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# Minimum Concave Cost Flow Over a Grid Network

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**Abstract** The minimum concave cost network flow problem (MCCNFP) is NP-hard, but efficient polynomial-time algorithms exist for some special cases such as the uncapacitated lot-sizing problem and many of its variants. We study the MCCNFP over a grid network with a general nonnegative separable concave cost function. We show that this problem is polynomially solvable when all sources are in the first echelon and all sinks are in two echelons, and when there is a single source but many sinks in multiple echelons. The polynomiality argument relies on a combination of a particular dynamic programming formulation and an investigation of the extreme points of the underlying flow polyhedron. We derive an analytical formula for the inflow into any node in an extreme point solution, which generalizes a result of Zangwill [20] for the multi-echelon lot-sizing problem.

**Keywords** Min concave cost flow · Grid network · Lot-sizing model

## 1 Introduction

We study the minimum concave cost flow problem over a grid network (CFG). A *grid network*  $G = (V, A)$  is a directed acyclic graph with the node set  $V = \{v_{l,t} | l \in \{0, \dots, L\}, t \in \{1, \dots, T\}\}$  and the arc set  $A = \{(v_{l,t}, v_{l,t+1}) | l \in \{0, \dots, L\}, t \in \{1, \dots, T -$

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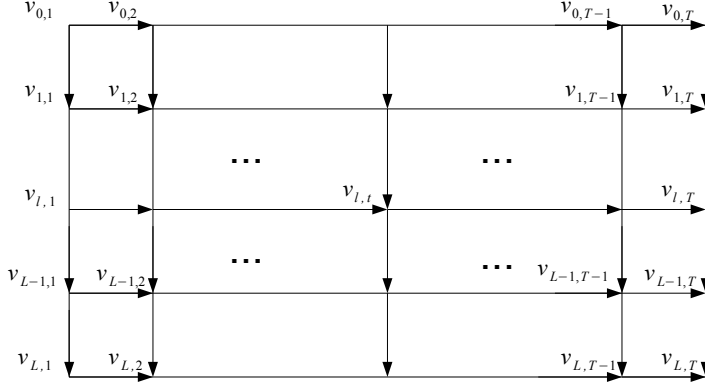
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$1\}} \cup \{(v_{l,t}, v_{l+1,t}) | l \in \{0, \dots, L-1\}, t \in \{1, \dots, T\}\}$ , as show in Figure 1. The nodes and/or arcs have associated numerical values such as supplies/demands, costs and capacities. We refer to the two subscripts  $l$  and  $t$  as the indices of *echelon* and *time period*, respectively, so the grid network we study has  $L+1$  echelons and  $T$  time periods.



**Fig. 1** The grid network

Given a grid network  $G$ , CFG is to find a vector  $x \in \mathbb{R}^{|A|}$  to

$$\begin{aligned} \min \quad & \sum_{a \in A} c_a(x_a) \\ \text{s.t.} \quad & \sum_{a \in \delta^+(v)} x_a - \sum_{a \in \delta^-(v)} x_a = b(v), \quad \forall v \in V, \\ & x_a \geq 0, \quad \forall a \in A, \end{aligned} \quad (1)$$

where  $c_a$  is the cost function for arc  $a$ ,  $b(v)$  is the supply at node  $v$ , and  $\delta^+(v)$  and  $\delta^-(v)$  are the set of outgoing and incoming arcs at node  $v$ , respectively. The node  $v$  is called a source if  $b(v) > 0$ , a sink if  $b(v) < 0$ , and a transshipment node if  $b(v) = 0$ . We assume that for each  $a \in A$ , the cost function  $c_a$  is a general nonnegative concave function represented by a value oracle.

The minimum concave cost network flow problem (MCCNFP) is NP-hard in general, as shown by a reduction from the subset sum problem [9], with a few known polynomially solvable special cases. There are two polynomially-solvable cases closely related to the problem we study. Zangwill [21] gave a polynomial-time dynamic programming (DP) algorithm for the multi-echelon lot-sizing problem, which can be formulated as a CFG with a single source and one echelon of sinks. Erickson et al. [6] proposed a DP algorithm for the general MCCNFP and showed that their algorithm runs in polynomial time when the graph is planar and all sources and sinks lie on a constant number of faces of the graph. The grid network is a planar graph, but in general the sources and sinks in CFG are not on a constant number of faces.

The main contributions of this paper are the following.

1. If all sources are at one echelon and all sinks are at two out of  $L$  echelons with  $L$  fixed, then CFG can be solved in polynomial time in  $T$  and the number of queries of a function-value oracle,

2. If there is one source and  $L$  echelons of sinks with  $L$  fixed, then CFG can be solved in polynomial time in  $T$  and the number of queries of a function-value oracle,

Our result unifies the complexity results of the uncapacitated lot-sizing problem (ULS) and many of its variants including the multi-echelon case in Zangwill [21] and two-echelon lot-sizing problem with intermediate demands in Zhang et al. [22], and gives new complexity results for the multi-echelon ULS with arbitrary intermediate demand and CFG with  $T$  sources and one echelon of intermediate demand. We provide a new DP framework for CFG focusing on the component of the state vector being the inflow into some node instead of flow on an arc. We then derive a new characterization for the optimal flow in a grid network with multiple sources. The characterization leads to a polynomial time DP algorithm, and also provides some insight on the optimal inventory and production quantities in a production planning setting. Our analysis was motivated by Zangwill's characterization of the optimal flow for the multi-echelon ULS. However, the presence of multiple sources introduces much more complexity on the structure of the optimal flow. Note that our result and the algorithm in Erickson et al. [6] both generalize the complexity result for multi-echelon ULS without intermediate demand, but we give new complexity results for CFG with intermediate demand.

Apart from its theoretical interest, polynomial solvability of special cases of MC-CNFP offers possibilities of deriving a tight or compact extended formulation for the original problem, which can help to solve the problem efficiently. For example, there is a general technique for deriving a compact extended formulation from a polytime DP algorithm in Martin [14]. As a first step of the study, the main focus of this paper is to discover general network topology over which the concave minimization problem can be efficiently solved, with few assumptions on the concave function itself. We are particularly interested in grid networks, which appear frequently as the underlying network structure or sub-structure in numerous business and engineering problems, such as the integrated supply chain management problem with coordination between manufacturer, distribution centers and retailers (Kaminsky and Simchi-Levi [13] and van Hoesel et al. [12]), production planning in a vertical production line (Pochet and Wolsey [15]), and remanufacturing and recycling problems in reverse logistics (Dekker et al. [5]), where the temporal dynamics of the system are modeled in one dimension of the grid network and the sequential actions in space are modeled in the other. The following three examples show how CFG generalizes other models.

**The ULS model.** The classical ULS is to determine an optimal production schedule given a sequence of deterministic non-stationary demand  $d_1, \dots, d_T$  over  $T$  time periods with fixed setup cost  $\alpha_t$ , unit production cost  $\beta_t$  and inventory holding cost  $i_t$  at period  $t \in \{1, \dots, T\}$ .

The ULS is a special case of CFG with  $L = 1$ ,  $b(v_{0,1}) = \sum_{t=1}^T d_t$ ,  $b(v_{1,t}) = -d_t$  for  $t \in \{1, \dots, T\}$ ,  $b(v) = 0$  for any other node  $v \in V$ , and the cost function

$$c_a(x_a) = \begin{cases} \alpha_t \mathbb{I}(\{x_a > 0\}) + \beta_t x_a, & a = (v_{0,t}, v_{1,t}), t \in \{1, \dots, T\}, \\ i_t x_a, & a = (v_{1,t}, v_{1,t+1}), t \in \{1, \dots, T-1\}, \\ 0, & a = (v_{0,t}, v_{0,t+1}), t \in \{1, \dots, T-1\}, \end{cases} \quad (2)$$

where  $\mathbb{I}(\{x_a > 0\})$  is an indicator function whose value is 1 if  $x_a > 0$  and 0 otherwise. Variants of ULS, such as assuming the production and holding costs to be general concave functions, the presence of multiple echelons and intermediate demands, can be also formulated as CFG. ULS with backlogging can also be formulated as CFG by adding additional backward arcs into the grid network.

