - 2}{\xi_2}}\varepsilon_{k+1}^{\frac{1}{\xi_2}} + \cdots + \varepsilon_{k+1}^{\frac{\xi_1 - 1}{\xi_2}}},
\end{aligned} \tag{2.3}$$

where the first equality comes from the fact $x^{-\alpha} - y^{-\alpha} = x^{-\alpha}y^{-\alpha}(y^{\alpha} - x^{\alpha})$ and the second one comes from $x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \cdots + y^{n-1})$. From (2.3), we have $\psi_{low}^j\varepsilon_k^{1+(j-1)\varrho} \leq [(j\xi_1 - \xi_2)/\xi_1]\varepsilon_{k+1}^{\frac{j\xi_1 - 1}{\xi_2}}\varepsilon_k^{\frac{\xi_2 - \xi_1 + 1}{\xi_2}} <| \psi_k^j |< [(j\xi_1 - \xi_2)/\xi_1]\varepsilon_k^{\frac{j\xi_1 - 1}{\xi_2}}\varepsilon_{k+1}^{\frac{\xi_2 - \xi_1 + 1}{\xi_2}} \leq \psi_{up}^j\varepsilon_k^{1+(j-1)\varrho}$.

Secondly, if $j\xi_1 < \xi_2$, then

$$
\begin{aligned}
& \varepsilon_k^{j\varrho}\varepsilon_{k+1}^{j\varrho}\frac{\varepsilon_k^{1-j\varrho}-\varepsilon_{k+1}^{1-j\varrho}}{\varepsilon_k^{\varrho}-\varepsilon_{k+1}^{\varrho}} \\
= \; & \varepsilon_k^{j\varrho}\varepsilon_{k+1}^{j\varrho}\frac{\varepsilon_k^{\frac{\xi_2-j\xi_1}{\xi_2}}-\varepsilon_{k+1}^{\frac{\xi_2-j\xi_1}{\xi_2}}}{(\varepsilon_k^{\frac{1}{\xi_2}}-\varepsilon_{k+1}^{\frac{1}{\xi_2}})\Big(\varepsilon_k^{\frac{\xi_1-1}{\xi_2}}+\varepsilon_k^{\frac{\xi_1-2}{\xi_2}}\varepsilon_{k+1}^{\frac{1}{\xi_2}}+\cdots+\varepsilon_{k+1}^{\frac{\xi_1-1}{\xi_2}}\Big)} \\
= \; & \varepsilon_k^{j\varrho}\varepsilon_{k+1}^{j\varrho}\frac{\varepsilon_k^{\frac{\xi_2-j\xi_1-1}{\xi_2}}+\varepsilon_k^{\frac{\xi_2-j\xi_1-2}{\xi_2}}\varepsilon_{k+1}^{\frac{1}{\xi_2}}+\cdots+\varepsilon_{k+1}^{\frac{\xi_2-j\xi_1-1}{\xi_2}}}{\varepsilon_k^{\frac{\xi_1-1}{\xi_2}}+\varepsilon_k^{\frac{\xi_1-2}{\xi_2}}\varepsilon_{k+1}^{\frac{1}{\xi_2}}+\cdots+\varepsilon_{k+1}^{\frac{\xi_1-1}{\xi_2}}},
\end{aligned}
$$

which follows that $\psi_{low}^j\varepsilon_k^{1+(j-1)\varrho} \le [(\xi_2-j\xi_1)/\xi_1]\varepsilon_{k+1}^{\frac{\xi_2-1}{\xi_2}}\varepsilon_k^{\frac{1+(j-1)\xi_1}{\xi_2}} < \psi_k^j < [(\xi_2-j\xi_1)/\xi_1]\varepsilon_k^{\frac{\xi_2-1}{\xi_2}}\varepsilon_{k+1}^{\frac{1+(j-1)\xi_1}{\xi_2}} \le \psi_{up}^j\varepsilon_k^{1+(j-1)\varrho}$. □

Now, we will deduce the monotonicity of $\Phi(X,\varepsilon)$ with respect to X and ε, respectively.

Theorem 2.4 *(I) For fixed $\varepsilon > 0$ and $X \in \mathcal{R}^{m\times n}$, the function $\Phi(tX,\varepsilon)$ is increased with respect to t.*

(II) Suppose the sequence $\{X_k\}$ has a limit point $X_\star$ with $rank(X_\star) = r \ge 1$, and ε_k approaches to zero with $\varepsilon_k > \varepsilon_{k+1} \ge \delta\varepsilon_k$. Then $\Phi(X_k,\varepsilon_k) > \Phi(X_k,\varepsilon_{k+1})$ for all k large enough if, for each $i \in \{r+1,\cdots,\iota\}$, one of the following conditions holds

(a) $\varepsilon_k^{1-\varrho} = \circ(\sigma_i^2(X_k))$;

(b) $\sigma_i^2(X_k) = \circ(\varepsilon_k^{1+\frac{\varrho}{\beta}})$.

Proof. The first conclusion comes from $\Phi(X,\varepsilon) = (1+\varepsilon^\varrho)\sum_{i=1}^{\iota}(\frac{\sigma_i^2(X)}{\sigma_i^2(X)+\varepsilon})^\beta = (1+\varepsilon^\varrho)\sum_{i=1}^{\iota}(1-\frac{\varepsilon}{\sigma_i^2(X)+\varepsilon})^\beta$.

For the second conclusion, it is easy to see that

$$
\begin{aligned}
& \Phi(X_k,\varepsilon_k)-\Phi(X_k,\varepsilon_{k+1}) \\
= & \sum_{i=1}^{\iota}\frac{\sigma_i^{2\beta}(X_k)\Big[(1+\varepsilon_k^\varrho)(\sigma_i^2(X_k)+\varepsilon_{k+1})^\beta-(1+\varepsilon_{k+1}^\varrho)(\sigma_i^2(X_k)+\varepsilon_k)^\beta\Big]}{(\sigma_i^2(X_k)+\varepsilon_k)^\beta(\sigma_i^2(X_k)+\varepsilon_{k+1})^\beta} \\
= & \sum_{i=1}^{\iota}\frac{\sigma_i^{2\beta}(X_k)\Big[(1+\varepsilon_k^\varrho)^{\beta^{-1}}(\sigma_i^2(X_k)+\varepsilon_{k+1})-(1+\varepsilon_{k+1}^\varrho)^{\beta^{-1}}(\sigma_i^2(X_k)+\varepsilon_k)\Big]}{(\sigma_i^2(X_k)+\varepsilon_k)^\beta(\sigma_i^2(X_k)+\varepsilon_{k+1})^\beta} \\
& \Big[(1+\varepsilon_k^\varrho)^{\beta^{-1}-1}(\sigma_i^2(X_k)+\varepsilon_{k+1})^{1-\beta}+(1+\varepsilon_k^\varrho)^{\beta^{-1}-2}(\sigma_i^2(X_k)+\varepsilon_{k+1})^{1-2\beta} \\
& (1+\varepsilon_{k+1}^\varrho)(\sigma_i^2(X_k)+\varepsilon_k)^\beta+\cdots+(1+\varepsilon_{k+1}^\varrho)^{\beta^{-1}-1}(\sigma_i^2(X_k)+\varepsilon_k)^{1-\beta}\Big]^{-1} \quad (2.4)
\end{aligned}
$$

hold for all k, where the second equality comes from the fact $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + y^{n-1})$. For each $i \in \{1, 2, \cdots, \iota\}$, we know

$$\begin{aligned}
& (1+\varepsilon_k^{\varrho})^{\beta^{-1}}(\sigma_i^2(X_k)+\varepsilon_{k+1}) - (1+\varepsilon_{k+1}^{\varrho})^{\beta^{-1}}(\sigma_i^2(X_k)+\varepsilon_k) \\
= \; & (1 + C_{\beta^{-1}}^1\varepsilon_k^{\varrho} + C_{\beta^{-1}}^2\varepsilon_k^{2\varrho} + \cdots + \varepsilon_k^{\beta^{-1}\varrho})(\sigma_i^2(X_k)+\varepsilon_{k+1}) - \\
& (1 + C_{\beta^{-1}}^1\varepsilon_{k+1}^{\varrho} + C_{\beta^{-1}}^2\varepsilon_{k+1}^{2\varrho} + \cdots + \varepsilon_{k+1}^{\beta^{-1}\varrho})(\sigma_i^2(X_k)+\varepsilon_k) \\
= \; & \sigma_i^2(X_k)[\beta^{-1}(\varepsilon_k^{\varrho} - \varepsilon_{k+1}^{\varrho}) + C_{\beta^{-1}}^2(\varepsilon_k^{2\varrho} - \varepsilon_{k+1}^{2\varrho}) + \cdots + \\
& (\varepsilon_k^{\beta^{-1}\varrho} - \varepsilon_{k+1}^{\beta^{-1}\varrho})] - [(\varepsilon_k - \varepsilon_{k+1}) + \beta^{-1}\varepsilon_k^{\varrho}\varepsilon_{k+1}^{\varrho}(\varepsilon_k^{1-\varrho} - \varepsilon_{k+1}^{1-\varrho}) + \\
& C_{\beta^{-1}}^2\varepsilon_k^{2\varrho}\varepsilon_{k+1}^{2\varrho}(\varepsilon_k^{1-2\varrho} - \varepsilon_{k+1}^{1-2\varrho}) + \cdots + \varepsilon_k^{\beta^{-1}\varrho}\varepsilon_{k+1}^{\beta^{-1}\varrho}(\varepsilon_k^{1-\beta^{-1}\varrho} - \varepsilon_{k+1}^{1-\beta^{-1}\varrho})] \\
= \; & \beta^{-1}(\varepsilon_k^{\varrho} - \varepsilon_{k+1}^{\varrho})\Big\{\sigma_i^2(X_k) + \sigma_i^2(X_k)\Big[\beta C_{\beta^{-1}}^2(\varepsilon_k^{\varrho} + \varepsilon_{k+1}^{\varrho}) + \cdots + \\
& \beta(\varepsilon_k^{(\beta^{-1}-1)\varrho} + \cdots + \varepsilon_{k+1}^{(\beta^{-1}-1)\varrho})\Big] - (\beta\psi_k^0 + \psi_k^1 + \cdots + \beta\psi_k^{\beta^{-1}})\Big\}, \qquad (2.5)
\end{aligned}$$

where C_a^b denotes the number of combination.

