METRIC REGULARITY OF COMPOSITION SET-VALUED MAPPINGS: METRIC SETTING AND CODERIVATIVE CONDITIONS

by

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Abstract: The paper concerns a new method to obtain a direct proof of the openness at linear rate/metric regularity of composite set-valued maps on metric spaces by the unification and refinement of several methods developed somehow separately in several works of the authors. In fact, this work is a synthesis and a precise specialization to a general situation of some techniques explored in the last years in the literature. In turn, these techniques are based on several important concepts (like error bounds, lower semicontinuous envelope of a set-valued map, local composition stability of multifunctions) and allow us to obtain two new proofs of a recent result having deep roots in the topic of regularity of mappings. Moreover, we make clear the idea that it is possible to use (co)derivative conditions as tools of proof for openness results in very general situations.

Keywords: regularity of multifunction · set-valued composition · coderivative conditions

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1 Introduction

The property of metric regularity has its origins in the open mapping principle for linear operators obtained in the 1930s by Banach and Schauder, and is one of the three basic and crucial principles of functional analysis, having various applications in many branches of mathematics. Later on, this principle was reinterpreted and generalized in two classical results: the tangent space of Lyusternik ([43]) and the surjection theorem of Graves ([25]). The next decisive step in this history was the extension of the Banach-Schauder principle to the case of set-valued maps with closed and convex graph given independently by Ursescu in [62] and Robinson in [58] (the celebrated Robinson-Ursescu Theorem). Moreover, it was observed in Dmitruk, Milyutin, and Osmolovsky [12] that the original proof of Lyusternik from [43] is applicable to a much more general setting: the sum of a covering at a rate $a > 0$ single-valued mapping and a Lipschitz one with a Lipschitz constant $b < a$ is covering at the rate $a - b$. Another remarkable insight given in the mentioned paper is that it clearly emphasizes the metric nature of openness and regularity properties. Afterwards, in 1996, Ursescu

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[63] was the first to obtain a fully set-valued extension of the above results, in the setting of Banach spaces. On the tracks of [12], the important work of Ioffe [30] made the crucial observation that the Lyusternik iteration process can be successfully used when the income space is a complete metric space and the outcome space has a linear structure with shift-invariant metric. Detailed studies on the case of the sum of a metrically regular set-valued mapping and a single-valued Lipschitz map appear, more recently, in works by Dontchev, Lewis, and Rockafellar ([17]), Dontchev, and Lewis ([16]), Arutyunov ([1], [2]), Mordukhovich ([45]). For a detailed account for the whole topic of regularity properties of mappings, as well as various applications the reader is referred to the books or works of many researchers: [4], [5], [6], [7], [8], [9], [10], [11], [41], [17], [26], [30], [31], [38], [39], [33], [34], [35], [36], [37], [43], [45], [47], [48], [49], [50], [54], [57], [61], [59], [64].

In the last years, the study of openness at linear rate (or equivalently metric regularity) of multifunctions obtained as operations with set-valued maps has received a new impetus coming from at least three connected issues: the link between Lyusternik-Graves type theorems and fixed point assertions ([1], [14]), the growing interest to generalized forms of compositions ([31], [23]) and the new developments of metric regularity results obtained under assumptions based on generalized differentiation calculus and especially on coderivative conditions ([52], [20]).

Included in this stream, the present paper concerns a new method to obtain a direct proof of the openness at linear rate/metric regularity of composite set-valued maps on metric spaces by the unification and refinement of several methods developed somehow separately in several works of the authors: [49], [51], [52], [53], [21], [20], [23], [24]. In some sense, this is a synthesis and a precise specialization to a general situation of some techniques explored in the quoted papers. In turn these techniques are based on several important concepts (like error bounds, strong slope associated to a function, lower semicontinuous envelope of a set-valued map, local composition stability of multifunctions) and allow us to obtain two new proofs of a recent result having deep roots in the literature on the topic of regularity of mappings. Moreover, we make clear the idea that it is possible to use (co)derivative conditions as tool of proof for openness results in very general situations.

More precisely, the corner stones that this work rely on are mainly the results in [49] on the error bounds for a nonlinear variational system, the main result in [24] concerning the openness of a composite multifunction, and also the coderivative conditions for metric regularity as these appear in [52], [20].

The paper contains a main result (Theorem 3.6) which is prepared by several propositions having their own interest. In that main result one obtains, under some already standard (hence expected) assumptions (see [23], [24]), the openness of an auxilliary multifunction associated to a composition set-valued map, and on this basis, a result of openness around the reference point for the considered composition. We want to emphasize here two main points both of them revealing the novelty and the relevance of our work. Firstly, the conclusion is significantly richer that the corresponding conclusions of the main results in [53] (from point of view of the generality of set-valued operations), [23], [24] (from point of view of the type of openness). Secondly, the proof is obtained using Ekeland Variational Principle (EVP, for short), a fact that answers a question A. Ioffe raised in a discussion with the fourth author of this paper: how to get direct proofs for openness results (and, also, for coincidence/fixed points results), on complete metric spaces, using EVP and not arguing by contradiction. In our knowlelge (see, for instance, [63], [14], [23]), in many cases, the proofs relying on EVP are made on normed vector spaces, and reasoning by contradiction. The supplemental structure of the space (i.e., its linear structure, but also the norm), which seems at first glance a little surprising, it is used essentially in the construction of the contradiction. In this work, by the careful analysis of some ideas spread in different articles (see [51], [52], [20]), we reached at the conclusion that the answer to the problem raised by Ioffe was already there, but not so easy to detect: one must appropriately apply EVP for the lower semicontinuous envelope of a certain distance function, by the appropriate choice of an auxiliary multifunction involved in the construction of this envelope. As consequence, by combining and extending some techniques from the quoted articles, we are able to give here a complete and direct proof based on EVP for the metric regularity of set-valued mappings of composition type. Moreover, in this way, we bring more light on the links between several tools used in getting regularity results for multifunctions.

As a by-product of the main result, a coincidence/fixed points assertion is obtained, a fact that contributes to a discussion on this subject initiated in [1], [2], [3] and continued in [14], [15], [23]. Futhermore, the important role of the assertions before the main result is again emphasized, as the (immediate) proof of the fixed point assertion relies on the appropriate application of one of them and of the main result (Theorem 3.6).

The last section deals with coderivative conditions for openness of composite mapping. Here we reconsider several ideas in [53] and [22] from a higher point of view and we employ a calculus rule for the Fréchet normal cone to a intersection of sets, passing through the concept of alliedness introduced and studied by Penot and his coauthors ([55], [42]). Finally, as an interesting fact which makes the link to the section before, we prove that one can obtain, on Asplund spaces, the conclusion of the main result of the paper by the use of the coderivative condition previously developed.