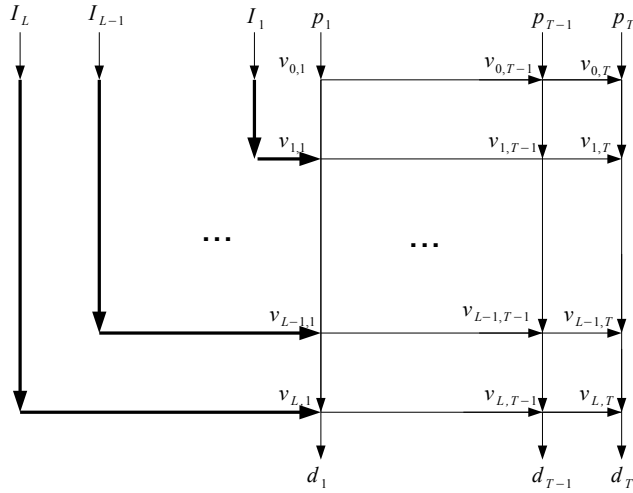
**The pure remanufacturing model.** Over  $T$  periods, a company recycles used products with quantities  $p_1, \dots, p_T$  and remanufactures them into new products to satisfy the demands  $d_1, \dots, d_T$ . The pure remanufacturing problem is to find an optimal production schedule to minimize the total production and holding costs, given the remanufacturing cost function  $\alpha_t$ , the holding cost functions  $i_{0,t}$  and  $i_{1,t}$  for used and new products, respectively. The model appears as a basic model in reverse logistics (Dekker et al. [5]). Although in van den Heuvel et al. [11] the model is shown to be equivalent to the ULS with inventory upper bounds when  $i_{0,t}$  and  $i_{1,t}$  are both linear functions, the transformation is not applicable when  $i_{0,t}$  and  $i_{1,t}$  are general concave functions. In fact, the model is more general since ULS can be seen as a special case of this model by letting  $p_1 = \sum_{i=1}^T d_i$  and  $p_t = 0$  for  $2 \leq t \leq T$ . The pure remanufacturing model is a special case of CFG with  $L = 1$ ,  $b(v_{0,t}) = p_t$  and  $b(v_{1,t}) = -d_t$  for  $t \in \{1, \dots, T\}$ , and the cost function

$$c_a(x_a) = \begin{cases} \alpha_t(x_a), & a = (v_{0,t}, v_{1,t}), t \in \{1, \dots, T\}, \\ i_{0,t}(x_a), & a = (v_{0,t}, v_{0,t+1}), t \in \{1, \dots, T-1\}, \\ i_{1,t}(x_a), & a = (v_{1,t}, v_{1,t+1}), t \in \{1, \dots, T-1\}. \end{cases} \quad (3)$$

To the best of our knowledge, the complexity of this problem was not known when there are multiple echelons with intermediate demands and general concave cost functions. In this paper, we show that the problem is polynomially solvable when there is one echelon of intermediate demands.

**Production planning in a rolling horizon model.** In a rolling horizon model for production planning where decisions are made periodically within a given time horizon, the presence of initial inventory is inevitable. However, the traditional setting for the lot-sizing problems assumes that the initial inventory is 0 at each echelon. The purpose of this critical assumption is more theoretical than practical, since it makes the underlying network a single-source network and greatly simplifies the analysis. When there are fixed initial inventory at each echelon in a multi-echelon setting, the analysis of the structure of the optimal solutions becomes cumbersome and difficult, as shown in van Hoesel et al. [12]. In fact, the rolling horizon model with initial inventory can be easily dealt with by transforming it to an equivalent CFG with  $L - 1$  additional time periods. Suppose that in the rolling horizon model the initial inventory at echelon  $l$  is  $I_l$  for  $1 \leq l \leq L$ . As shown in Figure 2, by attaching  $L - 1$  time periods before period 1, setting the supplies of the new sources at echelon 0 to be  $I_L, I_{L-1}, \dots, I_1$  and the cost to be large enough for each bold arc and 0 otherwise, the rolling horizon model with initial inventory is transformed into an equivalent CFG.

We end the introduction by reviewing MCCNFPs that can be solved in polynomial time. Most of them fall into the category of lot-sizing problems. The classical ULS model was first solved by Wagner and Whitin [19] in  $O(T^2)$  time with DP, and the complexity was later improved to  $O(T \ln T)$  in Aggarwal and Park [1],



**Fig. 2** An equivalent CFG for the rolling horizon model with initial inventory

Federgruen and Tzur [7] and Wagelmans et al. [18]. Zangwill [21] gave a DP algorithm for the multi-echelon ULS with demands in the last echelon, and his algorithm was later shown to run in  $O(LT^4)$  time for the  $L$ -echelon case in van den Heuvel et al. [12]. Other polynomial solvable variants include the constant capacitated lot-sizing problem (Florian and Klein [8]), ULS with backlogging, ULS with inventory upper bounds (Atamtürk and Küçükyavuz [4,3]), a multi-echelon ULS with constant production capacities at the first echelon (van den Heuvel et al. [12]), and recently a two-echelon ULS with intermediate demands (Zhang et al. [22]). Pochet and Wolsey [15] provides a detailed study of lot-sizing models that can be solved in polytime. Besides the lot-sizing problem, polynomially solvable cases include a single-source concave network flow problem with a single nonlinear arc cost (Guise-wite and Pardalos [10]), the network flow problem with a fixed number of sources and nonlinear arc costs (Tuy et al. [16]), and a production-transportation network flow problem where the concave cost function is defined on only a constant number of variables (Tuy et al. [17]).

The rest of the paper is organized as follows. In Section 2, we provide the DP framework for CFG, and introduce the terminology and notation used later in our analysis. In Section 3, we illustrate the idea of characterizing the extreme points for a simple case of CFG, where all sinks are at the last echelon. Then Section 4 deals with CFG with two echelons of sinks and Section 5 deals with CFG with one source and  $L$  echelons of sinks. We conclude in Section 6 by discussing some related open problems.

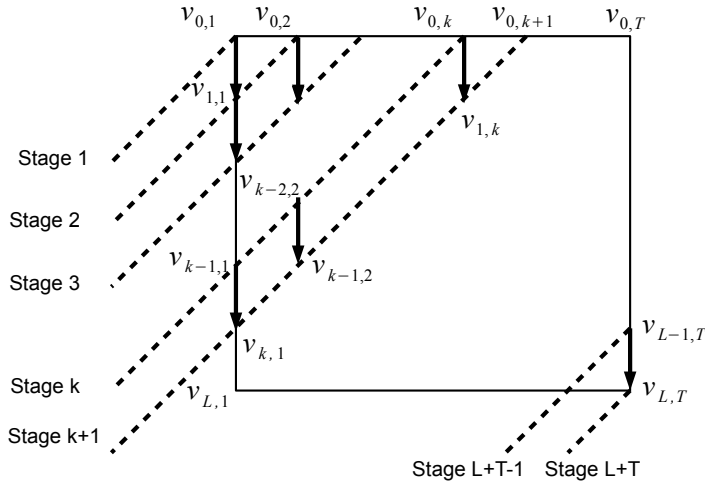
## 2 The solution approach.

In the rest of the paper, we assume that the number of echelons  $L$  is a constant and  $L \ll T$ , which is reasonable in practice since  $L$  is usually known in advance and we

are more interested in the complexity of CFG with respect to the number of time periods  $T$ . We propose to solve CFG by using a discrete time DP. The difficulty is that the state space at each stage of a natural DP formulation is an uncountable set. However, by analyzing the structure of the extreme points of the feasible set of (1), we are able to refine the state space to a set of size polynomial in  $T$ . For ease of exposition, we call the arcs of the form  $(v_{l,t}, v_{l,t+1})$  the *forward* arcs and the arcs of the form  $(v_{l,t}, v_{l+1,t})$  the *downward* arcs. To be consistent with the index of echelon in the lot-sizing model, we assume in our model that the first echelon is echelon 0. The elements of the DP are as follows.

