On RIC bounds of Compressed Sensing Matrices for Approximating Sparse Solutions Using ℓ_q Quasi Norms *

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Abstract

This paper follows the recent discussion on the sparse solution recovery with quasi-norms $\ell_q,\ q \in (0,1)$ when the sensing matrix possesses a Restricted Isometry Constant δ_{2k} (RIC). Our key tool is an improvement on a version of "the converse of a generalized Cauchy-Schwarz inequality" extended to the setting of quasi-norm. We show that, if $\delta_{2k} \leq 1/2$, any minimizer of the l_q minimization, at least for those $q \in (0, 0.9181]$, is the sparse solution of the corresponding underdetermined linear system. Moreover, if $\delta_{2k} \leq 0.4931$, the sparse solution can be recovered by any $l_q, q \in (0,1)$ minimization. The values 0.9181 and 0.4931 improves those reported previously in the literature.

Keywords: compressed sensing, restricted isometry constant, l_q minimization, quasi norm

1 Introduction

Given a $m \times n$ matrix Φ with $m \ll n$ and a nonzero vector $b \in \mathbb{R}^m$, one of the recently popular problem in compressed sensing is to find the sparsest solution of the underdetermined linear systems $\Phi x = b$. Here the sensing matrix Φ is assumed to obey a Uniform Uncertainty Principle (UUP) for every k sparse vector x, or to possess a Restricted Isometry Constant δ_{2k} (RIC) defined in [5] as follows:

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Definition 1 *For $k = 1, 2, \cdots$, the restricted isometry constant is the smallest number δ_k such that*

$$(1 - \delta_k)\|x\|_2^2 \le \|\Phi x\|_2^2 \le (1 + \delta_k)\|x\|_2^2$$

holds for all k-sparse vector $x \in \mathbb{R}^n$ with $\|x\|_0 \le k$, where $\|x\|_0$ denotes the number of nonzero elements of x.

For simplicity, we only discuss the sparse solution for noiseless recovery, as it can be easily extended to the noisy recovery case. See for example [4, 9].

As the sparsity is fundamental in signal processing, the problem can be mathematically formulated as to find

$$x^* = arg\min\{\|x\|_0 \mid \ \Phi x = b, x \in \mathbb{R}^n\}. \tag{1}$$

Unfortunately, the formulation (1) is practically intractable due to its NP-hardness [13]. For more information on the issue and related applications in signal and image processing, see [1] and references therein.

A common alternative is to consider the following convex problem using the ℓ_1 norm

$$x^{(1)} = arg\min\{\|x\|_1 \mid \ \Phi x = b, x \in \mathbb{R}^n\}. \tag{2}$$

Let x^* be the sparse solution to (1). Define

$$T_0 = \{i \in \{1, 2, \cdots, n\} \mid \ x_i^* \neq 0\}, \tag{3}$$

and define x_{T_0} for any $x \in \mathbb{R}^n$ as

$$(x_{T_0})_i = \begin{cases} x_i, & \text{if } i \in T_0, \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Then, a new ℓ_1 recovery result, which extended many existing ones in literature such as [4, 9, 2, 8], is stated below.

Theorem 1 ([12]) *Suppose $\delta_{2k} < \frac{77-\sqrt{1337}}{82} \approx 0.4931$, the solution $x^{(1)}$ of the problem (2) satisfies*

$$\|x - x^{(1)}\|_2 \le C_0 k^{-1/2} \|x - x_{T_0}\|_1, \quad \forall x \in \{x \mid \Phi x = b\},$$

where C_0 is a positive constant dependent on δ_{2k}. It follows that if x is k-sparse, the recovery is exact.

On the other hand, nonconvex ℓ_q quasi-norm minimization with $q \in (0, 1)$ is also considered to recover the sparse solution [10]. It is to solve, for a number $q \in (0, 1)$,

$$x^{(q)} = arg\min\{\|x\|_q \mid \ \Phi x = b, x \in \mathbb{R}^n\}. \tag{5}$$

More studies on the nonconvex ℓ_q minimization problem can be found in [6, 9, 14, 7]. In particular, the following two theorems will be strengthened in this paper.

Theorem 2 ([9]) *Suppose $\delta_{2k} < 2(3-\sqrt{2}) \approx 0.4531$. Then for any $q \in (0,1]$*

$$\|x - x^{(q)}\|_q \le C_0 \|x - x_{T_0}\|_q, \quad \forall x \in \{x \mid \Phi x = b\},$$

where C_0 is a positive constant dependent on δ_{2k}. In particular, if $x = x^*$ is k-sparse, the recovery is exact.

Theorem 3 ([14]) *Suppose $\delta_{2k} < 1/2$. There exists a number $q_0 \in (0,1]$ such that for any $q < q_0$, each minimizer x^q of the ℓ_q minimization (5) is the sparse solution of (5). Furthermore, there exists a positive constant C_q dependent on q and δ_{2k} such that*

$$\|x - x^{(q)}\|_q \le C_q \|x - x_{T_0}\|_q, \quad \forall x \in \{x \mid \Phi x = b\}.$$

More specifically, the upper bound of δ_{2k} in Theorem 2 can be improved to 0.4931 in Theorem 1 of the paper. Secondly, the sufficient condition $\delta_{2k} < 1/2$ in Theorem 3 can be extended to $\delta_{2k} \le 1/2$ and the threshold q_0 can be precisely estimated to be at least 0.9181 by Theorem 4. This is indeed a very surprising result since q_0, if it would have been computed by the analysis in [14], is only 0.0513. Our main tool to achieve these results is an improvement on a version of "the converse of a generalized Cauchy-Schwarz inequality" extended to the setting of ℓ_q quasi-norms. The key inequality is stated and proved in Section 2 below.

