A CONCENTRATED CAUCHY DISTRIBUTION WITH FINITE MOMENTS

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Abstract. The Cauchy distribution

$$\mathfrak{C}(a,b)(x) = \frac{1}{\pi\, b \left(1 + \left(\dfrac{x-a}{b}\right)^2\right)}, \quad -\infty < x < \infty,$$

with a, b real, $b > 0$, has no moments (expected value, variance, etc.), because the defining integrals diverge. An obvious way to "concentrate" the Cauchy distribution, in order to get finite moments, is by truncation, restricting it to a finite domain. An alternative, suggested by an elementary problem in mechanics, is the distribution

$$\mathfrak{C}_g(a,b)(x) = \frac{\sqrt{1+2\,b\,g}}{\pi\, b \left(1 + \left(\dfrac{x-a}{b}\right)^2\right) \sqrt{1 - 2\,b\,g \left(\dfrac{x-a}{b}\right)^2}}, \quad a - \sqrt{\tfrac{b}{2g}} < x < a + \sqrt{\tfrac{b}{2g}},$$

with a, b as above and a third parameter $g \geq 0$. It has the Cauchy distribution $C(a,b)$ as the special case with $g = 0$, and for any $g > 0$, $\mathfrak{C}_g(a,b)$ has finite moments of all orders, while keeping the useful "fat tails" property of $\mathfrak{C}(a,b)$.

1. Introduction

The **Cauchy** (also **Lorentz**) **distribution**, centered at a, has the density function

$$\mathfrak{C}(a,b)(x) = \frac{1}{\pi\, b \left(1 + \left(\dfrac{x-a}{b}\right)^2\right)}, \quad -\infty < x < \infty, \tag{1}$$

with two parameters, a (**location**) and $b > 0$ (**scale**), [2, p. 50]. The Cauchy distribution occurs in numerous applications, see, e.g., [3, p. 159]. A recent interest in $\mathfrak{C}(a,b)$ is due to its "fat tails", accounting for outliers (rare events) better than the more concentrated distributions commonly used, in particular the normal distribution. Indeed, these "rare" events with significant consequences (that may not be so rare, see, e.g., [4]), make the $6\,\sigma$ paradigm useless in certain applications.

The Cauchy gunman. The distribution (1) is illustrated by the following story: A gunman, located at the point $A(a,b)$, shoots at the x–axis in an angle θ, measured from the the vertical line $x = a$, see Fig. 1(a). The bullet hits the x–axis at the point $x = a + b\tan\theta$. Assuming the angle θ is uniformly distributed in $[-\pi/2, \pi/2]$, the point where the bullet hits is a random variable $\mathbf{X}$, with (1) as its density, a fact denoted by $\mathbf{X} \sim \mathfrak{C}(a,b)$. This follows from,

$$\frac{1}{\pi}\frac{d}{dx}\tan^{-1}\left(\frac{x-a}{b}\right) = \frac{1}{\pi\, b\left(1 + \left(\frac{x-a}{b}\right)^2\right)}, \quad -\infty < x < \infty.$$

Properties of the Cauchy distribution. These include

(a) If $\mathbf{X} \sim \mathfrak{C}(a,b)$ then all its moments (in particular, expected value and variance) do not exist because the defining integrals diverge. The median and mode of $\mathbf{X}$ are equal to a.