For simplicity, let

$$\begin{aligned}
\varphi_k^i = \; & \sigma_i^2(X_k)[\beta C_{\beta^{-1}}^2(\varepsilon_k^{\varrho} + \varepsilon_{k+1}^{\varrho}) + \cdots + \beta(\varepsilon_k^{(\beta^{-1}-1)\varrho} + \cdots \\
& + \varepsilon_{k+1}^{(\beta^{-1}-1)\varrho})] - (\beta\psi_k^0 + \psi_k^1 + \cdots + \beta\psi_k^{\beta^{-1}}),
\end{aligned}$$

$$\begin{aligned}
\tilde{\varphi}_k^i = \; & (1+\varepsilon_k^{\varrho})^{\beta^{-1}-1}(\sigma_i^2(X_k)+\varepsilon_{k+1})^{1-\beta} + (1+\varepsilon_k^{\varrho})^{\beta^{-1}-2}(\sigma_i^2(X_k)+\varepsilon_{k+1})^{1-2\beta} \\
& (1+\varepsilon_{k+1}^{\varrho})(\sigma_i^2(X_k)+\varepsilon_k)^{\beta} + \cdots + (1+\varepsilon_{k+1}^{\varrho})^{\beta^{-1}-1}(\sigma_i^2(X_k)+\varepsilon_k)^{1-\beta}
\end{aligned}$$

and $\sigma_\star := \sigma_1^2(X_\star) > 0$. It is easy to see that $\varphi_k^i \to 0$ as $k \to \infty$ and $\tilde{\varphi}_k^i$ is bounded for each i by Lemma 2.3. Hence, there exist a positive integer K and a positive real number R such that $\frac{2\sigma_\star}{3} < \sigma_1^2(X_k) < \frac{4\sigma_\star}{3}$, $\sigma_i^2(X_k) > \varphi_k^i$ $(i = 2, 3, \cdots, r)$, $\tilde{\varphi}_k^1 < R$ and $\max\{\varepsilon_k, |\varphi_k^1|\} < \frac{\sigma_\star}{3}$ for all $k \geq K$. Let $\varphi_\star = \frac{(2\sigma_\star/3)^{\beta}\sigma_\star/3}{(5\sigma_\star/3)^{2\beta}R}$, then $\varphi_\star > 0$.

After taking φ_k^i, $\tilde{\varphi}_k^i$ and (2.5) into (2.4), we have

$$\begin{aligned}
& \Phi(X_k, \varepsilon_k) - \Phi(X_k, \varepsilon_{k+1}) \\
= \; & \sum_{i=1}^{\iota} \frac{\beta^{-1}\sigma_i^{2\beta}(X_k)(\varepsilon_k^{\varrho} - \varepsilon_{k+1}^{\varrho})[\sigma_i^2(X_k) + \varphi_k^i]}{(\sigma_i^2(X_k)+\varepsilon_k)^{\beta}(\sigma_i^2(X_k)+\varepsilon_{k+1})^{\beta}\tilde{\varphi}_k^i} \\
> \; & \beta^{-1}(\varepsilon_k^{\varrho} - \varepsilon_{k+1}^{\varrho})\Big\{\varphi_\star + \sum_{i=r+1}^{\iota} \frac{\sigma_i^{2\beta}(X_k)[\sigma_i^2(X_k) + \varphi_k^i]}{(\sigma_i^2(X_k)+\varepsilon_k)^{\beta}(\sigma_i^2(X_k)+\varepsilon_{k+1})^{\beta}\tilde{\varphi}_k^i}\Big\} \qquad (2.6)
\end{aligned}$$

hold for all $k \geq K$. It is clear that we only need to prove that each term in the sum formula in (2.6) is non-negative for all k large enough or tends to zero with $k \to \infty$.

Firstly, if condition (a) hold for a certain i, then $\varphi_k^i = O(\varepsilon_k^{1-\varrho}) = \circ(\sigma_i^2(X_k))$ by Lemma 2.3, which deduce that (2.5) is positive for all k large enough. Consequently, the $i-$th term in sum formula in (2.6) is positive.

For all i, by using $\varepsilon_{k+1} \geq \delta\varepsilon_k$, we know there exists a real number $R' > 0$ such that $\frac{\tilde{\varphi}_k^i}{\varepsilon_k^{1-\beta}} \geq R'$ for all k large enough.

Secondly, if condition (b) hold for a certain i, we have

$$\begin{aligned}
& \frac{\sigma_i^{2\beta}(X_k)[\sigma_i^2(X_k)+\varphi_k^i]}{(\sigma_i^2(X_k)+\varepsilon_k)^\beta(\sigma_i^2(X_k)+\varepsilon_{k+1})^\beta\tilde{\varphi}_k^i} \\
= & \frac{\sigma_i^{2\beta}(X_k)[\sigma_i^2(X_k)+\varphi_k^i]/\varepsilon_k^{1+\beta}}{(\sigma_i^2(X_k)+\varepsilon_k)^\beta(\sigma_i^2(X_k)+\varepsilon_{k+1})^\beta\tilde{\varphi}_k^i/\varepsilon_k^{1+\beta}} \\
= & \frac{[\sigma_i^2(X_k)/\varepsilon_k^{1+\frac{\varrho}{\beta}}]^\beta[\sigma_i^2(X_k)+\varphi_k^i]/\varepsilon_k^{1-\varrho}}{[(\sigma_i^2(X_k)+\varepsilon_k)^\beta/\varepsilon_k^\beta][(\sigma_i^2(X_k)+\varepsilon_{k+1})^\beta/\varepsilon_k^\beta][\tilde{\varphi}_k^i/\varepsilon_k^{1-\beta}]} \to 0
\end{aligned}$$

by $\varphi_k^i = O(\varepsilon_k^{1-\varrho})$. □

Remark 2.5 *The conclusions in Theorem 2.4 still hold if $\beta = \frac{1}{k}$ for all $k = 1, 2, 3, \cdots$, and the corresponding proof is similar.*

From Theorem 2.4, we can deduce the following result.

Corollary 2.6 *Fix X and suppose the sequence $\{\varepsilon_k\}$ approaches to zero with $\varepsilon_k > \varepsilon_{k+1} \geq \delta\varepsilon_k$. Then $\Phi(X, \varepsilon_k) > \Phi(X, \varepsilon_{k+1})$ for all k large enough.*

Proof. Let $X_k \equiv X$, then the condition (b) in Theorem 2.4 always holds for all $i = r+1, \cdots, \iota$. □

In the following, we will deduce the (second order) Fréchet derivative of $\Phi(\cdot, \varepsilon)$. For the detail of the Fréchet derivative of the matrix-value function, please refer to [15].

Definition 2.7 *Let $\Psi : \mathcal{R}^{m_1\times n_1} \to \mathcal{R}^{m_2\times n_2}$ and $X, H \in \mathcal{R}^{m_1\times n_1}$. If $\nabla_X\Psi : \mathcal{R}^{m_1\times n_1} \to \mathcal{R}^{m_2\times n_2}$ is a linear operator satisfying*

$$\lim_{H\to 0} \frac{\| \Psi(X+H) - \Psi(X) - \nabla_X\Psi(H) \|_F}{\| H \|_F} = 0,$$

then Ψ is said to be Fréchet differentiable at X and $\nabla_X\Psi$ ($\nabla_{XX}^2\Psi$) is the (second order) Fréchet derivative of Ψ at X.

By X_{ij}, λ_i and ΔX, we denote the i-th row, j-th column element of a matrix X, the i-th diagonal element of a diag matrix Λ and the increment of X, respectively. In order to deduce the formulas of $\nabla_X\Phi$ and $\nabla_{XX}^2\Phi(\Delta X)$, which are the Fréchet derivative of $\Phi(\cdot, \varepsilon)$ and the second order Fréchet derivative of $\Phi(\cdot, \varepsilon)$ on ΔX, we need the following lemmas.

Lemma 2.8 *(a). For any nonsingular square matrix* X, $\nabla_X(X^{-1})(\Delta X) = -X^{-1}(\Delta X)X^{-1}$.

(b). $\nabla_X(XX^\top)(\Delta X) = X\Delta X^\top + \Delta XX^\top$.

(c). $\nabla_X(XX^\top XX^\top)(\Delta X) = XX^\top(X\Delta X^\top + \Delta XX^\top) + (X\Delta X^\top + \Delta XX^\top)XX^\top$.

(d). For any $X \succ 0,\ \Delta X \in \mathcal{S}^n$, *let* $X = U\Lambda U^\top, \Delta\tilde{X} = U^\top \Delta X U$, *then*

$$\nabla_X(X^{\frac{1}{2}})(\Delta X) = U\left(\frac{\Delta\tilde{X}_{ij}}{\lambda_i^{\frac{1}{2}}+\lambda_j^{\frac{1}{2}}}\right)_{nn} U^\top,$$

where $\left(\frac{\Delta\tilde{X}_{ij}}{\lambda_i^{\frac{1}{2}}+\lambda_j^{\frac{1}{2}}}\right)_{nn}$ *is* $n \times n$ *matrix with the* i*-th row,* j*-th column element being* $\Delta\tilde{X}_{ij}/(\lambda_i^{\frac{1}{2}} + \lambda_j^{\frac{1}{2}})$.

Suppose that the matrix function $\Psi : \mathcal{R}^{m\times n} \to \mathcal{S}^m$ *is Fréchet differentiable, and* $\Psi(X) = \tilde{U}\tilde{\Lambda}\tilde{U}^\top, \Delta\hat{X} = \tilde{U}^\top \nabla_X \Psi(\Delta X)\tilde{U}$. *Then*

(e). $\nabla_X(X^\top \Psi(X)X)(\Delta X) =$
$X^\top \Psi(X)\Delta X + \Delta X^\top \Psi(X)X + X^\top \nabla_X \Psi(\Delta X)X.$

(f). $\nabla_X(\Psi(X)^{-1})(\Delta X) =$
$-\Psi(X)^{-1}(\nabla_X \Psi(\Delta X))\Psi(X)^{-1}$ *if* $\Psi(X)$ *is invertible.*

(g). $\nabla_X(\Psi(X)^{\frac{1}{2}})(\Delta X) = \tilde{U}\left(\frac{\Delta\hat{X}_{ij}}{\tilde{\lambda}_i^{\frac{1}{2}}+\tilde{\lambda}_j^{\frac{1}{2}}}\right)_{mm}\tilde{U}^\top$ *if* $\Psi(X) \succ 0$.

Proof. The first three formulas come from

$$\begin{aligned}
& (X+\Delta X)^{-1} - X^{-1} + X^{-1}(\Delta X)X^{-1} \\
= \;& X^{-1}[(I+\Delta XX^{-1})^{-1} - (I - \Delta XX^{-1})] \\
= \;& X^{-1}[(I+\Delta XX^{-1})^{-1} - (I+\Delta XX^{-1})^{-1}(I - \Delta XX^{-1}\Delta XX^{-1})] \\
= \;& X^{-1}(I+\Delta XX^{-1})^{-1}\Delta XX^{-1}\Delta XX^{-1} \\
= \;& O(\| \Delta X \|_F^2),
\end{aligned}$$

$$\begin{aligned}
& (X+\Delta X)(X+\Delta X)^\top - XX^\top - (X\Delta X^\top + \Delta XX^\top) \\
= \;& \Delta X\Delta X^\top = O(\| \Delta X \|_F^2)
\end{aligned}$$

and

$$\begin{aligned}
& (X+\Delta X)(X+\Delta X)^\top(X+\Delta X)(X+\Delta X)^\top \\
= \;& XX^\top XX^\top + XX^\top(X\Delta X^\top + \Delta XX^\top) + (X\Delta X^\top \\
& +\Delta XX^\top)XX^\top + O(\| \Delta X \|_F^2),
\end{aligned}$$

respectively.