2 Preliminaries

This section contains some basic definitions and results used in the sequel. In what follows, we suppose that the involved spaces are metric spaces, unless otherwise stated. In this setting, $B(x,r)$ and $\overline{B}(x,r)$ denote the open and the closed ball with center x and radius r, respectively. On a product space we usually take the additive metric; when we choose another metric, this will be stated explicitly. If $x \in X$ and $A \subset X$, one defines the distance from x to A as $d(x,A) := \inf\{d(x,a) \mid a \in A\}$. As usual, we use the convention $d(x,\emptyset) = \infty$. The excess from a set A to a set B is defined as $e(A,B) := \sup\{d(a,B) \mid a \in A\}$. For a non-empty set $A \subset X$ we put $\operatorname{cl} A$ for its topological closure. One says that a set A is locally complete (closed) if there exists $r > 0$ such that $A \cap \overline{B}(x,r)$ is complete (closed). The symbol $\mathcal{V}(x)$ stands for the system of the neighborhoods of x.

Let X, Y, Z, P be metric spaces. For a multifunction $F : X \rightrightarrows Y$, the graph of F is the set $\operatorname{Gr} F := \{(x,y) \in X \times Y \mid y \in F(x)\}$. If $A \subset X$, then $F(A) := \bigcup_{x\in A} F(x)$. The inverse set-valued map of F is $F^{-1} : Y \rightrightarrows X$ given by $F^{-1}(y) = \{x \in X \mid y \in F(x)\}$. If $F_1 : X \rightrightarrows Y, F_2 : X \rightrightarrows Z$, we define the set-valued map $(F_1, F_2) : X \rightrightarrows Y \times Z$ by $(F_1,F_2)(x) := F_1(x) \times F_2(x)$. For a parametric multifunction $F : X \times P \rightrightarrows Y$, we use the notations: $F_p(\cdot) := F(\cdot, p)$ and $F_x(\cdot) := F(x, \cdot)$.

We recall now the concepts of openness at linear rate, metric regularity and Aubin property of a multifunction around the reference point.

Definition 2.1 *Let $F : X \rightrightarrows Y$ be a multifunction and $(\overline{x},\overline{y}) \in \operatorname{Gr} F$.*

(i) F is said to be open at linear rate $L > 0$ (or $L-$open) around $(\overline{x},\overline{y})$ if there exist a positive number $\varepsilon > 0$ and two neighborhoods $U \in \mathcal{V}(\overline{x})$, $V \in \mathcal{V}(\overline{y})$ such that, for every $\rho \in (0,\varepsilon)$ and every

$(x,y) \in \operatorname{Gr} F \cap [U \times V]$,

$$B(y, \rho L) \subset F(B(x, \rho)). \tag{2.1}$$

The supremum of $L > 0$ over all the combinations (L, U, V, ε) for which (2.1) holds is denoted by $\operatorname{lop} F(\overline{x}, \overline{y})$ and is called the exact linear openness bound, or the exact covering bound of F around $(\overline{x}, \overline{y})$.

(ii) F is said to have the Aubin property (or to be Lipschitz-like) around $(\overline{x}, \overline{y})$ with constant $L > 0$ if there exist two neighborhoods $U \in \mathcal{V}(\overline{x})$, $V \in \mathcal{V}(\overline{y})$ such that, for every $x, u \in U$,

$$e(F(x) \cap V, F(u)) \leq Ld(x, u). \tag{2.2}$$

The infimum of $L > 0$ over all the combinations (L, U, V) for which (2.2) holds is denoted by $\operatorname{lip} F(\overline{x}, \overline{y})$ and is called the exact Lipschitz bound of F around $(\overline{x}, \overline{y})$.

(iii) F is said to be metrically regular with constant $L > 0$ around $(\overline{x}, \overline{y})$ if there exist two neighborhoods $U \in \mathcal{V}(\overline{x})$, $V \in \mathcal{V}(\overline{y})$ such that, for every $(x, y) \in U \times V$,

$$d(x, F^{-1}(y)) \leq Ld(y, F(x)). \tag{2.3}$$

The infimum of $L > 0$ over all the combinations (L, U, V) for which (2.3) holds is denoted by $\operatorname{reg} F(\overline{x}, \overline{y})$ and is called the exact regularity bound of F around $(\overline{x}, \overline{y})$.

The links between the previous notions are as follows (see, e.g., [60, Theorem 9.43], [45, Theorems 1.52], [19, Section 3E]).

Theorem 2.2 *Let $F : X \rightrightarrows Y$ be a multifunction and $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$. Then F is open at linear rate around $(\overline{x}, \overline{y})$ iff F^{-1} has the Aubin property around $(\overline{y}, \overline{x})$ iff F is metrically regular around $(\overline{x}, \overline{y})$. Moreover, in every of the previous situations,*

$$(\operatorname{lop} F(\overline{x}, \overline{y}))^{-1} = \operatorname{lip} F^{-1}(\overline{y}, \overline{x}) = \operatorname{reg} F(\overline{x}, \overline{y}).$$

In the case of parametric set-valued maps one has the following partial notions of linear openness, metric regularity and Aubin property around the reference point.

Definition 2.3 *Let $F : X \times P \rightrightarrows Y$ be a multifunction and $((\overline{x}, \overline{p}), \overline{y}) \in \operatorname{Gr} F$.*

(i) F is said to be open at linear rate $L > 0$ (or $L-$open) with respect to x uniformly in p around $((\overline{x}, \overline{p}), \overline{y})$ if there exist a positive number $\varepsilon > 0$ and some neighborhoods $U \in \mathcal{V}(\overline{x})$, $V \in \mathcal{V}(\overline{p})$, $W \in \mathcal{V}(\overline{y})$ such that, for every $\rho \in (0, \varepsilon)$, every $p \in V$ and every $(x, y) \in \operatorname{Gr} F_p \cap [U \times W]$,

$$B(y, \rho L) \subset F_p(B(x, \rho)). \tag{2.4}$$

The supremum of $L > 0$ over all the combinations $(L, U, V, W, \varepsilon)$ for which (2.4) holds is denoted by $\widehat{\operatorname{lop}}_x F((\overline{x}, \overline{p}), \overline{y})$ and is called the exact linear openness bound, or the exact covering bound of F in x around $((\overline{x}, \overline{p}), \overline{y})$.

(ii) F is said to have the Aubin property (or to be Lipschitz-like) with respect to x uniformly in p around $((\overline{x}, \overline{p}), \overline{y})$ with constant $L > 0$ if there exist some neighborhoods $U \in \mathcal{V}(\overline{x})$, $V \in \mathcal{V}(\overline{p})$, $W \in \mathcal{V}(\overline{y})$ such that, for every $x, u \in U$ and every $p \in V$,

$$e(F_p(x) \cap W, F_p(u)) \leq Ld(x, u). \tag{2.5}$$

The infimum of $L > 0$ over all the combinations (L, U, V, W) for which (2.5) holds is denoted by $\widehat{\operatorname{lip}}_x F((\overline{x}, \overline{p}), \overline{y})$ and is called the exact Lipschitz bound of F in x around $((\overline{x}, \overline{p}), \overline{y})$.