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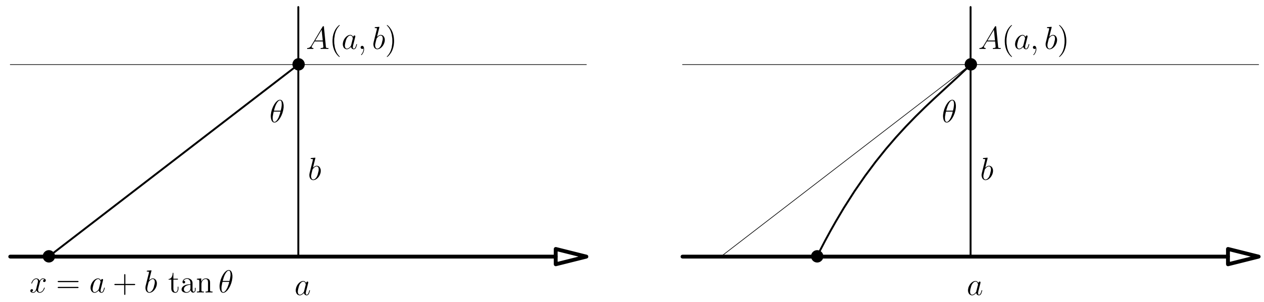


(a) The bullet hits x–axis at $x = a + b\,\tan\theta$ (b) With gravity, bullet hits closer to $x = a$

FIGURE 1. A gunman at $A(a, b)$ shoots the x–axis with a random angle θ

(b) The Cauchy distribution is stable, i.e. a sum of Cauchy RV's is a Cauchy RV. Let $\mathbf{X}$ and $\mathbf{Y}$ be independent RV's with $\mathbf{X} \sim \mathfrak{C}(a_1, b_1)$, and $\mathbf{Y} \sim \mathfrak{C}(a_2, b_2)$. Then $\mathbf{X} + \mathbf{Y} \sim \mathfrak{C}(a_1 + a_2, b_1 + b_2)$. This can be seen from the characteristic function (c.f.) of (1)

$$\widehat{\mathfrak{C}(a, b)}(w) = e^{-b|w|+i\,a\,w}, \tag{2}$$

from which it follows that the c.f. of $\mathbf{X} + \mathbf{Y}$ is $e^{-(b_1+b_2)|w|+i\,(a_1+a_2)w}$.

(c) If $\mathbf{X}_i \sim \mathfrak{C}(a, b)$, $i = 1\!:\!n$, so is their mean

$$\overline{\mathbf{X}} = \tfrac{1}{n} \sum_{i=1}^{n} \mathbf{X}_i \ \sim\ \mathfrak{C}(a, b),$$

making the sample mean unsuitable for estimation, see, e.g., [1, p. 490, Ex. 2].

(d) If $\mathbf{X}, \mathbf{Y}$ are independent and both $\sim \mathfrak{C}(a, b)$ then $2\,\mathbf{X} \sim \mathfrak{C}(2a, 2b)$, same as $\mathbf{X} + \mathbf{Y}$, although $2\,\mathbf{X}$ and $\mathbf{X}$ are not independent.

(e) If $\mathbf{X}, \mathbf{Y}$ are independent standard normal variables then their ratio $\mathbf{X}/\mathbf{Y}$ is a Cauchy RV.

(f) If $\mathbf{X}$ is a Cauchy RV so is its reciprocal $1/\mathbf{X}$. □

The lack of finite moments limits the usefulness of the Cauchy distribution. To remedy this situation it is possible to modify the Cauchy distribution, so as to retain its useful "fat tails" property, and still have finite moments of all orders. Two such distributions are considered, a truncated distribution in § 2, and a concentrated Cauchy distribution $\mathfrak{C}_g(a, b)$ in § 4. The latter is suggested by the Cauchy gunman story, in the presence of gravity, see § 3.

2. The truncated Cauchy distribution

The simplest way to concentrate the Cauchy distribution $\mathfrak{C}(a, b)$ is by truncation, restricting it to a finite interval I, symmetric w.r.t. $x = a$, say $I = [a - L, a + L]$, $L > 0$. The truncated Cauchy distribution has the density

$$\mathfrak{C}(a, b|I)(x) = \frac{1}{2\,\tan^{-1}\left(\dfrac{L}{b}\right)\,b\,\left(1 + \left(\dfrac{x-a}{b}\right)^2\right)}, \quad a - L \le x \le a + L, \tag{3}$$

the conditional density of (1) with the interval I as the underlying condition, see, e.g., [1, § 19.3].