For (d), it is easy to see that

$$
\begin{aligned}
& (X+\Delta X)^{\frac{1}{2}} - X^{\frac{1}{2}} - U\left(\frac{\Delta\tilde{X}_{ij}}{\lambda_i^{\frac{1}{2}}+\lambda_j^{\frac{1}{2}}}\right)_{nn} U^\top \\
= \;& U\Big[(\Lambda+\Delta\tilde{X})^{\frac{1}{2}} - \Lambda^{\frac{1}{2}} - \left(\frac{\Delta\tilde{X}_{ij}}{\lambda_i^{\frac{1}{2}}+\lambda_j^{\frac{1}{2}}}\right)_{nn}\Big]U^\top. \qquad (2.7)
\end{aligned}
$$

Let $\Upsilon := (\Lambda+\Delta\tilde{X})^{\frac{1}{2}} - \Lambda^{\frac{1}{2}} - \left(\frac{\Delta\tilde{X}_{ij}}{\lambda_i^{\frac{1}{2}}+\lambda_j^{\frac{1}{2}}}\right)_{nn}$, then

$$
\Lambda^{\frac{1}{2}}\Upsilon = \Lambda^{\frac{1}{2}}(\Lambda+\Delta\tilde{X})^{\frac{1}{2}} - \Lambda - \left(\frac{\lambda_i^{\frac{1}{2}}\Delta\tilde{X}_{ij}}{\lambda_i^{\frac{1}{2}}+\lambda_j^{\frac{1}{2}}}\right)_{nn} \qquad (2.8)
$$

and

$$
\Upsilon\Lambda^{\frac{1}{2}} = (\Lambda+\Delta\tilde{X})^{\frac{1}{2}}\Lambda^{\frac{1}{2}} - \Lambda - \left(\frac{\lambda_j^{\frac{1}{2}}\Delta\tilde{X}_{ij}}{\lambda_i^{\frac{1}{2}}+\lambda_j^{\frac{1}{2}}}\right)_{nn}. \qquad (2.9)
$$

Adding (2.8) with (2.9) yields

$$
\begin{aligned}
& \Lambda^{\frac{1}{2}}\Upsilon + \Upsilon\Lambda^{\frac{1}{2}} \\
= \;& [\Lambda^{\frac{1}{2}}(\Lambda+\Delta\tilde{X})^{\frac{1}{2}} + (\Lambda+\Delta\tilde{X})^{\frac{1}{2}}\Lambda^{\frac{1}{2}}] - 2\Lambda - \Delta\tilde{X}. \\
= \;& -[(\Lambda+\Delta\tilde{X})^{\frac{1}{2}} - \Lambda^{\frac{1}{2}}]^2 \\
= \;& \circ(\|\Delta\tilde{X}\|_F) = \circ(\|\Delta X\|_F), \qquad (2.10)
\end{aligned}
$$

where the third equality comes from

$$
\begin{aligned}
0 \;\preceq\;& [(\Lambda+\Delta\tilde{X})^{\frac{1}{2}} - \Lambda^{\frac{1}{2}}]^2/\|\Delta\tilde{X}\|_F \\
=\;& [(\Lambda/\|\Delta\tilde{X}\|_F + \Delta\tilde{X}/\|\Delta\tilde{X}\|_F)^{\frac{1}{2}} - (\Lambda/\|\Delta\tilde{X}\|_F)^{\frac{1}{2}}]^2 \\
\preceq\;& \max\Big\{[(\Lambda/\|\Delta\tilde{X}\|_F + I)^{\frac{1}{2}} - (\Lambda/\|\Delta\tilde{X}\|_F)^{\frac{1}{2}}]^2, \\
& [(\Lambda/\|\Delta\tilde{X}\|_F)^{\frac{1}{2}} - (\Lambda/\|\Delta\tilde{X}\|_F - I)^{\frac{1}{2}}]^2\Big\},
\end{aligned}
$$

$$
\lim_{x\to+\infty}(x+1)^{\frac{1}{2}} - x^{\frac{1}{2}} = \lim_{x\to+\infty}\frac{(1+x^{-1})^{\frac{1}{2}}-1}{x^{-\frac{1}{2}}} = \lim_{x\to+\infty}\frac{(1+x^{-1})^{-\frac{1}{2}}}{x^{\frac{1}{2}}} = 0
$$

and

$$
\lim_{x\to+\infty}x^{\frac{1}{2}} - (x-1)^{\frac{1}{2}} = \lim_{x\to+\infty}\frac{1-(1-x^{-1})^{\frac{1}{2}}}{x^{-\frac{1}{2}}} = \lim_{x\to+\infty}\frac{(1-x^{-1})^{-\frac{1}{2}}}{x^{\frac{1}{2}}} = 0.
$$

Together with (2.10), the positive definiteness of $\Lambda^{\frac{1}{2}}$ and the discussion between Lemma 6.1 and Lemma 6.2 in [16], we know $\Upsilon = \circ(\|\Delta X\|_F)$, which together with (2.7) complete the proof of (d).

The result (e) comes from

$$
\begin{aligned}
& (X^\top + \Delta X^\top)\Psi(X+\Delta X)(X+\Delta X) \\
= \; & (X^\top + \Delta X^\top)(\Psi(X) + \nabla_X\Psi(\Delta X) + \circ(\| \Delta X \|_F))(X+\Delta X) \\
= \; & X^\top\Psi(X)X + X^\top\Psi(X)\Delta X + \Delta X^\top\Psi(X)X + \\
& X^\top\nabla_X\Psi(\Delta X)X + \circ(\| \Delta X \|_F).
\end{aligned}
$$

For (f), it is easy to see that

$$
\begin{aligned}
& \Psi(X+\Delta X)^{-1} - (\Psi(X) + \nabla_X\Psi(\Delta X))^{-1} \\
= \; & [\Psi(X) + \nabla_X\Psi(\Delta X) + \circ(\| \Delta X \|_F)]^{-1} - (\Psi(X) + \nabla_X\Psi(\Delta X))^{-1} \\
= \; & [\Psi(X) + \nabla_X\Psi(\Delta X) + \circ(\| \Delta X \|_F)]^{-1}\Big[I - (\Psi(X) + \nabla_X\Psi(\Delta X)) \\
& (\Psi(X) + \nabla_X\Psi(\Delta X))^{-1} + \circ(\| \Delta X \|_F)(\Psi(X) + \nabla_X\Psi(\Delta X))^{-1}\Big] \\
= \; & [\Psi(X) + \nabla_X\Psi(\Delta X) + \circ(\| \Delta X \|_F)]^{-1}[\circ(\| \Delta X \|_F)] \\
& [\Psi(X) + \nabla_X\Psi(\Delta X)]^{-1} \\
= \; & \circ(\| \Delta X \|_F),
\end{aligned}
$$

which together with (a) deduces (f).

Let $\bar{\lambda}_i$ be the $i-$th eigenvalue of $\Psi(X) + \nabla_X\Psi(\Delta X)$, then there exists $R > 0$, such that $\bar{\lambda}_i \geq R$ for all $\| \Delta X \|_F$ small enough due to the positive definite of $\Psi(X)$. It follows that for any orthogonal matrix $\hat{U}$,

$$
\hat{U}\left(\frac{(\hat{U}^\top\circ(\|\Delta X\|_F)\hat{U})_{ij}}{\bar{\lambda}_i^{\frac{1}{2}} + \bar{\lambda}_j^{\frac{1}{2}}}\right)_{mm}\hat{U}^\top = \circ(\| \Delta X \|_F). \tag{2.11}
$$

For (g), we have

$$
\begin{aligned}
& (\Psi(X+\Delta X))^{\frac{1}{2}} \\
= \; & [\Psi(X) + \nabla_X\Psi(\Delta X) + \circ(\| \Delta X \|_F)]^{\frac{1}{2}} \\
= \; & [\Psi(X) + \nabla_X\Psi(\Delta X)]^{\frac{1}{2}} + \circ(\| \Delta X \|_F),
\end{aligned} \tag{2.12}
$$

where the second equality comes from (d) and (2.11). Together with (2.12) and (d), we can deduce (g). $\square$

Lemma 2.9 *Given $B \in \mathcal{R}^{m\times n}$, let operations $L_B : R^{m\times n} \to \mathcal{S}^n$, $\tilde{L}_B : \mathcal{R}^{m\times n} \to \mathcal{S}^m$ be defined as*

$$
L_B(X) = X^\top B + B^\top X, \quad \tilde{L}_B(X) = BX^\top + XB^\top,
$$

respectively, then the adjoints of the above two operations are

$$
L_B^*(Y) = B(Y + Y^\top), \quad \tilde{L}_B^*(Z) = (Z + Z^\top)B,
$$

where $Y \in R^{n\times n}, Z \in R^{m\times m}$.

Proof. For any $X \in R^{m\times n}, Y \in R^{n\times n}$, we have

$$\begin{aligned}\langle Y, L_B(X)\rangle &= \langle Y, X^\top B + B^\top X\rangle \\ &= \langle Y, X^\top B\rangle + \langle Y, B^\top X\rangle \\ &= \langle X, BY^\top\rangle + \langle BY, X\rangle = \langle X, B(Y^\top + Y)\rangle.\end{aligned}$$

The proof of the second formula is similar. $\square$

For $\beta = 1/2^k$ $(k = 1, 2, \cdots)$, let $\Pi_\beta = 1$ if $\beta = 1/2$, otherwise

$$\begin{aligned}\Pi_\beta &= \Big(2\sum_{i=1}^{\iota} \frac{(1+\varepsilon^\varrho)^{1/2}\sigma_i^{2\beta}(X)}{(\sigma_i^2(X)+\varepsilon)^\beta}\Big)^{-1}\cdots \\ &\Big(2\sum_{i=1}^{\iota} \frac{(1+\varepsilon^\varrho)^{1/2}\sigma_i^{1/4}(X)}{(\sigma_i^2(X)+\varepsilon)^{1/8}}\Big)^{-1}\Big(2\sum_{i=1}^{\iota} \frac{(1+\varepsilon^\varrho)^{1/2}\sigma_i^{1/2}(X)}{(\sigma_i^2(X)+\varepsilon)^{1/4}}\Big)^{-1}.\end{aligned}$$

Now, we are going to deduce the formulas of $\nabla_X\Phi$ and $\nabla^2_{XX}\Phi(\Delta X)$.

Theorem 2.10 *If $\beta = 1$, the Fréchet derivative of $\Phi(\cdot, \varepsilon)$ is*

$$\nabla_X\Phi = 2\varepsilon(1+\varepsilon^\varrho)(XX^\top + \varepsilon I)^{-2}X, \tag{2.13}$$

and for all $\Delta X \in \mathcal{R}^{m\times n}$, the second order Fréchet derivative of $\Phi(\cdot, \varepsilon)$ on ΔX is

$$\begin{aligned}\nabla^2_{XX}\Phi(\Delta X) = 2\varepsilon(1+\varepsilon^\varrho)\Big\{&(XX^\top+\varepsilon I)^{-2}\Delta X - (XX^\top+\varepsilon I)^{-2} \\ &\Big[2\varepsilon\tilde{L}_X(\Delta X) + L_{XX^\top}(\tilde{L}_X(\Delta X))\Big](XX^\top+\varepsilon I)^{-2}X\Big\}.\end{aligned} \tag{2.14}$$

Furthermore, $\nabla^2_{XX}\Phi$ is self-adjoint.

