NOTE: OPTIMAL NON-HOMOGENEOUS COMPOSITES FOR DYNAMIC LOADING REVISITED

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1. Problem formulation

Consider $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ as the spatial physical domain which is composed of two isotropic and linear materials α and β with overall volume fraction R and $1-R$, respectively. Assume the local volume fraction of phase α is denoted by function $w(\mathbf{x})$ and the effective mass density, ρ, damping and tensor of elasticity, $\mathbf{C}$, are functions of w, i.e., $\rho(\mathbf{x}) = \rho(w(\mathbf{x}))$ and $\mathbf{C}(\mathbf{x}) = \mathbf{C}(w(\mathbf{x}))$. Note that $\mathbf{C}$ is a forth order supersymmetric tensor (symmetric in both the right and the left Cartesian index pair, together with symmetry under the interchange of the pairs). For a given materials distribution w, it is assumed that the dynamics of displacement field is governed by the following PDE (note that our formulation is identical to that of (Turteltaub, 2005)):

$$\begin{cases} \rho\, \ddot{\mathbf{u}} &= \nabla\cdot(\mathbf{C}:\mathcal{D}(\mathbf{u})) \quad \texttt{in} \quad \Omega\times(0,T] \\ (\mathbf{C}:\mathcal{D}(\mathbf{u}))\cdot\mathbf{n} &= \hat{\mathbf{t}}(\mathbf{x},t) \quad \texttt{on} \quad \Gamma_t\times(0,T] \\ \mathbf{u}(\mathbf{x},t) &= \hat{\mathbf{u}}(\mathbf{x},t) \quad \texttt{on} \quad \Gamma_u\times(0,T] \\ \mathbf{u}(\mathbf{x},0) &= \mathbf{u}_0(\mathbf{x}) \quad \texttt{in} \quad \Omega\times\{t=0\} \\ \dot{\mathbf{u}}(\mathbf{x},0) &= \dot{\mathbf{u}}_0(\mathbf{x}) \quad \texttt{in} \quad \Omega\times\{t=0\} \end{cases} \tag{1.1}$$

where $[0, T]$ is the temporal domain, $\hat{\mathbf{t}}$ denotes the traction of structure with environment through traction boundaries, Γ_t, $\hat{\mathbf{u}}$ denotes the prescribed displacement on boundaries Γ_u, $\partial\Omega = \Gamma_t \cup \Gamma_u$, $\mathbf{u}_0$ and $\dot{\mathbf{u}}$ are initial displacement and velocity field respectively. Moreover, $\mathcal{D}(\mathbf{u}) := \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T) = \frac{1}{2}(\partial_i u_j + \partial_j u_i)_{i,j},\ i, j = 1\ldots, d$. Note that the double dot operator, :, denotes the usual contraction over two sets of indices. The objective functional, to be minimized, is defined as follows:

$$\mathcal{J}(w,\mathbf{u}) = \frac{1}{2}\int_0^T \int_\Omega H(\mathbf{x})\big(\zeta_k \rho\ \dot{\mathbf{u}}^2 + \zeta_p(\mathbf{C}:\mathcal{D}(\mathbf{u})) : \mathcal{D}(\mathbf{u})\big)\ d\mathbf{x}\ dt \tag{1.2}$$

where real coefficients $\zeta_k, \zeta_p \geqslant 0$ determine the importance of kinetic and potential parts of stored elastic energy respectively. Moreover, $0 \leqslant H(\mathbf{x}) \leqslant 1$ is a sufficiently smooth function which shows the importance of the objective function over Ω. For instance, to minimize J over $\Sigma \subseteq \Omega$ it is suffice to set $H(\mathbf{x}) \approx 1$ for $\mathbf{x} \in \Sigma$ and $H(\mathbf{x}) \approx 0$ for $\mathbf{x} \in \Omega\backslash\Sigma$. Note that $H(\mathbf{x})$ is in fact the regularized Heaviside function (cf. (Osher and Fedkiw, 2002)) corresponding to the characteristic function of Σ. When $\Sigma = \Omega$ we simply consider $H(\mathbf{x})$ as a constant function with values equal to 1 (in fact to ignore function H).