1. Decision stages. There are  $L + T$  stages, and the nodes with  $l + t = k$  are at stage  $k$ , as shown in Figure 3.
2. States. Define the state  $\mathbf{s}^k$  at stage  $k$  to be a vector whose component  $s_i^k$  denotes the amounts of inflow into some node  $v_{l,t}$  with  $l + t = k$ . For example,  $\mathbf{s}^1$  is a scalar that denotes the inflow into node  $v_{0,1}$  and  $\mathbf{s}^2$  is a two-dimensional vector whose first component is the inflow into node  $v_{1,1}$  and second component is the inflow into node  $v_{0,2}$ . In general, when  $k \leq L + 1$ ,  $\mathbf{s}^k$  is a  $k$ -dimensional vector whose components are the amounts of inflow into nodes  $v_{k-1,1}, v_{k-2,2}, \dots, v_{0,k}$ ; when  $L + 1 \leq k \leq T$ ,  $\mathbf{s}^k$  is a  $(L + 1)$ -dimensional vector whose components are the amount of inflow into nodes  $v_{L,k-L}, v_{L-1,k-L+1}, \dots, v_{0,k}$ ; when  $k \geq T + 1$ ,  $\mathbf{s}^k$  is a  $(L + T + 1 - k)$ -dimensional vector whose components are the amounts of inflow into nodes  $v_{L,k-L}, \dots, v_{k-T,T}$ .
3. Decision variables (or actions). Let the decision variable  $\mathbf{u}^k$  at stage  $k$  be a vector whose components are the amount of flow sent out by nodes at stage  $k$  through *downward* arcs. For example,  $\mathbf{u}^1$  is a scalar which denotes the flow sent along the arc  $(v_{0,1}, v_{1,1})$  and  $\mathbf{u}^2$  is a two-dimensional vector whose first component is the flow on arc  $(v_{1,1}, v_{2,1})$  and second component is the flow on arc  $(v_{0,2}, v_{1,2})$ . In general, when  $k \leq L + 1$ ,  $\mathbf{u}^k$  is a  $k$ -dimensional vector whose components are flows on arcs  $(v_{k-1,1}, v_{k,1}), (v_{k-2,2}, v_{k-1,2}), \dots, (v_{0,k}, v_{1,k})$ ; when  $L + 1 \leq k \leq T$ ,  $\mathbf{u}^k$  is a  $L$ -dimensional vector whose components are flows on arcs  $(v_{L-1,k+1-L}, v_{L,k+1-L}), \dots, (v_{0,k}, v_{1,k})$ ; when  $T + 1 \leq k \leq L + T - 1$ ,  $\mathbf{u}^k$  is a  $(L + T - k)$ -dimensional vector whose components are flows on arcs  $(v_{L-1,k+1-L}, v_{L,k+1-L}), \dots, (v_{k-T,T}, v_{k+1-T,T})$ .
4. The system equations. The state  $\mathbf{s}^{k+1}$  at stage  $k + 1$  can be easily calculated by the flow balance constraints once the state  $\mathbf{s}^k$  and the decision variable  $\mathbf{u}^k$  are known. Let the system equations be  $\mathbf{s}^{k+1} = H_k(\mathbf{s}^k, \mathbf{u}^k)$ , where  $H_k$  is the affine function representing the flow balance constraints for nodes at stage  $k + 1$ .
5. The cost function. The cost at stage  $k$  is the sum of all costs incurred by the downward arcs and forward arcs connecting nodes at stage  $k$  and nodes at stage  $k + 1$ , so it is a function of  $\mathbf{u}^k$  and  $\mathbf{s}^k$ . We use the function  $r_k(\mathbf{s}^k, \mathbf{u}^k)$  to denote the cost incurred at stage  $k$ .

Then CFG is formulated as a discrete time DP problem with the linear system  $\mathbf{s}^{k+1} = H_k(\mathbf{s}^k, \mathbf{u}^k)$  and cost function  $r_k$  over  $L + T$  stages. This DP formulation is difficult to solve directly since the state space at stage  $k$  is an uncountable set in general. However, by (1) CFG is equivalent to minimizing a concave function over the flow polyhedron  $P_F := \{x \in \mathbb{R}^{|A|} \mid x \text{ satisfies constraints in (1)}\}$ . It is well known that there exists an optimal solution which is an extreme point of  $P_F$  if  $P_F$  is not empty.



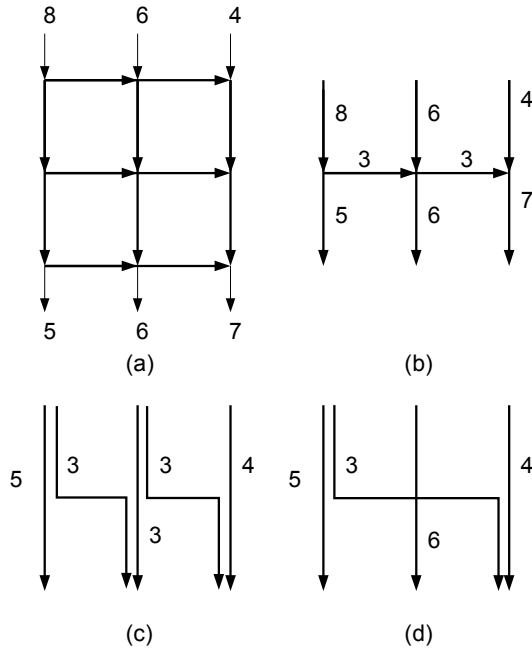
**Fig. 3** The dynamic programming formulation of CFG

Therefore in the DP formulation, it suffices to consider those states corresponding to the extreme points of  $P_F$ , the number of which is finite. To argue that this DP formulation can be solved in polynomial time, it remains to show that the cardinality of the state space at each stage is polynomial in  $T$ . Since the dimension of the state vector at each stage is at most  $L + 1$ , the task is reduced to show that each component of the state vector, namely the inflow into each node under all extreme points of  $P_F$ , can take on a finite set of values whose cardinality is polynomial in  $T$ .

Before proceeding to characterize the inflow under all extreme points, we introduce some terminology and notation which will be used throughout the paper. Let  $G = (V, A)$  be a directed graph or digraph. A *path* in  $G$  is an alternating sequence of distinct nodes and arcs  $\{v_1, a_1, v_2, a_2, \dots, v_l\}$  with  $a_i = (v_{i-1}, v_i)$  for  $1 \leq i \leq l-1$ . A *cycle* is a path  $\{v_1, a_1, v_2, a_2, \dots, v_l\}$  together with the arc  $(v_l, v_1)$ . The concepts of path and cycle in an undirected graph are similar to their directed versions except without specifying arc directions. We will use the same term “path” or “cycle” to refer to the object in a directed or undirected graph when the context is clear. The induced subgraph of  $G$  by the arc set  $A' \subseteq A$  is the subgraph  $G' = (V', A')$  where  $V'$  consists of nodes incident to any arc in  $A'$ . A vector  $\mathbf{f} \in \mathbb{R}^{|A|}$  is called a *flow* in  $G$  if  $\mathbf{f}$  satisfies the constraints in (1). A flow  $\mathbf{f}$  is called an *extreme flow* if it is an extreme point of the underlying flow polyhedron  $P_F$ . For any flow  $\mathbf{f}$ , let  $A_f = \{a \in A | f_a > 0\}$  be the set of arcs with nonzero flow. Let  $G_f$  denote the subgraph of  $G$  induced by the arc set  $A_f$ . The *underlying undirected graph* of  $G_f$  is an undirected graph obtained by replacing all directed arcs of  $G_f$  with undirected edges.

We begin to characterize the extreme flows with the following proposition.

**Proposition 1** *Each extreme flow  $\mathbf{f}$  in  $G$  can be decomposed into flows along paths each of which starts at one source and ends at one sink. In such a decomposition, there is at most one path with positive flow between each source-sink pair.*



**Fig. 4** An example showing two flow decompositions for the same extreme flow.

*Proof* Every flow can be decomposed into flows along paths and cycles, where each path starts from a source and ends at a sink; see Ahuja et al. [2]. When  $f$  is an extreme flow, the underlying undirected graph of  $G_f$  does not contain any cycles, so in any flow decomposition there will be no cycle with positive flow and at most one path with positive flow between each source-sink pair.  $\square$

Proposition 1 provides an alternative way to calculate the inflow into a node under an extreme flow  $f$  rather than summing up the flows over incoming arcs of that node: first decompose  $f$  into flows along paths between source-sink pairs, and then the inflow into a node is the summation of flows along paths that contain that node under that decomposition. For example, part (a) of Figure 4 is a CFG with  $L = 2$  and  $T = 3$  and part (b) is one extreme flow. To calculate the inflow into the central node (which is 9), we can use either the flow decomposition in part (c) to obtain  $9 = 3 + 3 + 3$ , or the flow decomposition in part (d) to obtain  $9 = 3 + 6$ .

Note that the flow decomposition is not unique for the extreme flow  $f$ . Our remaining job is to choose a *particular* flow decomposition under which it is easy to argue that the inflow can only attain a polynomial number of values.

### 3 CFG with sources in echelon 0 and sinks in echelon $L$ .

We illustrate the idea of choosing the particular flow decomposition by a special case of CFG, where all sources are at echelon 0, all sinks are at echelon  $L$  and all



other nodes are transshipment nodes. We call this case CFG-1. As mentioned earlier, CFG-1 generalizes the multi-echelon ULS, and is a special case of planar graphs with sources and sinks lying in a fixed number of faces studied in Erickson et al. [6]. For CFG-1, the inflow into any node under all extreme flows has a nice closed-form formula which generalizes the result for the multi-echelon ULS in Zangwill [21].

We introduce some notation first. Let the supply at node  $v_{0,t}$  be  $p_t$  and the demand at node  $v_{L,t}$  be  $d_t$  for every  $t \in \{1, \dots, T\}$ . Let  $P_t = \sum_{i=1}^t p_i$  and  $D_t = \sum_{i=1}^t d_i$  be the cumulative supply and demand up to period  $t \in \{1, \dots, T\}$ , respectively. Without loss of generality, we can always assume that  $P_T = D_T$ . Let  $\Gamma = \cup_{t=1}^T \{D_t, P_t\} \cup \{0\}$ . Then  $\Gamma$  contains at most  $2T$  elements. The main technical result of this section is given below and we prove it later in the section.