2 A Key Inequality

According to Cauchy-Schwarz inequality, we have the standard inequality

$$\|x\|_2 \ge \frac{\|x\|_1}{\sqrt{n}}.$$

The following converse of the above inequality is very recent:

Lemma 1 ([3]) *For any $x \in \mathbb{R}^n$,*

$$\|x\|_2 \le \frac{\|x\|_1}{\sqrt{n}} + \frac{\sqrt{n}}{4}\left(\max_{1\le i\le n}|x_i| - \min_{1\le i\le n}|x_i|\right).$$

On the other hand, Cauchy-Schwarz inequality can be extended to the setting of quasi-norm $\|x\|_q$, $q \in (0,1)$ with

$$\|x\|_2 \ge \frac{\|x\|_q}{n^{1/q-1/2}} \tag{6}$$

by using Hölder's inequality. The first converse of (6) was proposed in ([14]).

Lemma 2 ([14]) *Fix $0 < q < 1$. For any $x \in \mathbb{R}^n$,*

$$\|x\|_2 \le \frac{\|x\|_q}{n^{1/q-1/2}} + \sqrt{n}\left(\max_{1\le i\le n}|x_i| - \min_{1\le i\le n}|x_i|\right). \tag{7}$$

Our key result, Lemma 3 below, gives a sharpened estimation on the right hand side of (7). When p_q in (9) is replaced by 1 for any $q \in (0,1)$, Lemma 3 reduces to Lemma 2.

Lemma 3 *For $q \in (0,1)$ and $x \in \mathbb{R}^n$, there is*

$$\|x\|_2 \le \frac{\|x\|_q}{n^{1/q-1/2}} + p_q\sqrt{n}\left(\max_{1\le i\le n}|x_i| - \min_{1\le i\le n}|x_i|\right), \tag{8}$$

where

$$p_q := \left(\frac{q}{2}\right)^{\frac{q}{2-q}} - \left(\frac{q}{2}\right)^{\frac{2}{2-q}}. \tag{9}$$

Moreover, p_q is a decreasing convex function of $q \in (0,1)$ with

$$\lim_{q\to 0} p_q = 1 \quad \textit{and} \quad \lim_{q\to 1} p_q = 1/4.$$

Proof. Due to the symmetry of the inequality (8) in the components $|x_1|, |x_2|, \ldots, |x_n|$, we only have to prove the case for $x \in S = \{(x_1, x_2, \ldots, x_n) \ne 0 |\ x_1 \ge x_2 \ge \cdots \ge x_n \ge 0\}$ (notice that $x = 0$ is a trivial case). Furthermore, suppose the inequality (8) is true for $x \in S, x_1 = 1$. By substituting $\frac{x}{x_1}, x \in S$ into (8) and canceling the common factor $\frac{1}{x_1}$, we immediately generalize the result to all $x \in S$. In other words, our goal is to show

$$\|x\|_2 \le \frac{\|x\|_q}{n^{1/q-1/2}} + p_q\sqrt{n}\,(1 - x_n)\,,\ x \in S_1 = \{x \in S |\ x_1 = 1\} \tag{10}$$

where p_q is a function of q specified in (9).

Following the approach in [14], we define for any fixed $q \in (0,1)$

$$f(x) = \|x\|_2 - \frac{\|x\|_q}{n^{1/q-1/2}}$$

and compute the first order partial derivatives as

$$\frac{\partial f(x)}{\partial x_i} = \frac{x_i}{\|x\|_2} - \frac{\|x\|_q^{1-q} x_i^{q-1}}{n^{1/q-1/2}}.$$

Note that, when x_i increases with all other components fixed, the following two terms

$$\frac{\|x\|_2}{x_i} = \sqrt{\sum_{j=1}^{n}\left(\frac{x_j}{x_i}\right)^2} \tag{11}$$

and

$$\frac{\|x\|_q^{1-q} x_i^{q-1}}{n^{1/q-1/2}} = \frac{1}{n^{1/q-1/2}}\left(\sum_{j=1}^{n}(\frac{x_j}{x_i})^q\right)^{\frac{1-q}{q}} \tag{12}$$

are both decreasing. As the result, $\frac{\partial f(x)}{\partial x_i}$ is increasing and $f(x)$ is convex in each of the components x_i for $i = 1, 2, \cdots, n$. Analogously from (11) and (12), we can show that the composite function $g : R^{n-1} \to R$ such that

$$g(x_1, x_3, x_4, \ldots, x_n) = f(x_1, x_3, x_3, x_4, \ldots, x_n)$$

is also convex in the variable x_3 while all other components $x_1, x_4, \ldots, x_n$ remaining fixed. Likewise, we can conclude that $f(1, \ldots, 1, x_n, \ldots, x_n)$ is convex in the variable x_n where x_n is repeated for a couple of times.