This truncation corresponds to restricting the Cauchy gunman to angles θ in the interval

$$[-\tan^{-1}(\tfrac{L}{b}), \tan^{-1}(\tfrac{L}{b})], \tag{4}$$

of length $2\,\tan^{-1}(\frac{L}{b})$.

The distribution (3) has moments of all orders. In particular, if $\mathbf{X} \sim \mathfrak{C}(a, b|[a-L, a+L])$ then its expected value and variance are

$$\mathrm{E}\,\mathbf{X} = a, \tag{5}$$

$$\mathrm{Var}\,\mathbf{X} = \frac{b\,L}{\tan^{-1}\left(\frac{L}{b}\right)} - b^2, \tag{6}$$

as can be verified by integration. $\mathrm{Var}\,\mathbf{X}$ approaches its asymptote $(2\,b/\pi)\,L$ as $L \to \infty$, since then $\tan^{-1}(\frac{L}{b})$ tends to $\frac{\pi}{2}$.

Example 1. Consider the truncated Cauchy distribution $\mathfrak{C}(0, 1|I)$ with $(a, b) = (0, 1)$ and $I = [-L, L],\ L = 100\sqrt{5}$,

$$\mathfrak{C}(0, 1|I)(x) = \frac{1}{2\,\tan^{-1}(100\sqrt{5})\,(1+x^2)}, \quad -100\sqrt{5} \le x \le 100\sqrt{5}. \tag{7}$$

The x–interval has length $200\sqrt{5} = 447.2$, and the length of the corresponding θ–interval is, by (4), equal to $2\,\tan^{-1}(100\sqrt{5}) = 3.132648442$, almost π.

A RV $\mathbf{X}$ distributed according to (7) has a standard deviation $\sigma = 11.906$, by (6), and the $20\,\sigma$ tails probability is,

$$\mathrm{P}\,\{|\mathbf{X}| > 10\,\sigma = 119.06\} = 0.002507.$$

For comparison, if $\mathbf{Y} \sim \mathfrak{C}(0, 1)$, the Cauchy distribution (1) with $(a, b) = (0, 1)$, the corresponding probability is

$$\mathrm{P}\,\{|\mathbf{Y}| > 119.06\} = 0.005347.$$

3. The Cauchy gunman strikes again

Consider the Cauchy gunman in the presence of **gravity acceleration** $g \ge 0$ that causes the bullets to deviate from linear trajectories, see Fig. 1(b). To be specific, assume the initial velocity vector to have magnitude 1, with components

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix}, \text{ see Fig. 2(a).}$$

The velocity at time t is

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \sin\theta \\ -\cos\theta - g\,t \end{bmatrix}, \tag{8}$$

see Fig. 2(b), and the bullet position is then

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a + (\sin\theta)\,t \\ b - (\cos\theta)\,t - \frac{g}{2}\,t^2 \end{bmatrix}. \tag{9}$$

The positive solution of the quadratic equation

$$b - (\cos\theta)\,t - \frac{g}{2}\,t^2 = 0$$

gives the time for the bullet to reach the x–axis,

$$t_1 = \frac{1}{g}\left(\sqrt{\cos^2\theta + 2\,b\,g} - \cos\theta\right). \tag{10}$$

From (8) we also get

$$\frac{dx}{dy} = \frac{\dot{x}}{\dot{y}} = \frac{\sin\theta}{-\cos\theta - g\,t}. \tag{11}$$

