Lowest-rank Solutions of Continuous and Discrete Lyapunov Equations Over Symmetric Cone

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***Abstract*—The low-rank solutions of continuous and discrete Lyapunov equations are of great importance but generally difficult to achieve in control system analysis and design. Fortunately, Mesbahi and Papavassilopoulos [On the rank minimization problems over a positive semidefinite linear matrix inequality, IEEE Trans. Auto. Control, Vol. 42, No. 2 (1997), 239-243] showed that with the semidefinite cone constraint, the lowest-rank solutions of the discrete Lyapunov inequality can be efficiently solved by a linear semidefinite programming.**

In this paper, we further show that the lowest-rank solutions of both the continuous and discrete Lyapunov equations over symmetric cone are unique and can be exactly solved by their convex relaxations, the symmetric cone linear programming problems. Therefore, they are polynomial-time solvable. Since the underlying symmetric cone is a more general algebraic setting which contains the semidefinite cone as a special case, our results also answer an open question proposed by Recht, Fazel and Parrilo in [Guaranteed minimum-rank solutions of linear matrix equations via nuclear norm minimization, SIAM Review, Vol.52, No.3, (2010), 471-501].

***Index Terms*—Continuous Lyapunov Equation, Discrete Lyapunov Equation, Lowest-rank Solution, Symmetric Cone, Polynomial-time Solvability**

I. Introduction

THE continuous and discrete Lyapunov equations are fundamental matrix equations and play an significant role in control theory, model reduction and stochastic analysis of dynamical systems [1], [2], [4], [14]. The theoretical analysis and numerical solutions for these equations have been the topics of numerous publications, see [19], [24], [3], [41], [32] and the references therein. Among all solutions, the low-rank ones are of great importance in control system analysis and design. Here the concept of low-rank solution includes the low-rank constraint solution and the lowest-rank solution.

The idea of low-rank constraint solution comes from the so-called curse of dimensionality. As the explosion of the information in modern society and the growing complexity in practical control problems, the scale of problems is becoming larger and larger and the storage of these large-scale data eventually becomes problematic. A popular and reasonable algebraic technique for alleviating this is to approximate the solution by some low-rank matrix. In this way, there already exists a significant number of low-rank methods for solving Lyapunov equations using this principle. For example, by employing the matrix sign function, Larin and Aliev [23] has proposed a low-rank approximate solution for continuous Lyapunov equation; Based on the Krylov subspace, Jaimoukha and Kasenally [21], and Hochbruck and Starke [20] also designed some low-rank methods; A smaller subspace generated from the Krylov subspace is utilized by Simoncini [35] to design a projection method for solving large-scale continuous Lyapunov equations as well. Moreover, approximate solution of large sparse continuous Lyapunov equations were gained based on the power method by Gudmundsson and Laub in [12]. The low-rank Smith method was also considered for achieving low-rank approximation of continuous Lyapunov solutions as stated in [15], [33]; In 2004, Li and White [25] presented the Cholesky factor-alternating direction implicit (CF-ADI) algorithm and generated a low-rank approximation to the solution of the continuous Lyapunov equation with a respectable iteration complexity; Recently, Vandereycken and Vandewalle [40] introduced a new geometric framework for computing low-rank approximations to solutions of generalized continuous Lyapunov equations. The involved method was based on optimizing an objective function on the Riemannian manifold of symmetric positive semidefinite matrices of fixed rank and was implemented efficiently and scalably. Since the discrete Lyapunov equation can be derived to the continuous case by Cayley transformation under some mild condition, all aforementioned methods thereby work for the discrete case naturally. However, among all the above papers, two things need to be pointed out here: (1) the involved coefficient matrix is unavoidably imposed to satisfy some stable conditions to ensure the solution uniqueness of the original Lyapunov equation; (2) the generated low-rank matrix solution is an approximation to the original solution even though the accuracy of the best low-rank approximation increases rapidly with growing rank [32]. While in fact, the continuous or discrete Lyapunov equation may possess more than one solution in practical problems. In the case, how to get those minimal rank exact solutions remains essential as well.

The lowest-rank (i.e., minimal rank) solution has wide applications in the bilinear matrix inequality problem, static output feedback stabilization, reduced-order H^∞ synthesis, and μ-synthesis with constant scaling, see [29], [28], [11], [36], [37]. All these problems can be mathematically formulated as

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rank minimization problems. Technically, however, this type of optimization problems is NP-hard in general cases due to the non-continuity and non-convexity of the rank function. A common idea is to construct some easy-tackling relaxations. A variety of heuristic algorithms based on local optimization then emerged, such as the alternating projection and its variations [10], [31], the alternating matrix inequalities technique [39], linearization [13], augmented Lagrangian methods [6], and the nuclear-norm method [7]. Particularly, when the matrix variable is symmetric and positive semidefinite, the nuclear norm turns out to be the trace function and the corresponding heuristic is the so-called trace norm heuristic. This method has been observed to produce very low-rank solutions in practice and especially it can provide the exact solution for the discrete Lyapunov inequalities with semidefinite cone constraints as shown in [30]. This exact relaxation heavily relies on the special structure of the Stein-type operator involved in the discrete Lyapunov inequality.

Inspired by the aforementioned characteristic, we are concerned on the exact relaxations for both continuous and discrete Lyapunov equations. We will focus on the lowest-rank solutions of these two functions in a more general setting called Euclidean Jordan algebra which contains the symmetric matrices space as a special case (See Section 2 for details). This is primarily motivated by [34] in which Fazel et al. indicated that it would be a good try to extend rank minimization in the Euclidean Jordan algebra.

As we will introduce in the next section, the counterpart of the semidefinite matrix cone in symmetric matrices space is the symmetric cone in Euclidean Jordan algebra. Thus the corresponding optimization problems for achieving the lowest-rank solutions of continuous and discrete Lyapunov equations are termed as *the rank minimization problem of continuous (discrete, respectively) Lyapunov equation over symmetric cone* which take the following forms:
the rank minimization of continuous Lyapunov equation over symmetric cone:

$$(P_0) \qquad \begin{array}{ll} \min & rank(x) \\ \text{s.t.} & L_a(x) = b \\ & x \in K, \end{array}$$

the rank minimization of discrete Lyapunov equation over symmetric cone:

$$(\hat{P}_0) \qquad \begin{array}{ll} \min & rank(x) \\ \text{s.t.} & \mathcal{A}(x) = b \\ & x \in K, \end{array}$$

where a, b, x are elements in some Euclidean Jordan algebra and K is the corresponding symmetric cone, L_a is the Lyapunov transformation with respect to a, $\mathcal{A}$ is a Stein-type transformation of the form $\mathcal{A} := \alpha\mathcal{I} - S$ with $\alpha > 0$ and $S(K) \subseteq K$, $rank(x)$ is the rank of x. (More details see Section II). The trace relaxation models of these two rank minimization problems are

$$(P_1) \qquad \begin{array}{ll} \min & tr(x) \\ \text{s.t.} & L_a(x) = b \\ & x \in K. \end{array}$$

and

$$(\hat{P}_1) \qquad \begin{array}{ll} \min & tr(x) \\ \text{s.t.} & \mathcal{A}(x) = b \\ & x \in K, \end{array}$$

where $tr(x)$ is the trace of x.