**Theorem 1** *For CFG-1, the inflow into any node in  $G$  under all extreme flows is  $\gamma_2 - \gamma_1$ , where  $\gamma_1, \gamma_2 \in \Gamma$  with  $\gamma_2 \geq \gamma_1$ .*

*Remark 1* Consider the multi-echelon ULS where the supply  $p_1 = \sum_{i=1}^T d_i, p_2 = \dots = p_T = 0$ . The set  $\Gamma = \{D_1, D_2, \dots, D_T\}$ . By applying Theorem 1, the possible values for the inflow into any node under all extreme flows are  $D_{t_2} - D_{t_1} = \sum_{i=t_1+1}^{t_2} d_i$  with  $t_1 \leq t_2$ , a key result derived by Zangwill [21] in designing his DP algorithm for the multi-echelon ULS.

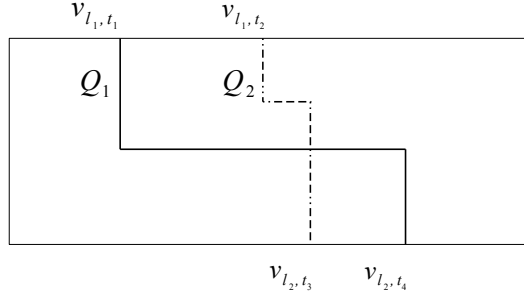
**Theorem 2** *For fixed  $L$ , CFG-1 can be solved in polynomial time in  $T$  and the number of queries of a function-value oracle.*

*Proof* By Theorem 1, the inflow into any node can attain  $O(T^2)$  values under all extreme flows in CFG-1. Then in the DP formulation for CFG-1, the cardinality of the state space at each stage is  $O(T^{2L})$ , so there are  $O(T^{4L})$  available actions at each stage. Since the DP has  $L + T$  stages, CFG-1 can be formulated as a shortest path problem over an acyclic graph with  $O(T^{4L+1})$  arcs, which can be solved in  $O(T^{4L+1})$  time; see Ahuja et al. [2].  $\square$

In the remainder of this section, we prove Theorem 1. As mentioned before, the idea is to choose a particular flow decomposition under which the inflow calculation is simple. By Proposition 1, there is at most one path with positive flow between any source-sink pair under any flow decomposition for a give extreme flow  $\mathbf{f}$ . Define  $\lambda_f(i, j)$  to be the amount of flow sent along the path between the source  $v_{0,i}$  and the sink  $v_{L,j}$  under some decomposition for  $\mathbf{f}$  (set  $\lambda_f(i, j) = 0$  if there is no path from  $v_{0,i}$  to  $v_{L,j}$  in  $G_f$ ), and consider the vector

$$\lambda_f = (\lambda_f(1, 1), \dots, \lambda_f(1, T), \lambda_f(2, 1), \dots, \lambda_f(2, T), \dots, \lambda_f(T, 1), \dots, \lambda_f(T, T)).$$

Then each flow decomposition for  $\mathbf{f}$  can be represented by a vector  $\lambda_f$ . For example, the flow decompositions in part (c) and (d) of Figure 4 can be represented by  $(5, 3, 0, 0, 3, 3, 0, 0, 4)$  and  $(5, 0, 3, 0, 6, 0, 0, 0, 4)$ , respectively. In the inflow calculation, we will choose the flow decomposition whose representation vector is the *lexicographically largest* among all flow decomposition vectors. We first give a formal definition of the lexicographical order between two vectors.



**Fig. 5**  $\mathcal{Q}_1$  and  $\mathcal{Q}_2$  must intersect.

**Definition 1** Given two vectors  $\mu, \nu \in \mathbb{R}^n$ ,  $\mu$  is lexicographically larger than  $\nu$ , denoted by  $\mu \succ \nu$ , if there exists  $i \in \{1, \dots, n\}$  such that  $\mu_j = \nu_j$  for  $j \leq i-1$  and  $\mu_i > \nu_i$ ;  $\mu$  is lexicographically no smaller than  $\nu$ , denoted by  $\mu \succeq \nu$ , if  $\mu \succ \nu$  or  $\mu = \nu$ ;  $\mu$  is lexicographically smaller than (no larger than)  $\nu$ , denoted by  $\mu \prec (\preceq) \nu$ , if  $-\mu \succ (\succeq) -\nu$ .

Let  $\pi_f$  be the lexicographically largest vector among all flow decomposition vectors for  $\mathbf{f}$ . For the example in Figure 4, the representation vector of the flow decomposition in part (c) is lexicographically largest for the extreme flow in part (b). We make the following simple observation, which will be useful to prove some nice properties of  $\pi_f$ .

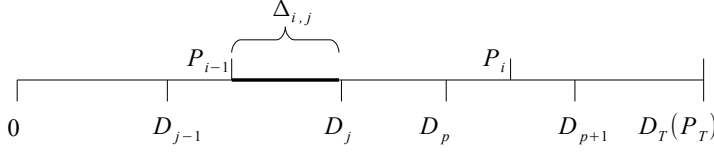
**Observation 1** As illustrated in Figure 5, given a grid network  $G$  and four nodes  $v_{l_1, t_1}$ ,  $v_{l_1, t_2}$ ,  $v_{l_2, t_3}$  and  $v_{l_2, t_4}$  in  $G$  with  $l_1 < l_2$ ,  $t_1 < t_2$  and  $t_3 < t_4$ , let  $\mathcal{Q}_1$  be any path from  $v_{l_1, t_1}$  to  $v_{l_2, t_4}$  and  $\mathcal{Q}_2$  be any path from  $v_{l_1, t_2}$  to  $v_{l_2, t_3}$ . Then  $\mathcal{Q}_1$  and  $\mathcal{Q}_2$  must intersect.

Now we begin to characterize the flow decomposition  $\pi_f$ .

### Proposition 2

1. For any  $i_1 < i_2$  and  $j_1 < j_2$ ,  $\pi_f(i_1, j_2) \cdot \pi_f(i_2, j_1) = 0$ .
2. If  $\pi_f(i_1, j_1) > 0$  and  $\pi_f(i_1, j_2) > 0$  with  $j_1 < j_2 - 1$ , then  $\pi_f(i_1, j) = d_j$  for any  $j \in \{j_1 + 1, \dots, j_2 - 1\}$ .
3. If  $\pi_f(i_1, j_1) > 0$  and  $\pi_f(i_2, j_1) > 0$  with  $i_1 < i_2 - 1$ , then  $\pi_f(i, j_1) = p_i$  for any  $i \in \{i_1 + 1, \dots, i_2 - 1\}$ .

*Proof* 1. Suppose that there exist  $i_1 < i_2$  and  $j_1 < j_2$  such that  $\pi_f(i_1, j_2) \cdot \pi_f(i_2, j_1) > 0$ . By Observation 1, the path from  $v_{0, i_1}$  to  $v_{L, j_2}$  must intersect with the path from  $v_{0, i_2}$  to  $v_{L, j_1}$ . If  $\pi_f(i_1, j_2) \geq \pi_f(i_2, j_1)$ , create a vector  $\tilde{\pi}_f$  in the following way:  $\tilde{\pi}_f(i_1, j_1) = \pi_f(i_1, j_1) + \pi_f(i_2, j_1)$ ,  $\tilde{\pi}_f(i_1, j_2) = \pi_f(i_1, j_2) - \pi_f(i_2, j_1)$ ,  $\tilde{\pi}_f(i_2, j_1) = 0$ ,  $\tilde{\pi}_f(i_2, j_2) = \pi_f(i_2, j_2) + \pi_f(i_2, j_1)$  and  $\tilde{\pi}_f(i, j) = \pi_f(i, j)$  for other  $(i, j)$  pairs. The vector  $\tilde{\pi}_f$  represents another flow decomposition of  $\mathbf{f}$  with  $\tilde{\pi}_f \succ \pi_f$ , a contradiction to the fact that  $\pi_f$  is the lexicographically largest flow decomposition vector. Similarly there is a contradiction when  $\pi_f(i_2, j_1) \geq \pi_f(i_1, j_2)$ .



**Fig. 6** The supply-demand partition.

2. Since  $\pi_f(i_1, j_1) > 0$ , by statement 1  $\pi_f(i, j) = 0$  for any  $i < i_1$  and  $j > j_1$ . Since  $\pi_f(i_1, j_2) > 0$ , by statement 1  $\pi_f(i, j) = 0$  for any  $i > i_1$  and  $j < j_2$ . Thus  $\pi_f(i, j) = 0$  for each  $j \in \{j_1 + 1, \dots, j_2 - 1\}$  and  $i \neq i_1$ . Then  $\pi_f(i_1, j) = d_j$  for any  $j \in \{j_1 + 1, \dots, j_2 - 1\}$  by the flow balance constraints.
3. Follows from a similar argument as in the proof of statement 2.  $\square$

Proposition 2 shows that under this particular flow decomposition  $\pi_f$ , supply at each period is decomposed to satisfy demand from consecutive periods (statement 2), demand at each period is decomposed to be fulfilled by supply from consecutive periods (statement 3), and demand at an early period is always served as much as possible by supply at an early period (follows from statement 1).