Suppose the singular value decomposition of X is $X = U\Lambda V^\top$, and $\Delta\tilde{X} = U^\top \Delta X V$. If $\beta = 2^{-k}$ $(k = 1, 2, \cdots)$, $m \geq n$ and X is full column rank, then

$$\nabla_X\Phi(\Delta X) = \Pi_\beta(1+\varepsilon^\varrho)\varepsilon\sum_{i=1}^{\iota}\frac{\Delta\tilde{X}_{ii}}{(\lambda_i^2+\varepsilon)^{\frac{3}{2}}}.$$

Proof. For the first conclusion, we have $\Phi(X, \varepsilon) = (1+\varepsilon^\varrho)\langle I, X^\top(XX^\top + \varepsilon I)^{-1}X\rangle$, which follows that for all $\Delta X \in \mathcal{R}^{m\times n}$,

$$\begin{aligned}&\nabla_X\Phi(\Delta X) \\ =\ &(1+\varepsilon^\varrho)\Big\langle I, \Delta X^\top(XX^\top+\varepsilon I)^{-1}X + X^\top(XX^\top+\varepsilon I)^{-1}\Delta X\Big\rangle \\ &-(1+\varepsilon^\varrho)\Big\langle I, X^\top(XX^\top+\varepsilon I)^{-1}(X\Delta X^\top + \Delta X X^\top)(XX^\top+\varepsilon I)^{-1}X\Big\rangle \\ =\ &(1+\varepsilon^\varrho)\Big\langle I, L_{(XX^\top+\varepsilon I)^{-1}X}(\Delta X)\Big\rangle\end{aligned}$$

$$
\begin{aligned}
& -(1+\varepsilon^{\varrho})\Big\langle (XX^\top+\varepsilon I)^{-1}XX^\top(XX^\top+\varepsilon I)^{-1}, \tilde{L}_X(\Delta X)\Big\rangle \\
= \;& 2(1+\varepsilon^{\varrho})\Big\langle (XX^\top+\varepsilon I)^{-1}X, \Delta X\Big\rangle \\
& -2(1+\varepsilon^{\varrho})\Big\langle (XX^\top+\varepsilon I)^{-1}XX^\top(XX^\top+\varepsilon I)^{-1}X, \Delta X\Big\rangle \\
= \;& 2(1+\varepsilon^{\varrho})\Big\langle (XX^\top+\varepsilon I)^{-1}X-(XX^\top+\varepsilon I)^{-1}XX^\top(XX^\top+\varepsilon I)^{-1}X, \Delta X\Big\rangle \\
= \;& \Big\langle 2(1+\varepsilon^{\varrho})\varepsilon(XX^\top+\varepsilon I)^{-2}X, \Delta X\Big\rangle,
\end{aligned}
$$

where the first equality comes from $(b), (e), (f)$ in Lemma 2.8, the third equality comes from Lemma 2.9, and the last one comes from $I=(XX^\top+\varepsilon I)(XX^\top+\varepsilon I)^{-1}=XX^\top(XX^\top+\varepsilon I)^{-1}+\varepsilon(XX^\top+\varepsilon I)^{-1}$.

The second order Fréchet derivative of $\Phi(\cdot,\varepsilon)$ on ΔX can be deduced directly from the first order derivative by using $(XX^\top+\varepsilon I)^{-2}=(XX^\top XX^\top+2\varepsilon XX^\top+\varepsilon^2 I)^{-1}$ and $(b), (c), (f)$ in Lemma 2.8.

Taking the self-adjoint of $\nabla^2_{XX}\Phi$ into consideration, we have for all $Z, Y \in R^{m\times n}$,

$$
\begin{aligned}
& \langle Y, \nabla^2_{XX}\Phi(Z)\rangle \\
= \;& 2\varepsilon(1+\varepsilon^{\varrho})\Big\langle Y, (XX^\top+\varepsilon I)^{-2}Z-(XX^\top+\varepsilon I)^{-2}\Big[2\varepsilon\tilde{L}_X(Z) \\
& +L_{XX^\top}(\tilde{L}_X(Z))\Big](XX^\top+\varepsilon I)^{-2}X\Big\rangle \\
= \;& 2\varepsilon(1+\varepsilon^{\varrho})\Big\{\langle Y, (XX^\top+\varepsilon I)^{-2}Z\rangle-\Big\langle Y, (XX^\top+\varepsilon I)^{-2}\Big[2\varepsilon\tilde{L}_X(Z) \\
& +L_{XX^\top}(\tilde{L}_X(Z))\Big](XX^\top+\varepsilon I)^{-2}X\Big\rangle\Big\}. \qquad (2.15)
\end{aligned}
$$

It is clear that

$$
\langle Y, (XX^\top+\varepsilon I)^{-2}Z\rangle = \langle Z, (XX^\top+\varepsilon I)^{-2}Y\rangle \qquad (2.16)
$$

and

$$
\begin{aligned}
& \Big\langle Y, (XX^\top+\varepsilon I)^{-2}\tilde{L}_X(Z)(XX^\top+\varepsilon I)^{-2}X\Big\rangle \\
= \;& \Big\langle (XX^\top+\varepsilon I)^{-2}YX^\top(XX^\top+\varepsilon I)^{-2}, \tilde{L}_X(Z)\Big\rangle \\
= \;& \Big\langle (XX^\top+\varepsilon I)^{-2}(XY^\top+YX^\top)(XX^\top+\varepsilon I)^{-2}, Z\Big\rangle \\
= \;& \Big\langle (XX^\top+\varepsilon I)^{-2}\tilde{L}_X(Y)(XX^\top+\varepsilon I)^{-2}, Z\Big\rangle, \qquad (2.17)
\end{aligned}
$$

where the second equality comes from Lemma 2.9. We also have

$$
\Big\langle Y, (XX^\top+\varepsilon I)^{-2}L_{XX^\top}(\tilde{L}_X(Z))(XX^\top+\varepsilon I)^{-2}X\Big\rangle
$$

$$
\begin{aligned}
&= \Big\langle (XX^\top+\varepsilon I)^{-2} Y X^\top (XX^\top+\varepsilon I)^{-2}, L_{XX^\top}(\tilde{L}_X(Z)) \Big\rangle \\
&= \Big\langle (XX^\top+\varepsilon I)^{-2}(XY^\top+YX^\top)(XX^\top+\varepsilon I)^{-2}XX^\top, \tilde{L}_X(Z) \Big\rangle \\
&= \Big\langle (XX^\top+\varepsilon I)^{-2}\Big[XX^\top(XY^\top+YX^\top)+(XY^\top+YX^\top)XX^\top\Big] \\
&\qquad (XX^\top+\varepsilon I)^{-2}X, Z \Big\rangle, \\
&= \Big\langle (XX^\top+\varepsilon I)^{-2} L_{XX^\top}(\tilde{L}_X(Y))(XX^\top+\varepsilon I)^{-2}X, Z \Big\rangle, \qquad (2.18)
\end{aligned}
$$

where the second and third equality come from Lemma 2.9 and the fact that $XX^\top(XX^\top+\varepsilon I) = XX^\top XX^\top + \varepsilon XX^\top = (XX^\top+\varepsilon I)XX^\top$. Together the formulas (2.15),(2.16),(2.17) with (2.18), we can deduce the self-adjoint of $\nabla^2_{XX}\Phi$.

Finally, for the case of $\beta = 1/2$, we know $X^\top(XX^\top+\varepsilon I)^{-1}X \succ 0$ since $m \geq n$ and X is full column rank. Let $\tilde{\Psi}(X, \Delta X, \varepsilon) = \Delta X^\top(XX^\top+\varepsilon I)^{-1}X + X^\top(XX^\top+\varepsilon I)^{-1}\Delta X - X^\top(XX^\top+\varepsilon I)^{-1}(X\Delta X^\top+\Delta XX^\top)(XX^\top+\varepsilon I)^{-1}X$ and $\Psi(X, \Delta X, \varepsilon) = X^\top(XX^\top+\varepsilon I)^{-1}X + \tilde{\Psi}(X, \Delta X, \varepsilon)$. It is easy to see that $\Psi(X, \Delta X, \varepsilon) \succ 0$ for all $\| \Delta X \|_F$ small enough and $X^\top(XX^\top+\varepsilon I)^{-1}X = U[\Lambda^\top(\Lambda\Lambda^\top+\varepsilon I)^{-1}\Lambda]U^\top$.

Similar to the proof of the first conclusion, we can deduce

$$
\Phi(X+\Delta X, \varepsilon) = (1+\varepsilon^\varrho)\langle I, [\Psi(X, \Delta X, \varepsilon) + \circ(\| \Delta X \|_F)]^{\frac{1}{2}}\rangle \qquad (2.19)
$$

by $(b), (e), (f)$ in Lemma 2.8.

It is easy to see that

$$
\begin{aligned}
& U^\top \tilde{\Psi}(X, \Delta X, \varepsilon) U \\
=\ & U^\top[\Delta X^\top(XX^\top+\varepsilon I)^{-1}X + X^\top(XX^\top+\varepsilon I)^{-1}\Delta X - \\
& X^\top(XX^\top+\varepsilon I)^{-1}(X\Delta X^\top+\Delta XX^\top)(XX^\top+\varepsilon I)^{-1}X]U \\
=\ & \Delta\tilde{X}^\top(\Lambda\Lambda^\top+\varepsilon I)^{-1}\Lambda + \Lambda^\top(\Lambda\Lambda^\top+\varepsilon I)^{-1}\Delta\tilde{X} - \\
& \Lambda^\top(\Lambda\Lambda^\top+\varepsilon I)^{-1}(\Lambda\Delta\tilde{X}^\top+\Delta\tilde{X}\Lambda^\top)(\Lambda\Lambda^\top+\varepsilon I)^{-1}\Lambda \\
=\ & \left(\left(\tfrac{\lambda_i}{\lambda_i^2+\varepsilon} + \tfrac{\lambda_j}{\lambda_j^2+\varepsilon}\right)\Delta\tilde{X}_{ij} \right)_{nn} - \left(\tfrac{\lambda_i}{\lambda_i^2+\varepsilon}\tfrac{\lambda_j}{\lambda_j^2+\varepsilon}(\lambda_i+\lambda_j)\Delta\tilde{X}_{ij} \right)_{nn} \\
=\ & \left(\tfrac{(\lambda_i+\lambda_j)\varepsilon}{(\lambda_i^2+\varepsilon)(\lambda_j^2+\varepsilon)}\Delta\tilde{X}_{ij} \right)_{nn}. \qquad (2.20)
\end{aligned}
$$

Together (g) in Lemma 2.8 with (2.19) and (2.20), we can deduce

$$
\begin{aligned}
\nabla_X\Phi(\Delta X) &= (1+\varepsilon^\varrho)\left\langle I, \left(\frac{\frac{(\lambda_i+\lambda_j)\varepsilon}{(\lambda_i^2+\varepsilon)(\lambda_j^2+\varepsilon)}}{\frac{\lambda_i}{(\lambda_i^2+\varepsilon)^{\frac{1}{2}}} + \frac{\lambda_j}{(\lambda_j^2+\varepsilon)^{\frac{1}{2}}}}\Delta\tilde{X}_{ij} \right)_{nn} \right\rangle \\
&= (1+\varepsilon^\varrho)\left\langle I, \left(\frac{(\lambda_i+\lambda_j)\varepsilon\Delta\tilde{X}_{ij}}{\lambda_i(\lambda_i^2+\varepsilon)^{\frac{1}{2}}(\lambda_j^2+\varepsilon)+\lambda_j(\lambda_j^2+\varepsilon)^{\frac{1}{2}}(\lambda_i^2+\varepsilon)} \right)_{nn} \right\rangle \\
&= (1+\varepsilon^\varrho)\varepsilon\sum_{i=1}^{\iota}\frac{\Delta\tilde{X}_{ii}}{(\lambda_i^2+\varepsilon)^{\frac{3}{2}}}. \qquad (2.21)
\end{aligned}
$$

For the case of $\beta = 2^{-k}$ $(k = 2, 3, \cdots)$, the corresponding conclusions can be deduced in a similar way by (g) in Lemma 2.8 with (2.21). $\Box$

Remark 2.11 *If $\beta = 2^{-j}$ $(j = 1, 2, \cdots)$ and X is rank-deficient, let $\{X_k\}$ be a full column rank matrix sequence with $X_k \to X$. Similar to that in reference [14], the B-subdifferentiable (C-subdifferentiable) of $\Phi(\cdot, \varepsilon)$ on X can be deduced from the third formula in (2.21) by taking the accumulated points (convex envelop of the accumulated points) of the singular values and the corresponding singular vectors of X_k when $k \to \infty$.*

3. Successive Projected Gradient Method

In this section, we will construct the successive projected gradient method for solving problem (1.1) with $\mathcal{C} = \{X \in \mathcal{R}^{m\times n} : \mathcal{A}(X) = b\}$, where $\mathcal{A} : \mathcal{R}^{m\times n} \longrightarrow \mathcal{R}^l$ is a linear operator.

