Improved Bounds for RIC in Compressed Sensing

Shenglong Zhou, Lingchen Kong, Naihua Xiu

***Abstract*—This paper improves bounds for restricted isometry constant (RIC) in compressed sensing. Let Φ be a $m \times n$ real matrix and k be a positive integer with $k \le n$. The main results of this paper show that if the restricted isometry constant of Φ satisfies**

$$\delta_{k+ak} < \frac{3}{2} - \frac{1+\sqrt{(4a+3)^2-8}}{8a} \text{ for } a > \frac{3}{8},$$

then k-sparse solution can be recovered exactly via l_1 minimization in the noiseless case. In particular, when $a = 1, 1.5, 2$ and 3, we have $\delta_{2k} < 0.5746$, $\delta_{2.5k} < 0.7074^*$, $\delta_{3k} < 0.7731$ and $\delta_{4k} < 0.8445$, which are the best bounds for RIC to our knowledge.

***Index Terms*—compressed sensing, restricted isometry constant, bound, l_1 minimization, exact recovery**

I. Introduction

THE concept of compressed sensing (CS) was first introduced by Donoho [9], Candès, Romberg and Tao [4] and Candès and Tao [5] with the involved essential idea–recovering some original n-dimensional but sparse signal\image from linear measurement with dimension far fewer than n. Recently, large numbers of researchers, including applied mathematicians, computer scientists and engineers, have begun to pay their attention to this area owing to its wide applications in signal processing, communications, astronomy, biology, medicine, seismology and so on, see, e.g., survey papers [1], [14] and a monograph [10].

The fundamental problem in compressed sensing is reconstructing a high-dimensional sparse signal from remarkably small number of measurements. We assume to recover a sparse solution $x \in \mathbb{R}^n$ of the underdetermined system of the form $\Phi x = y$, where $y \in \mathbb{R}^m$ is the available measurement and $\Phi \in \mathbb{R}^{m\times n}$ is a known measurement matrix (with $m \ll n$). The mathematical model would be to minimize the number of the non-zero components of x, i.e., to solve the following l_0-norm optimization problem:

$$\min \|x\|_0, \quad \text{s.t. } \Phi x = y, \tag{1}$$

where $\|x\|_0$ is l_0-norm of the vector $x \in \mathbb{R}^n$, i.e., the number of nonzero entries in x (this is not a true norm, as $\|\cdot\|_0$ is not positive homogeneous). However (1) is combinatorial and computationally intractable and one popular and powerful approach is to solve it via ℓ_1 *minimization* (its convex relaxation)

$$\min \|x\|_1, \quad \text{s.t. } \Phi x = y. \tag{2}$$

One of the most commonly used frameworks for sparse recovery via l_1 minimization is the *Restricted Isometry Property* (RIP) introduced by Candès and Tao [5]. For each integer $k \in \{1, 2, \cdots n\}$, the *restricted isometry constant* (RIC) δ_k of a matrix Φ is the smallest number such that

$$(1-\delta_k)\|x\|_2^2 \le \|\Phi x\|_2^2 \le (1+\delta_k)\|x\|_2^2 \tag{3}$$

holds for all k-sparse vectors. A vector with at most k nonzero entries is called *k-sparse*. For $x \in \mathbb{R}^n$, we define the *best k-sparse approximation* $x^{(k)} \in \mathbb{R}^n$ from x with all but the k largest entries (in absolute value) set to zero. If $k + k' \le n$, the *k, k'-Restricted Orthogonality Constant* (ROC) $\theta_{k,k'}$ is the smallest number that satisfies

$$|\langle \Phi x, \Phi x'\rangle| \le \theta_{k,k'}\|x\|_2\|x'\|_2 \tag{4}$$

for all k-sparse x and k'-sparse x' with disjoint supports. Candès and Tao [5] showed the link between RIC and ROC

$$\theta_{k,k'} \le \delta_{k+k'}. \tag{5}$$

The RIP condition is difficult to verify for a given matrix Φ because it essentially requires that ever subset of columns of Φ with certain cardinality approximately behaves like an orthonormal system. However, a widely used technique for avoiding checking the RIP condition directly is to generate the matrix randomly and to show that the resulting random matrix satisfies the RIP with high probability.