Since the maximum of a convex function always happens on the boundary, we have

$$\begin{aligned}
& \max_{1 \geq x_2 \geq x_3 \cdots \geq x_n} f(1, x_2, \ldots, x_n) \\
= & \max_{1 \geq x_3 \geq \cdots \geq x_n} \left\{ \max_{x_2 \in [x_3, 1]} f(1, x_2, \ldots, x_n) \right\} \\
= & \max_{1 \geq x_3 \geq \cdots \geq x_n} \max \left\{ f(1, 1, x_3, \ldots, x_n), f(1, x_3, x_3, \ldots, x_n) \right\} \\
= & \max \left\{ \max_{1 \geq x_3 \geq \cdots \geq x_n} f(1, 1, x_3, \ldots, x_n), \max_{1 \geq x_3 \geq \cdots \geq x_n} f(1, x_3, x_3, x_4 \ldots, x_n) \right\}. \qquad (13)
\end{aligned}$$

In (13), since $f(1, x_3, x_3, x_4 \ldots, x_n)$ is convex in x_3, it follows that

$$\max_{x_3 \in [x_4, 1]} f(1, x_3, x_3, x_4 \ldots, x_n) = \max\{f(1, 1, 1, x_4, \ldots, x_n), f(1, x_4, x_4, x_4, x_5, \ldots, x_n)\}.$$

Repeating the arguments iteratively, we can thus express the maximum of f only in terms of 1 and x_n as follows:

$$h(x_n) = \max_{1 \geq x_2 \geq x_3 \cdots \geq x_n} f(1, x_2, \ldots, x_n) = f(1, \ldots, 1, x_n, \ldots, x_n), \;\; x_n \in [0, 1].$$

Suppose the distribution of $x_1 = 1$ appears for r times ($1 \leq r \leq n$) in the maximum solution of f, we have

$$h(x_n) = \sqrt{r(1 - x_n^2) + n x_n^2} - \frac{(r(1 - x_n^q) + n x_n^q)^{1/q}}{n^{1/q - 1/2}}.$$

Since $h(x_n)$ is convex and $h(1) = 0$, we have

$$h(x_n) \leq (1 - x_n) h(0) + x_n h(1) = (1 - x_n) h(0).$$

Then it holds that

$$\begin{aligned}
f(x) & \leq h(x_n) \\
& \leq (1 - x_n) h(0) \\
& \leq (1 - x_n) \max_{r \in \{1, 2, \ldots, n\}} \left\{ \sqrt{r(1^2 - 0^2) + n0^2} - \frac{(r(1^q - 0^q) + n0^q)^{1/q}}{n^{1/q - 1/2}} \right\} \qquad (14) \\
& \leq (1 - x_n) \max_{r \in [1, n]} \left\{ \sqrt{r} - \frac{r^{1/q}}{n^{1/q - 1/2}} \right\} \qquad (15) \\
& = (1 - x_n) \left(\left(\frac{q}{2}\right)^{\frac{q}{2-q}} - \left(\frac{q}{2}\right)^{\frac{2}{2-q}} \right) \sqrt{n}.
\end{aligned}$$

where (14) is an upper bound estimation for $h(0)$ over the unknown parameter r (the number of times $x_1 = 1$ is repeated), and (15) is a concave maximization problem since $q \in (0, 1)$ and r is relaxed to a real number on $[1, n]$.

Finally, it is easy to verify that

$$\lim_{q\to 0} p_q = 1, \ \lim_{q\to 1} p_q = 1/4,$$

and p_q is a decreasing function of $q \in (0,1)$ since

$$\begin{aligned} \frac{d}{dq}p_q &= \frac{(q/2)^{\frac{q}{2-q}}}{(2-q)^2}(2\ln(q/2)+2-q) - \frac{(q/2)^{\frac{2}{2-q}}}{(2-q)^2}(2\ln(q/2)+\frac{4}{q}-2) \\ &= \frac{2\ln(q/2)}{(2-q)^2}p_q < 0. \end{aligned} \tag{16}$$

Moreover, the convexity of p_q over $q \in (0,1)$ can be verified by

$$\frac{d^2}{dq^2}p_q = \frac{(2\ln(q/2)+2-q)^2 + (2-q)^3/q}{(2-q)^4}p_q > 0.$$

and the proof is thus completed. □

In the following, we give an upper estimate of p_q for small q, which will be used later.

Lemma 4 *Denote Euler's number by* $e = 2.718\cdots$. *It holds that*

$$p_q < 1 + \frac{q\ln(q/2)}{2-q}, \ \forall q \in (0, 0.4797] \subset (0, 1-\sqrt{2}/e].$$

Proof. Since $e^x < 1 + x + x^2/2$ for $x < 0$, we have

$$p_q = (q/2)^{\frac{q}{2-q}} - (q/2)^{\frac{2}{2-q}} = e^{\frac{q\ln(q/2)}{2-q}} - (q/2)^{\frac{2}{2-q}} < 1 + \frac{q\ln(q/2)}{2-q} + \frac{q^2\ln^2(q/2)}{2(2-q)^2} - (q/2)^{\frac{2}{2-q}}.$$

To prove the lemma, it is sufficient to show that

$$\frac{-q\ln(q/2)}{\sqrt{2}(2-q)} \le (q/2)^{\frac{1}{2-q}}, \ \forall q \in (0, 1-\sqrt{2}/e]. \tag{17}$$

Since $\frac{1}{2-q} \le \frac{1}{1+\sqrt{2}/e}$ and $(q/2)^{\frac{1}{2-q}} \ge (q/2)^{\frac{1}{1+\sqrt{2}/e}}$, the inequality (17) can be confirmed by verifying

$$\sqrt{2}\frac{-(q/2)\ln(q/2)}{(1+\sqrt{2}/e)} \le (q/2)^{\frac{1}{1+\sqrt{2}/e}}, \ \forall q \in (0, 1-\sqrt{2}/e],$$

or equivalently, by verifying

$$-(q/2)^{\frac{\sqrt{2}/e}{1+\sqrt{2}/e}}\ln(q/2) \le \frac{1+\sqrt{2}/e}{\sqrt{2}}, \ \forall q \in (0, 1-\sqrt{2}/e].$$

Let $y = (q/2)^{\frac{\sqrt{2}/e}{1+\sqrt{2}/e}} \in (0,1)$. We then have the desired result by

$$-(q/2)^{\frac{\sqrt{2}/e}{1+\sqrt{2}/e}}\ln(q/2) = -\frac{1+\sqrt{2}/e}{\sqrt{2}/e}y\ln(y) \le \frac{1+\sqrt{2}/e}{\sqrt{2}}$$

because the negative entropy function $-y\ln(y)$ attains the maximum value of $1/e$. □

3 Main Results

Let Null(Φ) be the null space of Φ; $x^*, x^{(q)}$ be the solutions to (1) and (5), respectively; T_0 be defined in (3). Suppose $\|x^*\|_0 = k$ and define

$$T_0^c = \{1, 2, \cdots, n\} \setminus T_0.$$

The following null space property is essential. However, a refined version is immediately stated in Lemma 6.