(iii) F is said to be metrically regular with constant $L > 0$ with respect to x uniformly in p around $((\overline{x}, \overline{p}), \overline{y})$ if there exist some neighborhoods $U \in \mathcal{V}(\overline{x})$, $V \in \mathcal{V}(\overline{p})$, $W \in \mathcal{V}(\overline{y})$ such that, for every $(x, p, y) \in U \times V \times W$,

$$d(x, F_p^{-1}(y)) \leq L d(y, F_p(x)). \tag{2.6}$$

The infimum of $L > 0$ over all the combinations (L, U, V, W) for which (2.6) holds is denoted by $\widehat{\operatorname{reg}}_x F((\overline{x}, \overline{p}), \overline{y})$ and is called the exact regularity bound of F in x around $((\overline{x}, \overline{p}), \overline{y})$.

Interchanging the roles of p and x one gets a similar set of concepts.

Let now X be a metric space, and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a function. Set

$$S := \{x \in X : f(x) \leq 0\}.$$

One denotes the quantity $\max\{f(x), 0\}$ by $[f(x)]_+$.

Here and in what follows the convention $0 \cdot (+\infty) = 0$ is used. The next result gives an error bound estimation and it will be useful as an intermediate step in the proof of the main result of the paper.

Theorem 2.4 *([51, Theorem 2.1]) Let (X, d) be a complete metric space, and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function, and $\overline{x} \in X$ such that $f(\overline{x}) > 0$. Setting*

$$m(\overline{x}) := \inf \left\{ \sup_{u \in X, u \neq \overline{x}} \frac{f(x) - [f(u)]_+}{d(x, u)} : \begin{array}{c} d(x, \overline{x}) < d(\overline{x}, S) \\ f(x) \leq f(\overline{x}) \end{array} \right\},$$

one has

$$m(\overline{x}) \cdot d(\overline{x}, S) \leq f(\overline{x}).$$

In what follows we shall use the lower semicontinuous envelope associated to a multifunction $F : X \rightrightarrows Y$ as $\varphi_F : X \times Y \to \mathbb{R} \cup \{+\infty\}$,

$$\varphi_F(x, y) := \liminf_{(u,v) \to (x,y)} d(v, F(u)) = \liminf_{u \to x} d(y, F(u)).$$

This concept, with appropriate choices of the base multifunction, proved to be an useful tool in proving results on metric regularity of set-valued mappings and in vector optimization, using the error bound approach (see, e.g., [52], [20]). Observe that $\varphi_F(x, y) \geq 0$ and $\varphi_F(x, y) \leq d(y, F(x))$ for every $(x, y) \in X \times Y$.

Proposition 2.5 *([20, Lemma 1]) Let $F : X \rightrightarrows Y$ be a multifunction. Then, for every $y \in Y$,*

$$\{x \in X : \varphi_F(x, y) = 0\} = (\operatorname{cl} F)^{-1}(y),$$

where $\operatorname{cl} F$ is the multifunction whose graph is $\operatorname{cl} \operatorname{Gr} F$. In particular, if F has closed graph, then

$$F^{-1}(y) = \{x \in X : \varphi_F(x, y) = 0\}.$$

3 Metric regularity of compositions

Let $F_1 : X \rightrightarrows Y_1$, $F_2 : X \rightrightarrows Y_2$ and $G : Y_1 \times Y_2 \rightrightarrows Z$ be set-valued mappings, where X, Y_1, Y_2, Z are metric spaces. Consider the following composition multifunctions $H : X \rightrightarrows Z$ defined by

$$H(x) := G(F_1(x), F_2(x)). \tag{3.1}$$

Our aim in this article is to investigate the metric regularity of this multifunction H. As in the case of the sum of multifunctions (that is, $Y_1 = Y_2 = Y$, where Y is a normed vector space, and $G(y_1, y_2) = y_1 + y_2$), in general, H is not necessarily closed. Thus almost all the results on metric regularity known in the literature, that need the closedness of the multifunction under consideration, could not be directly applied for H. For this purpose, let us consider, associated to F_1, F_2 and G the following multifunction: $R : X \times Y_1 \times Y_2 \rightrightarrows Z$, given as

$$R(x, y_1, y_2) = \begin{cases} G(y_1, y_2), & \text{if } (y_1, y_2) \in F_1(x) \times F_2(x) \\ \emptyset, & \text{otherwise.} \end{cases} \tag{3.2}$$

Obviously, R is a closed-graph multifunction whenever F_1, F_2, G are closed-graph ones.

Next, we want to see how the associated function φ_R looks like. We know by definition that

$$\varphi_R((x, y_1, y_2), z) := \liminf_{(x', y_1', y_2', z') \to (x, y_1, y_2, z)} d(z', R(x', y_1', y_2')).$$

Clearly, if $(y_1, y_2) \notin F_1(x) \times F_2(x)$, say $y_1 \notin F_1(x)$, and $F_1(x)$ is closed, then $\varphi_R((x, y_1, y_2), z) = +\infty$. As consequence, if F_1, F_2 are closed-graph multifunctions and $(y_1, y_2) \notin F_1(x) \times F_2(x)$, one has that $\varphi_R((x, y_1, y_2), z) = +\infty$.

Otherwise, we have

$$\begin{aligned} \varphi_R((x, y_1, y_2), z) &= \liminf_{\substack{(x', y_1', y_2', z') \to (x, y_1, y_2, z) \\ (y_1', y_2') \in F_1(x') \times F_2(x')}} d(z', R(x', y_1', y_2')) \\ &= \liminf_{\substack{(x', y_1', y_2') \to (x, y_1, y_2) \\ (y_1', y_2') \in F_1(x') \times F_2(x')}} d(z, R(x', y_1', y_2')) \\ &= \liminf_{\substack{(x', y_1', y_2') \to (x, y_1, y_2) \\ (y_1', y_2') \in F_1(x') \times F_2(x')}} d(z, G(y_1', y_2')). \end{aligned}$$

In conclusion, if F_1, F_2 are closed-graph multifunctions, one has that

$$\varphi_R((x, y_1, y_2), z) = \begin{cases} \liminf\limits_{\substack{(x', y_1', y_2') \to (x, y_1, y_2) \\ (y_1', y_2') \in F_1(x') \times F_2(x')}} d(z, G(y_1', y_2')), & \text{if } (y_1, y_2) \in F_1(x) \times F_2(x) \\ +\infty, & \text{otherwise.} \end{cases}$$

We are ready to prove a result which will become very useful in the sequel. As mentioned, in the next result we take the additive distance on product spaces.