By exploiting the features of the Lyapunov and Stein-type transformations in Euclidean Jordan algebra, together with our development on properties of symmetric cone, we obtain that the above two trace norm minimization problems are exact relaxations to problems (P_0) and $(\hat{P}_0)$ respectively when $b \in K$. Moreover, uniqueness of the solution is also achieved under the same condition once the feasible region is nonempty. In this regard, the lowest-rank solution of either continuous or discrete Lyapunov equation over symmetric cone can be efficiently solved by some convex program methods such as the interior-point method in polynomial time, and the generated solution is exact, rather than approximate or local optimal as obtained in the aforementioned low-rank methods and relaxation approaches. Additionally, the coefficient a in the continuous case here is no longer be restricted to ensure uniqueness of the solution to the corresponding equation. All we need is the feasibility of the problem and $b \in K$. In this case, the feasible set here may not be singleton. Even though, by developing the algebraic features of symmetric cone, we can prove that the minimal rank ones among all solutions to these two equations constraint are unique. As a byproduct, we investigate the relation between the minimal rank solution and the ranks of the the coefficient a and right-hand side element b, and provide the lower and upper bounds of the optimal rank.

The outline of this paper is as follows. In Section II, we review some basic concepts and theorems of Euclidean Jordan algebra, develop some useful features of symmetric cone and investigate some essential properties on rank function in the underlying setting. In Sections III and IV, we study the exact relaxation for lowest-rank solutions of Lyapunov equation over symmetric cone in continuous and discrete cases respectively. Some concluding remarks are made in Section V.

II. Preliminaries

The Euclidean Jordan algebra and its fundamental concepts and properties are reviewed in this section to help get familiar with this more general algebraic setting. Additionally, the structure features of Lyapunov transformation and quadratic representation and more useful properties on symmetric cone and rank function are also exploited here for the sequel main analysis.

A. Euclidean Jordan Algebras

Let $(\mathcal{J}, \langle\cdot,\cdot\rangle)$ be an n-dimensional inner product space over real field $\mathbb{R}$ endowed with a bilinear mapping $\circ : (x, y) \mapsto x \circ y$ from $\mathcal{J} \times \mathcal{J}$ to $\mathcal{J}$. The triple $(\mathcal{J}, \circ, \langle\cdot,\cdot\rangle)$ ($\mathcal{J}$ for short) is called a *Euclidean Jordan algebra* if the following conditions hold:

(1) $x \circ y = y \circ x$ for all x, $y \in \mathcal{J}$;

(2) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$ for all $x, y \in \mathcal{J}$, where $x^2 := x \circ x$;

(3) $\langle x \circ y, z\rangle = \langle y, x \circ z\rangle$ for all x, y, $z \in \mathcal{J}$.

We call $x \circ y$ the *Jordan product* of x and y. We also assume that there exists an element $e \in \mathcal{J}$, called the *unit element*, such that $x \circ e = x$ for all $x \in \mathcal{J}$.

For any given Euclidean Jordan algebra $\mathcal{J}$, its *cone of squares* is defined as $K := \{x^2 : x \in \mathcal{J}\}$. This cone is closed convex self-dual and homogeneous and hence exactly the *symmetric cone* of $\mathcal{J}$ as shown in [8].

Some basic concepts about the Euclidean Jordan algebra $\mathcal{J}$ are supplied as follows. For any $x \in \mathcal{J}$, the integer k is called the *degree* of x, denoted by $deg(x)$, if k is the smallest integer such that the set $\{e, x, x^2, \cdots, x^k\}$ is linearly dependent. The *rank* of $\mathcal{J}$ is defined as $r(\mathcal{J}) := \max\{deg(x) : x \in \mathcal{J}\}$. We write r instead of $r(\mathcal{J})$ throughout the remainder of the paper for simplicity. An element $v \in \mathcal{J}$ is said to be *idempotent* if $v \neq 0$ and $v^2 = v$. Two idempotents e_1 and e_2 are called *orthogonal* if $e_1 \circ e_2 = 0$. A complete system of orthogonal idempotents is a set $\{e_1, e_2, \cdots, e_k\}$ where for each distinct i and j, e_i, e_j are orthogonal and $\sum_{j=1}^{k} e_j = e$. An idempotent is *primitive* if it cannot be written as the sum of two other idempotents. A complete system of orthogonal primitive idempotents is called a *Jordan frame*, where the number of idempotents in this system is exactly the rank of $\mathcal{J}$.

By employing the tool of Jordan frame, we are in a position to state the spectral decomposition theorem as follows.

Theorem 2.1: (Theorem III.1.2, [8]) Let $\mathcal{J}$ be a Euclidean Jordan algebra with rank r. Then for any $x \in \mathcal{J}$, there exists a Jordan frame $\{e_1, e_2, \cdots, e_r\}$ and real numbers $\lambda_1(x)$, $\lambda_2(x)$, $\cdots$, $\lambda_r(x)$ such that

$$x = \sum_{i=1}^{r} \lambda_i(x) e_i. \tag{2.1}$$

The numbers $\lambda_1(x), \lambda_2(x), \cdots, \lambda_r(x)$ (counting multiplicities) are called the eigenvalues of x and (2.1) the spectral decomposition of x.

With the help of Theorem 2.1, the symmetric cone K and its interior $intK$, can be described as

$$K = \{x \in \mathcal{J} : \text{all eigenvalues of } x \text{ are nonnegative}\},$$

$$intK = \{x \in \mathcal{J} : \text{all eigenvalues of } x \text{ are positive}\}.$$

Base on the spectral decomposition of x in (2.1), the square x^2, the square root $x^{1/2}$, trace, inertia, and rank of x are, respectively, defined by

$$x^2 = \sum_{i=1}^{r} \lambda_i^2(x) e_i,$$

$$x^{1/2} = \sum_{i=1}^{r} \sqrt{\lambda_i(x)} e_i, \quad \text{if } x \in K$$

$$tr(x) := \sum_{i=1}^{r} \lambda_i(x),$$

$$In(x) := (\pi(x), \nu(x), \delta(x))$$

$$rank(x) := \pi(x) + \nu(x),$$

Here $\pi(x)$, $\nu(x)$, and $\delta(x)$ are, respectively, the number of positive, negative, and zero eigenvalues of x. See [16] for details. Relying on the function $tr(\cdot)$, an inner product can be defined as

$$\langle x, y \rangle := tr(x \circ y), \quad \forall x, y \in \mathcal{J}.$$

The norm induced by this inner product is called the Frobenius norm which has the expression

$$\|x\|_F := \sqrt{\langle x, x \rangle} = \left(\sum_{j=1}^{r} \lambda_j^2(x) \right)^{1/2}, \quad \forall x \in \mathcal{J}.$$

Just like the eigen-value decomposition for real symmetric matrices, the spectral decomposition heavily relies on the Jordan frame of the corresponding decomposed element and may fail to hold for any given Jordan frame. To make up this deficiency, some other decompositions are also proposed.

Theorem 2.2: (Theorem VI.2.1, [8]) Let $\mathcal{J}$ be a Euclidean Jordan algebra with rank r, and $\{e_1, e_2, \cdots, e_r\}$ be some given Jordan frame. Then we have $\mathcal{J} = \bigoplus_{i \leq j} \mathcal{J}_{ij}$, where $\mathcal{J}_{ii} = \{x \in \mathcal{J} | x \circ e_i = x\}$, $\mathcal{J}_{ij} = \{x \in \mathcal{J} | x \circ e_i = x \circ e_j = \frac{x}{2}\}$ are Pierce spaces of $\mathcal{J}$. Furthermore,

(i) $\mathcal{J}_{ij} \circ \mathcal{J}_{ij} \subseteq \mathcal{J}_{ii} + \mathcal{J}_{jj}$;
(ii) $\mathcal{J}_{ij} \circ \mathcal{J}_{jk} \subseteq \mathcal{J}_{ik}$, if $i \neq k$;
(iii) $\mathcal{J}_{ij} \circ \mathcal{J}_{kl} = \{0\}$, if $\{i, j\} \cap \{k, l\} = \emptyset$.