Lemma 5 ([10]) *$x^{(q)}$ is the unique sparse solution x^* if and only if*

$$\|h_{T_0}\|_q < \|h_{T_0^c}\|_q, \ \forall h \in \mathrm{Null}(\Phi), \ h \neq 0, \tag{18}$$

where h_{T_0} and $h_{T_0^c}$ are similarly defined as in (4).

Lemma 6 *$x^{(q)}$ is the unique sparse solution x^* if and only if*

$$\|h_{T_0}\|_q < \|h_{T_0^c}\|_q, \ \forall h \in \mathrm{Null}(\Phi), \ h_{T_0^c} \neq 0. \tag{19}$$

Proof. It is sufficient to study the difference between (18) and (19). Suppose $0 \neq h \in \mathrm{Null}(\Phi)$ and $h_{T_0^c} = 0$. It follows that $h_{T_0} \neq 0$ and $\Phi h_{T_0} = 0$. Therefore,

$$\Phi(x^* + t h_{T_0}) = b, \ \forall t \in \mathbb{R},$$

and

$$\min_{t \in \mathbb{R}} \|x^* + t h_{T_0}\|_0 \leq k - 1,$$

which contradicts the optimality of x^*. $\square$

The purpose of this research is to establish sufficient conditions for (19) with the help of Lemmas 3 and 4 so that

$$\tau(h, q) := \frac{\|h_{T_0}\|_q}{\|h_{T_0^c}\|_q} < 1, \ \forall h \in \mathrm{Null}(\Phi), \ h_{T_0^c} \neq 0.$$

To this end, let

$$h = h_{T_0} + h_{T_1} + h_{T_2} + \cdots,$$

where T_1 corresponds to the locations of the k largest entries of $h_{T_0^c}$, T_2 the locations of the next k largest entries of $h_{T_0^c}$ and so on. Without loss of generality, we assume

$$h = (h_{T_0}, h_{T_1}, h_{T_2}, \cdots)^T$$

with the cardinality of T_i being equal to k for $i = 0, 1, 2, \cdots$. Define a ratio

$$t := t(h, q) \in (0, 1]$$

such that

$$\|h_{T_1}\|_q^q = t \sum_{i \geq 1} \|h_{T_i}\|_q^q.$$

According to Lemma 6, we only focus on nonzero $h_{T_0^c}$, which immediately implies that $h_{T_1} \neq 0$, i.e., $t > 0$. Several technique lemmas are needed.

Lemma 7 ([14]) *For $q \in (0,1)$, we have*

$$\sum_{i \geq 2} \|h_{T_i}\|_2^2 \leq \frac{1}{k^{(2-q)/q}} (1-t) t^{(2-q)/q} \left(\sum_{i \geq 1} \|h_{T_i}\|_q^q \right)^{2/q}$$

Lemma 8 *For $q \in (0,1)$, $h \in \mathrm{Null}(\Phi)$, $h_{T_0^c} \neq 0$, we have*

$$\sum_{i \geq 2} \|h_{T_i}\|_2 \leq \frac{1 + (p_q - 1) t^{1/q}}{k^{1/q - 1/2}} \left(\sum_{i \geq 1} \|h_{T_i}\|_q^q \right)^{1/q}$$

Proof. We first apply Lemma 3 to each h_{T_i} to get

$$k^{1/q-1/2} \|h_{T_i}\|_2 \leq \|h_{T_i}\|_q + p_q k^{1/q} (|h_{ik+1}| - |h_{ik+k}|), \quad i = 2, 3, \ldots \tag{20}$$

and sum up (20) over all $i \geq 2$. Then,

$$\begin{aligned}
k^{1/q-1/2} \sum_{i \geq 2} \|h_{T_i}\|_2 &\leq \sum_{i \geq 2} \|h_{T_i}\|_q + p_q k^{1/q} \left\{ |h_{2k+1}| - (|h_{2k+k}| - |h_{3k+1}|) - \cdots \right\} \\
&\leq \sum_{i \geq 2} \|h_{T_i}\|_q + p_q k^{1/q} |h_{2k+1}| \\
&= \sum_{i \geq 2} \|h_{T_i}\|_q + p_q k^{1/q} (|h_{2k+1}|^q)^{1/q} \\
&\leq \sum_{i \geq 2} \|h_{T_i}\|_q + p_q k^{1/q} (\|h_{T_1}\|_q^q / k)^{1/q} \\
&= \sum_{i \geq 1} \|h_{T_i}\|_q + (p_q - 1) \left(\|h_{T_1}\|_q^q \right)^{1/q} \\
&\leq \left(\sum_{i \geq 1} \|h_{T_i}\|_q^q \right)^{1/q} + (p_q - 1) \left(t \sum_{i \geq 1} \|h_{T_i}\|_q^q \right)^{1/q} \quad \text{(since } q < 1\text{)} \\
&= (1 + (p_q - 1) t^{1/q}) \left(\sum_{i \geq 1} \|h_{T_i}\|_q^q \right)^{1/q}.
\end{aligned}$$