Although the RIP condition is difficult to check, it is of independent interesting to study the bounds for RIC in CS since l_1-norm minimization can recover a sparse signal under various conditions on δ_k and $\theta_{k,k'}$, such as, the condition $\delta_k + \theta_{k,k} + \theta_{k,2k} < 1$ in [5], $\delta_{2k} + \theta_{k,2k} < 1$ in [6], and $\delta_{1.25k} + \theta_{k,1.25k} < 1$ in [3]. In the literature [7], [12], [11], [3], [13], the bounds for δ_{2k} in CS seems to attract much attention. The details are listed in the Table 1 below. To the best of our knowledge, the bound for δ_{2k} on sparse recovery is gradually improved from $\sqrt{2}-1 (\approx 0.4142)$ to 0.4931 in recent years.

The main contribution of the present paper is to give the new bounds for RIC in CS in the following theorem.

Theorem I.1. *Let x be a feasible solution to* (1) *and $x^{(k)}$ be the best k-sparse approximation of x. If the following inequality holds*

$$\delta_{k+ak} < \frac{3}{2} - \frac{1+\sqrt{(4a+3)^2-8}}{8a} \text{ for } a > \frac{3}{8}, \tag{6}$$

then the solution $\hat{x}$ to the l_1 minimization problem (2) *satisfies*

$$\|\hat{x} - x\|_1 \le \frac{2\,(1+C_0)}{1-C_0}\|x - x^{(k)}\|_1, \tag{7}$$

for some positive constant $C_0 < 1$ given explicitly by (13). *In particular, if x is k-sparse, the recovery is exact.*

From Theorem I.1, when $a = 1, 1.5, 2$ and 3, we get that $\delta_{2k} < 0.5746$, $\delta_{2.5k} < 0.7047$, $\delta_{3k} < 0.7731$ and $\delta_{4k} <$

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$^*\delta_{2.5k}$ means $\delta_{\lceil 2.5k\rceil}$. In whole paper, for a positve real number a, the δ_{ak} and $\theta_{k,ak}$ are defined as $\delta_{\lceil ak\rceil}$ and $\theta_{k,\lceil ak\rceil}$, respectively.

0.8445. Observing the Table 1, it is clear that our conditions are all weaker than the ones known in the literature.

	δ_{2k}	$\delta_{2.5k}$	δ_{3k}	δ_{4k}
Candès [7]	0.4142	—	—	—
Foucart, Lai [12]	0.4531	—	—	—
Foucart [11]	0.4652	—	—	—
Cai, Wang, Xu [3]	0.4721	—	0.535	0.585
Mo and Li [13]	0.4931	—	—	—
Our Results	0.5746	0.7074	0.7731	0.8445

TABLE I
DIFFERENT BOUNDS FOR RIC.

The organization of this paper is as follows. In the next section, we establish some key inequalities. In the section III, we prove our main result. In section IV, we conclude this paper with some remarks .

II. KEY INEQUALITIES

In this section, we will give some inequalities, which play an important role in improving the RIC bound for sparse recovery in this paper.

We begin with the following interesting and important inequality, which states the connection between several norms of $l_0, l_1, l_2, l_\infty$and $l_{-\infty}$. Here, we define $\|x\|_{-\infty}$ norm as $\|x\|_{-\infty} := \min_i \{|x_i|\}$ (In fact, $l_{-\infty}$ is not a norm since the triangle inequality does not hold). For convenience, we call (8) the *Norm Inequality* formally.

Proposition II.1. (**Norm Inequality**) *For any $x \in \mathbb{R}^n$,*

$$\begin{aligned} \frac{\|x\|_1}{\sqrt{\|x\|_0}} &\leq \|x\|_2 \\ &\leq \frac{\|x\|_1}{\sqrt{\|x\|_0}} + \frac{\sqrt{\|x\|_0}}{4}(\|x\|_\infty - \|x\|_{-\infty}). \quad (8) \end{aligned}$$

Furthermore, we obtain the following general inequality

$$\frac{\|x\|_1}{\sqrt{\|x\|_0}} \leq \|x\|_2 \leq \frac{\|x\|_1}{\sqrt{\|x\|_0}} + \frac{\sqrt{\|x\|_0}}{4}\|x\|_\infty.$$