This is called *the Pierce decomposition*. In this case, for any $x \in \mathcal{J}$, with respect to the given Jordan frame $\{e_1, e_2, \cdots, e_r\}$, its Pierce decomposition can be expressed as

$$x = \sum_{i=1}^{r} x_i e_i + \sum_{1 \leq i < j \leq r} x_{ij}, \tag{2.2}$$

where $x_i \in \mathbb{R}$ and $x_{ij} \in \mathcal{J}_{ij}$.

Combining the above two types of decomposition techniques, Kong, Tunçel and Xiu [22] introduced a new decomposition technique for Euclidean Jordan algebras as follows.

Theorem 2.3: (Theorem 2.3, [22]) For any $a \in \mathcal{J}$ with its spectral decomposition $a = \sum_{i=1}^{r} a_i e_i$, where $a_1 \geq a_2 \geq \cdots \geq a_r$. Denote

$$\alpha := \{i : a_i > 0\}, \;\; \beta := \{i : a_i = 0\} \; and \;\; \gamma := \{i : a_i < 0\}. \tag{2.3}$$

In this case, $\mathcal{J}$ can be expressed as the orthogonal direct sum of $\mathcal{J}_{\alpha\alpha}$, $\mathcal{J}_{\alpha\beta}$, $\mathcal{J}_{\alpha\gamma}$, $\mathcal{J}_{\beta\beta}$, $\mathcal{J}_{\beta\gamma}$ and $\mathcal{J}_{\gamma\gamma}$, where $\mathcal{J}_{st} := \bigoplus_{i \leq j, i \in s, j \in t} \mathcal{J}_{ij}$ for any s, $t \in \{\alpha, \beta, \gamma\}$ with respect to the Jordan frame $\{e_1, e_2, \cdots, e_r\}$. Moreover, for any distinct s, $t \in \{\alpha, \beta, \gamma\}$, $(\mathcal{J}_{ss}, \circ, \langle \cdot, \cdot \rangle)$, $(\mathcal{J}_{ss} \bigoplus \mathcal{J}_{st} \bigoplus \mathcal{J}_{tt}, \circ, \langle \cdot, \cdot \rangle)$ are Euclidean Jordan algebras.

Invoking the above theorem, the projection of a onto the symmetric cone K, written as a_+, is exactly $a_+ := \sum_{i \in \alpha} a_i c_i$. Particularly, by taking $a = c$ for some nonzero idempotent c in the above theorem, we can get the following decomposition:

$$\mathcal{J} = \mathcal{J}(c, 1) \oplus \mathcal{J}(c, \frac{1}{2}) \oplus \mathcal{J}(c, 0), \tag{2.4}$$

with $\mathcal{J}(c, \gamma) := \{x : x \circ c = \gamma x\}$, $\gamma = 1$, $\frac{1}{2}$ and 0.

Recall that a Euclidean Jordan algebra is said to be *simple* if it is not the direct sum of two other Euclidean Jordan algebras. The classification theorem in [8] says that every simple Euclidean Jordan algebra is isomorphic to one of the following:

(i) The algebra $\mathbb{S}^n$ of all $n\times n$ real symmetric matrices with trace inner product and $X\circ Y=\frac{1}{2}(XY+YX)$;
(ii) The algebra $\Lambda^n:=(\mathbb{R}\times\mathbb{R}^{n-1},\circ,\langle\cdot,\cdot\rangle)$, with $\langle\cdot,\cdot\rangle$ the usual inner product and the Jordan product $(x_0,\bar{x})\circ(y_0,\bar{y}):=(x_0y_0+\langle\bar{x},\bar{y}\rangle,x_0\bar{y}+y_0\bar{x})$;
(iii) The algebra $\mathcal{H}^n$ of all $n\times n$ complex Hermitian matrices with trace inner product and $X\circ Y=\frac{1}{2}(XY+YX)$;
(iv) The algebra $\mathcal{Q}^n$ of all $n\times n$ quaternion Hermitian matrices with trace inner product and $X\circ Y=\frac{1}{2}(XY+YX)$;
(v) The algebra $\mathcal{O}^3$ of all 3×3 octonion Hermitian matrices with trace inner product and $X\circ Y=\frac{1}{2}(XY+YX)$.

As in [8], any Euclidean Jordan algebra is, in a unique way, a direct sum of simple Euclidean Jordan algebras. Thus, in this paper, we just discuss in the setting of simple Euclidean Jordan algebra, since the related results can be extended to the non-simple case almost in a parallel manner. The following are two properties on symmetric cone of any simple Euclidean Jordan algebra $\mathcal{J}$.

Lemma 2.4: (Exercise 7, [8]) For any given Jordan frame $\{e_1,e_2,\cdots,e_r\}$ and any $x\in K$ with the Pierce decomposition $x=\sum_{i=1}^r x_ie_i+\sum_{1\le i<j\le r}x_{ij}$, we have $x_ix_j\ge\frac{\|x_{ij}\|^2}{2}$ for any i, j with $i<j$.

Lemma 2.5: ([18]) Let x and y be any two elements in $\mathcal{J}$, $\lambda_1(x)$, $\cdots$, $\lambda_r(x)$ and $\lambda_1(y)$, $\cdots$, $\lambda_r(y)$ be their eigenvalues in a non-increasing order. If $x\succeq y$, then $\lambda_i(x)\ge\lambda_i(y)$ for all $i\in\{1,\cdots,r\}$.

Lemma 2.6: ([18]) If $y\succeq x\succ 0$, then $x^{-1}\succeq y^{-1}$.

Noting that the Jordan product "$\circ$" defined in the Euclidean Jordan Algebra $\mathcal{J}$ is a bilinear mapping, thus for any $a\in\mathcal{J}$, *Lyapunov transformation* is defined by $L_a(x):=a\circ x$ for all $x\in\mathcal{J}$. For any a, $b\in\mathcal{J}$, we can also define a corresponding transformation of x and y as follows:

$$P_{a,b}:=L_aL_b+L_bL_a-L_{a\circ b}.$$

$P_{a,a}$ is always abbreviated as P_a and called the *quadratic representation* of a, i.e., $P_a:=2L_a^2-L_{a^2}$. Both L_a and P_a are self-adjoint and have the following decomposition forms respectively.

Theorem 2.7: (Theorem 3.1, [38]) For any $a\in\mathcal{J}$ with its spectral decomposition $a=\sum\limits_{i=1}^r a_ic_i$, the corresponding Lyapunov transformation has the following decomposition

$$L_a=\sum_{i=1}^r a_iC_{ii}+\sum_{i<j}\frac{a_i+a_j}{2}C_{ij},$$

$$P_a=\sum_{i\le j}a_ia_jC_{ij},$$

where C_{ij} is the projection operator onto the eigen-space $\mathcal{J}_{ij}$, for any $i\le j$.

Some properties on the Lyapunov transformation and the quadratic representation are reviewed in the following.

Lemma 2.8: For any $a\in\mathcal{J}$, we have $P_a(K)\subseteq K$. Moreover, if $a\in intK$, then L_a is invertible and $L_a^{-1}(K)\subseteq K$, $L_a^{-1}(intK)\subseteq intK$.

B. *Properties on Rank*

This subsection is devoted to some properties on rank which plays an important role in the main analysis.

Lemma 2.9: ([16]) Let c be an idempotent in $\mathcal{J}$. For any $x\in\mathcal{J}(c,1)$ and any $y\in\mathcal{J}(c,0)$, we have $rank(x+y)=rank(x)+rank(y)$.

By employing the decomposition (2.4) for some idempotent $c\in\mathcal{J}$, we denote the symmetric cone of Euclidean Jordan sub-algebra $\mathcal{J}(c,\gamma)$ by $K(c,\gamma):=\{u^2:u\in\mathcal{J}(c,\gamma)\}$ for $\gamma=0$ and 1. If u is invertible in $\mathcal{J}(c,1)$, we denote the inverse of u in $\mathcal{J}(c,1)$ by u_*^{-1}.