□

Lemma 9 *For $q \in (0,1)$, $h \in \mathrm{Null}(\Phi)$, $h_{T_0^c} \neq 0$, we have*

$$\|\Phi(h_{T_0}+h_{T_1})\|_2^2 = \|\Phi(\sum_{j\geq 2} h_{T_j})\|_2^2 \leq \left(\frac{(1-t)t^{(2-q)/q}}{k^{2/q-1}} + \frac{\delta_{2k}(1+(p_q-1)t^{1/q})^2}{k^{2/q-1}}\right)\left(\sum_{i\geq 1}\|h_{T_i}\|_q^q\right)^{2/q}.$$

Proof. According to Lemma 1.2 in [5], we have

$$\langle \Phi(h_{T_i}), \Phi(h_{T_j})\rangle \leq \delta_{2k}\|h_{T_i}\|_2\|h_{T_j}\|_2.$$

Therefore,

$$\begin{aligned}
\|\Phi(\sum_{j\geq 2} h_{T_j})\|_2^2 &= \sum_{i,j\geq 2}\langle \Phi(h_{T_i}), \Phi(h_{T_j})\rangle \\
&= \sum_{j\geq 2}\langle \Phi(h_{T_j}), \Phi(h_{T_j})\rangle + 2\sum_{2\leq i<j}\langle \Phi(h_{T_i}), \Phi(h_{T_j})\rangle \\
&\leq (1+\delta_k)\sum_{i\geq 2}\|h_{T_i}\|_2^2 + 2\delta_{2k}\sum_{i>j\geq 2}\|h_{T_i}\|_2\|h_{T_j}\|_2 \\
&\leq \sum_{i\geq 2}\|h_{T_i}\|_2^2 + \delta_{2k}(\sum_{i\geq 2}\|h_{T_i}\|_2)^2 \\
&\leq \left(\frac{(1-t)t^{(2-q)/q}}{k^{2/q-1}} + \frac{\delta_{2k}(1+(p_q-1)t^{1/q})^2}{k^{2/q-1}}\right)\left(\sum_{i\geq 1}\|h_{T_i}\|_q^q\right)^{2/q}.
\end{aligned}$$

where the last inequality follows from Lemma 7 and Lemma 8. □

Lemma 10 ([14]) *For $q \in (0,1)$, we have*

$$\|\Phi(h_{T_0}+h_{T_1})\|_2^2 \geq \frac{1-\delta_{2k}}{k^{2/q-1}}(\tau(h,q)^2 + t^{2/q})\left(\sum_{i\geq 1}\|h_{T_i}\|_q^q\right)^{2/q}. \tag{21}$$

Combining Lemma 9 with Lemma 10, we obtain

$$(1-\delta_{2k})(\tau(h,q)^2 + t^{2/q}) \leq (1-t)t^{(2-q)/q} + \delta_{2k}(1+(p_q-1)t^{1/q})^2.$$

After rearrangement of the terms, it implies that

$$\tau(h,q)^2 \leq (\delta_{2k}(1+(p_q-1)t^{1/q})^2 + t^{(2-q)/q} - (2-\delta_{2k})t^{2/q})/(1-\delta_{2k}). \tag{22}$$

Then, an immediate sufficient condition for x^q being the sparse solution of (1) is to require the right hand side of (22) being less than 1 for all $t \in (0,1]$. If we focus on $\delta_{2k} < 1$, the sufficient condition that the right hand side of (22) being less than 1 can be expressed as

$$r(t,q,\delta_{2k}) := 2\delta_{2k} + (p_q-1)t^{1/q}(2+(p_q-1)t^{1/q})\delta_{2k} + t^{(2-q)/q} - (2-\delta_{2k})t^{2/q} < 1,\ \forall t \in (0,1]. \tag{23}$$

The problem then becomes to estimate the range of $q \in (0,1)$ and $\delta_{2k} \in (0,1)$ for which (23) is true.

Notice that by setting $p_q = 1,\ \forall q \in (0,1)$ and by the fact (shown in [14]) that when $q \to 0^+$,

$$\sup_{t\in(0,1]} \left\{ t^{(2-q)/q} - (2-\delta_{2k})t^{2/q} \right\} \le q/e \to 0,$$

one can immediately obtain Theorem 3. In general, as $p_q < 1$ for $q \in (0,1)$, we expect an improvement over Theorem 3.

To begin with, we rewrite

$$r(t,q,\delta_{2k}) = r_1(t,q,\delta_{2k}) + r_2(t,q,\delta_{2k}) + r_3(t,q,\delta_{2k}).$$

where

$$\begin{aligned} r_1(t,q,\delta_{2k}) &= \delta_{2k} + \delta_{2k}(1-(1-p_q)t^{1/q})^2; \\ r_2(t,q,\delta_{2k}) &= t^{(2-q)/q}; \\ r_3(t,q,\delta_{2k}) &= -(2-\delta_{2k})t^{2/q} \end{aligned}$$

with the ranges $\delta_{2k}, q \in (0,1)$ and $t \in (0,1]$.

We first discuss the monotonicity of functions $r,\ r_1,\ r_2,\ r_3$. Here is the summary:

(a) $r_1(t,q,\delta_{2k})$ and $r_3(t,q,\delta_{2k})$ are decreasing functions of $t \in (0,1]$ whereas $r_2(t,q,\delta_{2k})$ is increasing.

(b) $r_1(t,q,\delta_{2k})$ is decreasing in terms of q since both $1-p_q$ and $t^{1/q}$ are increasing functions of q.