Proof Firstly, from the Cauchy-Schwarz Inequality, we have $\|x\|_1 \leq \sqrt{n}\|x\|_2$. From (6) in [2] by Cai, Wang, Xu, we have

$$\|x\|_2 \leq \frac{\|x\|_1}{\sqrt{n}} + \frac{\sqrt{n}}{4}(\|x\|_\infty - \|x\|_{-\infty}).$$

Then we get

$$\frac{\|x\|_1}{\sqrt{n}} \leq \|x\|_2 \leq \frac{\|x\|_1}{\sqrt{n}} + \frac{\sqrt{n}}{4}(\|x\|_\infty - \|x\|_{-\infty}).$$

Without loss of generality, we decompose x into $x = (\bar{x}, 0)^T$, where $\bar{x} \in \mathbb{R}^{\|x\|_0}$, so we have

$$\frac{\|\bar{x}\|_1}{\sqrt{\|x\|_0}} \leq \|\bar{x}\|_2 \leq \frac{\|\bar{x}\|_1}{\sqrt{\|x\|_0}} + \frac{\sqrt{\|x\|_0}}{4}(\|\bar{x}\|_\infty - \|\bar{x}\|_{-\infty}).$$

It is easily to check that $\|\bar{x}\|_1 = \|x\|_1, \|\bar{x}\|_2 = \|x\|_2, \|\bar{x}\|_\infty = \|x\|_\infty$ and $\|\bar{x}\|_{-\infty} \geq \|x\|_{-\infty}$. Our conclusion holds immediately. □

Throughout the paper, let $\hat{x}$ be a solution to the minimization problem (2), and $x \in \mathbb{R}^n$ be a feasible one, i.e., $\Phi x = y$. Clearly, $\|\hat{x}\|_1 \leq \|x\|_1$. We let $x^{(k)} \in \mathbb{R}^n$ define as above again. Without loss of generality we assume that the support of $x^{(k)}$ is T_0.

Denote that $h = \hat{x} - x$ and h_T is the vector equal to h on an index set T and zero elsewhere. We decompose h into a sum of vectors $h_{T_0}, h_{T_1}, h_{T_2}, \ldots$, where T_1 corresponds to the locations of the ak largest coefficients of $h_{T_0^C}$ ($T_0^C = T_1 \cup T_2 \cup \cdots$); T_2 to the locations of the $4ak$ largest coefficients of $h_{(T_0 \cup T_1)^C}$, T_3 to the locations of the next $4ak$ largest coefficients of $h_{(T_0 \cup T_1)^C}$, and so on. That is

$$h = h_{T_0} + h_{T_1} + h_{T_2} + \cdots . \quad (9)$$

Here, the sparsity of h_{T_0} is at most k; the sparsity of h_{T_1} is at most ak; the sparsity of $h_{T_j} (j \geq 2)$ are at most $4ak$.

In order to get a improved bound on RIC, for the above decomposition (9), we define

$$\rho := \frac{\|h_{T_1}\|_1}{\sum_{j \geq 1} \|h_{T_j}\|_1}. \quad (10)$$

Obviously, $\rho \in [0, 1]$ and

$$\sum_{j \geq 2} \|h_{T_j}\|_1 = (1 - \rho) \sum_{j \geq 1} \|h_{T_j}\|_1.$$

Applying the Norm Inequality, we can give some inequalities of h, which are very useful in the proof of our main results.

Lemma II.2. *Let $h_{T_0}, h_{T_1}, h_{T_2}, \cdots$, and ρ be given by* (9) *and* (10)*, respectively. Then*

$$\sum_{j \geq 2} \|h_{T_j}\|_2^2 \leq \frac{\rho(1 - \rho)}{ak} \left(\sum_{j \geq 1} \|h_{T_j}\|_1 \right)^2 \quad (11)$$

and

$$\sum_{j \geq 2} \|h_{T_j}\|_2 \leq \frac{1}{\sqrt{4ak}} \sum_{j \geq 1} \|h_{T_j}\|_1. \quad (12)$$

Proof By the definitions of $h_{T_j} (j = 1, 2, \cdots)$ and ρ, direct calculation yields

$$\begin{aligned} \sum_{j \geq 2} \|h_{T_j}\|_2^2 &\leq \sum_{j \geq 2} \|h_{T_j}\|_1 \|h_{T_2}\|_\infty \\ &\leq \frac{1}{ak} \sum_{j \geq 2} \|h_{T_j}\|_1 \|h_{T_1}\|_1 \\ &= \frac{\rho(1 - \rho)}{ak} \left(\sum_{j \geq 1} \|h_{T_j}\|_1 \right)^2 . \end{aligned}$$