Proposition 3.1 *Let $F_1 : X \rightrightarrows Y_1$, $F_2 : X \rightrightarrows Y_2$ and $G : Y_1 \times Y_2 \rightrightarrows Z$ be multifunctions and $\bar{z} \in G(\bar{y}_1, \bar{y}_2), (\bar{y}_1, \bar{y}_2) \in F_1(\bar{x}) \times F_2(\bar{x})$. Consider the following statements:*

(i) there exist a neighborhood $\mathcal{U} \times \mathcal{V}_1 \times \mathcal{V}_2 \times \mathcal{W} \subset X \times Y_1 \times Y_2 \times Z$ of $(\bar{x}, \bar{y}_1, \bar{y}_2, \bar{z})$ and $\tau > 0$ such that

$$d((x, y_1, y_2), R^{-1}(z)) \leq \tau \varphi_R((x, y_1, y_2), z) \quad \textit{for all } (x, y_1, y_2, z) \in \mathcal{U} \times \mathcal{V}_1 \times \mathcal{V}_2 \times \mathcal{W}. \tag{3.3}$$

(ii) there exist a neighborhood $\mathcal{U} \times \mathcal{V}_1 \times \mathcal{V}_2 \times \mathcal{W} \subset X \times Y_1 \times Y_2 \times Z$ of $(\overline{x}, \overline{y}_1, \overline{y}_2, \overline{z})$ and $\tau > 0$ such that

$$d(x, H^{-1}(z)) \leq \tau d(z, G(F_1(x) \cap \mathcal{V}_1, F_2(x) \cap \mathcal{V}_2)) \quad \text{for all } (x, z) \in \mathcal{U} \times \mathcal{W}. \tag{3.4}$$

(iii) there exists $\varepsilon > 0$ such that for every $\rho \in (0, \varepsilon)$ and for every $(x, y_1, y_2, z) \in B(\overline{x}, \varepsilon) \times B(\overline{y}_1, \varepsilon) \times B(\overline{y}_2, \varepsilon) \times B(\overline{z}, \varepsilon)$ such that $z \in G(y_1, y_2), (y_1, y_2) \in F_1(x) \times F_2(x)$,

$$B(z, \rho\tau^{-1}) \in H(B(x, \rho)).$$

Then, we have the following implications: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

Proof. *(i)* $\Rightarrow$ *(ii)*. Using *(i)*, we get that there exist a neighborhood $\mathcal{U} \times \mathcal{V}_1 \times \mathcal{V}_2 \times \mathcal{W} \subset X \times Y_1 \times Y_2 \times Z$ of $(\overline{x}, \overline{y}_1, \overline{y}_2, \overline{z})$ and $\tau > 0$ such that

$$d((x, y_1, y_2), R^{-1}(z)) \leq \tau d(z, G(y_1, y_2)) \tag{3.5}$$

for all $(x, y_1, y_2, z) \in \mathcal{U} \times \mathcal{V}_1 \times \mathcal{V}_2 \times \mathcal{W}$ with $(y_1, y_2) \in F_1(x) \times F_2(x)$.

Take $z \in \mathcal{W}$. If for $x \in \mathcal{U}$, $F_1(x) \cap \mathcal{V}_1 = \emptyset$ or $F_2(x) \cap \mathcal{V}_2 = \emptyset$, then $d(z, G(F_1(x) \cap \mathcal{V}_1, F_2(x) \cap \mathcal{V}_2)) = +\infty$. Suppose now that there is $(y_1, y_2) \in [F_1(x) \cap \mathcal{V}_1] \times [F_2(x) \cap \mathcal{V}_2]$ such that $G(y_1, y_2) \neq \emptyset$, because otherwise, again, relation (3.4) trivially holds.

Take now arbitrary $(y_1, y_2) \in [F_1(x) \cap \mathcal{V}_1] \times [F_2(x) \cap \mathcal{V}_2]$ such that $G(y_1, y_2) \neq \emptyset$. From (3.5), it follows that for every $\varepsilon > 0$, there exists $(x', y_1', y_2') \in R^{-1}(z)$, i.e., $z \in G(y_1', y_2')$, $(y_1', y_2') \in F_1(x') \times F_2(x')$, such that

$$d((x, y_1, y_2), (x', y_1', y_2')) \leq \tau d(z, G(y_1, y_2)) + \varepsilon. \tag{3.6}$$

Consequently,

$$d(x, x') \leq \tau d(z, G(y_1, y_2)) + \varepsilon, \tag{3.7}$$

and, since $z \in G(y_1', y_2')$, $(y_1', y_2') \in F_1(x') \times F_2(x')$, we have that $x' \in H^{-1}(z)$, hence

$$d(x, H^{-1}(z)) \leq \tau d(z, G(y_1, y_2)) + \varepsilon,$$

and, finally,

$$d(x, H^{-1}(z)) \leq \tau d(z, G(F_1(x) \cap \mathcal{V}_1, F_2(x) \cap \mathcal{V}_2)) + \varepsilon \quad \text{for all } (x, z) \in \mathcal{U} \times \mathcal{W}.$$

Making $\varepsilon \to 0$, one gets the conclusion.

(ii) $\Rightarrow$ *(iii)*. By *(ii)*, there are $\delta, \tau > 0$ such that for every $(x, y_1, y_2, z) \in B(\overline{x}, \delta) \times B(\overline{y}_1, \delta) \times B(\overline{y}_2, \delta) \times B(\overline{z}, \delta)$, one has

$$d(x, H^{-1}(z)) \leq \tau d(z, G(F_1(x) \cap B(\overline{y}_1, \delta), F_2(x) \cap B(\overline{y}_2, \delta))) \quad \text{for all } (x, z) \in B(\overline{x}, \delta) \times B(\overline{z}, \delta).$$

Choose $\varepsilon < \delta \dfrac{\tau}{\tau + 1}$. Then, for every $(x, y_1, y_2, z) \in B(\overline{x}, \varepsilon) \times B(\overline{y}_1, \varepsilon) \times B(\overline{y}_2, \varepsilon) \times B(\overline{z}, \varepsilon)$ such that $z \in G(y_1, y_2)$, $(y_1, y_2) \in F_1(x) \times F_2(x)$ and every $\rho \in (0, \varepsilon)$, take

$$y \in B(z, \rho\tau^{-1}).$$

Then

$$d(y, \overline{z}) \leq d(y, z) + d(z, \overline{z}) < \rho\tau^{-1} + \varepsilon < \varepsilon\tau^{-1} + \varepsilon < \frac{\tau + 1}{\tau} \delta \frac{\tau}{\tau + 1} = \delta.$$

It follows that

$$d(x, H^{-1}(y)) \leq \tau d(y, z) < \tau\rho\tau^{-1} = \rho.$$

Thus, there is $u \in H^{-1}(y)$ or, equivalently, $y \in H(u)$ such that $d(x, u) < \rho$. It means that

$$y \in H(B(x, \rho).$$

In conclusion,

$$B(z, \rho\tau^{-1}) \in H(B(x, \rho)).$$

The proof is complete. □

The next concept of local composition stability, initially introduced in [24] in order to conserve the Aubin property for compositions, as a natural extension of its corresponding predecesor, the local sum-stability ([22]), will allow us to obtain regularity results around the reference point.