Lemma 2.10: (Corollary 5, [16]) Let c be any idempotent in $\mathcal{J}$ and $x=u+v+w$ with $u\in\mathcal{J}(c,1)$, $v\in\mathcal{J}(c,\frac{1}{2})$ and $w\in\mathcal{J}(c,0)$. If u is invertible in $\mathcal{J}(c,1)$, then

(i) $x\in K$ if and only if $u\in int(K(c,1))$ and $w-P_v(u_*^{-1})\in K$;

(ii) $rank(x)=rank(u)+rank(w-P_v(u_*^{-1}))$

Corollary 2.11: Let c be any idempotent in $\mathcal{J}$. For any $z\in\mathcal{J}(c,\frac{1}{2})$ and $x\in int(K(c,1))$, if $P_z(x)=0$, we have $z=0$.

Proof: Set $y:=x_*^{-1}+z$. Note that $x_*^{-1}\in int(K(c,1))$ and $0-P_z(x)=0\in K$. It follows from Lemma 2.10 (i) that $y\in K$. Combining with Lemma 2.4, we can immediately get $z=0$. ■

Proposition 2.12: Let $\{c_1,\cdots,c_r\}$ be any Jordan frame and $x=\sum_{i=1}^s x_ic_i$ with $x_i>0$ and $1\le s<r$, and $y\in\mathcal{J}$. If $y\succeq x$ and $rank(y)=s$, then $y\in\mathcal{J}(\sum_{i=1}^s c_i,1)$.

Proof: For simplicity, we denote $c:=\sum_{i=1}^s c_i$. Let $y=u+v+w$ with $u\in\mathcal{J}(c,1)$, $v\in\mathcal{J}(c,\frac{1}{2})$ and $w\in\mathcal{J}(c,0)$. It is equivalent to showing that $v=0$ and $w=0$. Noting that $x\in int(K(c,1))$, we can choose sufficiently small $\epsilon>0$ such that $\tilde{x}:=x-\epsilon c\in int(K(c,1))$. Henceforth, $rank(x-\epsilon c)=s$ and $u-\tilde{x}\in int(K(c,1))$. Invoking Lemma 2.10 (ii), we have

$$rank(y)=rank(u)+rank(w-P_v(u_*^{-1})).$$

Together with

$$rank(y)=rank(u)=s \tag{2.5}$$

we can get

$$w-P_v(u_*^{-1})=0. \tag{2.6}$$

Note that $y-\tilde{x}\in K$ and $u-\tilde{x}\in int(K(c,1))$, it follows from Lemma 2.10 (i) that $w-P_v((u-\tilde{x})_*^{-1})\in K$. Using (2.6), this further implies that

$$P_v(u_*^{-1}-(u-\tilde{x})_*^{-1})\in K.$$

On the other hand, noting that $u-\tilde{x}$, $u\in int(K(c,1))$ and $u\succeq u-\tilde{x}$, Lemma 2.6 implies that

$$(u-\tilde{x})_*^{-1}-u_*^{-1}\in int(K(c,1)) \tag{2.7}$$

from the fact that $u - \tilde{x}$, $\tilde{x} \in int(K(c,1))$. By the property that $P_a(K) \subseteq K$ for any a, we have

$$P_v((u-\tilde{x})_*^{-1} - u_*^{-1}) \in K.$$

Therefore, $P_v((u-\tilde{x})_*^{-1} - u_*^{-1}) = 0$. By employing Corollary 2.11, together with (2.7), we can obtain $v = 0$. Utilizing (2.6), it follows immediately that $w = 0$. This completes the proof. ∎

Lemma 2.13: Let x, $y \in K$. If $y \succeq x$, then $rank(y) \geq rank(x)$.

Proof: Let $x = \sum_{i=1}^r \lambda_i(x) c_i$ and $y = \sum_{i=1}^r \lambda_i(y) e_i$ be their spectral decompositions respectively, with $\lambda_1(x) \geq \cdots \geq \lambda_r(x)$ and $\lambda_1(y) \geq \cdots \geq \lambda_r(y)$. From Lemma 2.5, we have $\lambda_i(x) \leq \lambda_i(y)$, for all $i \in \{1, \cdots, r\}$. Together with x, $y \in K$, we have

$$0 \leq \lambda_i(x) \leq \lambda_i(y), \quad \forall i = 1, \cdots, r.$$

This immediately implies that $\pi(y) \geq \pi(x)$. Noting that x, $y \in K$, we have $\nu(x) = \nu(y) = 0$. By definition of rank, the desired inequality $rank(y) \geq rank(x)$ arrives. ∎

Lemma 2.14: If $a \in intK$ and $b \in K$, then $rank(L_a^{-1}(b)) \geq rank(b)$.

Proof: From the property that $L_d^{-1}(K) \subseteq K$ for any $d \in intK$, we can immediately get that $L_a^{-1}(b) \in K$. Let $x := L_a^{-1}(b)$ and $x = \sum_{i=1}^{r} x_i f_i$ be its spectral decomposition with $\beta_x := \{i : x_i > 0\}$. It is evident that for any $i \notin \beta_x$, $x_i = 0$ from the fact $x \in K$. Now, we will show the desired inequality by considering the following three cases.

Case 1: $|\beta_x| = 0$. That is, $x = b = 0$. Trivially, $rank(x) = rank(b) = 0$.

Case 2: $|\beta_x| = r$. That is, $x \in intK$. Obviously, the desired inequality holds since $rank(x) = r \leq rank(b)$.

Case 3: $0 < |\beta_x| < r$. That is, $x \in K \setminus (intK \cup \{0\})$. Denote $f := \sum_{i \in \beta_x} f_i$. Let

$$a = \sum_{i=1}^{r} a_i f_i + \sum_{i<j} a_{ij}, \quad b = \sum_{i=1}^{r} b_i f_i + \sum_{i<j} b_{ij}$$

be their corresponding Pierce decompositions. By the property that $L_a(x) = L_x(a)$, together with Theorem 2.7, we have

$$b = \sum_{i \in \beta_x} x_i a_i f_i + \sum_{i<j, i,j \in \beta_x} \frac{x_i + x_j}{2} a_{ij} + \sum_{i \in \beta_x, j \notin \beta_x} \frac{x_i}{2} a_{ij}.$$

This implies that

$$\begin{cases} b_i = 0, & \forall i \notin \beta_x; \\ b_{ij} = 0, & \forall i, j \notin \beta_x, \ i < j. \end{cases} \tag{2.8}$$

Noticing that $b \in K$, it follows from Lemma 2.4 that

$$b_{ij} = 0, \quad \forall i \in \beta_x, \forall j \notin \beta_x. \tag{2.9}$$

By employing (2.8) and (2.9), we obtain that $b = \sum_{i \in \beta_x} b_i f_i + \sum_{i<j, i,j \in \beta_x} b_{ij}$, that is $b \in \mathcal{J}(f, 1)$. This further implies that

$$rank(b) \leq rank(\mathcal{J}(f,1)) = rank(x).$$

∎

III. Continuous Case

In this section, we study the lowest-rank solutions of continuous Lyapunov equations over symmetric cones. For simplicity, we denote

$$\mathcal{F} := \{x : L_a(x) = b, x \in K\}, \ \hat{\mathcal{F}} := \{x : L_a(x) \succeq b, x \in K\}.$$

The main result of this section is stated in the following theorem.

Theorem 3.1: Let $a \in \mathcal{J}$ and $b \in K$. If $\mathcal{F} \neq \emptyset$, then problems (P_0) and (P_1) have the same unique solution x^*. Moreover, $rank(b) \leq rank(x^*) \leq rank(a_+)$.