Thus, (11) holds. We remain to show (12). Applying the Norm Inequality (8), we obtain that

$$\begin{aligned} \|h_{T_j}\|_2 &\leq \frac{\|h_{T_j}\|_1}{\sqrt{4ak}} + \frac{\sqrt{4ak}}{4} \left(\|h_{T_j}\|_\infty - \|h_{T_j}\|_{-\infty} \right) \\ &\leq \frac{\|h_{T_j}\|_1}{\sqrt{4ak}} + \frac{\sqrt{4ak}}{4} \left(\|h_{T_j}\|_\infty - \|h_{T_{j+1}}\|_\infty \right) , \end{aligned}$$

for $j=2,3,\cdots$, where for the last h_{T_j}, we set $\|h_{T_{j+1}}\|_\infty := 0$. Adding up all the inequalities for $j=2,3,\cdots$, we get that

$$\begin{aligned}\sum_{j\geq 2}\|h_{T_j}\|_2 &\leq \frac{\sum_{j\geq 2}\|h_{T_j}\|_1}{\sqrt{4ak}}+\frac{\sqrt{4ak}}{4}\|h_{T_2}\|_\infty\\ &\leq \frac{\sum_{j\geq 2}\|h_{T_j}\|_1}{\sqrt{4ak}}+\frac{\sqrt{4ak}}{4ak}\|h_{T_1}\|_1\\ &= \left(\frac{1-\rho}{\sqrt{4ak}}+\frac{\rho}{\sqrt{4ak}}\right)\sum_{j\geq 1}\|h_{T_j}\|_1\\ &= \frac{1}{\sqrt{4ak}}\sum_{j\geq 1}\|h_{T_j}\|_1.\end{aligned}$$

The desired conclusion holds immediately. □

In the end of this section, we give two lemmas which give us the connection about the norms of Φh_{T_j} and h_{T_j}.

Lemma II.3. *Let h_{T_0} and h_{T_1} be given by* (9) *. Then*

$$\|\Phi\left(h_{T_0}+h_{T_1}\right)\|_2^2 \geq \frac{1-\delta_{k+ak}}{k}\left(\|h_{T_0}\|_1^2+\frac{1}{a}\|h_{T_1}\|_1^2\right).$$

Proof From (3), we obtain

$$\|\Phi\left(h_{T_0}+h_{T_1}\right)\|_2^2 \geq (1-\delta_{k+ak})\,\|h_{T_0}+h_{T_1}\|_2^2.$$

Because the supports T_0 and T_1 are disjoint, the following equality holds

$$\|h_{T_0}+h_{T_1}\|_2^2=\|h_{T_0}\|_2^2+\|h_{T_1}\|_2^2.$$

Therefore

$$\begin{aligned}\|\Phi\left(h_{T_0}+h_{T_1}\right)\|_2^2 &\geq (1-\delta_{k+ak})\left(\|h_{T_0}\|_2^2+\|h_{T_1}\|_2^2\right)\\ &\geq \frac{1-\delta_{k+ak}}{k}\left(\|h_{T_0}\|_1^2+\frac{1}{a}\|h_{T_1}\|_1^2\right),\end{aligned}$$

where the second inequality is derived from (8). □

Lemma II.4. *Let $h_{T_0}, h_{T_1}, h_{T_2}, \cdots,$ and ρ be given by* (9) *and* (10) *, respectively. Then*

$$\|\sum_{j\geq 2}\Phi h_{T_j}\|_2^2 \leq \frac{4\rho(1-\rho)+\delta_{8ak}}{4ak}\left(\sum_{j\geq 1}\|h_{T_j}\|_1\right)^2.$$