Definition 3.2 *Let $F : X \rightrightarrows Y$, $G : Y \rightrightarrows Z$ be multifunctions and $(\overline{x}, \overline{y}, \overline{z}) \in X \times Y \times Z$ such that $\overline{y} \in F(\overline{x})$, $\overline{z} \in G(\overline{y})$. We say that the multifunctions F, G are locally composition-stable around $(\overline{x}, \overline{y}, \overline{z})$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for every $x \in B(\overline{x}, \delta)$ and every $z \in (G \circ F)(x) \cap B(\overline{z}, \delta)$, there is $y \in F(x) \cap B(\overline{y}, \varepsilon)$ such that $z \in G(y)$.*

A simple case which ensures the locally composition-stable property of H is as follows.

Proposition 3.3 *Let X, Y_1, Y_2, Z be metric spaces, $F_1 : X \rightrightarrows Y_1$, $F_2 : X \rightrightarrows Y_2$ and $G : Y_1 \times Y_2 \rightrightarrows Z$ be three multifunctions, and $(\overline{x}, \overline{y}_1, \overline{y}_2, \overline{z}) \in X \times Y_1 \times Y_2 \times Z$ be such that $F_1(\overline{x}) = \{\overline{y}_1\}$, $F_2(\overline{x}) = \{\overline{y}_2\}$ and F_1, F_2 are upper semicontinuous at $\overline{x}$. Then $(F_1, F_2), G$ are locally composition-stable around $(\overline{x}, (\overline{y}_1, \overline{y}_2), \overline{z})$.*

Proof. Since F_1, F_2 are upper semicontinuous at $\overline{x}$, for every $\varepsilon > 0$ there exist $\delta_1, \delta_2 > 0$ such that

$$\begin{aligned} F_1(x) &\subset B(\overline{y}_1, 2^{-1}\varepsilon), \quad \text{for all } x \in B(\overline{x}, \delta_1), \\ F_2(x) &\subset B(\overline{y}_2, 2^{-1}\varepsilon), \quad \text{for all } x \in B(\overline{x}, \delta_2). \end{aligned} \tag{3.8}$$

Set $\delta := \min\{\delta_1, \delta_2, \varepsilon\}$, and take $x \in B(\overline{x}, \delta)$, $z \in (G \circ (F_1, F_2))(x) \cap B(\overline{z}, \delta)$. Then, there are $y_1 \in F_1(x)$, $y_2 \in F_2(x)$ such that $z \in G(y_1, y_2)$. By (3.8), one gets that $y_1 \in B(\overline{y}_1, 2^{-1}\varepsilon)$, $y_2 \in B(\overline{y}_2, 2^{-1}\varepsilon)$, i.e. the conclusion. □

Another interesting case which ensures the locally composition-stable property of H is given in the following proposition.

Proposition 3.4 *Let X, Y_1, Y_2, Z be metric spaces, $F_1 : X \rightrightarrows Y_1$, $F_2 : X \rightrightarrows Y_2$ be two set-valued mappings and $G : Y_1 \times Y_2 \to Z$ be a single-valued mapping, and $(\overline{x}, \overline{y}_1, \overline{y}_2, \overline{z}) \in X \times Y_1 \times Y_2 \times Z$ be such that $\overline{z} = G(\overline{y}_1, \overline{y}_2)$, $(\overline{y}_1, \overline{y}_2) \in F_1(\overline{x}) \times F_2(\overline{x})$ and*
(i) $F_1(\overline{x}) = \{\overline{y}_1\}$, and F_1 is upper semicontinuous at $\overline{x}$;
(ii) $G(\cdot, \overline{y}_2)$ is continuous at $\overline{y}_1$;
(iii) $G(y_1, \cdot)$ is an isometry for all y_1 near $\overline{y}_1$.
Then $(F_1, F_2), G$ are locally composition-stable around $(\overline{x}, (\overline{y}_1, \overline{y}_2), \overline{z})$.

Proof. Let $\varepsilon > 0$ be given. By (ii), there exists $\delta_1 > 0$ such that

$$d(G(y_1, \overline{y}_2), G(\overline{y}_1, \overline{y}_2)) < \varepsilon/2, \text{ for all } y_1 \in B(\overline{y}_1, \delta_1). \tag{3.9}$$

By (i), there is $\delta_2 > 0$ such that

$$F_1(x) \subset B(\overline{y}_1, \delta_1) \text{ for all } x \in B(\overline{x}, \delta_2). \tag{3.10}$$

Let $\delta := \{\delta_1, \delta_2, \varepsilon/2\}$, and let $x \in B(\overline{x}, \delta)$, $z \in G(F_1(x), F_2(x)) \cap B(\overline{z}, \delta)$.

Then, there are $y_1 \in F_1(x), y_2 \in F_2(x)$ such that $z = G(y_1, y_2) \in B(\overline{z}, \delta)$. Moreover, by (3.10), one has $y_1 \in B(\overline{y}_1, \delta_1)$. Thus, by (3.9), $d(G(y_1, \overline{y}_2), G(\overline{y}_1, \overline{y}_2)) < \varepsilon/2$. Consequently, by (iii), we obtain that

$$\begin{aligned} \delta > d(z, \overline{z}) &= d(G(y_1, y_2), G(\overline{y}_1, \overline{y}_2)) \geq \\ &\geq d(G(y_1, y_2), G(y_1, \overline{y}_2)) - d(G(y_1, \overline{y}_2), G(\overline{y}_1, \overline{y}_2)) \geq d(y_2, \overline{y}_2) - \varepsilon/2. \end{aligned}$$

So,

$$d(y_2, \overline{y}_2) < \delta + \varepsilon/2 < \varepsilon.$$

The proof is complete. □

Proposition 3.5 *Let X, Y_1, Y_2, Z be metric spaces, $F_1 : X \rightrightarrows Y_1$, $F_2 : X \rightrightarrows Y_2$ and $G : Y_1 \times Y_2 \rightrightarrows Z$ be multifunctions and $(\overline{x}, \overline{y}_1, \overline{y}_2, \overline{z}) \in X \times Y_1 \times Y_2 \times Z$ be such that $\overline{z} \in G(\overline{y}_1, \overline{y}_2), (\overline{y}_1, \overline{y}_2) \in F_1(\overline{x}) \times F_2(\overline{x})$. If there exist $\delta > 0$ and $\tau > 0$ such that*

$$d(x, H^{-1}(z)) \leq \tau d(z, G(F_1(x) \cap B(\overline{y}_1, \delta), F_2(x) \cap B(\overline{y}_2, \delta))) \quad \text{for all } (x, z) \in B(\overline{x}, \delta) \times B(\overline{z}, \delta) \tag{3.11}$$

and $(F_1, F_2), G$ are locally composition-stable around $(\overline{x}, (\overline{y}_1, \overline{y}_2), \overline{z})$, then H is metrically regular around $(\overline{x}, \overline{z})$ with modulus τ.

Moreover, if $F_1(\overline{x}) = \{\overline{y}_1\}$, $F_2(\overline{x}) = \{\overline{y}_2\}$, F_1, F_2 are upper semicontinuous at $\overline{x}$ and (3.11) holds, then H is metrically regular around $(\bar{x}, \bar{z})$ with modulus τ.