Proof: Let $a = \sum_{i=1}^{r} a_i c_i$ be its spectral decomposition and α, β, γ be defined as in Theorem 2.3. Let $\mathcal{J}_{st}$ be the corresponding eigen-space for any given s, $t \in \{\alpha, \beta, \gamma\}$ and K_t be the symmetric cone in the Euclidean Jordan subalgebra $\mathcal{J}_{tt}$. For any $x \in \mathcal{F}$, with its Pierce decomposition $x = \sum_{i=1}^{r} a_i c_i + \sum_{i<j} x_{ij}$, by employing Theorem 2.3, we can rewrite x as follows:

$$x = x_{\alpha\alpha} + x_{\beta\beta} + x_{\gamma\gamma} + x_{\alpha\beta} + x_{\alpha\gamma} + x_{\beta\gamma},$$

with $x_{st} \in \mathcal{J}_{st}$ for any s, $t \in \{\alpha, \beta, \gamma\}$. Similarly, we can decompose b as

$$b = b_{\alpha\alpha} + b_{\beta\beta} + b_{\gamma\gamma} + b_{\alpha\beta} + b_{\alpha\gamma} + b_{\beta\gamma}.$$

It follows from Theorem 2.7 that

$$L_a(x) = \sum_{i=1}^{r} a_i x_i c_i + \sum_{1 \leq i < j \leq r} \frac{a_i + a_j}{2} x_{ij} = b. \tag{3.1}$$

Mentioning that $x \in K$ and $b \in K$, Lemma 2.4 tells us that

$$x_i \geq 0, \ a_i x_i \geq 0, \ \forall i \tag{3.2}$$

$$x_i x_j \geq \frac{1}{2} \|x_{ij}\|_F^2, \ a_i x_i a_j x_j \geq \frac{1}{2} \|\frac{a_i + a_j}{2} x_{ij}\|_F^2 \ 1 \leq i < j \leq r. \tag{3.3}$$

It is obvious from (3.2) that for any $i \in \gamma$, $x_i = 0$. This together with (3.3) implies that $x_{jk} = 0$ for any $\{j,k\} \cap \gamma \neq \emptyset$, which leads to

$$x_{\alpha\gamma} = 0, \ x_{\beta\gamma} = 0, \ x_{\gamma\gamma} = 0. \tag{3.4}$$

Combining with (3.1), we have

$$b_{\alpha\gamma} = 0, \ b_{\beta\gamma} = 0, \ b_{\gamma\gamma} = 0.$$

In addition, for any $i \in \alpha$ and $j \in \beta$, (3.3) indicates that $x_{ij} = 0$, that is, $x_{\alpha\beta} = 0$. Therefore,

$$x = x_{\alpha\alpha} + x_{\beta\beta}. \tag{3.5}$$

In this case, $x \in K$ is equivalent to $x_{\alpha\alpha} \in K_\alpha$ and $x_{\beta\beta} \in K_\beta$. Let $\bar{L}_{a_{\alpha\alpha}} : \mathcal{J}_{\alpha\alpha} \to \mathcal{J}_{\alpha\alpha}$ and $\bar{L}_{a_{\beta\beta}} : \mathcal{J}_{\beta\beta} \to \mathcal{J}_{\beta\beta}$ be the corresponding restrictions of L_a into $\mathcal{J}_{\alpha\alpha}$ and $\mathcal{J}_{\beta\beta}$ respectively. (3.5) implies that

$$b = L_a(x) = \bar{L}_{a_{\alpha\alpha}}(x_{\alpha\alpha}) + \bar{L}_{a_{\beta\beta}}(x_{\beta\beta}). \tag{3.6}$$

Note that $a_{\beta\beta} = 0$ Thus $b = b_{\alpha\alpha}$ and $x_{\beta\beta}$ can be arbitrarily chosen from K_β. Moreover, since $a_{\alpha\alpha} \in intK_\alpha$, we have

$\bar{L}_{a_{\alpha\alpha}}$ is invertible in $\mathcal{J}_{\alpha\alpha}$ and $\bar{L}_{a_{\alpha\alpha}}^{-1}(K_\alpha) \subseteq K_\alpha$. This implies that $x_{\alpha\alpha} = \bar{L}_{a_{\alpha\alpha}}^{-1}(b) \in K_\alpha$. Henceforth,

$$\mathcal{F} = \bar{L}_{a_{\alpha\alpha}}^{-1}(b) + K_\beta. \tag{3.7}$$

This shows that for any $z \in \mathcal{F}$, we can find some $u \in K_\beta$ such that $z = \bar{L}_{a_{\alpha\alpha}}^{-1}(b) + u$. Therefore,

$$tr(z) \geq tr(\bar{L}_{a_{\alpha\alpha}}^{-1}(b)), \quad \forall z \in \mathcal{F} \tag{3.8}$$

which means that $\bar{L}_{a_{\alpha\alpha}}^{-1}(b)$ is exactly the unique solution of problem (P_1). On the other hand, noting that $\bar{L}_{a_{\alpha\alpha}}^{-1}(b) \in \mathcal{J}_{\alpha\alpha}$ and $u \in \mathcal{J}_{\beta\beta}$, by invoking Lemma 2.9, for all $z \in \mathcal{F}$, we have

$$rank(z) = rank(\bar{L}_{a_{\alpha\alpha}}^{-1}(b)) + rank(u) \geq rank(\bar{L}_{a_{\alpha\alpha}}^{-1}(b)).$$

This tells us that $\bar{L}_{a_{\alpha\alpha}}^{-1}(b)$ is also the unique solution of problem (P_0). Let $x^* := \bar{L}_{a_{\alpha\alpha}}^{-1}(b)$. This comes to the first part of the theorem.

To get the second part, we note that $a_+ = a_{\alpha\alpha}$ and $rank(a_{\alpha\alpha}) = rank(\mathcal{J}_{\alpha\alpha}) = |\alpha|$. It follows readily that $rank(x^*) \leq rank(a_+)$ since $x^* \in \mathcal{J}_{\alpha\alpha}$. On the other hand, by utilizing the facts that $b \in K_\alpha$, $a_{\alpha\alpha} \in intK_\alpha$ and $\bar{L}_{a_{\alpha\alpha}}(x^*) = b$, we have $rank(x^*) \geq rank(b)$ from Lemma 2.14. This completes the whole proof. ■

Remark 3.2: The feasibility in the aforementioned problems are well studied in [26] where several necessary and\or sufficient conditions for $\mathcal{F} \neq \emptyset$ are proposed. Moreover, the least-squares solution of the constraint system are also established in [27].

The Lyapunov transformation can lead to some hyper-lattice structure in K. Before we introducing the definition of the hyper-lattice in K, we first employ the following notions for the sake of simplicity:

(i) $U(x, y) := \{z : 0 \preceq z \preceq x, 0 \preceq z \preceq y\}$;

(ii) The set of the maximal points of $U(x, y)$, denoted by $U_{sup}(x, y)$, is defined in the following way: for any $u \in U(x, y)$, there exists some $z \in U_{sup}(x, y)$ such that $z \succeq u$, $z \in U(x, y)$, and there exists no $v \in U(x, y)$ such that $v \neq z$ and $v \succeq z$. Evidently, the element $z \in U_{sup}(x, y)$ not only depends on x and y, but also on the specific matrix element u.

Definition 3.3: A set $\mathcal{M} \subseteq K$ is called a hyper-lattice in K if for any x and $y \in \mathcal{M}$, there always exists some $z \in U(x, y)$ such that $z \in \mathcal{M}$.