After taking $X = \sum_{i=1}^m \sum_{j=1}^n X_{i,j} E_i^j$ (where, E_i^j is the $m \times n$ matrix with all entries equal to zero but the i-th row, j-th column component equal to one) into consideration, we can reformulate $\mathcal{A}(X) = b$ as $Avec(X) = b$ with $A \in R^{l\times mn}$ (where, $A_{i,j} = \mathcal{A}(E_i^j)$), and $vec(X) = (X_{1,1}, X_{2,1}, \cdots, X_{m,1}, X_{1,2}, X_{2,2}, \cdots, X_{m,2}, \cdots, X_{1,n}, X_{2,n}, \cdots, X_{m,n})^\top \in \mathcal{R}^{mn}$. Hence, without loss of generality, we suppose the linear operator $\mathcal{A}$ has the following formula

$$\mathcal{A}(X) = (\langle A_1, X\rangle,\ \langle A_2, X\rangle,\ \cdots,\ \langle A_l, X\rangle)^\top$$

where $A_k = mat(A(k; :)) \in \mathcal{R}^{m\times n},\ k = 1, 2, \cdots, l$ (mat is the invert operate of vec and $A(k; :)$ denotes the k-th row of A). Hence the adjoint operator of $\mathcal{A}$ can be reformulated as: for all $\lambda \in \mathcal{R}^l,\ \mathcal{A}^*\lambda = \sum_{i=1}^l \lambda_i A_i$. We suppose $b \neq 0$, since the optimal solutions of the problem (1.1) and (2.2) are all zero if $b = 0$ by the first result of Theorem 2.4.

From (b) in Proposition 2.2, we know the optimal solution and the optimal value of the approximation problem (2.2) will approach to that of the original problem (1.1) when ε approaches to zero under certain conditions. So, in this section, we will find the solution of the problem (1.1) by using the projected gradient (PG) method to solve the approximation problem (2.2) with certain ε. Let $P_{\mathcal{C}}[\cdot]$ be the orthogonal projection from matrix space into the convex set $\mathcal{C}$, i.e., $P_{\mathcal{C}}[X] = argmin\{\| Z - X \|_F\!: Z \in \mathcal{C}\}$. It is easy to see that $P_{\mathcal{C}}[X] = mat\{[I - A^\top(AA^\top)^{-1}A]vec(X) + A^\top(AA^\top)^{-1}b\}$ in this section.

The projected gradient method was originally proposed by Goldstein[7], and Levitin and Polyak[9] in 1960s, even since then, there have been various extensions which make the PG method more widely applicable and more efficient in computation, e.g., [2, 5, 17]. In reference [17], Zhao and Chen solved the continuous non-smooth optimization problem by the smoothing PG method, which is designed by using a series of the classical PG method of Calamai and More [2] to find the stationary point of the smooth subproblems. As far as the ability of solving problem be concerned, the smoothing PG method is better than the classical PG method because the smoothing PG method can be used for solving the problem comprising the non-smooth (locally Lipschitzian continuous) function but the classical PG method can only be used for solving the smooth problem.

However, the function $rank(\cdot)$ is not even continuous, which follows that the smoothing PG and the classical PG method are not suitable for solving the problem (1.1) directly. In order to solve the problem (1.1), we design the successive PG method by using the construction of the smoothing PG. The main difference between the smoothing PG method and the successive PG method is the update of ε_k. The updating formula of ε_k in the successive PG method is designed to guarantee the decrease of $\Phi(X_k, \varepsilon_k)$ for each k.

For the approximation problem (2.2), in what follows, we only consider the special case of $\beta = 1$. By $\nabla_{X_k}\Phi$ and r_k, we denote the Fréchet derivative of $\Phi(\cdot, \varepsilon_k)$ on X_k and the rank of X_k.

Algorithm 3.1 *Successive Projected Gradient Method*

(S.0) *Let* $\hat{\gamma}, \gamma_1, \gamma_3$ *and* γ_4 *be positive constants, where* $\gamma_4 \ll 1$ *and* $\gamma_1 \ll \gamma_3$. *Let* $\gamma_2, \tau, \tau_1, \tau_2, \tau_3$ *and* τ_4 *be constants in* $(0, 1)$, *where* $\tau_1 \leq \tau_2$. *Choose* $X_0 \in \mathcal{C}$ *and* $\varepsilon_0 > 0$. *Set* ϵ, $k = 0$, $j = 0$.

(S.1) *If* $\| P_{\mathcal{C}}[X_k - \nabla_{X_k}\Phi] - X_k \|_F = 0$, *Let* $X_{k+1} := X_k$ *and go to (S.3). Otherwise, let* $Y_k^0 = X_k$, *and go to (S.2).*

(S.2) *(PG method) Compute*

$$Y_k^j(\alpha) = P_{\mathcal{C}}[Y_k^j - \alpha\nabla_{Y_k^j}\Phi],$$

and set $Y_k^{j+1} := Y_k^j(\alpha_k^j)$ *where* α_k^j *is chosen so that,*

$$\Phi(Y_k^{j+1}, \varepsilon_k) \leq \Phi(Y_k^j, \varepsilon_k) + \tau_1\langle\nabla_{Y_k^j}\Phi, Y_k^{j+1} - Y_k^j\rangle \tag{3.1}$$

and

$$\gamma_3 \geq \alpha_k^j \geq \gamma_1, \text{ or } \alpha_k^j \geq \gamma_2\tilde{\alpha}_k^j > 0, \tag{3.2}$$

such that $\tilde{Y}_k^{j+1} = Y_k^j(\tilde{\alpha}_k^j)$ *satisfies*

$$\Phi(\tilde{Y}_k^{j+1}, \varepsilon_k) > \Phi(Y_k^j, \varepsilon_k) + \tau_2\langle\nabla_{Y_k^j}\Phi, \tilde{Y}_k^{j+1} - Y_k^j\rangle. \tag{3.3}$$

If $\frac{\|Y_k^{j+1} - Y_k^j\|_F}{(\alpha_k^j)^{\tau_4}} \geq \hat{\gamma}\varepsilon_k$, *j:=j+1 and go to (S.2); else* $X_{k+1} := Y_k^{j+1}$, *and go to (S.3).*

(S.3) Compute $\tilde{\tau} = \max\left\{\tau_3, \frac{\sigma_{r_{k+1}}^{\frac{2}{1-\varrho}+\gamma_4}(X_{k+1})}{\sigma_{r_k}^{\frac{2}{1-\varrho}+\gamma_4}(X_k)}\right\}$, *and set* $\varepsilon_{k+1} = min\{\tilde{\tau}, \tau\}\varepsilon_k$. *If* $\min\{\mid \Phi(X_k, \varepsilon_k) - \Phi(X_{k+1}, \varepsilon_{k+1}) \mid, \| X_k - X_{k+1} \|_F\} < \epsilon$, *stop; else* $k = k+1$ *and return to step (S.1).*

It is clear that $\| P_{\mathcal{C}}[X_k - \nabla_{X_k}\Phi] - X_k \|_F = 0$ if and only if X_k is a stationary point of minimizing $\Phi(X, \varepsilon_k)$ such that $X \in \mathcal{C}$. From the discussion in [2, 5, 17], we know that the accepted regulation of α_k^j, i.e., the structure of (3.1), (3.2) and (3.3) is reasonable.

Because $b \neq 0$, the origin would not be a feasible point of problem (1.1) and (2.2), which follows that there exists a positive number $\xi = \| P[0] \|_F = \| A^\top(AA^\top)^{-1}b \|_F$ such that $\| X \|_F \geq \xi$ holds for all $X \in \mathcal{C}$. So we have for all $X \in \mathcal{C}$, $\sum_{i=1}^{\iota} \sigma_i^2(X) \geq \xi^2 \Longrightarrow \sigma_1(X) \geq \frac{\xi}{\sqrt{\iota}} \Longleftrightarrow \sigma_1(X)^{-1} \leq \frac{\sqrt{\iota}}{\xi}$. It follows that there exists a number $\theta_X \in \{1, 2, \cdots, \iota\}$ such that $\sigma_i(X)^{-1} \leq \frac{\sqrt{\iota}}{\xi}$, $\forall i : 1 \leq i \leq \theta_X$ and $\sigma_i(X) < \frac{\xi}{\sqrt{\iota}}$, $\forall i : \theta_X < i \leq \iota$. From that was mentioned above, we can deduce the following lemma.

Lemma 3.2 *(a). Let* $a_2 > a_1 > 0$, *then* $\nabla_X \Phi$ *is bounded for all* $\varepsilon \in [a_1, a_2]$ *and* $X \in \mathcal{C}$.

(b). For fixed ε, $\nabla^2_{XX}\Phi$ *is bounded above for all* $X \in \mathcal{C}$, *i.e., there exists a positive number* R_ε *such that*

$$\mid \langle \nabla^2_{XX}\Phi(Y), Y \rangle \mid \leq R_\varepsilon$$

holds for all $X \in \mathcal{C}$ *and* $Y \in \mathcal{R}^{m\times n}$ *with* $\| Y \|_F \leq 1$.