Proof. Suppose that (3.11) holds. Since $(F_1, F_2), G$ are locally composition-stable around the point $(\overline{x}, (\overline{y}_1, \overline{y}_2), \overline{z})$, there exists $\eta > 0$ such that for every $x \in B(\overline{x}, \eta)$ and every $z \in (G \circ (F_1, F_2))(x) \cap B(\overline{z}, \eta)$, there is $(y_1, y_2) \in (F_1(x) \cap B(\overline{y}_1, \delta)) \times (F_2(x) \cap B(\overline{y}_2, \delta))$ such that $z \in G(y)$.

Taking η smaller if necessary, we can assume that $\eta < \delta$. Consider the following two cases:

Case 1. $d(z, H(x)) < \eta/2$.

Fix $(x, z) \in B(\overline{x}, \eta/2) \times B(\overline{z}, \eta/2)$ and $\gamma > 0$, small enough in order to have

$$d(z, H(x)) + \gamma < \eta/2.$$

Then, there exists $t \in G(y_1, y_2), (y_1, y_2) \in F_1(x) \times F_2(x)$ such that $d(z, t) < d(z, G(F_1(x), F_2(x)) + \gamma < \eta/2$. Since $d(z, \overline{z}) < \eta/2$, one has that

$$d(t, \overline{z}) \leq d(z, t) + d(z, \overline{z}) < \eta/2 + \eta/2 = \eta.$$

Thus,

$$t \in (G \circ (F_1, F_2))(x) \cap B(\overline{z}, \eta).$$

Then, by the locally composition-stable property of $(F_1, F_2), G$ around $(\overline{x}, (\overline{y}_1, \overline{y}_2), \overline{z})$, there is $(y_1, y_2) \in (F_1(x) \cap B(\overline{y}_1, \delta)) \times (F_2(x) \cap B(\overline{y}_2, \delta))$ such that $t \in G(y_1, y_2)$.

Consequently,

$$d(x,H^{-1}(z)) \le \tau d(z,G(F_1(x)\cap B(\overline{y}_1,\delta),F_2(x)\cap B(\overline{y}_2,\delta))) \le \tau d(z,t) < \tau(d(z,G(F_1(x),F_2(x)))+\gamma).$$

As $\gamma > 0$ is arbitrarily small, one obtains that

$$d(x,H^{-1}(z)) \le \tau d(z,G(F_1(x),F_2(x))) = \tau d(z,H(x)).$$

Since (x,z) is arbitrary in $B(\overline{x},\eta/2)\times B(\overline{z},\eta/2)$, this yields the conclusion.

Case 2. $d(z,H(x)) \ge \eta/2$.

Without loosing the generality, choose η sufficiently small so that $\tau\eta/2 < \delta$. Fix now $(x,z) \in B(\overline{x},\tau\eta/4)\times B(\overline{z},\eta/4)$.

If $d(\overline{x},H^{-1}(z)) = 0$, one can succesively write

$$d(x,H^{-1}(x)) \le d(x,\overline{x}) + d(\overline{x},H^{-1}(z)) \le \frac{\eta}{2}\cdot\frac{\tau}{2} \le \frac{\tau}{2}d(z,H(x)) < \tau d(z,H(x)).$$

Suppose now $d(\overline{x},H^{-1}(z)) > 0$. Then, for any $\varepsilon > 0$, by (3.11), there exists $u \in H^{-1}(z)$ such that

$$d(\overline{x},u) < (1+\varepsilon)\tau d(z,H(\overline{x})) \le (1+\varepsilon)\tau d(z,\overline{z}) < (1+\varepsilon)\tau\frac{\eta}{4} \le (1+\varepsilon)\frac{\tau}{2}d(z,H(x)).$$

Consequently,

$$\begin{aligned} d(x,u) &\le d(x,\overline{x}) + d(\overline{x},u) < \tau\frac{\eta}{4} + (1+\varepsilon)\frac{\tau}{2}d(z,H(x)) \\ &< \frac{\tau}{2}d(z,H(x)) + (1+\varepsilon)\frac{\tau}{2}d(z,H(x)). \end{aligned}$$

Making $\varepsilon \longrightarrow 0$, it follows that

$$d(x,H^{-1}(z)) \le \tau d(z,H(x)),$$

i.e. the conclusion. □

Theorem 3.6 *Let X,Y_1,Y_2 be complete metric spaces, and Z be a metric space. Suppose that $F_1 : X \rightrightarrows Y_1$, $F_2 : X \rightrightarrows Y_2$ and $G : Y_1\times Y_2 \rightrightarrows Z$ are closed-graph multifunctions, satisfying the following conditions for some $(\overline{x},\overline{y}_1,\overline{y}_2,\overline{z}) \in X\times Y_1\times Y_2\times Z$ with $(\overline{x},\overline{y}_1)\in \operatorname{Gr} F_1$, $(\overline{x},\overline{y}_2)\in \operatorname{Gr} F_2$, $((\overline{y}_1,\overline{y}_2),\overline{z})\in \operatorname{Gr} G$:*

(i) F_1 is metrically regular around $(\overline{x},\overline{y}_1)$;

(ii) F_2 has the Aubin property around $(\overline{x},\overline{y}_2)$;

(iii) G is metrically regular around $((\overline{y}_1,\overline{y}_2),\overline{z})$ with respect to y_1, uniformly in y_2;

(iv) G has the Aubin property around $((\overline{y}_1,\overline{y}_2),\overline{z})$ with respect to y_2, uniformly in y_1;

(v) $0 < \operatorname{reg} F_1(\overline{x},\overline{y}_1)\cdot \operatorname{lip} F_2(\overline{x},\overline{y}_2)\cdot \widehat{\operatorname{reg}}_{y_1} G((\overline{y}_1,\overline{y}_2),\overline{z})\cdot \widehat{\operatorname{lip}}_{y_2} G((\overline{y}_1,\overline{y}_2),\overline{z}) < 1$.

Denote

$$\rho_0 := \frac{\operatorname{reg} F_1(\overline{x},\overline{y}_1)\cdot \widehat{\operatorname{reg}}_{y_1} G((\overline{y}_1,\overline{y}_2),\overline{z})}{1-\operatorname{reg} F_1(\overline{x},\overline{y}_1)\cdot \operatorname{lip} F_2(\overline{x},\overline{y}_2)\cdot \widehat{\operatorname{reg}}_{y_1} G((\overline{y}_1,\overline{y}_2),\overline{z})\cdot \widehat{\operatorname{lip}}_{y_2} G((\overline{y}_1,\overline{y}_2),\overline{z})} > 0. \tag{3.12}$$

Then exist a neighborhood $\mathcal{U}\times\mathcal{V}_1\times\mathcal{V}_2\times\mathcal{W} \subset X\times Y_1\times Y_2\times Z$ of $(\overline{x},\overline{y}_1,\overline{y}_2,\overline{z})$, $\rho > 0$ arbitrarily close to ρ_0 such that $\rho > \rho_0$, and an equivalent metric d_0 on $X\times Y_1\times Y_2$ such that

$$d_0((x,y_1,y_2),R^{-1}(z)) \le \rho\cdot\varphi_R((x,y_1,y_2),z) \quad \forall (x,y_1,y_2,z)\in \mathcal{U}\times\mathcal{V}_1\times\mathcal{V}_2\times\mathcal{W}. \tag{3.13}$$