Proof By direct calculations, we obtain that

$$\begin{aligned}&\|\sum_{j\geq 2}\Phi h_{T_j}\|_2^2\\ \leq\;& \sum_{j\geq 2}|\langle\Phi h_{T_j},\Phi h_{T_j}\rangle|+2\sum_{j>i\geq 2}|\langle\Phi h_{T_j},\Phi h_{T_i}\rangle|\\ \leq\;& (1+\delta_{8ak})\sum_{j\geq 2}\|h_{T_j}\|_2^2+2\delta_{8ak}\sum_{j>i\geq 2}\|h_{T_j}\|_2\|h_{T_i}\|_2\\ =\;& \sum_{j\geq 2}\|h_{T_j}\|_2^2+\delta_{8ak}\left(\sum_{j\geq 2}\|h_{T_j}\|_2\right)^2\\ \leq\;& \left[\frac{\rho(1-\rho)}{ak}+\frac{\delta_{8ak}}{4ak}\right]\left(\sum_{j\geq 1}\|h_{T_j}\|_1\right)^2\\ =\;& \frac{4\rho(1-\rho)+\delta_{8ak}}{4ak}\left(\sum_{j\geq 1}\|h_{T_j}\|_1\right)^2,\end{aligned}$$

where the first inequality holds by the triangle inequality, the second holds due to (3) and (5), the third is from (11) and (12); and the first equality holds from

$$\sum_{j\geq 2}\|h_{T_j}\|_2^2+2\sum_{j>i\geq 2}\|h_{T_j}\|_2\|h_{T_i}\|_2=\left(\sum_{j\geq 2}\|h_{T_j}\|_2\right)^2.$$

Hence, the desired result follows. □

III. PROOF OF THE MAIN RESULT

In this section, we will prove our main result. For simplicity, we first define a quadric function of variable ρ,

$$f(\rho) := -4(2-\delta_{k+ak})\rho^2+4\rho+\delta_{8ak},$$

Clearly, it is a strictly concave function. We can easily obtain the optimal maximum value of $f(\rho)$ through demanding its derivative, that is

$$f(\rho^*)=\max_{0\leq\rho\leq 1}f(\rho)=\frac{1+(2-\delta_{k+ak})\delta_{8ak}}{2-\delta_{k+ak}}>0.$$

where

$$\rho^* := \frac{1}{2(2-\delta_{k+ak})}\in[0,1].$$

Moreover, we denote that

$$\begin{aligned}C_0 :\;&= \frac{\sqrt{f(\rho^*)}}{\sqrt{4a(1-\delta_{k+ak})}}\\ &= \sqrt{\frac{1+(2-\delta_{k+ak})\delta_{8ak}}{4a(2-\delta_{k+ak})(1-\delta_{k+ak})}}.\end{aligned}\tag{13}$$

Before proving our main results, we show that the RIP bound in (6) is a sufficient condition for $C_0<1$.

Lemma III.1. *If* (6) *holds, then $C_0<1$.*

Proof From (6), it is easy to verify that

$$-4a\delta_{k+ak}^2+(12a-1)\delta_{k+ak}+3-8a<0,$$

which is equivalent to

$$\frac{3-\delta_{k+ak}}{4a(2-\delta_{k+ak})(1-\delta_{k+ak})}<1.$$

Since $0\leq\delta_{8ak}\leq 1$, and by (13), we have

$$\begin{aligned}C_0^2 &= \frac{1+(2-\delta_{k+ak})\delta_{8ak}}{4a(2-\delta_{k+ak})(1-\delta_{k+ak})}\\ &\leq \frac{3-\delta_{k+ak}}{4a(2-\delta_{k+ak})(1-\delta_{k+ak})}<1.\end{aligned}$$

Thus, if (6) holds, we ensure $C_0<1$. □

Now we begin to prove our main result.

Proof of Theorem I.1 The proof proceeds in two steps, which are common approach in literature [7], [3]: the first step is to prove that

$$\|h_{T_0}\|_1 \leq C_0\sum_{j\geq 1}\|h_{T_j}\|_1.\tag{14}$$

The second step shows that $\|\hat{x}-x\|_1$ is appropriately small.