In fact, the value of  $\pi_f$  can be computed exactly. As shown in Figure 6, put all the cumulative demand and supply points on the real line. Let  $E_{i,j} = [P_{i-1}, P_i] \cap [D_{j-1}, D_j]$  for  $i, j \in \{1, \dots, T\}$  and  $\Delta_{i,j} = |E_{i,j}|$  denote the length of the interval  $E_{i,j}$ .

**Proposition 3** *The vector  $\pi_f$  is fixed for any extreme flow  $\mathbf{f}$ , and  $\pi_f(i, j) = \Delta_{i,j}$  for  $1 \leq i, j \leq T$ .*

*Proof* The proof is based on induction on the pair  $(i, j)$ . For the base case where  $(i, j) = (1, 1)$ ,  $\pi_f(1, 1) = d_1$  since the sink  $v_{L,1}$  can only be reached from the source  $v_{0,1}$ . Meanwhile,  $P_1 \geq D_1$  if CFG-1 is feasible. Then  $\Delta_{1,1} = d_1$  by the definition of  $\Delta_{1,1}$ . We have  $\pi_f(1, 1) = \Delta_{1,1}$ .

Now suppose that  $\pi_f(i, j) = \Delta_{i,j}$  holds for all pairs  $(i, j) \preceq (i_1, j_1)$ . If  $j_1 = T$ , the next lexicographically larger pair is  $(i_1 + 1, 1)$ . Since there is no path from  $v_{0,i_1+1}$  to  $v_{L,1}$ ,  $\pi_f(i_1 + 1, 1) = 0$ . Since  $P_{i_1} \geq P_1 \geq D_1$ ,  $\Delta_{i_1+1,1} = |[P_{i_1}, P_{i_1+1}] \cap [0, D_1]| = 0$ . Then  $\pi_f(i_1 + 1, 1) = \Delta_{i_1+1,1}$ . If  $j_1 < T$ , the next lexicographically larger pair is  $(i_1, j_1 + 1)$ . We will show that  $\pi_f(i_1, j_1 + 1) = \Delta_{i_1, j_1+1}$  in four different cases. Without loss of generality, we assume that the supply  $p_t > 0$  and demand  $d_t > 0$  for  $1 \leq t \leq T$ . Recall that  $\Delta_{i,j} = 0$  implies that either  $P_{i-1} \geq D_j$  or  $D_{j-1} \geq P_i$  and  $\Delta_{i,j} > 0$  implies that  $P_{i-1} < D_j$  or  $D_{j-1} < P_i$ .

1. If  $\Delta_{i_1, j_1} > 0$  and  $\Delta_{i_1, j_1+1} = 0$ , by the definition of  $\Delta_{i,j}$  we have  $D_{j_1-1} < P_{i_1} \leq D_{j_1}$ . Then

$$p_{i_1} = |[P_{i_1-1}, P_{i_1}]| = |[P_{i_1-1}, P_{i_1}] \cap \bigcup_{j=1}^{j_1} [D_{j-1}, D_j]| = \sum_{j=1}^{j_1} \Delta_{i_1, j} = \sum_{j=1}^{j_1} \pi_f(i_1, j).$$

The third equality follows from the definition of  $\Delta_{i,j}$  and the last equality follows from the induction hypothesis. By the flow balance constraint at the node  $v_{0,i_1}$ , we have  $\pi_f(i_1, j_1 + 1) = 0$ . Then  $\pi_f(i_1, j_1 + 1) = \Delta_{i_1, j_1+1}$ .

2. If  $\Delta_{i_1, j_1} > 0$  and  $\Delta_{i_1, j_1+1} > 0$ , by the definition of  $\Delta_{i, j}$  we have  $P_{i_1-1} < D_{j_1} < P_{i_1}$ .  
 (a) If  $P_{i_1} > D_{j_1+1}$ , then  $\Delta_{i_1, j_1+1} = d_{j_1+1}$ . Meanwhile,

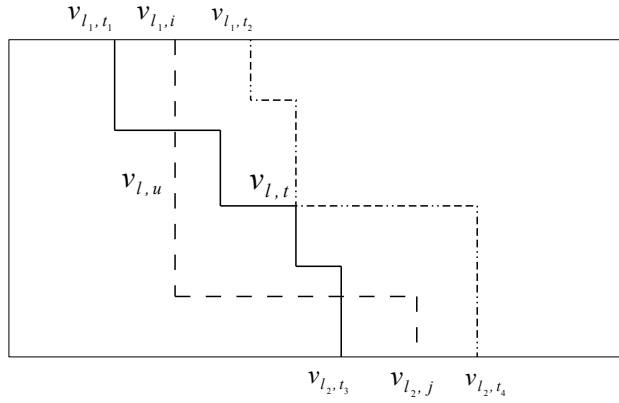
$$\sum_{j=j_1+1}^T \pi_f(i_1, j) = p_{i_1} - \sum_{j=1}^{j_1} \pi_f(i_1, j) = p_{i_1} - \sum_{j=1}^{j_1} \Delta_{i_1, j} > \Delta_{i_1, j_1+1} = d_{j_1+1}. \quad (4)$$

The first equality is the flow balance constraint at the node  $v_{0, i_1}$  and the inequality follows from the assumption that  $P_{i_1} > D_{j_1+1}$ . Since  $\pi_f(i_1, j_1+1) \leq d_{j_1+1}$ , by (4) there must exist  $j > j_1+1$  such that  $\pi_f(i_1, j) > 0$ . By applying statement 2 in Proposition 2 with  $\pi_f(i_1, j_1) > 0$ , we have  $\pi_f(i_1, j_1+1) = d_{j_1+1}$ . Then  $\pi_f(i_1, j_1+1) = \Delta_{i_1, j_1+1}$ .

- (b) If  $P_{i_1} \leq D_{j_1+1}$ ,  $p_{i_1} = \sum_{j=1}^{j_1+1} \Delta_{i_1, j}$ . Then  $\sum_{j=j_1+1}^T \pi_f(i_1, j) = p_{i_1} - \sum_{j=1}^{j_1} \pi_f(i_1, j) = \Delta_{i_1, j_1+1}$ . If  $\pi_f(i_1, j_1+1) < \Delta_{i_1, j_1+1}$ , then  $\exists j > j_1+1$  such that  $\pi_f(i_1, j) > 0$ . By applying statement 2 in Proposition 2 with  $\pi_f(i_1, j_1) > 0$ ,  $\pi_f(i_1, j_1+1) = d_{j_1+1}$ . Then  $\pi_f(i_1, j_1+1) \geq \Delta_{i_1, j_1+1}$ , a contradiction.
3. If  $\Delta_{i_1, j_1} = 0$  and  $\Delta_{i_1, j_1+1} = 0$ , either  $P_{i_1-1} \geq D_{j_1+1}$  or  $D_{j_1-1} \geq P_{i_1}$ . If  $P_{i_1-1} \geq D_{j_1+1}$ ,  $d_{j_1+1} = \sum_{i=1}^{i_1-1} \Delta_{i, j_1+1} = \sum_{i=1}^{i_1-1} \pi_f(i, j_1+1)$ . Then  $\pi_f(i_1, j_1+1) = 0$  by the flow balance constraint at node  $v_{L, j_1+1}$ . If  $D_{j_1-1} \geq P_{i_1}$ , then  $p_{i_1} = \sum_{j=1}^{j_1-1} \Delta_{i_1, j} = \sum_{j=1}^{j_1-1} \pi_f(i_1, j)$ . Then  $\pi_f(i_1, j_1+1) = 0$  by the flow conservation constraint at node  $v_{0, i_1}$ .
4. If  $\Delta_{i_1, j_1} = 0$  and  $\Delta_{i_1, j_1+1} > 0$ , then  $D_{j_1} \leq P_{i_1-1} < D_{j_1+1}$ .  
 (a) If  $P_{i_1} > D_{j_1+1}$ ,  $d_{j_1+1} = \sum_{i=i_1}^i \Delta_{i, j_1+1} = \sum_{i=i_1}^{i_1-1} \pi_f(i, j_1+1) + \Delta_{i_1, j_1+1}$ . Since  $d_{j_1+1} = \sum_{i=1}^{i_1} \pi_f(i, j_1+1)$ ,  $\sum_{i=i_1} \pi_f(i, j_1+1) = \Delta_{i_1, j_1+1}$ . Suppose that  $\exists i > i_1$  such that  $\pi_f(i, j_1+1) > 0$ . Since  $P_{i_1} > D_{j_1+1}$ ,  $\exists j > j_1+1$  such that  $\pi_f(i_1, j) > 0$ . We find  $i > i_1$  and  $j > j_1+1$  such that  $\pi_f(i, j_1+1) \cdot \pi_f(i_1, j) > 0$ , a contradiction to statement 1 in Proposition 2. Then  $\pi_f(i, j_1+1) = 0$  for all  $i > i_1$  and  $\pi_f(i_1, j_1+1) = \Delta_{i_1, j_1+1}$ .  
 (b) If  $P_{i_1} \leq D_{j_1+1}$ , we have  $D_{j_1} \leq P_{i_1-1} \leq P_{i_1} \leq D_{j_1+1}$ . Then  $\Delta_{i_1, j_1+1} = p_{i_1}$  and  $\Delta_{i_1, j} = 0$  for each  $j \leq j_1$ , so  $\pi_f(i_1, j) = 0$  for any  $j \leq j_1$  by the induction hypothesis. Then  $p_{i_1} = \sum_{j=j_1+1}^T \pi_f(i_1, j)$ . If  $\pi_f(i_1, j_1+1) < \Delta_{i_1, j_1+1} = p_{i_1}$ , then  $\exists j > j_1+1$  such that  $\pi_f(i_1, j) > 0$  by the flow conservation constraint at node  $v_{0, i_1}$  and  $\exists i > i_1$  such that  $\pi_f(i, j_1+1) > 0$  by the flow conservation constraint at node  $v_{L, j_1+1}$ . We find  $i > i_1$  and  $j > j_1+1$  such that  $\pi_f(i, j_1+1) \cdot \pi_f(i_1, j) > 0$ , a contradiction to statement 1 in Proposition 2. Then  $\pi_f(i_1, j_1+1) = \Delta_{i_1, j_1+1}$ .  $\square$