Proof. (a) It is easy to see from Theorem 2.10 that

$$\begin{aligned}
\| \nabla_X \Phi \|_F &= 2\varepsilon(1+\varepsilon^\varrho)\Big(\sum_{i=1}^{\iota} \frac{\sigma_i^2(X)}{(\sigma_i^2(X)+\varepsilon)^4}\Big)^{\frac{1}{2}} \\
&\leq 2\varepsilon(1+\varepsilon^\varrho)\Big(\sum_{i=1}^{\theta_X}\Big(\frac{\sqrt{\iota}}{\xi}\Big)^6 + \sum_{i=\theta_X+1}^{\iota} \frac{\xi^2}{\varepsilon^4 \iota}\Big)^{\frac{1}{2}} \\
&\leq 2a_2(1+a_2^\varrho)\Big(\frac{\iota^4}{\xi^6} + \frac{\xi^2}{a_1^4}\Big)^{\frac{1}{2}}.
\end{aligned}$$

(b) For all $X \in \mathcal{C}$ and $Y \in \mathcal{R}^{m\times n}$ with $\| Y \|_F \leq 1$, by Theorem 2.10, we have

$$\begin{aligned}
& \mid \langle \nabla^2_{XX}\Phi(Y), Y \rangle \mid \\
= \ & 2\varepsilon(1+\varepsilon^\varrho) \mid \langle (XX^\top + \varepsilon I)^{-2}Y - (XX^\top + \varepsilon I)^{-2} \\
& [2\varepsilon \tilde{L}_X(Y) + L_{XX^\top}(\tilde{L}_X(Y))](XX^\top + \varepsilon I)^{-2}X, Y \rangle \mid \\
\leq \ & 2\varepsilon(1+\varepsilon^\varrho)\{\langle (XX^\top + \varepsilon I)^{-2}Y, Y \rangle + \\
& 2\varepsilon \mid \langle (XX^\top + \varepsilon I)^{-2}\tilde{L}_X(Y)(XX^\top + \varepsilon I)^{-2}X, Y \rangle \mid + \\
& \mid \langle (XX^\top + \varepsilon I)^{-2}L_{XX^\top}(\tilde{L}_X(Y))(XX^\top + \varepsilon I)^{-2}X, Y \rangle \mid\}. \qquad (3.4)
\end{aligned}$$

Suppose the full singular decomposition of X is $X = U\Lambda V^\top$, and $\hat{Y} = U^\top Y V$. It is clear that $\| Y \|_F = \| \hat{Y} \|_F \leq 1$, and then

$$\begin{aligned}\langle (XX^\top + \varepsilon I)^{-2} Y, Y\rangle &= \langle (\Lambda\Lambda^\top + \varepsilon I)^{-2}\hat{Y}, \hat{Y}\rangle \\ &\leq \| \hat{Y} \|_F^2 \| (\Lambda\Lambda^\top + \varepsilon I)^{-2} \|_F \leq \frac{\sqrt{\iota}}{\varepsilon}. \end{aligned} \quad (3.5)$$

The first inequality in (3.5) comes from the fact that $| \langle X, Y\rangle | \leq \| X \|_F \| Y \|_F$ holds for all X, Y. Similarly, we can obtain the following two inequalities.

$$\begin{aligned} & | \langle (XX^\top + \varepsilon I)^{-2} \tilde{L}_X(Y)(XX^\top + \varepsilon I)^{-2} X, Y\rangle | \\ \leq\ & | \langle (\Lambda\Lambda^\top + \varepsilon I)^{-2}\Lambda\hat{Y}^\top(\Lambda\Lambda^\top + \varepsilon I)^{-2}\Lambda, \hat{Y}\rangle | + \\ & | \langle (\Lambda\Lambda^\top + \varepsilon I)^{-2}\hat{Y}\Lambda^\top(\Lambda\Lambda^\top + \varepsilon I)^{-2}\Lambda, \hat{Y}\rangle | \\ \leq\ & \| (\Lambda\Lambda^\top + \varepsilon I)^{-2}\Lambda \|_F^2 \| \hat{Y} \|_F^2 + \\ & \| (\Lambda\Lambda^\top + \varepsilon I)^{-2} \|_F \| \hat{Y} \|_F^2 \| \Lambda^\top(\Lambda\Lambda^\top + \varepsilon I)^{-2}\Lambda \|_F \\ \leq\ & \sum_{i=1}^{\iota} \frac{\sigma_i^2(X)}{(\sigma_i^2(X)+\varepsilon)^4} + (\sum_{i=1}^{\iota} \frac{1}{(\sigma_i^2(X)+\varepsilon)^4})^{\frac{1}{2}} (\sum_{i=1}^{\iota} \frac{\sigma_i^4(X)}{(\sigma_i^2(X)+\varepsilon)^4})^{\frac{1}{2}} \\ \leq\ & \Big(\sum_{i=1}^{\theta_X} (\frac{\sqrt{\iota}}{\xi})^6 + \sum_{i=\theta_X+1}^{\iota} \frac{\xi^2}{\varepsilon^4 \iota} \Big) + \frac{\sqrt{\iota}}{\varepsilon^2} \Big(\sum_{i=1}^{\theta_X} (\frac{\sqrt{\iota}}{\xi})^4 + \sum_{i=\theta_X+1}^{\iota} \frac{\xi^4}{\varepsilon^4 \iota^2} \Big)^{\frac{1}{2}} \\ \leq\ & \frac{\iota^4}{\xi^6} + \frac{\xi^2}{\varepsilon^4} + \frac{\sqrt{\iota}}{\varepsilon^2} (\frac{\iota^3}{\xi^4} + \frac{\xi^4}{\varepsilon^4 \iota})^{\frac{1}{2}} \end{aligned} \quad (3.6)$$

and

$$\begin{aligned} & | \langle (XX^\top + \varepsilon I)^{-2} L_{XX^\top}(\tilde{L}_X(Y))(XX^\top + \varepsilon I)^{-2} X, Y\rangle | \\ \leq\ & | \langle (\Lambda\Lambda^\top + \varepsilon I)^{-2}\Lambda\hat{Y}^\top\Lambda\Lambda^\top(\Lambda\Lambda^\top + \varepsilon I)^{-2}\Lambda, \hat{Y}\rangle | + \\ & | \langle (\Lambda\Lambda^\top + \varepsilon I)^{-2}\hat{Y}\Lambda^\top\Lambda\Lambda^\top(\Lambda\Lambda^\top + \varepsilon I)^{-2}\Lambda, \hat{Y}\rangle | + \\ & | \langle (\Lambda\Lambda^\top + \varepsilon I)^{-2}\Lambda\Lambda^\top\Lambda\hat{Y}^\top(\Lambda\Lambda^\top + \varepsilon I)^{-2}\Lambda, \hat{Y}\rangle | + \\ & | \langle (\Lambda\Lambda^\top + \varepsilon I)^{-2}\Lambda\Lambda^\top\hat{Y}\Lambda^\top(\Lambda\Lambda^\top + \varepsilon I)^{-2}\Lambda, \hat{Y}\rangle | \\ \leq\ & 2(\sum_{i=1}^{\iota} \frac{\sigma_i^2(X)}{(\sigma_i^2(X)+\varepsilon)^4})^{\frac{1}{2}} (\sum_{i=1}^{\iota} \frac{\sigma_i^6(X)}{(\sigma_i^2(X)+\varepsilon)^4})^{\frac{1}{2}} + (\sum_{i=1}^{\iota} \frac{1}{(\sigma_i^2(X)+\varepsilon)^4})^{\frac{1}{2}} \\ & (\sum_{i=1}^{\iota} \frac{\sigma_i^8(X)}{(\sigma_i^2(X)+\varepsilon)^4})^{\frac{1}{2}} + \sum_{i=1}^{\iota} \frac{\sigma_i^4(X)}{(\sigma_i^2(X)+\varepsilon)^4} \\ \leq\ & 2\Big(\sum_{i=1}^{\theta_X} (\frac{\sqrt{\iota}}{\xi})^6 + \sum_{i=\theta_X+1}^{\iota} \frac{\xi^2}{\varepsilon^4 \iota} \Big)^{\frac{1}{2}} \Big(\sum_{i=1}^{\theta_X} (\frac{\sqrt{\iota}}{\xi})^2 + \sum_{i=\theta_X+1}^{\iota} \frac{\xi^6}{\varepsilon^4 \iota^3} \Big)^{\frac{1}{2}} + \\ & \frac{\sqrt{\iota}}{\varepsilon^2}\sqrt{\iota} + \sum_{i=1}^{\theta_X} (\frac{\sqrt{\iota}}{\xi})^4 + \sum_{i=\theta_X+1}^{\iota} \frac{\xi^4}{\varepsilon^4 \iota^2} \\ \leq\ & 2(\frac{\iota^4}{\xi^6} + \frac{\xi^2}{\varepsilon^4})^{\frac{1}{2}} (\frac{\iota^2}{\xi^2} + \frac{\xi^6}{\varepsilon^4 \iota^2})^{\frac{1}{2}} + \frac{\iota}{\varepsilon^2} + \frac{\iota^3}{\xi^4} + \frac{\xi^4}{\iota \varepsilon^4}. \end{aligned} \quad (3.7)$$

Together with (3.4),(3.5),(3.6) and (3.7), we have

$$
\begin{aligned}
& \mid \langle \nabla^2_{XX}\Phi(Y), Y\rangle \mid \\
\leq\ & 2\varepsilon(1+\varepsilon^{\varrho})\Big\{\frac{\sqrt{\iota}}{\varepsilon} + 2\varepsilon\Big[\frac{\iota^4}{\xi^6} + \frac{\xi^2}{\varepsilon^4} + \frac{\sqrt{\iota}}{\varepsilon^2}(\frac{\iota^3}{\xi^4} + \frac{\xi^4}{\varepsilon^4\iota})^{\frac{1}{2}}\Big] + \\
& 2(\frac{\iota^4}{\xi^6} + \frac{\xi^2}{\varepsilon^4})^{\frac{1}{2}}(\frac{\iota^2}{\xi^2} + \frac{\xi^6}{\varepsilon^4\iota^2})^{\frac{1}{2}} + \frac{\iota}{\varepsilon^2} + \frac{\iota^3}{\xi^4} + \frac{\xi^4}{\iota\varepsilon^4}\Big\} := R_\varepsilon
\end{aligned}
$$

holds for all $X \in \mathcal{C}$ and Y with $\| Y \|_F \leq 1$. $\square$

It is easy to see that R_ε in the above lemma is a number related to ε, and R_ε increases to infinity when ε approaches to zero. If ε is fixed, which implies that R_ε is a constant, we know $\nabla_X\Phi$ is uniformly continuous from (b) in Lemma 3.2, which together with the fact that $\Phi(\cdot,\varepsilon) \geq 0$ deduce that the assumptions in [2, 17] hold for $\Phi(\cdot,\varepsilon)$. So, for k fixed, the inner iteration sequence $\{Y_k^j\}$ converges to the stationary point of $\Phi(X,\varepsilon_k)$ when $j \to \infty$ by the Theorem 3.2 in [2]. It follows that the singular value of Y_k^j which correspond to the zero singular value of the stationary point will converge to zero. And, from Theorem 2.3 in [2] and the boundary of α_k^j, we have $\frac{\|Y_k^{j+1}-Y_k^j\|_F}{(\alpha_k^j)^{\tau_4}} = \frac{\|Y_k^{j+1}-Y_k^j\|_F}{\alpha_k^j}(\alpha_k^j)^{1-\tau_4} \to 0$ when $j \to \infty$. Hence, Algorithm 3.1 is reasonable, i.e., the inner loop in Algorithm 3.1 (related to Y_k^j) stops after finite iterations.