The displacements $x - a$ and $y - b$ are therefore related by

$$x - a = -\frac{\sin\theta}{\cos\theta + g\,t}\,(y - b).$$

Substituting $y = 0$ and t_1 from (10), we get

$$x = a + b\,\frac{\sin\theta}{\sqrt{\cos^2\theta + 2\,b\,g}}, \tag{12}$$

where a bullet shot in angle θ will hit the x–axis. In particular, all hits fall in the interval

$$I_g(a,b) = \left[a - \sqrt{\tfrac{b}{2\,g}}, a + \sqrt{\tfrac{b}{2\,g}}\ \right], \tag{13}$$

corresponding to $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

4. The concentrated Cauchy distribution

Theorem 1. Let $\mathbf{X}$ be the random variables with values (12), where θ is uniformly distributed in $[-\pi/2, \pi/2]$. Then $\mathbf{X}$ has the probability density

$$\mathfrak{C}_g(a,b)(x) = \frac{\sqrt{1+2\,b\,g}}{\pi\,b\left(1+\left(\dfrac{x-a}{b}\right)^2\right)\sqrt{1-2\,b\,g\left(\dfrac{x-a}{b}\right)^2}}, \quad a - \sqrt{\tfrac{b}{2\,g}} < x < a + \sqrt{\tfrac{b}{2\,g}}, \tag{14}$$

with expected value

$$\mathrm{E}\,\mathbf{X} = a, \tag{15}$$

and variance

$$\mathrm{Var}\,\mathbf{X} = \frac{b^{3/2}\sqrt{1+2\,b\,g} - b^2\sqrt{2g}}{\sqrt{2g}}. \tag{16}$$

Proof. First express θ as function of x,

$$(x-a)^2 = \frac{b^2\,\sin^2\theta}{1-\sin^2\theta + 2\,b\,g}, \text{ by (12)}$$

$$\therefore\ (b^2 + (x-a)^2)\,\sin^2\theta = (1+2\,b\,g)\,(x-a)^2$$

$$\therefore\ \sin\theta = \frac{(x-a)\sqrt{1+2\,b\,g}}{\sqrt{b^2+(x-a)^2}},$$

and finally

$$\theta = \sin^{-1}\left(\frac{(x-a)\sqrt{1+2\,b\,g}}{\sqrt{b^2+(x-a)^2}}\right). \tag{17}$$

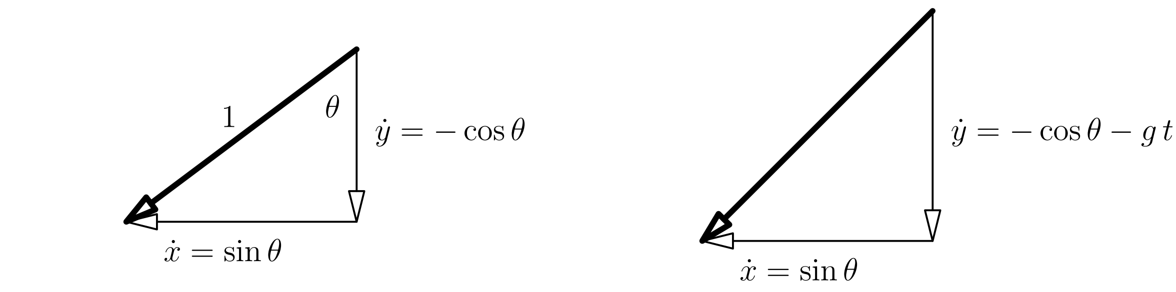
Then

$$\begin{aligned}\mathfrak{C}_g(a,b)(x) &= \frac{1}{\pi}\,\frac{d\theta}{dx}, \quad \text{for } \theta \text{ given by (17)},\\ &= \frac{b^2\sqrt{1+2\,b\,g}}{\pi\,(b^2+(x-a)^2)\,\sqrt{b^2-2\,b\,g\,(x-a)^2}},\end{aligned}$$

which is defined for $x \in I_g(a,b)$, and simplifies to (14). The claims (15)–(16) are verified by integration. □