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The admissible design domain could be defined as follows:

$$\mathcal{A} = \{q \in L^\infty(\Omega) \mid \int_\Omega q(\mathbf{x}) d\mathbf{x} = R|\Omega|, \quad 0 < R < 1, \quad 0 \leqslant q \leqslant 1\}.$$

Therefore, the optimal design problem in the present study could be expressed as follows:

$$\min_{w \in \mathcal{A}} \mathcal{J}(w, \mathbf{u}), \ \texttt{such that } \mathbf{u} \texttt{ solves } 1.1 \tag{1.3}$$

2. Adjoint sensitivity analysis of wave equation

To solve 1.3 by a gradient based minimization algorithm, the gradients of objective function and constraints with respect to design vector should be computed. This procedure is commonly called as the design sensitivity analysis (cf. (Tortorelli and Michaleris, 1994)). Because of high dimensionality of the design space and implicit dependence of $\mathbf{u}$ to w, the direct differentiation of the objective function is not economic in practice. Instead, the adjoint approach is commonly employed to cope this problem. In this method the reduced gradient of objective function with respect to design vector is computed in expense of solving an additional PDE, called adjoint PDE. The reduction here understood the reduction with respect to the PDE constraint. The reduced gradient of objective function based on adjoint approach will be computed in this section. The methodology used here is in a close connection to the first odered nessesary optimality conditions based on KKT conditions (cf. (Borzi and Schulz, 2012; Nocedal and Wright, 1999; Tröltzsch, 2010)).

Consider sufficiently regular (adjoint) functions $\mathbf{p}$, $\mathbf{p}_s$ and $\mathbf{p}_v$ defined in $\Omega \times (0, T]$, $\Omega \times \{t = 0\}$ and $\Omega \times \{t = 0\}$ respectively. Similarly consider functions $\mathbf{p}_t$ and $\mathbf{p}_u$ defined on boundaries $\Gamma_t \times (0, T]$ and $\Gamma_t \times (0, T]$ respectively. Ignoring design constraint, the augmented lagrangian corresponding to problem 1.3 could be expressed as follows (note that design constraints could be managed straightforwardly, e.g., using the projected gradient approach (see: (Tavakoli and Zhang, 2012))):

$$\begin{aligned}
\mathcal{L}(w, \mathbf{u}, \mathbf{p}, \mathbf{p}_t, \mathbf{p}_u, \mathbf{p}_s, \mathbf{p}_v) = & \frac{1}{2} \int_0^T \int_\Omega H(\mathbf{x}) \big(\zeta_k \rho \ \dot{\mathbf{u}}^2 + \zeta_p (\mathbf{C} : \mathcal{D}(\mathbf{u})) : \mathcal{D}(\mathbf{u})\big) \ d\mathbf{x} \ dt \\
& + \int_0^T \int_\Omega \mathbf{p} \cdot \big(\rho \ \ddot{\mathbf{u}} - \nabla \cdot (\mathbf{C} : \mathcal{D}(\mathbf{u}))\big) \ d\mathbf{x} \ dt \\
& + \int_0^T \int_{\Gamma_t} \mathbf{p}_t \cdot \big((\mathbf{C} : \mathcal{D}(\mathbf{u})) \cdot \mathbf{n} - \hat{\mathbf{t}}(\mathbf{x}, t)\big) d\mathbf{x} \ dt \\
& + \int_0^T \int_{\Gamma_u} \mathbf{p}_u \cdot \big(\mathbf{u}(\mathbf{x}, t) - \hat{\mathbf{u}}(\mathbf{x}, t)\big) d\mathbf{x} \ dt \\
& + \int_\Omega \mathbf{p}_s \cdot \big(\mathbf{u}(\mathbf{x}, 0) - \mathbf{u}_0(\mathbf{x})\big) d\mathbf{x} \\
& + \int_\Omega \mathbf{p}_v \cdot \big(\dot{\mathbf{u}}(\mathbf{x}, 0) - \dot{\mathbf{u}}_0(\mathbf{x})\big) d\mathbf{x}
\end{aligned} \tag{2.1}$$

Note that introducing lagrange multipliers for boundary and initial conditions and direct augmentation of these constraints into lagrangian is not common in engineering literature. However it is quite sound and essential to rigourously drive the adjoint PDE and its corresponding initial and boundary conditions. This treatment will be well understood considering KKT conditions (cf. (Tröltzsch, 2010) and chapter 2 of (Borzi and Schulz, 2012)). In our opinion this way helps a lot to avoid mistake during the derivation of adjoint PDE and its corresponding boundary conditions. For a comprehensive account on this approach see: (Tavakoli, 2012).