(c) The sum of the latter two functions $(r_2+r_3)(t,q,\delta_{2k})$ is an increasing function of q for $t \le \frac{1}{2-\delta_{2k}}$ since

$$\frac{\partial(r_2+r_3)(t,q,\delta_{2k})}{\partial q} = -2/q^2(\ln t)t^{2/q-1}(1-(2-\delta_{2k})t) \ge 0.$$

(d) $r(t,q,\delta_{2k})$ is an increasing function of δ_{2k} since we can rewrite (23) as

$$\begin{aligned} r(t,q,\delta_{2k}) &= \delta_{2k}(2+2(p_q-1)t^{1/q} + (p_q-1)^2t^{2/q} + t^{2/q}) + t^{(2-q)/q} - 2t^{2/q} \\ &= \delta_{2k}(1+(1+(p_q-1)t^{1/q})^2) + t^{(2-q)/q} - 2t^{2/q} \end{aligned}$$

with $1+(1+(p_q-1)t^{1/q})^2 > 0$.

Moreover, we have

Lemma 11 *Suppose $\delta_{2k} \le 1/2$, it holds that*

$$r(t,q,\delta_{2k}) < 1,\ \forall t \in (\frac{1}{2-\delta_{2k}}, 1],\ \forall q \in (0,1).$$

Proof. Suppose $t > \frac{1}{2-\delta_{2k}}$, we always have

$$\begin{aligned} r(t,q,\delta_{2k}) &= 2\delta_{2k} + (p_q-1)t^{1/q}(2+(p_q-1)t^{1/q})\delta_{2k} + (1/t-2+\delta_{2k})t^{2/q} \\ &\leq 2\delta_{2k} + (p_q-1)t^{1/q}(2+(p_q-1)t^{1/q})\delta_{2k} \\ &< 2\delta_{2k} \leq 1. \end{aligned}$$

□

Now we present our main result.

Theorem 4 *Suppose $\delta_{2k} \leq 1/2$. For any $q \in (0, 0.9181]$, each minimizer x^q of the ℓ_q minimization (5) is the sparse solution of (1).*

Proof. Since $r(t,q,\delta_{2k})$ is an increasing function of δ_{2k} by monotonicity (d), for any $\delta'_{2k} < \delta_{2k}$, we have

$$r(t,q,\delta_{2k}) < 1,\ \forall t \in (0,1] \implies r(t,q,\delta'_{2k}) < 1,\ \forall t \in (0,1].$$

Hence, it is sufficient to assume that $\delta_{2k} = 1/2$. Then we have

$$r(t,q,1/2) = 1 + (p_q-1)t^{1/q}(2+(p_q-1)t^{1/q})/2 + t^{(2-q)/q} - 3/2t^{2/q},$$

which is less than 1, by Lemma 11, for all $t \in (\frac{2}{3}, 1]$ and $q \in (0,1)$. The rest of the proof is to show that $r(t,q,1/2) < 1$ on $t \in (0, 2/3]$ and $q \in (0, 0.9181]$ by incorporating monotonicity (a) - (c) on sufficiently fine meshes.

First, for any $q \in (0,1)$,

$$\begin{aligned} \frac{\partial r(t,q,1/2)}{\partial t} &= \frac{t^{(2/q)-2}}{q}\left(\frac{p_q-1}{t^{(1/q)-1}} + 2 - q - (3-(p_q-1)^2)t\right) \\ &\leq \frac{p_q-1}{q}t^{(1/q)-1} + \frac{2-q}{q}t^{(2/q)-2} \\ &= (p_q-1+(2-q)t^{(1/q)-1})\frac{t^{(1/q)-1}}{q}, \\ &< 0,\quad \forall\, 0 < t^{(1/q)-1} < \frac{1-p_q}{2-q}. \end{aligned}$$

In other words, $r(t,q,1/2)$ is strictly decreasing on $t \in (0, (\frac{1-p_q}{2-q})^{\frac{q}{1-q}})$. Moreover, $r(0,q,1/2) = 1$. Hence, for any $q \in (0,1)$, we have

$$r(t,q,1/2) < 1,\ \forall\, 0 < t < \left(\frac{1-p_q}{2-q}\right)^{\frac{q}{1-q}}, \tag{24}$$

which specifies the range on which $r(t,q,1/2) < 1$ by a function of $q \in (0,1)$.

Secondly, to analyze the function $\left(\frac{1-p_q}{2-q}\right)^{\frac{q}{1-q}}$, we can compute to get

$$\lim_{q\to 0^+}\left(\frac{1-p_q}{2-q}\right)^{\frac{q}{1-q}} = 1;\ \lim_{q\to 1^-}\left(\frac{1-p_q}{2-q}\right)^{\frac{q}{1-q}} = 0,$$

and its first derivative

$$\frac{d}{dq}\left(\left(\frac{1-p_q}{2-q}\right)^{\frac{q}{1-q}}\right) = \left(\frac{1-p_q}{2-q}\right)^{\frac{q}{1-q}}\left(\frac{1}{(1-q)^2}\ln\frac{1-p_q}{2-q}+\frac{q(1-p_q-p_q'(2-q))}{(1-q)(1-p_q)(2-q)}\right)$$

$$= \left(\frac{1-p_q}{2-q}\right)^{\frac{q}{1-q}}\left(\frac{1}{(1-q)^2}\ln\frac{1-p_q}{2-q}+\frac{q(1-p_q-\frac{2\ln(q/2)}{(2-q)}p_q)}{(1-q)(1-p_q)(2-q)}\right) \quad (25)$$

where (16) is used for p_q'. We consider the following two cases.