Proposition 3 is somewhat surprising. It states that the amount of flow sent between each source-sink pair is a constant under this particular flow decomposition, no matter what the flow  $\mathbf{f}$  is. Thus we have an invariant quantity among all extreme flows, which is key to showing that the inflow can only attain a polynomial number of values in  $T$ . We present one more result before proving Theorem 1.

**Proposition 4** *Given any extreme flow  $\mathbf{f}$ , let  $\mathcal{Q}_1$  be a path from  $v_{l_1, t_1}$  to  $v_{l_2, t_3}$  and  $\mathcal{Q}_2$  be a path from  $v_{l_1, t_2}$  to  $v_{l_2, t_4}$  in  $G_f$  with  $l_1 < l_2, t_1 \leq t_2$  and  $t_3 \leq t_4$ . If  $\mathcal{Q}_1$  and  $\mathcal{Q}_2$  both contain  $v_{l, t}$ , then any path from  $v_{l_1, i}$  to  $v_{l_2, j}$  in  $G_f$  with  $t_1 \leq i \leq t_2$  and  $t_3 \leq j \leq t_4$  also contains the node  $v_{l, t}$ .*



**Fig. 7** The cycle created in  $G_f$  if the path from  $v_{l_1, i}$  to  $v_{l_2, j}$  bypasses the node  $v_{l_1, t}$ .

*Proof* Proof by contradiction. As shown in Figure 7, the node  $v_{l_1, t}$  is contained in the path from  $v_{l_1, t_1}$  to  $v_{l_2, t_3}$  and the path from  $v_{l_1, t_2}$  to  $v_{l_2, t_4}$  in  $G_f$ . Suppose that there exists some pair  $(i, j)$  with  $t_1 \leq i \leq t_2$  and  $t_3 \leq j \leq t_4$  such that the path from  $v_{l_1, i}$  to  $v_{l_2, j}$  bypasses the node  $v_{l_1, t}$ . Then the path must contain some node  $v_{l_1, u}$  with either  $u < t$  or  $u > t$ . If  $u < t$ , by Observation 1, the path from  $v_{l_1, t_1}$  to  $v_{l_1, t}$  must intersect with the path from  $v_{l_1, i}$  to  $v_{l_1, u}$ , and the path from  $v_{l_1, t}$  to  $v_{l_2, t_3}$  must intersect with the path  $v_{l_1, u}$  to  $v_{l_2, j}$  in  $G_f$ . The two intersections create a cycle in the underlying undirected graph of  $G_f$ , a contradiction. The argument is essentially the same if  $u > t$ .  $\square$

*Proof (Proof of Theorem 1.)* Given an extreme flow  $\mathbf{f}$ , the inflow to node  $v_{l_1, t}$  can be calculated under the flow decomposition  $\pi_f$  as a summation of flows along paths that contain  $v_{l_1, t}$ . Let  $(i_1, j_1)$  and  $(i_2, j_2)$  be the lexicographically smallest and largest pairs  $(i, j)$  such that  $\pi_f(i, j) > 0$  and the path from  $v_{0, i}$  to  $v_{L, j}$  in  $G_f$  contain  $v_{l_1, t}$ . Since  $i_1 \leq i_2$  and  $\pi_f(i_1, j_1), \pi_f(i_2, j_2) > 0$ , by statement 1 of Proposition 2 we have  $j_1 \leq j_2$ . By applying Proposition 4 with  $l_1 = 0$  and  $l_2 = L$ , any path from  $v_{0, i}$  to  $v_{L, j}$  in  $G_f$  with  $i_1 \leq i \leq i_2$  and  $j_1 \leq j \leq j_2$  will also contain  $v_{l_1, t}$ . In addition, since  $\pi_f(i_1, j_1) > 0$  and  $\pi_f(i_2, j_2) > 0$ ,  $\pi_f(i, j) = 0$  for any  $(i, j)$  pair such that  $i > i_1, j < j_1$  or  $i < i_2, j > j_2$ . Therefore,

$$\begin{aligned} \text{Inflow into } v_{l_1, t} &= \sum_{i=i_1}^{i_2} \sum_{j=j_1}^{j_2} \pi_f(i, j) = \sum_{i=i_1}^{i_2} \sum_{j=j_1}^{j_2} \Delta_{i, j} \\ &= \sum_{(i, j) \preceq (i_2, j_2)} \Delta_{i, j} - \sum_{(i, j) \prec (i_1, j_1)} \Delta_{i, j} \\ &= \min\{P_{i_2}, D_{j_2}\} - \max\{P_{i_1-1}, D_{j_1-1}\} \\ &= \gamma_2 - \gamma_1, \end{aligned}$$

where  $\gamma_1, \gamma_2 \in \Gamma$  and  $\gamma_2 \geq \gamma_1$ . The penultimate equality follows from the definition of  $\Delta_{i, j}$ .  $\square$

#### 4 Sources in echelon 0 and two echelons of sinks.

We consider the case of CFG where there are  $T$  sources at echelon 0 and two echelons of sinks, which we call CFG-2. CFG-2 generalizes the two-echelon ULS with

intermediate demands in Zhang et al. [22] and multi-echelon pure remanufacturing problems with intermediate demands. It is significantly harder to prove the polynomiality of the inflow values in CFG-2 than that of CFG-1, since there is no such invariant quantity  $\pi_f$  as in CFG-1. Our strategy is for each extreme flow in CFG-2 to calculate the inflow under a flow decomposition that satisfies certain properties similar to the properties of  $\pi_f$  in Proposition 2. In this way we are able to show that the inflow into any node can attain only a polynomial number of values under all extreme flows in CFG-2.

To simplify the analysis, we assume that in CFG-2 the sinks are at the last two echelons, echelon  $L-1$  and echelon  $L$ . When sinks are at other two echelons, the flow decomposition needs to be adjusted accordingly, but the inflow calculation and the complexity result are not affected. Let the supply at source  $v_{0,t}$  be  $p_t$  and the demand at sink  $v_{l,t}$  be  $d_{l,t}$  for  $t \in \{1, \dots, T\}$  and  $l \in \{L-1, L\}$ . Let  $P_t = \sum_{i=1}^t p_i$  be the cumulative supply up to period  $t$  as in CFG-1 and  $D_{l,t} = \sum_{i=1}^t d_{l,i}$  be the cumulative demand up to period  $t$  at echelon  $l$  for  $l = L-1, L$ . As in CFG-1, each flow decomposition for an extreme flow  $\mathbf{f}$  can be represented by a vector whose components are the amount of flow sent along the paths between source-sink pairs. The difference is that we need *three* indices instead of two for a source-sink pair in CFG-2. Let  $\mu_f(i, j, l)$  denote the amount of flow along the path from the source  $v_{0,i}$  to the sink  $v_{l,j}$  in  $G_f$  under some flow decomposition. Then each flow decomposition for  $\mathbf{f}$  can be represented by a vector

$$\mu_f = (\mu_f(1, 1, L), \mu_f(1, 1, L-1), \mu_f(1, 2, L), \mu_f(1, 2, L-1), \dots, \mu_f(T, T, L), \mu_f(T, T, L-1)).$$

Let  $\chi_f$  be the lexicographically largest vector among all flow decomposition vectors for the extreme flow  $\mathbf{f}$ . Then  $\chi_f$  satisfies some properties similar to those of  $\pi_f$  in CFG-1.