For the first step, we note that $\Phi h = 0$, which implies that

$$\|\Phi(h_{T_0}+h_{T_1})\|_2^2 = \|\sum_{j\geq 2}\Phi h_{T_j}\|_2^2.$$

From Lemmas 2.3 and 2.4, the following inequality holds

$$\begin{aligned} &\frac{1-\delta_{k+ak}}{k}\left(\|h_{T_0}\|_1^2+\frac{1}{a}\|h_{T_1}\|_1^2\right) \\ \leq\ &\frac{4\rho(1-\rho)+\delta_{8ak}}{4ak}\left(\sum_{j\geq 1}\|h_{T_j}\|_1\right)^2. \end{aligned}$$

Then we get

$$\begin{aligned} \|h_{T_0}\|_1^2 &\leq \left(\frac{4\rho(1-\rho)+\delta_{8ak}}{4a(1-\delta_{k+ak})}-\frac{\rho^2}{a}\right)\left(\sum_{j\geq 1}\|h_{T_j}\|_1\right)^2 \\ &= \frac{f(\rho)}{4a(1-\delta_{k+ak})}\left(\sum_{j\geq 1}\|h_{T_j}\|_1\right)^2 \\ &\leq \frac{f(\rho^*)}{4a(1-\delta_{k+ak})}\left(\sum_{j\geq 1}\|h_{T_j}\|_1\right)^2, \end{aligned}$$

where the first inequality is derived from (10). Combining with (13), we get (14).

For the second step, we have

$$\begin{aligned} \sum_{j\geq 1}\|h_{T_j}\|_1 &= \|h-h_{T_0}\|_1 \\ &\leq \|h_{T_0}\|_1+2\|x-x^{(k)}\|_1 \\ &\leq C_0\sum_{j\geq 1}\|h_{T_j}\|_1+2\|x-x^{(k)}\|_1, \end{aligned}$$

where the first inequality holds from (12) in [7]. Then

$$\sum_{j\geq 1}\|h_{T_j}\|_1 \leq \frac{2}{1-C_0}\|x-x^{(k)}\|_1.$$

This together with (14) yields

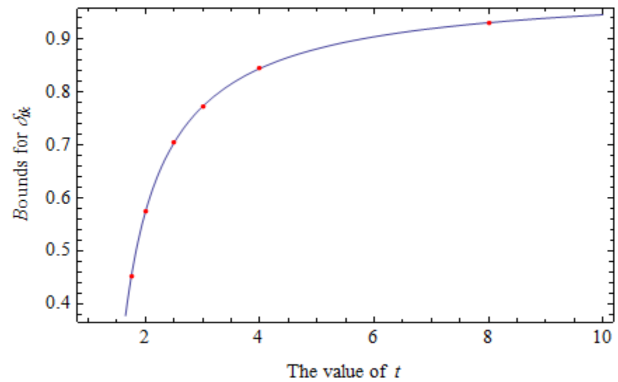
$$\begin{aligned} \|\hat{x}-x\|_1=\|h\|_1 &= \|h_{T_0}\|_1+\sum_{j\geq 1}\|h_{T_j}\|_1 \\ &\leq \frac{2(1+C_0)}{1-C_0}\|x-x^{(k)}\|_1. \end{aligned}$$

We complete to prove (7).

In particular, if x is k-sparse, then $x-x^{(k)}=0$, and hence $x=\hat{x}$ from (7). □

IV. CONCLUSION

In this paper, we have gotten some improved bounds for RIC in CS. Clearly, it holds that $\frac{3}{2}-\frac{1+\sqrt{(4a+3)^2-8}}{8a}\leq 0$ when $a\leq\frac{3}{8}$, thus we cannot yield the bound for δ_k. Fortunately, when $a>\frac{3}{8}$, the condition (6) enables us to obtain several interesting RIC bounds for measurement matrices, such as δ_{2k}, $\delta_{2.5k}$, δ_{3k}, δ_{4k} and δ_{8k} and so on. For intuitionistic analysis, we draw the curve about the connection between $t(:=a+1)$ and the bound for δ_{tk}.



From the graph above, it is easy to see that the bounds for δ_{tk} increase fast between $1.75\leq t\leq 3$ and the bounds for δ_{tk} are larger than 0.9 when $t\geq 6$. In addition, Davies and Gribonval [8] has given detailed counter-examples to show that the bound of δ_{2k} cannot exceed $1/\sqrt{2}\approx 0.7071$. In special case of $n\leq 4k$, Mo and Li [13] gave the bound $\delta_{2k}<0.656$, which is closed to 0.7071. In our paper, we have $\delta_{2k}<0.5746$ in general case. Based on $0.5746<0.7071$, we wonder whether there is a better way to improve the bound 0.5746 for δ_{2k}. So the further research topic we can do is to reduce the gap.

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