Remark 3.3 *In Algorithm 3.1, the condition $\gamma_3 \geq \alpha_k^j$ in (3.2) can be omitted indeedly. For k fixed, it is easy to see that*

$$
P_{\mathcal{C}}[Y_k^j - \alpha\nabla_{Y_k^j}\Phi] = Y_k^j - \alpha mat\{[I - A^\top(AA^\top)^{-1}A]vec(\nabla_{Y_k^j}\Phi)\}
$$

by the feasibility of Y_k^j, which follows that the last item in (3.1) can be reformulated as

$$
\begin{aligned}
& \langle \nabla_{Y_k^j}\Phi, Y_k^{j+1} - Y_k^j\rangle \\
=\ & vec(\nabla_{Y_k^j}\Phi)'[I - A^\top(AA^\top)^{-1}A]vec(\nabla_{Y_k^j}\Phi).
\end{aligned}
$$

We have

$$
\begin{aligned}
& [I - A^\top(AA^\top)^{-1}A]vec(\nabla_{Y_k^j}\Phi) = 0 \\
\Longleftrightarrow\ & vec(\nabla_{Y_k^j}\Phi)'[I - A^\top(AA^\top)^{-1}A]vec(\nabla_{Y_k^j}\Phi) = 0
\end{aligned}
$$

by the positive semidefinite of $I - A^\top(AA^\top)^{-1}A$. From the proof of Theorem 2.3 in [2], we know $\langle \nabla_{Y_k^j}\Phi, Y_k^{j+1} - Y_k^j\rangle \to 0$ without the condition $\gamma_3 \geq \alpha_k^j$, which deduces $[I - A^\top(AA^\top)^{-1}A]vec(\nabla_{Y_k^j}\Phi) \to 0$. By a simple computation, we know the "projected gradient" $\nabla_{\mathcal{C}}\Phi(Y_k^j,\varepsilon_k)$ in [2] can be reformulated as $-[I-$

$A^\top(AA^\top)^{-1}A]vec(\nabla_{Y_k^j}\Phi)$ *in this work, which will approach to zero. Hence we can conclude that any accumulated point of* $\{Y_k^j\}$ *is a stationary point by Lemma 3.1 in [2].*

From (a) in Lemma 3.2, we know that $\{\nabla_{X_k}\Phi\}$ is bounded if $\{\varepsilon_k\}$ has a low boundary. Taking the operator $\lceil\cdot\rceil$ into consideration, we have $rank(X) = \lceil\Phi(X,\varepsilon)\rceil$ for all $\varepsilon \in (0, \varepsilon_\star]$ from Lemma 2.1. Hence, $\varepsilon_\star$ can be considered as the low boundary of $\{\varepsilon_k\}$ in this case, which follows that the iteration matrices $\{\nabla_{X_k}\Phi\}$ are bounded and have at least one accumulated point.

Now we will introduce the following conclusion.

Theorem 3.4 *Suppose* $X_\star$ *(*$rank(X_\star) = r$*) is any accumulated point of iteration points* $\{X_k\}$ *and* $\lim_{j\to\infty} X_{k_j} = X_\star$*. If* $\sigma_i^2(X_{k_j}) = \circ(\varepsilon_{k_j})$ *for* $i = r+1, \cdots, \iota$ *and the condition of* (II) *in Theorem 2.4 holds, then* $\lim_{k\to\infty}\Phi(X_k, \varepsilon_k) = r$*.*

Moreover, if the rational number $\varrho \in [1/3, 1)$ *and the condition* (b) *in Theorem 2.4 is enforced by* $\sigma_i^2(X_k) = \circ(\varepsilon_k^2)$*, then any accumulated point of* $\{\nabla_{X_k}\Phi\}$ *is zero.*

Proof. For the subset $\{X_{k_j}\} \subseteq \{X_k\}$, we have

$$\begin{aligned}
\lim_{j\to\infty}\Phi(X_{k_j},\varepsilon_{k_j}) &= \lim_{j\to\infty}\sum_{i=1}^{\iota}\frac{(1+\varepsilon_{k_j}^{\varrho})\sigma_i^2(X_{k_j})}{\sigma_i^2(X_{k_j})+\varepsilon_{k_j}} \\
&= \lim_{j\to\infty}\sum_{i=1}^{r}\frac{(1+\varepsilon_{k_j}^{\varrho})\sigma_i^2(X_{k_j})}{\sigma_i^2(X_{k_j})+\varepsilon_{k_j}} + \lim_{j\to\infty}\sum_{i=r+1}^{\iota}\frac{(1+\varepsilon_{k_j}^{\varrho})\sigma_i^2(X_{k_j})}{\sigma_i^2(X_{k_j})+\varepsilon_{k_j}} \\
&= r + \lim_{j\to\infty}\sum_{i=r+1}^{\iota}\frac{(1+\varepsilon_{k_j}^{\varrho})\sigma_i^2(X_{k_j})/\varepsilon_{k_j}}{\sigma_i^2(X_{k_j})/\varepsilon_{k_j}+1} = r
\end{aligned}$$

because $\sigma_i^2(X_{k_j}) = \circ(\varepsilon_{k_j})$ for $i = r+1, \cdots, \iota$.

From Theorem 2.4, we know that $\Phi(X_{k+1}, \varepsilon_{k+1}) < \Phi(X_{k+1}, \varepsilon_k)$ for k large enough, which, together with $\Phi(X_{k+1}, \varepsilon_k) < \Phi(X_k, \varepsilon_k)$ guaranteed by (3.1), deduces that $\Phi(X_{k+1}, \varepsilon_{k+1}) < \Phi(X_{k+1}, \varepsilon_k) < \Phi(X_k, \varepsilon_k)$. From the formula (2.1), we know $\Phi(X, \varepsilon) \geq 0$, which together with the monotonicity of $\Phi(X_k, \varepsilon_k)$ deduces that $\{\Phi(X_k, \varepsilon_k)\}$ has a limit point. Hence,

$$\lim_{k\to\infty}\Phi(X_k,\varepsilon_k) = \lim_{j\to\infty}\Phi(X_{k_j},\varepsilon_{k_j}) = r.$$

For the second conclusion, without loss of generality, suppose

$$\lim_{k\to\infty}\nabla_{X_k}\Phi = \nabla_{X_\star}\Phi.$$

Let $X_k = U_k\Lambda_k V_k^\top$, we then have

$$\nabla_{X_\star}\Phi = 2\lim_{k\to\infty}(1+\varepsilon_k^{\varrho})U_k[\varepsilon_k(\Lambda_k\Lambda_k^\top + \varepsilon_k I)^{-2}\Lambda_k]V_k^\top.$$

The sets $\{U_k\}$, $\{V_k\}$ are bounded, and the corresponding accumulated points are denoted by $U_\star$ and $V_\star$, respectively.

It is easy to see that the limit value of the i-th diagonal of $\varepsilon_k(\Lambda_k\Lambda_k^\top + \varepsilon_k I)^{-2}\Lambda_k$ is zero for $i = 1, 2, \cdots, r$. And, for all $i \in \{r+1, \cdots, \iota\}$, there are two cases should be discussed.

Firstly, if the condition (a) in Theorem 2.4 holds, then $\varepsilon_k = \circ(\sigma_i^{\frac{2}{1-\varrho}}(X_k)) = \circ(\sigma_i^3(X_k))$ for $\varrho \in [1/3, 1)$. Hence, we have $\frac{\varepsilon_k\sigma_i(X_k)}{(\sigma_i^2(X_k)+\varepsilon_k)^2} = \frac{\varepsilon_k}{\sigma_i^3(X_k)(1+\frac{\varepsilon_k}{\sigma_i^2(X_k)})^2} \to 0$.

Secondly, if $\sigma_i^2(X_k) = \circ(\varepsilon_k^2)$, we have $\frac{\varepsilon_k\sigma_i(X_k)}{(\sigma_i^2(X_k)+\varepsilon_k)^2} = \frac{\sigma_i(X_k)}{\varepsilon_k(\frac{\sigma_i^2(X_k)}{\varepsilon_k}+1)^2} \to 0$ by using $\lim_{k\to\infty} \frac{\sigma_i(X_k)}{\varepsilon_k} \to 0$.

So $\nabla_{X_\star}\Phi = 0$, i.e., $\lim_{k\to\infty} \nabla_{X_k}\Phi = 0$. $\square$

Remark 3.5 *Suppose $r_k = r_{k+1}$ and $\frac{\sigma_{r_k}^{\frac{2}{1-\varrho}+\gamma_4}(X_{k+1})}{\sigma_{r_k}^{\frac{2}{1-\varrho}+\gamma_4}(X_k)} \geq \tau_3$ in Algorithm 3.1 hold together for all k large enough, then we have*

$$\varepsilon_{k+1} \leq \frac{\sigma_{r_{k+1}}^{\frac{2}{1-\varrho}+\gamma_4}(X_{k+1})}{\sigma_{r_k}^{\frac{2}{1-\varrho}+\gamma_4}(X_k)}\varepsilon_k \Longrightarrow \frac{\varepsilon_{k+1}}{\sigma_{r_{k+1}}^{\frac{2}{1-\varrho}+\gamma_4}(X_{k+1})} \leq \frac{\varepsilon_k}{\sigma_{r_k}^{\frac{2}{1-\varrho}+\gamma_4}(X_k)},$$

which deduces that

$$\varepsilon_k = O(\sigma_{r_k}^{\frac{2}{1-\varrho}+\gamma_4}(X_k)) = \circ(\sigma_{r_k}^{\frac{2}{1-\varrho}}(X_k)) \Longrightarrow \varepsilon_k^{1-\varrho} = \circ(\sigma_{r_k}^2(X_k)).$$

Subsequently, the condition (a) in Theorem 2.4 is satisfied.

With the help of the continuity of $\Phi(X, 0)$, Theorem 2.6 in [17] guarantees $\langle\nabla_{X_\star}\Phi, Z - X_\star\rangle \geq 0$ for all $Z \in \mathcal{C}$ under certain conditions. In this paper, if $\{X_k\}$ is bounded and the conditions in Theorem 3.4 hold, we know $\lim_{k\to\infty}\langle\nabla_{X_k}\Phi,\, Z - X_k\rangle = 0$ for all $Z \in \mathcal{C}$.

4. Numerical result

Without loss of generality, we suppose $m \leq n$ and A is of full row rank. Some preliminary results of the successive projected gradient method for the problem (1.1) will be shown in this section. In general, the successive projected gradient method can only find the approximate solution (i.e., the limited point of the stationary point of approximation problem) of the original problem (1.1), so the result of our algorithm may be larger than the optimal value of problem (1.1). But, our algorithm has its own advantages. Firstly, our algorithm can be used for the matrix rank minimization problem with small number of constraints, and the result generated by our algorithm is a feasible point with rather low rank. Secondly, the results of our algorithm are exact when the number of the constraint is large enough. We have implement the successive projected gradient method in Matlab(R2009b) with Window XP. All runs are performed on an Intel Pentium(R) Dual-Core 2.6GHz PC with 1.99G memory.

For the orthogonal projection

$$P_{\mathcal{C}}[Y_k^j - \alpha \nabla_{Y_k^j}\Phi] = Y_k^j - \alpha mat\{[I - A^\top (AA^\top)^{-1}A]vec(\nabla_{Y_k^j}\Phi)\},$$

we take the singular decomposition of Y_k^j as $Y_k^j = U_k^j \Lambda_k^j (V_k^j)^\top$, and then,

$$\begin{aligned} \nabla_{Y_k^j}\Phi &= 2\varepsilon_k(1+\varepsilon_k^{\varrho})(Y_k^j(Y_k^j)^\top + \varepsilon_k I)^{-2}Y_k^j \\ &= 2\varepsilon_k(1+\varepsilon_k^{\varrho})U_k^j[(\Lambda_k^j(\Lambda_k^j)^\top + \varepsilon_k I)^{-2}\Lambda_k^j](V_k^j)^\top. \end{aligned} \tag{4.1}$$

From (4.1), we know that $vec(\nabla_{Y_k^j}\Phi) = 2\varepsilon_k(1+\varepsilon_k^{\varrho})(V_k^j \otimes U_k^j)vec\{[\Lambda_k^j(\Lambda_k^j)^\top + \varepsilon_k I]^{-2}\Lambda_k^j\}$, where '$\otimes$' denotes Kronecker product. For details, please refer to [8]. Hence, we only need to determine the 1-th column, $(m+2)$-th column, $\cdots$, $[(r-1)m+r]$-th column of $V_k^j \otimes U_k^j$ owe to the special structure of $vec\{[\Lambda_k^j(\Lambda_k^j)^\top + \varepsilon_k I]^{-2}\Lambda_k^j\}$, where r is the rank of Y_k^j.