Assume the set of points that satisfy the necessary optimality conditions corresponding to problem 1.3 is denoted by $\mathcal{S}$. Obviously local solutions of 1.3 are members of $\mathcal{S}$. Ignoring control constraints, $\mathcal{S}$ is in fact equivalent to set of stationary points of lagrangian $\mathcal{L}$. Therefore it is suffice to differentiate (in the sense of Gâteaux cf. chapter 10 of (Allaire, 2007)) $\mathcal{L}$ with respect to its input argument and equate them to zeros. Partial differentiation of $\mathcal{L}$ with respect to $\mathbf{p}$, $\mathbf{p}_t$, $\mathbf{p}_u$, $\mathbf{p}_s$ and $\mathbf{p}_v$ and equating terms to zeros results direct problem 1.1.

$$\begin{aligned}\mathcal{L}_w(w,\mathbf{u},\mathbf{p},\mathbf{p}_t,\mathbf{p}_u,\mathbf{p}_s,\mathbf{p}_v)(\delta w) &= \frac{1}{2}\int_0^T\int_\Omega H(\mathbf{x})\big(\zeta_k\rho_w\ \dot{\mathbf{u}}^2+\zeta_p(\mathbf{C}_w:\mathcal{D}(\mathbf{u})):\mathcal{D}(\mathbf{u})\big)\ \delta w\ d\mathbf{x}\ dt\\ &+\int_0^T\int_\Omega \rho_w\mathbf{p}\cdot\ddot{\mathbf{u}}\ \delta w\ d\mathbf{x}\ dt\\ &-\int_0^T\int_\Omega \mathbf{p}\cdot\big(\nabla\cdot(\mathbf{C}_w\delta w:\mathcal{D}(\mathbf{u}))\big)\ d\mathbf{x}\ dt\\ &+\int_0^T\int_{\Gamma_t}\mathbf{p}_t\cdot(\mathbf{C}_w:\mathcal{D}(\mathbf{u}))\cdot\mathbf{n}\ \delta w\ d\mathbf{x}\ dt\\ &:= \mathtt{I}_1+\mathtt{I}_2-\mathtt{I}_3+\mathtt{I}_4=0\end{aligned} \tag{2.2}$$

where the subscript w indicates partial differentiation with respect to w, i.e., $(\cdot)_w = \partial(\cdot)/\partial w$. Using divergence theorem, term $\mathtt{I}_3$ simplifies to:

$$\begin{aligned}\mathtt{I}_3 &= \int_0^T\int_{\Gamma_t}\mathbf{p}\cdot(\mathbf{C}_w:\mathcal{D}(\mathbf{u}))\cdot\mathbf{n}\ \delta w\ d\mathbf{x}\ dt+\int_0^T\int_{\Gamma_u}\mathbf{p}\cdot(\mathbf{C}_w:\mathcal{D}(\mathbf{u}))\cdot\mathbf{n}\ \delta w\ d\mathbf{x}\ dt\\ &-\int_0^T\int_\Omega\nabla\mathbf{p}:\mathbf{C}_w:\mathcal{D}(\mathbf{u})\ \delta w\ d\mathbf{x}\ dt := \mathtt{I}_{3,1}+\mathtt{I}_{3,2}-\mathtt{I}_{3,3}\end{aligned} \tag{2.3}$$