Lemma 3.4: Let $a \in \mathcal{J}$ and $b \in K$. If $\mathcal{F} \neq \emptyset$, then both $\mathcal{F}$ and $\hat{\mathcal{F}}$ are hyper-lattices and they share the same least element $\bar{L}_{a_{\alpha\alpha}}^{-1}(b)$.

Proof: From the proof of Theorem 3.1, we know that $\mathcal{F} = \bar{L}_{a_{\alpha\alpha}}^{-1}(b) + K_\beta$. This immediately derives that $\bar{L}_{a_{\alpha\alpha}}^{-1}(b)$ is the least element of $\mathcal{F}$ and for any given x, $y \in \mathcal{F}$ and any $z \in U(x, y)$, we can always find some u, v and $w \in U(u, v)$ such that

$$x = \bar{L}_{a_{\alpha\alpha}}^{-1}(b) + u, y = \bar{L}_{a_{\alpha\alpha}}^{-1}(b) + v, z = \bar{L}_{a_{\alpha\alpha}}^{-1}(b) + w.$$

Evidently, we have $z \in \mathcal{F}$, which implies that $\mathcal{F}$ is a hyper-lattice. To achieve the hyper-lattice structure of $\hat{\mathcal{F}}$, we assume that $\hat{x}$ is an arbitrary element in $\hat{\mathcal{F}}$. That is, $\hat{x} \in K$, and $\hat{u} := L_a(\hat{x}) - b \in K$. Since $\mathcal{F} \neq \emptyset$, from the proof of Theorem 3.1, we can easily verify that $L_a(K) \subseteq K_\alpha$ and $b \in K_\alpha$. Henceforth, $\hat{u} \in K_\alpha$. By direct calculation, we can obtain that $\hat{x} \in \bar{L}_{a_{\alpha\alpha}}^{-1}(b + \hat{u}) + K_\beta$. This further implies that

$$\hat{\mathcal{F}} = \{x : x = \bar{L}_{a_{\alpha\alpha}}^{-1}(b + u) + K_\beta, u \in K_\alpha\}. \tag{3.9}$$

By applying (3.9), together with $\bar{L}_{a_{\alpha\alpha}}^{-1}(K_\alpha) \subseteq K_\alpha$, we have

$$z \succeq \bar{L}_{a_{\alpha\alpha}}^{-1}(b) \in K, \forall z \in \hat{\mathcal{F}}.$$

This shows that $\bar{L}_{a_{\alpha\alpha}}^{-1}(b)$ is the least element of $\hat{\mathcal{F}}$. Moreover, for any $\hat{x}_1$, $\hat{x}_2 \in \hat{\mathcal{F}}$, (3.9) allows us to find some u_1, $u_2 \in K_\alpha$ and v_1, $v_2 \in K_\beta$ such that

$$\hat{x}_1 = \bar{L}_{a_{\alpha\alpha}}^{-1}(b + u_1) + v_1, \ \hat{x}_2 = \bar{L}_{a_{\alpha\alpha}}^{-1}(b + u_2) + v_2.$$

For any $w \in U(u_1, u_2)$ and $h \in U(v_1, v_2)$, it is easy to verify that $\hat{x}_3 := \bar{L}_{a_{\alpha\alpha}}^{-1}(b + w) + h \in \hat{\mathcal{F}}$ and $\hat{x}_3 \in U(\hat{x}_1, \hat{x}_2)$. By definition, the desired hyper-lattice structure of $\hat{\mathcal{F}}$ arrives. This completes the proof. ■

Applying the hyper-lattice structure exploited in Lemma 3.4, we can similarly get the following equivalence.

Theorem 3.5: Let $a \in \mathcal{J}$ and $b \in K$. If $\mathcal{F} \neq \emptyset$, then problem (P_0) is equivalent to

$$(\tilde{P_0}) \qquad \begin{array}{ll} \min & rank(x) \\ \text{s.t.} & L_a(x) \succeq b \\ & x \in K. \end{array}$$

Some specific examples are presented as follows for illustration.

Example 3.6: Let $a \in \mathcal{J}$ and L_a be the corresponding Lyapunov transformation.

(i) If $\mathcal{J} = \mathbb{R}^n$ and $K = \mathbb{R}^n_+$, then L_a is exactly the diagonal matrix $Diag(a)$. The results in Theorems 3.1 and 3.5 follows easily by direct calculation.

(ii) If $\mathcal{J} = \Lambda^n$ and $K = \Lambda^n_+$, then L_a can be expressed as the matrix $\begin{pmatrix} a_1 & a_2^T \\ a_2 & a_1 I \end{pmatrix}$, where $a = (a_1, a_2)$ with $a_1 \in \mathbb{R}$ and $a_2 \in \mathbb{R}^{n-1}$. Theorems 3.1 and 3.5 tell us that the minimal rank solution of Lyapunov equation/inequality over second order cone is unique and can be equivalently solved by linear programming over second order cone in polynomial time.

(iii) If $\mathcal{J} = \mathbb{S}^n$ and $K = \mathbb{S}^n_+$, then $L_a(X) = \frac{1}{2}(AX + XA)$ with $a = A \in \mathbb{S}^n$. In this case, Theorems 3.1 and 3.5 indicate that the minimal rank solution of continuous Lyapunov equation/inequality over semidefinite cone is unique and can equivalently solved by semidefinite programming problem (SDP for short) in polynomial time.

IV. Discrete Case

In this section, we study the lowest-rank solutions of discrete Lyapunov equation over symmetric cone. We introduce the definition of Stein-type transformation as a start.

Definition 4.1: Let $\mathcal{A} : \mathcal{J} \to \mathcal{J}$ be a linear transformation. We say that $\mathcal{A}$ is a Stein-type transformation if it can be expressed as $\mathcal{A} = \alpha\mathcal{I} - S$ with some $\alpha > 0$ and a linear transformation $S : \mathcal{J} \to \mathcal{J}$ satisfying $S(K) \subseteq K$.

We call it the Stein-type transformation since it is of a similar type as the Stein transformation $S_H(X) := \mathcal{A}(X) = X - HXH^\top$ in matrix space. It is easy to verify that the Stein-type transformation includes the Stein transformation S_H with $H \in \mathbb{R}^{n\times n}$ and also the type-Z-transformation $L(X) := X - \sum_{i=1}^{k} M_i X M_i^\top$ with $M_i \in \mathbb{R}^{n\times n}$ in matrix space $\mathbb{S}^n$ as its special cases. Some interesting properties of the Stein-type transformation are exploited as follows.

Lemma 4.2: Let $\mathcal{A} : \mathcal{J} \to \mathcal{J}$ be any Stein-type transformation and $b \in K$, the following two system are equivalent:
(a) $\mathcal{A}(x) = b$, $x \in K$;
(b) $\mathcal{A}(x) - b \in K$, $x \in K$, $\langle x, \mathcal{A}(x) - b\rangle = 0$.

Proof: By definition, we can find some positive α and some linear transformation S with $S(K) \subseteq K$ such that $\mathcal{A} = \alpha\mathcal{I} - S$. Apparently, any solution to system (a) is a solution to system (b). To get the converse part, let x^* be any solution to system (b). From the self-duality of symmetric cone K and the condition $S(K) \subseteq K$, it follows that

$$\begin{aligned} & \langle \mathcal{A}(x^*), \mathcal{A}(x^*) - b\rangle \\ = \; & \langle \alpha x^*, \mathcal{A}(x^*) - b\rangle - \langle S(x^*), \mathcal{A}(x^*) - b\rangle \\ = \; & -\langle S(x^*), \mathcal{A}(x^*) - b\rangle \\ \le \; & 0. \end{aligned}$$

On the other hand,

$$\langle \mathcal{A}(x^*), \mathcal{A}(x^*)-b\rangle = \langle \mathcal{A}(x^*)-b, \mathcal{A}(x^*)-b\rangle + \langle b, \mathcal{A}(x^*)-b\rangle \ge 0 \quad (4.1)$$

by the condition $b \in K$. Henceforth, $\langle \mathcal{A}(x^*), \mathcal{A}(x^*) - b\rangle = 0$. Together with (4.1). we further obtain that $\mathcal{A}(x^*) - b = 0$. This arrives at the desired equivalence. ■

Similar to the Lyapunov transformation, the Stein-type transformation can lead to some hyper-lattice structure in K as follows. The proof is similar to the one of Lemma II.1 in [30], and we state it here for completeness.