As consequence, R is metrically regular around $((\overline{x},\overline{y}_1,\overline{y}_2),\overline{z})$ with respect to the metric d_0 on $X\times Y_1\times Y_2$, and

$$\operatorname{reg} R((\overline{x},\overline{y}_1,\overline{y}_2),\overline{z}) \le \rho_0.$$

Moreover, if $(F_1,F_2),G$ are locally composition-stable around $(\overline{x},(\overline{y}_1,\overline{y}_2),\overline{z})$, then H is metrically regular around $(\overline{x},\overline{z})$, and

$$\operatorname{reg} H(\overline{x},\overline{z}) \le \rho_0.$$

Proof. Take $l,m,\eta,\lambda>0$ such that, for $\xi>0$ arbitrary small,

$$\begin{aligned} m-\xi &> \operatorname{reg} F_1(\overline{x},\overline{y}_1),\\ l-\xi &> \operatorname{lip} F_2(\overline{x},\overline{y}_2),\\ \eta-\xi &> \widehat{\operatorname{lip}}_{y_2} G((\overline{y}_1,\overline{y}_2),\overline{z}),\\ \lambda &> \widehat{\operatorname{reg}}_{y_1} G((\overline{y}_1,\overline{y}_2),\overline{z}),\\ ml\lambda\gamma &< 1. \end{aligned} \tag{3.14}$$

Endow the product space $X\times Y_1\times Y_2$ with the equivalent metric defined by

$$d_0((x,y_1,y_2),(u,v_1,v_2)) = \max\{d(x,u), md(y_1,v_1), l^{-1}d(y_2,v_2)\},$$

and set

$$\rho := \frac{m\lambda}{1-ml\lambda\eta} > \rho_0.$$

Notice that $(X\times Y_1\times Y_2,d_0)$ is a complete metric space.
Fix $\xi>0$ arbitrary small such that (3.14) holds. By *(ii)*, there is $\delta_1>0$ such that

$$e(F_2(x)\cap B(\overline{y}_2,\delta_1),F_2(x')) \le (l-\xi)d(x,x') \quad \forall x,x'\in B(\overline{x},\delta_1). \tag{3.15}$$

By *(iv)*, there is $\delta_2>0$ such that

$$e(G(y_1,y_2)\cap B(\overline{z},\delta_2),G(y_1,y_2')) \le (\eta-\xi)d(y_2,y_2') \quad \forall y_1\in B(\overline{y}_1,\delta_2),(y_2,y_2')\in B(\overline{y}_2,\delta_2). \tag{3.16}$$

Moreover, by *(iii)*, there is $\delta_3>0$ such that

$$d(y_1,G_{y_2}^{-1}(z)) \le \lambda d(z,G(y_1,y_2)) \quad \forall y_1\in B(\overline{y}_1,\delta_3), y_2\in B(\overline{y}_2,\delta_3), z\in B(\overline{z}.\delta_3). \tag{3.17}$$

Finally, by *(i)*, there is $\delta_4>0$ such that

$$d(x,F_1^{-1}(y_1)) \le (m-\xi)d(y_1,F_1(x)) \quad \forall x\in B(\overline{x},\delta_4), y_1\in B(\overline{y}_1,\delta_4). \tag{3.18}$$

Set

$$\begin{aligned} \delta &:= \min\{2^{-1}\delta_1, 2^{-1}\delta_2, \delta_3, 2^{-1}\delta_4\},\\ \gamma &:= \min\left\{\frac{\delta}{\lambda},\frac{\delta}{m\lambda},\frac{\delta}{m\lambda l}\right\}, \end{aligned}$$

and take arbitrary $(x,y_1,y_2,z)\in B(\overline{x},\delta)\times B(\overline{y}_1,\delta)\times B(\overline{y}_2,\delta)\times B(\overline{z},\delta)$, such that $z\notin R(x,y_1,y_2)$, $\varphi_R((x,y_1,y_2),z)<\gamma$. Then $(y_1,y_2)\in F_1(x)\times F_2(x)$, because otherwise $\varphi_R((x,y_1,y_2),z)=+\infty$. Hence $z\notin G(y_1,y_2)$.

We want to prove next that for every $\tau > 0$, there exists $(u, v_1, v_2) \in X \times Y_1 \times Y_2$ such that

$$0 < d_0((x, y_1, y_2), (u, v_1 v_2)) < (\rho + \tau) \left[\varphi_R((x, y_1, y_2), z) - \varphi_R((u, v_1, v_2), z) \right]. \tag{3.19}$$

For this, fix $\tau > 0$ and take arbitrary $\{(x_n, y_{1n}, y_{2n})\} \subset X \times Y_1 \times Y_2$ converging to (x, y_1, y_2) such that

$$\lim_{n \to \infty} d(z, R(x_n, y_{1n}, y_{2n})) = \varphi_R((x, y_1, y_2), z) < \gamma.$$

Hence, as above, for every n sufficiently large, $(y_{1n}, y_{2n}) \in F_1(x_n) \times F_2(x_n)$. Also, using the closedness of $\operatorname{Gr} G$, it follows that, for every n sufficiently large,

$$z \notin G(y_{1n}, y_{2n}) \text{ and } d(z, G(y_{1n}, y_{2n})) < \gamma.$$

Now, because $(y_{1n}, y_{2n}) \to (y_1, y_2)$, one deduces that $y_{1n} \in B(\overline{y}_1, \delta)$ and $y_{2n} \in B(\overline{y}_2, \delta)$, for every n sufficiently large. We can use now (3.17) to get that

$$d(y_{1n}, G^{-1}_{y_{2n}}(z)) \leq \lambda d(z, G(y_{1n}, y_{2n})) < \lambda\gamma \leq \delta. \tag{3.20}$$

Remark that the closedness of $\operatorname{Gr} G$ assures the fact that $d(y_{1n}, G^{-1}_{y_{2n}}(z)) > 0$ for every n large enough. Also, by (3.20), one may suppose that $G^{-1}_{y_{2n}}(z) \neq \emptyset$, so for every $\varepsilon > 0$, there exists $v_{1n} \in G^{-1}_{y_{2n}}(z)$, or $z \in G(v_{1n}, y_{2n})$, such that

$$\begin{aligned} d(y_{1n}, v_{1n}) &< \left(1 + \frac{\varepsilon}{2\lambda}\right) d(y_{1n}, G^{-1}_{y_{2n}}(z)) \leq \left(\lambda + \frac{\varepsilon}{2}\right) d(z, G(y_{1n}, y_{2n})) \\ &= \left(\lambda + \frac{\varepsilon}{2}\right) [d(z, G(y_{1n}, y_{2n})) - d(z, G(v_{1n}, y_{2n})]. \end{aligned} \tag{3.21}$$

Observe that we may suppose without loosing the generality that $\lim\limits_{n \to \infty} d(v_{1n}, y_{1n})$ exists and $\lim\limits_{n \to \infty} d(v_{1n}, y_{1n}) = \lim\limits_{n \to \infty} d(v_{1n}, y_1) > 0$, because otherwise we would obtain using the closedness of $\operatorname{Gr} G$ that $z \in G(y_1, y_2)$.