$$\begin{aligned}\mathcal{L}_{\mathbf{u}}(w,\mathbf{u},\mathbf{p},\mathbf{p}_t,\mathbf{p}_u,\mathbf{p}_s,\mathbf{p}_v)(\delta\mathbf{u}) &= \zeta_k\int_0^T\int_\Omega H(\mathbf{x})\ \rho\ \dot{\mathbf{u}}\cdot\delta\dot{\mathbf{u}}\ d\mathbf{x}\ dt\\ &+\frac{\zeta_p}{2}\int_0^T\int_\Omega H(\mathbf{x})\big[(\mathbf{C}:\mathcal{D}(\delta\mathbf{u})):\mathcal{D}(\mathbf{u})+(\mathbf{C}:\mathcal{D}(\mathbf{u})):\mathcal{D}(\delta\mathbf{u})\big]\ d\mathbf{x}\ dt\\ &+\int_0^T\int_\Omega\rho\ \mathbf{p}\cdot\delta\ddot{\mathbf{u}}\ d\mathbf{x}\ dt-\int_0^T\int_\Omega\mathbf{p}\cdot(\nabla\cdot(\mathbf{C}:\mathcal{D}(\delta\mathbf{u})))\ d\mathbf{x}\ dt\\ &+\int_0^T\int_{\Gamma_t}\mathbf{p}_t\cdot((\mathbf{C}:\mathcal{D}(\delta\mathbf{u}))\cdot\mathbf{n})\ d\mathbf{x}\ dt+\int_0^T\int_{\Gamma_u}\mathbf{p}_u\cdot\delta\mathbf{u}(\mathbf{x},t)\ d\mathbf{x}\ dt\\ &+\int_\Omega\mathbf{p}_s\cdot\delta\mathbf{u}(\mathbf{x},0)\ d\mathbf{x}+\int_\Omega\mathbf{p}_v\cdot\delta\dot{\mathbf{u}}(\mathbf{x},0)\ d\mathbf{x}\\ &:= \mathtt{J}_1+\mathtt{J}_2+\mathtt{J}_3-\mathtt{J}_4+\mathtt{J}_5+\mathtt{J}_6+\mathtt{J}_7+\mathtt{J}_8=0\end{aligned} \tag{2.4}$$

The expression of $\mathtt{J}_1$ simplifies to the following form by means of integration by part:

$$\mathtt{J}_1 = \zeta_k \int_\Omega H(\mathbf{x})\ \rho\ \dot{\mathbf{u}}(\mathbf{x},T)\cdot\delta\mathbf{u}(\mathbf{x},T)\ d\mathbf{x} - \zeta_k \int_\Omega H(\mathbf{x})\ \rho\ \dot{\mathbf{u}}(\mathbf{x},0)\cdot\delta\mathbf{u}(\mathbf{x},0)\ d\mathbf{x}$$
$$- \zeta_k \int_0^T \int_\Omega H(\mathbf{x})\ \rho\ \ddot{\mathbf{u}}\cdot\delta\mathbf{u}\ d\mathbf{x}\ dt := \mathtt{J}_{1,1} - \mathtt{J}_{1,2} - \mathtt{J}_{1,3} \tag{2.5}$$

Considering the symmetry of tensor $\mathbf{C}$ results:

$$\mathtt{J}_2 = \zeta_p \int_0^T \int_\Omega H(\mathbf{x})(\mathbf{C} : \mathcal{D}(\mathbf{u})) : \mathcal{D}(\delta\mathbf{u})\ d\mathbf{x}\ dt$$

Prior to continuing the derivation, it is worth to introduce a useful identity, helpful to derive dual operator corresponding to strain tensor. For a symmetric tensor $\mathbf{A}$ and an arbitrary tensor $\mathbf{B}$ the following identity holds:

$$\mathbf{A} : \mathbf{B} = \mathbf{A} : \mathbf{B}^T = \frac{1}{2}\ \mathbf{A} : (\mathbf{B} + \mathbf{B}^T) \tag{2.6}$$

The proof of 2.6 is evident using elementary tensor algebra. Using the divergence theorem and identity 2.6, $\mathtt{J}_2$ is simplified to:

$$\mathtt{J}_2 = \zeta_p \int_0^T \int_{\Gamma_t} (H(\mathbf{x})(\mathbf{C} : \mathcal{D}(\mathbf{u}))\cdot\mathbf{n})\cdot\delta\mathbf{u}\ d\mathbf{x}\ dt + \zeta_p \int_0^T \int_{\Gamma_u} (H(\mathbf{x})(\mathbf{C} : \mathcal{D}(\mathbf{u}))\cdot\mathbf{n})\cdot\delta\mathbf{u}\ d\mathbf{x}\ dt$$
$$- \zeta_p \int_0^T \int_\Omega (\nabla\cdot(H(\mathbf{x})(\mathbf{C} : \mathcal{D}(\mathbf{u}))))\cdot\delta\mathbf{u}\ d\mathbf{x}\ dt := \mathtt{J}_{2,1} + \mathtt{J}_{2,2} - \mathtt{J}_{2,3} \tag{2.7}$$