The function in the right hand side (RHS) of (14) is real in the interval $I_g(a,b)$, is undefined at the endpoints $a \pm \sqrt{\frac{b}{2\,g}}$, and is imaginary elsewhere. The density function $\mathfrak{C}_g(a,b)$ can therefore be defined as the real part of RHS(14),

$$\mathfrak{C}_g(a,b)(x) = \Re\left(\frac{\sqrt{1+2\,b\,g}}{\pi\,b\left(1+\left(\frac{x-a}{b}\right)^2\right)\sqrt{1-2\,b\,g\left(\frac{x-a}{b}\right)^2}}\right), \quad -\infty < x < \infty, \tag{18}$$

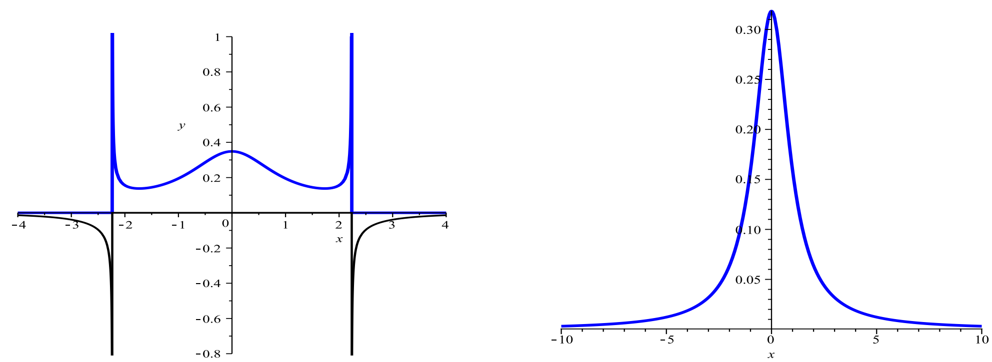


(a) The initial velocity (time $t = 0$) (b) The velocity at time t

FIGURE 2. The bullet velocity

which is positive in the interval (13), blows up as x approaches the endponits of the interval, and is zero elsewhere. The imaginary part of RHS(14), not relevant for the present discussion, behaves analogously outside the interval (13) where it is negative.

The real and imaginary parts of RHS(14) are illustrated in Fig. 3(a) for the values $(a, b) = (0, 1)$ and $g = 0.1$, with the interval $I_g(a, b) = [-\sqrt{5}, \sqrt{5}\,]$, by (13). The "skirt effect" near the endpoints of the interval becomes less noticeable as g decreases, and the interval widens.



(a) $g = 0.1$ (b) $g = 10^{-5}$

FIGURE 3. The density $\mathfrak{C}_g(a, b)$ of (14), with $(a, b) = (0, 1)$ and 2 values of g.

Example 2. Consider the concentrated Cauchy distribution $\mathfrak{C}_g(0, 1)$ with $(a, b) = (0, 1)$. The choice $g = 10^{-5}$ gives, by (13), the interval $I_{0.00001}(0, 1) = [-100\sqrt{5}, 100\sqrt{5}\,]$, the same as the interval in Ex. 1. The density (14) is

$$\mathfrak{C}_{0.00001}(0, 1)(x) = \frac{\sqrt{1 + 2 \cdot 10^{-5}}}{\pi\,(1 + x^2)\,\sqrt{1 - 2 \cdot 10^{-5} \cdot x^2}}, \quad -100\sqrt{5} < x < 100\sqrt{5}, \tag{19}$$

see Fig. 3(b). A RV $\mathbf{X}$ distributed according to (19) has a standard deviation $\sigma = 14.920$, by (16), and the $20\,\sigma$ tails probability is

$$\mathrm{P}\,\{|\mathbf{X}| > 10\,\sigma = 149.2\} = 0.003178.$$

For comparison, if $\mathbf{Y} \sim \mathfrak{C}(0, 1)$, the Cauchy distribution (1) with $(a, b) = (0, 1)$, the corresponding probability is

$$\mathrm{P}\,\{|\mathbf{Y}| > 149.2\} = 0.004267.$$

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