In order to improve the convergence rate, many researchers use the PROPA CK package to compute a partial SVD. For the detail of PROPACK package, please refer to [10] and its references. In this work, we will use the exact singular value decomposition instead of the partial SVD during the running, because (4.1) cannot be used if we use PROPACK package to compute partial SVD in this paper. And, in the proceeding of running, we will set the number with its abstract value less than 1e-8 as zero.

We will use the successive projected gradient method for solving the matrix completion problem and its generalization. The construction of the matrix completion problem is similar to that in [3, 10]: Firstly, matrices $M_1 \in \mathcal{R}^{m\times r}$, $M_1 \in \mathcal{R}^{r\times n}$ are generated randomly with $r < m$ (where, $M_1 = 15*rand(1,1)*randn(m,r)+2*rand(1,1)*randn(m,r)$, $M_2 = 5*rand(1,1)*randn(r,n) - 20*rand(1,1)*randn(r,n)$). After taking $M = M_1M_2$, we will find the solution of the following problem

$$\min\{rank(X) :\ X_{i,j} = M_{i,j},\ (i,j) \in \Omega\} \tag{4.2}$$

where the elements in Ω, which is the set of indices of the observed entries, are generated randomly (by using order '*randint*'). It is easy to see that the optimal value of the above problem is less than r.

For the parameters of the approximation function, we select $\beta = 1,\ \varrho = 0.8$, and the parameters in Algorithm 3.1 are set as follows.

$\hat{\gamma} = \max\{1e3, mn\},\ \gamma_1 = 0.5,\ \gamma_2 = 0.25,\ \gamma_3 = 1e3,\ \gamma_4 = 0.05,\ \tau = 0.9,\ \tau_1 = 1e-6,\ \tau_2 = 1e-6,\ \tau_3 = 0.6,\ \tau_4 = 1/3,\ X_0 = P[0],\ \varepsilon_0 = 1e4 * mn,\ \alpha_k^0 = \gamma_3$ and $\epsilon = 1e-7$. Moreover, $\alpha_k^{j+1} = 0.2\alpha_k^j$ if $\alpha_k^j \geq \gamma_1$, otherwise $\alpha_k^{j+1} = \gamma_2\alpha_k^j$. The number of iterations is not larger than 500.

In the following tables, m, n, r, l denote the number of row, column, rank of X and the number of the row of A, $ite, sol, val, time$ denote the number of iterations, the optimal solution, the rank of sol and the spending time. The value $d_r = r(m+n-r)$ denotes the freedom of an $m \times n$ rank-r matrix.

The data in table 1 show the results of the successive projected gradient method for solving problem (4.2) with small number of constraints. All these results, which may be not the exact optimal solutions of problem (4.2), are just the approximate solutions of the original problem. In the following table, the parameters $l \le 3(m+n)$ and $r \le (m+n)/6$.

Table 1: Numerical results for (4.2) with a small number of constraints

Parameters					Successive PG		
(m,n)	r	l	d_r	l/d_r	ite	val	$time(sec)$
(20,30)	6	80	264	0.303	83	8	0.681
	6	90	264	0.340	91	6	0.897
	6	100	264	0.378	96	7	0.807
	7	90	301	0.299	85	6	0.681
	7	100	301	0.332	93	6	0.834
(50,60)	12	280	1176	0.238	138	11	9.358
	12	290	1176	0.246	126	12	10.894
	12	300	1176	0.255	143	13	9.781
	12	310	1176	0.263	111	10	13.313
	12	320	1176	0.272	94	12	8.481
(80,100)	20	500	3200	0.156	124	14	56.814
	20	510	3200	0.159	104	15	56.936
	20	520	3200	0.162	120	15	57.845
	20	530	3200	0.165	133	16	67.928
	20	540	3200	0.168	147	19	65.158
(120,150)	25	500	6125	0.081	173	22	387.36
	25	600	6125	0.098	148	25	274.22
	25	700	6125	0.114	131	20	288.97
	30	600	7200	0.083	138	19	327.63
	30	700	7200	0.097	109	23	255.25
(200,240)	60	400	22800	0.017	110	63	3804.99
	60	480	22800	0.021	127	54	4836.16
	60	500	22800	0.022	145	50	4769.02
	60	900	22800	0.039	130	31	4952.41
	60	1000	22800	0.043	140	30	6783.96

It is easy to see that the data in the above table are reasonable, i.e., most of results in the random experiments are not more than r in general.

The data in table 2 are generated by using Algorithm 3.1 for finding the solution of the problem (4.2) with a relatively larger number of constraints, i.e., $l \in (mn/4, 4mn/5)$ and $r \in (0, (m+n)/6)$.

Table 2: Numerical results for (4.2) with a relatively larger number of constraints

Parameters					Successive PG			
(m,n)	r	l	d_r	l/d_r	ite	val	$time(sec)$	$\parallel sol - M \parallel_F$
(20,30)	3	400	141	2.836	93	3	0.807	9.229e-7
	4	450	184	2.445	55	4	0.339	7.490e-8
	5	480	225	2.133	115	5	0.832	3.778e-7
	6	480	264	1.818	128	6	0.896	5.667e-7
	7	480	301	1.594	82	7	0.823	3.749e-7
(50,60)	3	1800	321	5.607	41	3	1.902	1.355e-7
	4	2000	424	4.717	42	4	1.935	4.204e-8
	5	2000	525	3.809	82	5	3.969	6.455e-8
	10	2100	1000	2.100	85	10	6.305	5.573e-7
	15	2300	1425	1.614	83	15	10.099	4.797e-7
(80,100)	3	3600	531	6.779	70	3	22.913	1.550e-7
	4	4000	704	5.681	55	4	16.367	4.056e-8
	5	4000	875	4.571	80	5	24.427	4.578e-7
	15	5000	2475	2.020	95	15	66.893	5.640e-7
	25	6000	3875	1.548	86	25	68.492	6.985e-8
(100,120)	3	4000	651	6.144	91	3	99.693	6.844e-7
	4	5000	864	5.787	67	4	49.542	2.571e-7
	5	6000	1075	5.581	85	5	54.891	1.270e-7
	20	8000	4000	2.000	100	20	120.98	2.922e-7
	30	9000	5700	1.578	89	30	89.313	6.072e-7
(200,230)	3	10200	1281	7.962	110	3	1560.3	6.926e-7
	4	11800	1704	6.924	106	4	2856.9	4.343e-7
	10	23000	4200	5.476	81	10	1371.3	1.031e-7
	40	30000	15600	1.923	92	40	3314.3	4.394e-7
	60	36000	22200	1.621	89	60	3296.7	3.419e-7
(600,600)	6	50000	7164	6.979	136	6	6.75e5	2.274e-6
	10	70000	11900	5.882	141	10	6.97e5	8.664e-7
	18	100000	21276	4.700	121	18	4.08e5	1.031e-7

From all these data, we know the optimal solution of problem (4.2) agree with M with high possibility in this case.

The efficiency of the successive PG method for the matrix completion may be lower than the methods based on the partial SVD, but most of small-dimension matrix completion problems can be solved by the successive PG method if the number of constraints is large enough. Hence, it is a possibly effective way to separate the large-dimension matrix, which need to be recovered, into some small-dimension sub-matrices, and to solve the sub-matrices by using successive PG method together with parallel computation.

In the following, we will find the solution of the following problem

$$\min\{rank(X) : \ Avec(X) = b\} \tag{4.3}$$

where A is generated randomly (where, $A = 10 * randn(l, m * n)$) and $b = Avec(M)$. It is easy to see that the matrix completion problem is a special case of problem (4.3).

Table 3: Numerical results for problem (4.3)

Parameters					Successive PG			
(m,n)	r	l	d_r	l/d_r	ite	val	$time(sec)$	$\Vert sol - M \Vert_F$
(20,30)	3	400	141	2.836	71	3	0.833	1.056e-7
	4	450	184	2.445	41	4	0.599	6.615e-8
	5	480	225	2.133	42	5	0.632	1.127e-7
	6	480	264	1.818	48	6	0.993	3.392e-8
	7	480	301	1.594	70	7	1.145	2.968e-7
(50,60)	3	1800	321	5.607	42	3	12.153	8.734e-8
	4	2000	424	4.717	76	4	21.153	5.187e-8
	5	2000	525	3.809	38	5	14.613	4.023e-8
	10	2100	1000	2.1	52	10	19.599	1.405e-7
	15	2300	1425	1.614	70	15	30.714	2.113e-7
(80,100)	3	3600	531	6.779	47	3	92.987	1.221e-7
	4	4000	704	5.681	45	4	110.588	1.902e-7
	5	4000	875	4.571	97	5	153.310	1.400e-7
	15	5000	2475	2.020	72	15	239.370	2.135e-7
(100,120)	3	3800	651	5.837	57	3	171.994	1.526e-7
	4	4000	864	4.629	60	4	178.234	2.688e-7
	5	4100	1075	3.813	74	5	236.084	2.993e-7

The data in table 3 show that efficiency of Algorithm 3.1 for solving problem (4.3).

If we take $M = M_1 M_1^\top \succeq 0$, then M, the solution of problem (4.3) with l large enough, is exactly the solution of the following problem

$$\min_{X \in \mathcal{S}^n} \ rank(X) \ \ s.t. \ \ \mathcal{A}(X) = b, \ X \succeq 0. \tag{4.4}$$

So the condition $X \succeq 0$ in (4.4) can be omitted if the information in constrain $\mathcal{A}(X) = b$ is sufficient, i.e., l is large enough. The data in table 4 show the efficiency of successive PG method to solve problem (4.4) without using the condition $X \succeq 0$.

Table 4: Numerical results for problem (4.4) with l large enough

Parameters					Successive PG			
(m,n)	r	l	d_r	l/d_r	ite	val	$time(sec)$	$\Vert sol - M \Vert_F$
(25,25)	3	400	141	2.836	52	3	0.826	5.978e-8
	4	450	184	2.445	88	4	1.996	8.143e-8
	5	480	225	2.133	54	5	0.809	6.943e-8
	6	480	264	1.818	58	6	0.880	1.783e-7
	7	480	301	1.594	68	7	1.113	3.822e-7
(55,55)	3	1800	321	5.607	41	3	12.148	8.282e-8
	4	2000	424	4.717	56	4	17.993	7.291e-8
	5	2000	525	3.809	44	5	15.537	6.773e-8
	10	2100	1000	2.1	67	10	23.174	6.448e-8
	15	2300	1425	1.614	86	15	33.762	2.578e-7
(90,90)	3	3600	531	6.779	45	3	87.853	7.686e-8
	4	4000	704	5.681	44	4	107.044	1.270e-7
	5	4000	875	4.571	54	5	114.217	5.502e-8
	15	5000	2475	2.020	117	15	260.940	4.589e-7
	25	5200	3875	1.341	118	25	514.068	8.931e-7
(110,110)	3	3900	651	5.990	59	3	150.039	7.547e-8
	4	4000	864	4.629	71	4	178.376	2.409e-7
	5	4100	1075	3.813	80	5	207.778	2.484e-7

From all the data in the above tables, we know the successive PG method is effective.

5. Conclusions

In this paper, we present a new kind of approximation functions to approximate the rank of matrix, and analyze the properties of the new approximation functions and the corresponding approximation problems. With the help of these properties, we construct the successive projected gradient method for solving the matrix rank minimization problem. The convergence theory and the preliminary numerical result of the new method are also presented.

The successive PG method cannot be used with the partial SVD, so, how to improve the efficiency of the successive PG method is the direction of our research in the future.

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