Applying the integration by part on term $\mathtt{J}_3$ gives:

$$\begin{aligned} \mathtt{J}_3 = & \int_\Omega \rho\ \mathbf{p}(\mathbf{x},T)\cdot\delta\dot{\mathbf{u}}(\mathbf{x},T)\ d\mathbf{x} - \int_\Omega \rho\ \mathbf{p}(\mathbf{x},0)\cdot\delta\dot{\mathbf{u}}(\mathbf{x},0)\ d\mathbf{x} \\ & - \int_\Omega \rho\ \dot{\mathbf{p}}(\mathbf{x},T)\cdot\delta\mathbf{u}(\mathbf{x},T)\ d\mathbf{x} + \int_\Omega \rho\ \dot{\mathbf{p}}(\mathbf{x},0)\cdot\delta\mathbf{u}(\mathbf{x},0)\ d\mathbf{x} \\ & + \int_0^T \int_\Omega \rho\ \ddot{\mathbf{p}}\cdot\delta\mathbf{u}\ d\mathbf{x}\ dt := \mathtt{J}_{3,1} - \mathtt{J}_{3,2} - \mathtt{J}_{3,3} + \mathtt{J}_{3,4} + \mathtt{J}_{3,5} \end{aligned} \tag{2.8}$$

Applying the divergence theorem on term $\mathtt{J}_4$ and considering identity 2.6 results:

$$\mathtt{J}_4 = \int_0^T \int_{\Gamma_t} \mathbf{p}\cdot((\mathbf{C} : \mathcal{D}(\delta\mathbf{u}))\cdot\mathbf{n})\ d\mathbf{x}\ dt + \int_0^T \int_{\Gamma_u} \mathbf{p}\cdot((\mathbf{C} : \mathcal{D}(\delta\mathbf{u}))\cdot\mathbf{n})\ d\mathbf{x}\ dt +$$
$$- \int_0^T \int_{\Gamma_t} ((\mathbf{C} : \mathcal{D}(\mathbf{p}))\cdot\mathbf{n})\cdot\delta\mathbf{u}\ d\mathbf{x}\ dt - \int_0^T \int_{\Gamma_u} ((\mathbf{C} : \mathcal{D}(\mathbf{p}))\cdot\mathbf{n})\cdot\delta\mathbf{u}\ d\mathbf{x}\ dt$$
$$+ \int_0^T \int_\Omega (\nabla\cdot(\mathbf{C} : \mathcal{D}(\mathbf{p})))\cdot\delta\mathbf{u}\ d\mathbf{x}\ dt := \mathtt{J}_{4,1} + \mathtt{J}_{4,2} - \mathtt{J}_{4,3} - \mathtt{J}_{4,4} + \mathtt{J}_{4,5} \tag{2.9}$$

Collection of all terms in $\mathcal{L}_{\mathbf{u}}$ in an abstract form results:

$$\mathcal{L}_{\mathbf{u}} = \underbrace{(\mathtt{J}_{3,5} - \mathtt{J}_{4,5} - \mathtt{J}_{1,3} - \mathtt{J}_{2,3})}_{\texttt{in } \Omega\times(0,T]} + \underbrace{(\mathtt{J}_{5} + \mathtt{J}_{2,1} - \mathtt{J}_{4,1} + \mathtt{J}_{4,3})}_{\texttt{on } \Gamma_t\times(0,T]} + \underbrace{(\mathtt{J}_{6} + \mathtt{J}_{2,2} - \mathtt{J}_{4,2} + \mathtt{J}_{4,4})}_{\texttt{on } \Gamma_u\times(0,T]}$$
$$+ \underbrace{(\mathtt{J}_{7} + \mathtt{J}_{8} - \mathtt{J}_{1,2} - \mathtt{J}_{3,2} + \mathtt{J}_{3,4})}_{\texttt{in } \Omega\times\{t=0\}} + \underbrace{(\mathtt{J}_{1,1} + \mathtt{J}_{3,1} - \mathtt{J}_{3,3})}_{\texttt{in } \Omega\times\{t=T\}} = 0 \tag{2.10}$$

Note that the terms inside each parenthesis of 2.10 are defined on the same spatiotemporal domain. Taking $\mathbf{p}_t = \mathbf{p}$ on boundary Γ_t results $\mathtt{J}_5 = \mathtt{J}_{4,1}$. Therefore $\mathtt{J}_5$ cancels $\mathtt{J}_{4,1}$ in the second parenthesis of 2.10. Since the Dirichlet boundary condition on Γ_u is independent from $\mathbf{u}$, $\delta\mathbf{u} = 0$ on Γ_u. Therefore $\mathtt{J}_6 = \mathtt{J}_{2,2} = \mathtt{J}_{4,4} = 0$. Setting $p = 0$ on Γ_u results in $\mathtt{J}_{4,2} = 0$. Because the initial conditions are independent from $\mathbf{u}$, $\delta\mathbf{u} = \delta\dot{\mathbf{u}} = 0$. Therefore, $\mathtt{J}_7 = \mathtt{J}_8 = \mathtt{J}_{1,2} = \mathtt{J}_{3,2} = \mathtt{J}_{3,4} = 0$. Taking $\mathbf{p}(\mathbf{x}, T) = 0$, results $\mathtt{J}_{3,1} = 0$. Putting it altogether, setting $\mathcal{L}_{\mathbf{u}}$ equal to zero leads to the following adjoint PDE:

$$\left\{\begin{array}{rcll} \rho\, \ddot{\mathbf{p}} - \nabla\cdot(\mathbf{C} : \mathcal{D}(\mathbf{p})) & = & \zeta_k\rho\, \ddot{\mathbf{u}}\, H(\mathbf{x}) + \zeta_p\nabla\cdot(H(\mathbf{x})(\mathbf{C} : \mathcal{D}(\mathbf{u}))) & \texttt{in} \quad \Omega\times(T,0] \\ -(\mathbf{C} : \mathcal{D}(\mathbf{p}))\cdot\mathbf{n} & = & \zeta_p(H(\mathbf{x})(\mathbf{C} : \mathcal{D}(\mathbf{u})))\cdot\mathbf{n} & \texttt{on} \quad \Gamma_t\times(T,0] \\ \mathbf{p}(\mathbf{x}, t) & = & 0 & \texttt{on} \quad \Gamma_u\times(T,0] \\ \mathbf{p}(\mathbf{x}, T) & = & 0 & \texttt{in} \quad \Omega\times\{t=T\} \\ \dot{\mathbf{p}}(\mathbf{x}, T) & = & \zeta_k\dot{\mathbf{u}}(\mathbf{x}, T) & \texttt{in} \quad \Omega\times\{t=T\} \end{array}\right. \tag{2.11}$$

Note that the initial conditions are available at $t = T$ for the adjoint PDE and it should be integrated in reverse time direction, i.e., interval $(T, 0]$, in contrast to the direct PDE. Collection of all terms in $\mathcal{L}_w$ in abstract form results:

$$\mathcal{L}_w = \underbrace{(\mathtt{I}_1 + \mathtt{I}_2 + \mathtt{I}_{3,3})}_{\texttt{in } \Omega\times(0,T]} + \underbrace{\mathtt{I}_4 - \mathtt{I}_{3,1}}_{\texttt{on } \Gamma_t\times(0,T]} - \underbrace{\mathtt{I}_{3,2}}_{\texttt{on } \Gamma_u\times(0,T]} = 0 \tag{2.12}$$

Since $\mathbf{p}_t = \mathbf{p}$ on Γ_t, $\mathtt{I}_4$ is equal to $\mathtt{I}_{3,1}$. Because $\mathbf{p} = 0$ on Γ_u, $\mathtt{I}_{3,2} = 0$. Considering these facts, the reduced gradient of objective function, $\mathbf{g}(\mathbf{x})$, could be expressed as follows:

$$\mathbf{g}(\mathbf{x}) = \int_0^T j(\mathbf{x}, t)\; dt \tag{2.13}$$

where

$$j(\mathbf{x}, t) = \zeta_k\; \rho_w\; H(\mathbf{x})\; \dot{\mathbf{u}}^2/2 + \zeta_p\; H(\mathbf{x})(\mathbf{C}_w : \mathcal{D}(\mathbf{u})) : \mathcal{D}(\mathbf{u})/2 + \rho_w\; \mathbf{p}\cdot\ddot{\mathbf{u}} + \nabla\mathbf{p} : \mathbf{C}_w : \mathcal{D}(\mathbf{u})$$

using the identify 2.6, we will have:

$$j(\mathbf{x}, t) = \zeta_k\; \rho_w\; H(\mathbf{x})\; \dot{\mathbf{u}}^2/2 + \zeta_p\, H(\mathbf{x})\,(\mathbf{C}_w : \mathcal{D}(\mathbf{u})) : \mathcal{D}(\mathbf{u})/2 + \rho_w\; \mathbf{p}\cdot\ddot{\mathbf{u}} + \mathcal{D}(\mathbf{p}) : \mathbf{C}_w : \mathcal{D}(\mathbf{u})$$

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