**Proposition 5**

1. For any  $i_1 < i_2$ ,  $j_1 < j_2$  and  $l \in \{L-1, L\}$ ,  $\chi_f(i_1, j_2, l) \cdot \chi_f(i_2, j_1, l) = 0$ .
2. If  $\chi_f(i_1, j_1, l) > 0$  and  $\chi_f(i_1, j_2, l) > 0$  with  $j_1 < j_2 - 1$  and  $l \in \{L-1, L\}$ , then  $\chi_f(i_1, j, l) = d_{l,j}$  for any  $j \in \{j_1 + 1, \dots, j_2 - 1\}$ .

*Proof* Similar to the proof of Proposition 2. □

Note that there is no similar result in CFG-2 to the statement 3 of Proposition 2. Given a sink  $v_{L,j_1}$ , even if  $\chi_f(i_1, j_1, L), \chi_f(i_2, j_1, L) > 0$  with  $i_1 < i_2$ ,  $\chi_f(i, j_1, L)$  can be 0 instead of  $p_i$  for  $i_1 < i < i_2$ , since the source  $v_{0,i}$  can satisfy demand only on echelon  $L-1$ . Let

$$\begin{aligned} X_L &= \cup_j \{D_{L,j}\} \cup \cup_{i,j} \{P_i - D_{L-1,j}\}, \\ X_{L-1} &= \cup_j \{D_{L-1,j}\} \cup \cup_{i,j} \{P_i - D_{L,j}\} \cup \cup_{i,j,k} \{P_i + D_{L-1,j} - P_k\}. \end{aligned} \quad (5)$$

**Proposition 6** For CFG-2, the inflow into any node under any extreme flow is

$$\sum_{m=L-1}^L (\gamma_{m,2} - \gamma_{m,1}),$$

where  $\gamma_{m,1}, \gamma_{m,2} \in X_m$  with  $\gamma_{m,2} \geq \gamma_{m,1}$  and  $m \in \{L-1, L\}$ .

*Proof* Given an extreme flow  $\mathbf{f}$ , the inflow into  $v_{l,t}$  can be calculated under the flow decomposition  $\chi_f$  as a summation of flows along paths that contain  $v_{l,t}$ . Let  $(i_1, j_1, L)$  and  $(i_2, j_2, L)$  be the lexicographically smallest and largest  $(i, j, L)$  tuples such that  $\chi_f(i, j, L) > 0$  and the path from source  $v_{0,i}$  to sink  $v_{L,j}$  contains  $v_{l,t}$  in  $G_f$ . Let  $(i_3, j_3, L-1)$  and  $(i_4, j_4, L-1)$  be the lexicographically smallest and largest  $(i, j, L-1)$  tuples such that  $\chi_f(i, j, L-1) > 0$  and the path from source  $v_{0,i}$  to sink  $v_{L-1,j}$  contains  $v_{l,t}$  in  $G_f$ . Then by an argument similar to the one used in the proof of Theorem 1,

$$\begin{aligned} \text{Inflow into } v_{l,t} &= \sum_{i=i_1}^{i_2} \sum_{j=j_1}^{j_2} \chi_f(i, j, L) + \sum_{i=i_3}^{i_4} \sum_{j=j_3}^{j_4} \chi_f(i, j, L-1) \\ &= \left[ \sum_{(i,j,L) \preceq (i_2, j_2, L)} \chi_f(i, j, L) - \sum_{(i,j,L) \prec (i_1, j_1, L)} \chi_f(i, j, L) \right] + \\ &\quad \left[ \sum_{(i,j,L-1) \preceq (i_4, j_4, L-1)} \chi_f(i, j, L-1) - \sum_{(i,j,L-1) \prec (i_3, j_3, L-1)} \chi_f(i, j, L-1) \right] \end{aligned}$$

It remains to show that the term  $\sum_{(i,j,m) \preceq (I,J,m)} \chi_f(i, j, m) \in X_m$  for any  $I, J$  and  $m = L-1, L$ . The proof is based on induction on  $I$ .

The base case  $I = 1$ . Since  $\sum_{(i,j,m) \preceq (1,J,m)} \chi_f(i, j, m) = \sum_{j=1}^J \chi_f(1, j, m)$  for  $m \in \{L-1, L\}$ , we have to show that  $\sum_{j=1}^J \chi_f(1, j, L-1) \in X_{L-1}$  and  $\sum_{j=1}^J \chi_f(1, j, L) \in X_L$  for any  $J$  under any extreme flow  $\mathbf{f}$ . First fix the extreme flow  $\mathbf{f}$ , let  $\alpha$  be the largest time index  $j$  such that  $\chi_f(1, j, L-1) > 0$  and  $\beta$  be the largest time index  $j$  such that  $\chi_f(1, j, L) > 0$ . Then  $\sum_{j=1}^J \chi_f(1, j, L-1) = \sum_{j=1}^J d_{L-1,j} = D_{L-1,J}$  for any  $J < \alpha$  and  $\sum_{j=1}^J \chi_f(1, j, L) = \sum_{j=1}^J d_{L,j} = D_{L,J}$  for any  $J < \beta$ , according to statement 2 in Proposition 5. It remains to show that  $\sum_{j=1}^\alpha \chi_f(1, j, L-1) \in X_{L-1}$  and  $\sum_{j=1}^\beta \chi_f(1, j, L) \in X_L$ . By the flow balance constraint at node  $v_{0,1}$ , we have

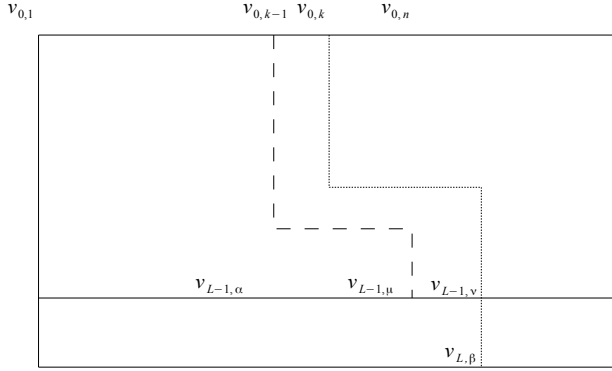
$$p_1 = \sum_{j=1}^{\alpha} \chi_f(1, j, L-1) + \sum_{j=1}^{\beta} \chi_f(1, j, L). \quad (6)$$

If either  $\sum_{j=1}^{\alpha} \chi_f(1, j, L-1) = D_{L-1,\alpha}$  or  $\sum_{j=1}^{\beta} \chi_f(1, j, L) = D_{L,\beta}$ , then we are done. Otherwise we must have  $\sum_{j=1}^{\alpha} \chi_f(1, j, L-1) < D_{L-1,\alpha}$  and  $\sum_{j=1}^{\beta} \chi_f(1, j, L) < D_{L,\beta}$ , implying that both the demand  $d_{L-1,\alpha}$  and demand  $d_{L,\beta}$  are only partially satisfied by the supply  $p_1$ . Let  $k$  be the smallest time index  $i > 1$  such that  $\chi_f(i, \beta, L) > 0$ . It implies that under the flow decomposition  $\chi_f$ , supply  $p_k$  is the first supply after  $p_1$  to satisfy the demand  $D_{L,\beta}$ , and sources  $v_{0,2}, \dots, v_{0,k-1}$  make no contribution to demands at echelon  $L$ , as shown in Figure 8.

**Claim** There exists some  $i_1 \in \{2, \dots, k-1\}$  and  $j_1 \in \{1, \dots, T\}$  such that

$$\sum_{(i,j,L-1) \preceq (i_1, T, L-1)} \chi_f(i, j, L-1) = D_{L-1, j_1}.$$

The claim indicates that under the flow decomposition  $\chi_f$ , supply  $p_1, \dots, p_{i_1}$  are decomposed to satisfy demand  $d_{L-1,1}, \dots, d_{L-1, j_1}$  on echelon  $L-1$ , demand  $d_{L,1}, \dots, d_{L, \beta-1}$



**Fig. 8** The case that  $d_{L-1,\alpha}$  and  $d_{L,\beta}$  are partially satisfied by  $p_1$ .

on echelon  $L$  and part of demand  $d_{L,\beta}$ . In addition,  $p_2, \dots, p_{i_1}$  only satisfy the demand on echelon  $L-1$ . Then

$$P_{i_1} = \sum_{(i,j,L-1) \preceq (i_1,T,L-1)} \chi_f(i,j,L-1) + \sum_{j=1}^{\beta} \chi_f(1,j,L). \quad (7)$$

By the claim and (7),  $\sum_{j=1}^{\beta} \chi_f(1,j,L) = P_{i_1} - D_{L-1,j_1} \in X_L$ . Then by (6),

$$\sum_{j=1}^{\alpha} \chi_f(1,j,L-1) = p_1 + D_{L-1,j_1} - P_{i_1} \in X_{L-1}.$$