(i) Let $q \in (0, 0.3]$. In this case, $p_q \geq p_{0.3} > 0.6081$. According to Lemma 4, we have

$$\frac{-q\ln(q/2)}{(1-p_q)(2-q)} < 1.$$

Therefore,

$$\begin{aligned}\frac{1}{(1-q)^2}\ln\frac{1-p_q}{2-q}+\frac{q(1-p_q-\frac{2\ln(q/2)}{(2-q)}p_q)}{(1-q)(1-p_q)(2-q)} &< \frac{1}{1-q}\left\{\frac{\ln(\frac{1-p_q}{2-q})}{1-q}+\frac{q}{2-q}+\frac{2p_q}{2-q}\right\}\\ &\leq \frac{1}{1-q}\left\{\ln\left(\frac{1-0.6081}{2-0.3}\right)+\frac{0.3}{2-0.3}+\frac{2}{2-0.3}\right\}\\ &< 0,\end{aligned}$$

implying that $\left(\frac{1-p_q}{2-q}\right)^{\frac{q}{1-q}}$ is strictly decreasing for $q \in (0, 0.3]$.

(ii) Let $q \in [0.3, 0.99]$. Define a partition of $q \in (0, 0.3]$ by $q_s = 0.3 + 0.01s$ for $s = 0, 1, \cdots, 69$ and try to estimate the right hand side of (25) on each mesh $q \in [q_s, q_{s+1}]$ where q_s, q_{s+1} are points in the partition. Then,

$$\begin{aligned}&\frac{\ln\frac{1-p_q}{2-q}}{(1-q)^2}+\frac{q(1-p_q-\frac{2\ln(q/2)}{(2-q)}p_q)}{(1-q)(1-p_q)(2-q)}\\ =\ &\frac{\ln\frac{1-p_q}{2-q}}{(1-q)^2}+\frac{q}{(1-q)(2-q)}+\frac{-2q\ln(q/2)p_q}{(1-q)(1-p_q)(2-q)^2}\\ \leq\ &\frac{\ln\frac{1-p_{q_{s+1}}}{2-q_{s+1}}}{(1-q_s)^2}+\frac{q_{s+1}}{(1-q_{s+1})(2-q_{s+1})}+\frac{-2q_{s+1}\ln(q_s/2)p_{q_s}}{(1-q_{s+1})(1-p_{q_s})(2-q_{s+1})^2}.\end{aligned} \quad (26)$$

With the aid of computer, the evaluation of (26) shows that they are negative on all mesh points. That is, $\left(\frac{1-p_q}{2-q}\right)^{\frac{q}{1-q}}$ is strictly decreasing for $q \in [0.3, 0.99]$.

Together with (i) and (ii), we conclude that $\left(\frac{1-p_q}{2-q}\right)^{\frac{q}{1-q}}$ is strictly decreasing for $q \in (0, 0.99]$, which leads to the following estimation:

$$\left(\frac{1-p_q}{2-q}\right)^{\frac{q}{1-q}} \geq \left(\frac{1-p_{0.9181}}{2-0.9181}\right)^{\frac{0.9181}{1-0.9181}} > 0.0105, \qquad \forall q \in (0, 0.9181]; \quad (27)$$

$$\left(\frac{1-p_q}{2-q}\right)^{\frac{q}{1-q}} \geq \left(\frac{1-p_{0.17}}{2-0.17}\right)^{\frac{0.17}{1-0.17}} > 0.6821, \qquad \forall q \in (0, 0.17]. \quad (28)$$

It follows from (**??**) and (24) that

$$r(t,q,1/2) < 1, \qquad \forall t \in (0, 0.0105],\ \forall q \in (0, 0.9181];$$

from (**??**) and Lemma 11 that

$$r(t,q,1/2) < 1, \qquad \forall t \in (0, 1],\ \forall q \in (0, 0.17].$$

That is, to prove (23), it is left to verify that $r(t,q,1/2) < 1$ on $(t,q) \in [0.0105, 2/3] \times [0.17, 0.9181]$. Our idea is to subdivide the region $[0.0105, 2/3] \times [0.17, 0.9181]$ into the union of small squares with the type $[t_r, t_{r+1}] \times [q_s, q_{s+1}]$. Then, by the monotonicity (a), (b) and (c), we can estimate $r(t,q,1/2)$ on this square by the corner points as follows:

$$\begin{aligned} & \max_{t\in[t_r,t_{r+1}]} \max_{q\in[q_s,q_{s+1}]} r(t,q,1/2) \\ \leq\ & r_1(t_r, q_s, 1/2) + r_2(t_{r+1}, q_{s+1}, 1/2) + r_3(t_r, q_{s+1}, 1/2). \end{aligned} \tag{29}$$

If the evaluation by computer shows that (**??**) is less than one, we are done with the square. Otherwise, the estimation might not be tight enough so that we have to subdivide the square into finer meshes.