Lemma 4.3: Let $\mathcal{A}$ be a Stein-type transformation and $b \in K$. Then the set $\tilde{\mathcal{F}} := \{x : x \in K, \mathcal{A}(x) \succeq b\}$ is a hyper-lattice.

Proof: The assertion trivially holds if $\tilde{\mathcal{F}} = \emptyset$. For the case of $\tilde{\mathcal{F}} \neq \emptyset$, let x, $y \in \tilde{\mathcal{F}}$ and z be any element in $U(x, y)$. It follows by definition that

$$\alpha x \succeq S(x) + b \succeq S(z) + b \succeq 0,$$

$$\alpha y \succeq S(y) + b \succeq S(z) + b \succeq 0,$$

which implies that $\frac{S(z)+b}{\alpha} \in U(x, y)$ for any $z \in U(x, y)$. Let g be a set-valued mapping from $U(x, y)$ to itself such that

$$g(z) := \{\omega \in U(x, y) : \alpha\omega \succeq S(z) + b\}.$$

By the definition of $U_{sup}(x, y)$, there exists $h \in U_{sup}(x, y)$ such that $h \succeq \frac{S(z)+b}{\alpha}$. Thus $h \in g(z)$ which means that $g(z) \neq \emptyset$ for any $z \in U(x, y)$. Let $\{z_k\} \subseteq U(x, y)$ and $\{\omega_k\}$ be the corresponding sequence satisfying $\alpha\omega_k \succeq S(z_k) + b$ such that $z_k \to z^*$ and $\omega_k \to \omega^*$. It follows from the fact $\alpha\omega_k - S(z_k) - b \succeq 0$ and the closedness of symmetric cone K that $\alpha\omega^* - S(z^*) - b \succeq 0$. This derives that $\omega^* \in g(z^*)$ which means that g is upper semicontinuous. This together with the convexity of $U(x, y)$ can further deduce that there exists $\bar{z} \in U(x, y)$ such that $\alpha\bar{z} \succeq S(\bar{z})+b$ by employing the well-known Kakutani's Fixed Point Theorem [9]. From above, we know that for any x, $y \in \tilde{\mathcal{F}}$, there exists $\bar{z} \in U(x, y)$ such that $\bar{z} \in \tilde{\mathcal{F}}$. Therefore, the desired assertion arrives. ■

Proposition 4.4: Let $\mathcal{A}$ be a Stein-type transformation and $b \in K$. If $\tilde{\mathcal{F}}$ defined as in Lemma 4.3 is nonempty, then it has a unique least element x^*, i.e., $y \succeq x^*$ for any $y \in \tilde{\mathcal{F}}$. Moreover, x^* is the unique least element of $\mathcal{F}^0 := \{x \in \mathcal{J} : \mathcal{A}(x) = b, x \in K\}$.

Proof: It follows from the hyper-lattice structure of $\tilde{\mathcal{F}}$ that the least element, if exists, must be unique. Now we are going to show the existence with a constructive proof. We adopt the following auxiliary linear program over symmetric cones:

$$(SCLP) \qquad \begin{array}{ll} \min & tr(x) \\ \text{s.t.} & x \in \tilde{\mathcal{F}} \\ & tr(x) \le tr(u) \end{array}$$

where u is some point in $\tilde{\mathcal{F}}$. Note that the feasible set of the above linear program is bounded and closed and hence has a solution, says x^*. For any $y \in \tilde{\mathcal{F}}$, by the hyper-lattice structure of $\tilde{\mathcal{F}}$, there exists $\omega \in U(x^*, y)$ such that $\omega \in \tilde{\mathcal{F}}$. By definition, we know that $x^* \succeq \omega \succeq 0$. Hence, $tr(x^* - \omega) \ge 0$. Combining with the fact x^* is of the minimal trace in $\tilde{\mathcal{F}}$, we get $tr(x^* - \omega) = 0$ which further implies that $x^* = \omega$. Therefore, $x^* = \omega \preceq y$ for any $y \in \tilde{\mathcal{F}}$. This shows that x^* is the unique least element of $\tilde{\mathcal{F}}$.

For the "moreover" part, we first claim that x^* is a solution to the linear complementarity problem SCLCP($\mathcal{A}, -b$):

$$x \in K, \mathcal{A}(x) - b \in K, \langle x, \mathcal{A}(x) - b\rangle = 0.$$

Assume on the contrary that x^* is not a solution of the above system, that is,

$$\langle x^*, \mathcal{A}(x^*) - b\rangle > 0. \quad (4.2)$$

Let $x^* = \sum_{i=1}^{r} x_i^* c_i$ be the spectral decomposition. Denote $\beta := \{i : x_i^* > 0\}$ and $\bar{\beta} := \{1, \cdots, r\} \setminus \beta$, $\mathcal{J}_\beta := \bigoplus_{i\le j, i,j\in\beta} \mathcal{J}_{\{c_i,c_j\}}$, $\mathcal{J}_{\bar{\beta}} := \bigoplus_{i\le j, i,j\in\bar{\beta}} \mathcal{J}_{\{c_i,c_j\}}$ and $\mathcal{J}_{\beta\bar{\beta}} := \bigoplus_{i\in\beta, j\in\bar{\beta}} \mathcal{J}_{\{c_i,c_j\}}$. From Theorem 2.3, we have $\mathcal{J} = \mathcal{J}_\beta \bigoplus \mathcal{J}_{\bar{\beta}} \bigoplus \mathcal{J}_{\beta\bar{\beta}}$ and J_β, $\mathcal{J}_{\bar{\beta}}$ are sub-Euclidean Jordan algebras. Let K_β and $K_{\bar{\beta}}$ be the corresponding symmetric cones. Now we claim that $\mathcal{A}(x^*) - b \in K_\beta$. Suppose $A(x^*) - b = \alpha x^* - (S(x^*) + b) = \alpha x^* - (y_\beta + y_{\bar{\beta}} + y_{\beta\bar{\beta}}$ with some $y_\beta \in \mathcal{J}_\beta$, $y_{\bar{\beta}} \in \mathcal{J}_{\bar{\beta}}$ and $y_{\beta\bar{\beta}} \in \mathcal{J}_{\beta\bar{\beta}}$. From the properties of symmetric cones, we have $y_{\bar{\beta}} \in K_{\bar{\beta}}$ from the fact $S(x^*) + b \in K$. On the other hand, from the fact $\mathcal{A}(x^*) - b \in K$ and $x^* \in K_\beta$, we have $-y_{\bar{\beta}} \in K_{\bar{\beta}}$. This implies that $y_{\bar{\beta}} = 0$. By Lemma 2.4, we can immediately get $y_{\beta\bar{\beta}} = 0$. The aforementioned claim is therefore proven. Combining with (4.2), we know that $\mathcal{A}(x^*) - b \in K_\beta \setminus \{0\}$.