The proof of the claim is based on contradiction. Suppose that the claim does not hold, then we show that there will be a cycle in the underlying undirected graph of  $G_f$ . If such  $i_1$  and  $j_1$  in the claim do not exist, then under the flow decomposition  $\chi_f$  each supply  $p_i$  ( $2 \leq i \leq k-1$ ) ends up only fulfilling part of some demand on echelon  $L-1$ , which implies that  $v_{0,i}$  is connected to  $v_{0,i+1}$  through some sink on echelon  $L-1$  since they both contribute to the demand at that sink. The source  $v_{0,1}$  is connected to  $v_{0,2}$  through the sink  $v_{L-1,\alpha}$  by the same reason. In addition,  $v_{0,k-1}$  is connected to  $v_{0,k}$  through some sink on echelon  $L-1$  as well due to the following reason. As shown in Figure 8, let  $\mu$  be the largest time index  $j$  such that  $\chi_f(k-1,\mu,L-1) > 0$ , which implies that  $v_{0,k-1}$  is connected to  $v_{L-1,\mu}$  in  $G_f$  and there exists  $n > k-1$  such that  $\chi_f(n,\mu,L-1) > 0$ . If  $n = k$ , then  $v_{0,k-1}$  and  $v_{0,k}$  are connected through the sink  $v_{L-1,\mu}$ . If  $n > k$ , then the path from  $v_{0,k}$  to  $v_{L,\beta}$  will intersect with either the path from  $v_{0,k-1}$  to  $v_{L-1,\mu}$  or the path from  $v_{0,n}$  to  $v_{L-1,\mu}$ . Either case contradicts statement 1 of Proposition 5. Now  $v_{0,1}$  is connected to  $v_{0,k}$  through two different undirected paths, one through nodes  $v_{0,2}, \dots, v_{0,k-1}$  and sinks at echelon  $L-1$ , the other through the sink  $v_{L,\beta}$  at echelon  $L$ . Thus there is a cycle containing  $v_{0,1}$  and  $v_{0,k}$  in the underlying undirected graph of  $G_f$ . We proved here the claim.

The induction step. Suppose that  $\sum_{(i,j,m) \preceq (I,J,m)} \chi_f(i,j,m) \in X_m$  under all extreme flows for any  $I \leq I_0$ . We want to show that  $\sum_{(i,j,m) \preceq (I_0+1,J,m)} \chi_f(i,j,m) \in X_m$  for each  $J$  and  $m \in \{L-1, L\}$ . The proof is similar as in the base case. Let  $\alpha$  be the largest



time index  $j$  such that  $\chi_f(I_0 + 1, i, L - 1) > 0$  and  $\beta$  be the largest time index  $j$  such that  $\chi_f(I_0 + 1, j, L) > 0$ .

We first show that  $\sum_{(i,j,L-1) \preceq (I_0+1, J, L-1)} \chi_f(i, j, L-1) \in X_{L-1}$  each  $J < \alpha$ .

1. If  $\alpha$  is the only time index  $j$  such that  $\chi_f(I_0 + 1, j, L - 1) > 0$ , which indicates that the supply  $p_{I_0+1}$  only contributes to demand  $d_{L-1, \alpha}$  on echelon  $L - 1$ , then for any  $J < \alpha$

$$\sum_{(i,j,L-1) \preceq (I_0+1, J, L-1)} \chi_f(i, j, L-1) = \sum_{(i,j,L-1) \preceq (I_0, T, L-1)} \chi_f(i, j, L-1) \in X_{L-1}.$$

2. If there are at least two  $j$ 's such that  $\chi_f(I_0 + 1, j, L - 1) > 0$ , then for each  $J < \alpha$ ,  $\sum_{(i,j,L-1) \preceq (I_0+1, J, L-1)} \chi_f(i, j, L-1)$  either equals  $\sum_{(i,j,L-1) \preceq (I_0, T, L-1)} \chi_f(i, j, L-1)$  or  $D_{L-1, j_1}$  for some  $j_1$  on echelon  $L - 1$ .

Similarly,  $\sum_{(i,j,L) \preceq (I_0+1, J, L)} \chi_f(i, j, L) \in X_L$  for each  $J < \beta$ .

Now it remains to show that the result holds for both

$$\sum_{(i,j,L-1) \preceq (I_0+1, \alpha, L-1)} \chi_f(i, j, L-1) \text{ and } \sum_{(i,j,L) \preceq (I_0+1, \beta, L)} \chi_f(i, j, L).$$

By the flow balance constraint,

$$P_{I_0+1} = \sum_{(i,j,L-1) \preceq (I_0+1, \alpha, L-1)} \chi_f(i, j, L-1) + \sum_{(i,j,L) \preceq (I_0+1, \beta, L)} \chi_f(i, j, L). \quad (8)$$

If either  $\sum_{(i,j,L-1) \preceq (I_0+1, \alpha, L-1)} \chi_f(i, j, L-1) = D_{L-1, \alpha}$  or  $\sum_{(i,j,L) \preceq (I_0+1, \beta, L)} \chi_f(i, j, L) = D_{L, \beta}$ , then we are done. Otherwise the demand  $d_{L-1, \alpha}$  and  $d_{L, \beta}$  is partially satisfied by the supply up to period  $I_0 + 1$ . By an argument similar to the one in the base case, we can show that  $\sum_{(i,j,L-1) \preceq (I_0+1, \alpha, L-1)} \chi_f(i, j, L-1) \in X_{L-1}$  and  $\sum_{(i,j,L) \preceq (I_0+1, \beta, L)} \chi_f(i, j, L) \in X_L$ .  $\square$

**Theorem 3** For fixed  $L$ , CFG-2 can be solved in polynomial time in  $T$  and the number of queries of a function-value oracle.

*Proof* By (5), the cardinality of  $X_{L-1}$  is  $O(T^3)$  and the cardinality of  $X_L$  is  $O(T^2)$ . By Proposition 6, the inflow into any node is a summation of the difference of two elements in  $X_{L-1}$  and the difference of two elements in  $X_L$ , so it can attain  $O(T^{3+3} \cdot T^{2+2}) = O(T^{10})$  values under all extreme flows in CFG-2. Then in the DP formulation for CFG-2, the cardinality of the state space at each stage is  $O(T^{10L})$ . Thus CFG-2 can be solved in polynomial time by solving a shortest path problem over an acyclic network.  $\square$

## 5 One source and $L$ echelons of sinks.

In this section we deal with another special case of CFG where there are only one source  $v_{0,1}$  at echelon 0 and  $L$  echelons of sinks at echelon 1 to  $L$ . This distribution of sources and sinks is exactly the case in multi-echelon ULS with intermediate demands. To our knowledge, the complexity of multi-echelon ULS with intermediate demands was not known except for  $L = 2$ . We will show that this problem is polynomially solvable for any constant  $L$ . The idea of the proof is also based on the inflow calculation by flow decomposition, except that the characterization here is much simpler in the one-source case than in CFG-1 and CFG-2.

Let the demand at echelon  $l$  time  $t$  be  $d_{l,t}$  for  $1 \leq l \leq L$  and  $1 \leq t \leq T$ , and the supply at  $v_{0,1}$  to be  $p_1 = \sum_{l=1}^L \sum_{t=1}^T d_{l,t}$ . Given an extreme flow  $\mathbf{f}$ , let  $\omega_f(i, j)$  be the flow sent along the path from  $v_{0,1}$  to  $v_{i,j}$  in  $G_f$ . Since there is only one source,  $\omega_f(i, j) = d_{i,j}$  for any  $i, j$  under any extreme flow  $\mathbf{f}$ . To calculate the inflow into the node  $v_{l,t}$ , let  $v_{m,i_1}$  and  $v_{m,i_2}$  be the two nodes on echelon  $m$  with  $i_1$  smallest and  $i_2$  largest among those nodes  $v_{m,i}$ 's such that the path from  $v_{0,1}$  to  $v_{m,i}$  contains  $v_{l,t}$  in  $G_f$ . Then

$$\text{Inflow into } v_{l,t} = \sum_{m=l}^L \sum_{i=i_1}^{i_2} \omega_f(m, i) = \sum_{m=l}^L \sum_{i=i_1}^{i_2} d_{m,i}, \text{ for } 1 \leq l \leq L,$$

$$\text{Inflow into } v_{l,t} = \sum_{m=1}^L \sum_{i=i_1}^{i_2} d_{m,i}, \text{ for } l = 0.$$

Thus the inflow into each node in  $G$  can attain only  $O(T^{2L})$  values under all extreme flows, which is polynomial for constant  $L$ , and the problem can be solved by DP in polynomial time.

## 6 Future work and open problems.

There are certainly more questions left open than we have answered in this paper. The critical one is whether CFG with  $T$  sources at echelon 0 and sinks in all echelons 1 to  $L$  is polynomially solvable for constant  $L$ . A positive answer to this question would unify all complexity results in this paper. Our conjecture is that it is the case. Another interesting direction is to see how one can generalize the inflow calculation with particular flow decomposition to argue the complexity of MCCNFP on capacitated networks. With arc capacity and multiple sources, the flow decomposition for extreme flows is more complicated. Finally, can we derive any extend formulation from the polynomial DP algorithm, for example, if we assume a fixed-charge cost function?

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