Our calculation shows that, by covering $[0.0105, 2/3] \times [0.17, 0.9181]$ with

$$[0.0105, 0.6667]\times[0.17, 0.9172] \bigcup [0.0105, 0.66667]\times[0.9172, 0.91809] \bigcup [0.0105, 0.666667]\times[0.91809, 0.9181],$$

we can get the desired result as

$$\begin{aligned} \max_{r\in\{105,106,\cdots,6666\}} \{ \max_{s\in\{1700,1701,\cdots,9171\}} & \{r_1(t_r, q_s, 1/2) \\ & + r_2(t_{r+1}, q_{s+1}, 1/2) + r_3(t_{r+1}, q_{s+1}, 1/2)\}\} < 1, \end{aligned}$$

where $t_r = 0.0001r$ and $q_s = 0.0001s$;

$$\begin{aligned} \max_{r\in\{1050,1051,\cdots,66666\}} \{ \max_{s\in\{91720,91721,\cdots,91808\}} & \{r_1(t_r, q_s, 1/2) \\ & + r_2(t_{r+1}, q_{s+1}, 1/2) + r_3(t_{r+1}, q_{s+1}, 1/2)\}\} < 1, \end{aligned}$$

where $t_r = 0.00001r$ and $q_s = 0.00001s$ in this partition; and

$$\begin{aligned} \max_{r\in\{10500,10501,\cdots,666666\}} & \{r_1(t_l, q_{91809}, 1/2) \\ & + r_2(t_{r+1}, q_{91810}, 1/2) + r_3(t_r, q_{91810}, 1/2)\} < 1. \end{aligned}$$

where $t_r = 0.000001r$.

All the above calculations were carried out by computer and the proof is thus complete. □

Remark 1 *Let $q \to 0^+$. Then $p_q \to 1$ and it follows from (23) that $\delta_{2k} \leq \frac{1}{2}$. In other words, $\delta_{2k} \leq \frac{1}{2}$ can not be further improved based on the sufficient condition (22). Furthermore, one can verify that*

$$r(0.064, 0.9182, 1/2) > 1.0000002,$$

implying that the condition $q \in (0, 0.9181]$ in Theorem 4 is tight up to the fourth decimal digit.

Remark 2 *Based on the analysis in [14], the threshold q_0 in Theorem 3 can be estimated to be around* 0.0513*, far less than the number 0.9181 reported in our Theorem 4.*

From the above analysis, if the sparse recovery is to be exact for any $q \in (0, 1)$, we may require a tighter restricted isometric constant than $\delta_{2k} = 0.5$. This is the spirit of Theorem 2, which we shall show immediately an improvement on their result.

To investigate the issue, we look into the case when $q = 1, p_q = 1/4$. The sufficient condition (23) becomes

$$r(t, 1, \delta_{2k}) = 2\delta_{2k} + (1 - 3\delta_{2k}/2)t - (2 - 25\delta_{2k}/16)t^2 < 1, \ \forall t \in (0, 1]. \tag{30}$$

Since $2 - 25\delta_{2k}/16 > 0$, $r(t, 1, \delta_{2k})$ is a concave parabola of $t \in (0, 1]$. The sufficient condition in (**??**) holds if and only if the equation $r(t, 1, \delta_{2k}) = 1$ has no solution. Namely, we need

$$(1 - 3\delta_{2k}/2)^2 + 4(2 - 25\delta_{2k}/16)(2\delta_{2k} - 1) < 0.$$

It follows that

$$\delta_{2k} < \frac{77 - \sqrt{1337}}{82} \ \ (> 0.493109)\,.$$

We therefore have the following theorem.

Theorem 5 *Suppose $\delta_{2k} \leq 0.4931$. Then for any $q \in (0, 1)$ each minimizer x^q of the ℓ_q minimization (5) is the sparse solution of (1).*

Proof. Since $r(t, q, \delta_{2k})$ is an increasing function of δ_{2k}, we fix δ_{2k} at 0.4931. According to Lemma 11, $r(t, q, 0.4931) < 1$ for $t \in (0.663614, 1]$. Therefore, it is sufficient to check the maximum of $r(t, q, 0.4931)$ over the region $(t, q) \in [0, 0.6637] \times [0.9181, 1]$. This is done by dividing the region into two parts

$$[0, 0.6637] \times [0.9181, 1] = [0, 0.6637] \times [0.9181, 0.9992] \bigcup [0, 0.6637] \times [0.9992, 1],$$

the first of which with a mesh size 0.0001 whereas the latter with 0.00001. The computation has been verified by computer as follows

$$
\begin{aligned}
& \max_{t\in[0,0.6637]} \max_{q\in[0.9181,1]} r(t,q,0.4931) \\
\leq \; & \max\{\{ \max_{r=0,1,2,\cdots,6636} \max_{s=9181,9182,\cdots,9991} \{r_1(t_r,q_s,0.4931) \\
& \qquad + r_2(t_{r+1},q_{s+1},0.4931) + r_3(t_r,q_{s+1},0.4931)\}\}; \\
& \max_{u=0,1,2,\cdots,66369} \{ \max_{v=99920,99921,\cdots,99999} \{r_1(t_u,q_v,0.4931) \\
& \qquad + r_2(t_{u+l},q_{v+1},0.4931) + r_3(t_u,q_{v+1},0.4931)\}\}\} \\
< \; & 1
\end{aligned}
$$

where $t_r = 0.0001r, q_s = 0.0001s, t_u = 0.00001u, q_v = 0.00001v$ and the proof is thus complete. □

4 Conclusion

In this paper, based on the extended "converse of a generalized Cauchy-Schwarz inequality" of quasi-norm, we establish new sufficient conditions under which the minimizer of the ℓ_q minimization is the sparse solution of the corresponding underdetermined linear system. More precisely, we show that if the restricted isometry constant $\delta_{2k} \leq 1/2$, then for any $q \leq 0.9181$ the solution of ℓ_q minimization also solves the ℓ_0 minimization. Furthermore, if $\delta_{2k} \leq 0.4931$, then for any $q \in (0,1]$, the minimizer of ℓ_q minimization remains optimal in ℓ_0 minimization. Our results strongly improves those reported previously in the literature. We believe the value 0.4931 can be improved to $\frac{77-\sqrt{1337}}{82}$ with a new proof, but now it is still open. The other future research is to extend the presented results to the rank minimization problem.

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