For any $\epsilon > 0$ and $z \in K_\beta$, it follows from direct calculation that

$$\begin{aligned}\mathcal{A}(x^*-\epsilon z)-b &= \alpha(x^*-\epsilon z)-S(x^*-\epsilon z)-b\\ &\succeq \alpha(x^*-\epsilon z)-S(x^*)-b\\ &= \mathcal{A}(x^*)-b-\alpha\epsilon z\end{aligned}$$

where the inequality is derived from the fact $S(K) \subseteq K$ and $z \in K$. Noting that $\mathcal{A}(x^*)-b \in K_\beta \setminus \{0\}$ and $x^* \in intK_\beta$, by choosing sufficiently small $\epsilon > 0$ and let $z := \mathcal{A}(x^*) - b$, we can get $\mathcal{A}(x^*) - b - \alpha\epsilon z \in K_\beta$ and $x^* - \epsilon z \in K_\beta$. It further implies that $\mathcal{A}(x^* - \epsilon z) - b \in K$. Thus $x^* - \epsilon z \in \tilde{\mathcal{F}}$ and $x^* \succeq x^* - \epsilon z, x^* \neq x^* - \epsilon z$. This contradicts that x^* is the least element of $\tilde{\mathcal{F}}$. Therefore, x^* is a solution to SCLCP($\mathcal{A}, -b$). By invoking the equivalence established in Lemma 4.2, the desired result follows. ■

Now we are in a position to propose the main result of this section.

Theorem 4.5: Let $\mathcal{A} : \mathcal{J} \to \mathcal{J}$ is a Stein-type transformation and $b \in K$. If the feasible set $\mathcal{F}^0 \neq \emptyset$, then problems $(\hat{P}_0)$ and $(\hat{P}_1)$ share the common unique solution x^*. Moreover, $rank(x^*) \geq rank(b)$.

Proof: From Proposition 4.4, we know that the feasible set $\mathcal{F}^0$ has a unique least element x^*. By definition, $x \succeq x^*$ for any $x \in \mathcal{F}^0$. This further implies that

$$tr(x) > tr(x^*), \quad \forall x \in \mathcal{F}^0, x \neq x^*, \tag{4.3}$$

$$rank(x) \geq rank(x^*), \quad \forall x \in \mathcal{F}^0, \tag{4.4}$$

from the fact $tr(u) > 0$ for any $u \in K \setminus \{0\}$ and Lemma 2.13. Furthermore, (4.3) indicates that x^* is exactly the unique solution of problem $(\hat{P}_1)$, and (4.4) implies that x^* is also a solution to problem $(\hat{P}_0)$.

Now it remains to show that x^* is also the unique solution of problem $(\hat{P}_0)$. We assume on the contrary that there exists some other solution y to problem (P_0) with the its Pierce decomposition $y = \sum_{i=1}^{r} y_i c_i + \sum_{i<j} y_{ij}$. It follows readily that $rank(y) = rank(x^*)$ and $y \succeq x \succeq 0$. Let $x^* = \sum_{i=1}^{s} x_i^* c_i$ be its spectral decomposition with the minimal rank s. It can be derived from Proposition 2.12 that $y \in \mathcal{J}(\sum_{i=1}^{s} c_i, 1)$. Thus, $y - x^* \in K(\sum_{i=1}^{s} c_i, 1)$. Mentioning that $x^* \in int(K(\sum_{i=1}^{s} c_i, 1))$, we can always get some $\hat{x} := x^* - \delta(y - x^*) \in K$ by choosing some sufficiently small $\delta > 0$. Evidently, $\mathcal{A}(\hat{x}) = b$ from the feasibility of y. Henceforth, $\hat{x} \in \mathcal{F}^0$ and $tr(\hat{x}) < tr(x^*)$. This consequently leads to a contradiction to the fact x^* of the minimal trace in $\mathcal{F}^0$ as we have claimed previously. Therefore, the solution uniqueness of problem $(\hat{P}_0)$ follows. Noting that $\alpha x^* = S(x^*) + b \succeq b \succeq 0$, it follows from Lemma 2.13 that $rank(x^*) = rank(\alpha x^*) \geq rank(b)$. This completes the proof. ■

Remark 4.6: From Proposition 4.4, we can similarly show that an exact lowest-rank solution to discrete Lyapunov inequality over symmetric cone can be obtained by the unique solution to its trace minimization relaxation problem. This can be regarded as a generalization of Theorem II.2 in [30] from a matrix space to a Euclidean Jordan algebra. However, the equivalence of these two problems may not hold in the inequality case. Unlike in the equation case in Theorem 4.5, solutions to the low rank problem could be many. A counterexample is presented as follows.

Example 4.7: Let $B := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathcal{A}(X) := 3X - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} X \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. By direct calculation, we can obtain that $X^* := \begin{bmatrix} \frac{3}{8} & 0 \\ 0 & \frac{1}{8} \end{bmatrix}$ is the unique solution to

$$\begin{array}{ll} \min & tr(X) \\ \text{s.t.} & \mathcal{A}(X) \succeq B \\ & X \in \mathbb{S}^n_+, \end{array}$$

while there many solutions such as X^* and $\begin{bmatrix} \frac{1}{2} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{4} \end{bmatrix}$ to

$$\begin{array}{ll} \min & rank(X) \\ \text{s.t.} & \mathcal{A}(X) \succeq B \\ & X \in \mathbb{S}^n_+. \end{array}$$

The illustration of Theorem 4.5 and Remark 4.6 for the specific symmetric cones $\mathbb{R}^n_+$, and $\mathbb{S}^n_+$ are listed in the following example.

Example 4.8: Let $\mathcal{A}$ be a Stein-type transformation from $\mathcal{J}$ to itself with K be its corresponding symmetric cone.

(i) Let $\mathcal{J} = \mathbb{R}^n$ and $K = \mathbb{R}^n_+$. In this case, A is exactly a Z-matrix whose off-diagonal entries are non-positive. The aforementioned equivalence and solution uniqueness in Theorem 4.5 and Remark 4.6 follows from the least element theory for Z-matrix which is elaborated in [5].

(ii) Let $\mathcal{J} = \mathbb{S}^n$ and $K = \mathbb{S}^n_+$. In this case, More examples of Stein-type transformations can be found.

   (a) the type-Z-transformation $\mathcal{A}(X) := X - \sum_{i=1}^{k} M_i X M_i^\top$ (See [30]),
   (b) $\mathcal{A}(X) = A \circ_H X$ for any $A \in \mathbb{S}^n_+$ with $\circ_H$ the Hadamard product,
   (c) $\mathcal{A}(X) = L_A^{-1}(X)$ for any $A \in \mathbb{S}^n_{++}$,

   are special Stein-type transformations. In these cases, we can also get the same results as in Theorem 4.5.

V. Concluding Remarks

In this paper, we have studied the equivalence of the rank minimization and trace minimization problems for continuous and discrete Lyapunov equations over symmetric cone under some mild condition. The contributions are threefold: (1) The lowest-rank solutions of both the continuous and discrete Lyapunov equations over symmetric cones can be equivalently solved via linear programming over symmetric cone in polynomial time; (2) The solution uniqueness is achieved; (3) The upper and lower bounds of the minimal rank are proposed. All these results can be served as theoretical support for some real problems incurred in control theory, stability analysis and system design due to the wide applications of continuous and discrete Lyapunov equations in this community. Note that the trace minimization is indeed the nuclear norm relaxation problem for the original rank minimization problem since the trace can be regarded as the nuclear norm for any element in symmetric cone. It is worth pointing out that our results can also be applied to the so-called Schattern p-norm (i.e., the l_p norm of the vector generated by all eigenvalues) minimization case by using the similar technique. There are some challenging issues left for future research. For example, it is worth mentioning that both the Lyapunov transformation and the Stein-type transformation are Z transformations as defined in [17]. *Can we generalized the above results to the case of general Z transformation?* Even to consider the L_A with nonsymmetric $A \in \mathbb{R}^{n\times n}$ would be very useful.

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