Also, for every $\varepsilon > 0$ sufficiently small such that $\left(1 + \dfrac{\varepsilon}{2\lambda}\right) d(z, G(y_{1n}, y_{2n})) < \gamma$, one has that $d(y_{1n}, v_{1n}) < \lambda\gamma \leq \delta$, so

$$d(v_{1n}, \overline{y}_1) \leq d(v_{1n}, y_{1n}) + d(y_{1n}, \overline{y}_1) < 2\delta \leq \delta_4.$$

As $x_n \in B(\overline{x}, \delta_4)$, $v_{1n} \in B(\overline{y}_1, \delta_4)$ and $d(y_{1n}, v_{1n}) > 0$ for every n sufficiently large, by (3.18), we have that

$$d(x_n, F_1^{-1}(v_{1n}) \leq (m - \xi) d(v_{1n}, F_1(x_n)) < m d(v_{1n}, y_{1n}) < m\lambda\gamma \leq \delta,$$

so there is $u_n \in F_1^{-1}(v_{1n})$ such that

$$d(x_n, u_n) < m d(v_{1n}, y_{1n}) < \delta. \tag{3.22}$$

Hence,

$$d(u_n, \overline{x}) \leq d(u_n, x_n) + d(x_n, \overline{x}) < 2\delta \leq \delta_1,$$

or $u_n \in B(\overline{x}, \delta_1)$. Since $x_n \in B(\overline{x}, \delta_1)$, $y_{2n} \in F_2(x_n) \cap B(\overline{y}_2, \delta_1)$, using (3.15), we have that

$$d(y_{2n}, F_2(u_n)) \leq e(F_2(x_n) \cap B(\overline{y}_2, \delta_1), F_2(u_n)) \leq (l - \xi) d(x_n, u_n),$$

so there exists $v_{2n} \in F_2(u_n)$ such that

$$d(y_{2n}, v_{2n}) \le l d(x_n, u_n) < lmd(v_{1n}, y_{1n}) < lm\lambda\gamma \le \delta. \tag{3.23}$$

Then,

$$d(v_{2n}, \overline{y}_2) \le d(y_{2n}, v_{2n}) + d(v_{2n}, \overline{y}_2) < 2\delta \le \delta_2.$$

Finally, as $d(v_{1n}, \overline{y}_1) < 2\delta \le \delta_1$, by (3.16), one has

$$\begin{aligned} d(z, G(v_{1n}, v_{2n})) &\le d(z, G(v_{1n}, y_{2n})) + e(G(v_{1n}, y_{2n}) \cap B(\overline{z}, \delta_2), G(v_{1n}, v_{2n})) \\ &\le d(z, G(v_{1n}, y_{2n})) + (\eta - \xi) d(y_{2n}, v_{2n}). \end{aligned} \tag{3.24}$$

Remark that, by (3.22) and (3.23), we have that

$$l^{-1} d(y_{2n}, v_{2n}) \le d(x_n, u_n) \le md(v_{1n}, y_{1n}),$$

hence

$$d_0((x_n, y_{1n}, y_{2n}), (u_n, v_{1n}, v_{2n})) = md(y_{1n}, v_{1n}) > 0 \quad \forall n \text{ sufficiently large.}$$

Without loosing the generality, we may suppose that $\varepsilon > 0$ is chosen sufficiently small such that

$$\frac{1}{m(\lambda + \varepsilon/2)} - l\eta > \frac{1}{\rho + \tau/2}.$$

Also, because $(y_{1n}, y_{2n}) \in F_1(x_n) \times F_2(x_n)$ and $(v_{1n}, v_{2n}) \in F_1(u_n) \times F_2(u_n)$, and using also (3.24), (3.21), one gets that

$$\begin{aligned} &\limsup_{n\to\infty} \frac{d(z, R(x_n, y_{1n}, y_{2n})) - d(z, R(u_n, v_{1n}, v_{2n}))}{d_0((x_n, y_{1n}, y_{2n}), (u_n, v_{1n}, v_{2n}))} \\ &\ge \limsup_{n\to\infty} \frac{d(z, G(y_{1n}, y_{2n})) - d(z, G(v_{1n}, v_{2n}))}{d_0((x_n, y_{1n}, y_{2n}), (u_n, v_{1n}, v_{2n}))} \\ &\ge \limsup_{n\to\infty} \frac{d(z, G(y_{1n}, y_{2n})) - d(z, G(v_{1n}, y_{2n})) - \eta d(y_{2n}, v_{2n})}{\max\{d(x_n, u_n), md(y_{1n}, v_{1n}), l^{-1} d(y_{2n}, v_{2n})\}} \\ &\ge \limsup_{n\to\infty} \frac{d(z, G(y_{1n}, y_{2n})) - d(z, G(v_{1n}, y_{2n}))}{md(y_{1n}, v_{1n})} - \frac{\eta d(y_{2n}, v_{2n})}{l^{-1} d(y_{2n}, v_{2n})} \\ &\ge \frac{1}{m(\lambda + \varepsilon/2)} - l\eta > \frac{1}{\rho + \tau/2}. \end{aligned}$$

Remark that

$$\begin{aligned} \lim_{n\to\infty} d_0((u_n, v_{1n}, v_{2n}), (x, y_1, y_2)) &= \lim_{n\to\infty} d_0((u_n, v_{1n}, v_{2n}), (x_n, y_{1n}, y_{2n})) \\ &= \lim_{n\to\infty} md(y_{1n}, v_{1n}) > 0. \end{aligned}$$

Take now $r \in (0, \lim_{n\to\infty} d_0((u_n, v_{1n}, v_{2n}), (x, y_1, y_2)))$. Because $\lim_{\theta\to 0_+} \dfrac{1+\theta}{1-\theta}\left(\rho + \dfrac{\tau}{2}\right) = \rho + \dfrac{\tau}{2} <$

$\rho + \tau$, there exists $\theta > 0$ sufficiently small such that, for every n sufficiently large,

$$\frac{1+\theta}{1-\theta}\left(\rho + \frac{\tau}{2}\right) < \rho + \tau,$$
$$d_0((u_n, v_{1n}, v_{2n}), (x_n, y_{1n}, y_{2n})) \geq r, \quad d_0((x_n, y_{1n}, y_{2n}), (x, y_1, y_2)) < \theta r,$$
$$d(z, R(x_n, y_{1n}, y_{2n})) < \varphi_R((x, y_1, y_2), z) + \frac{\theta}{\rho + \frac{\tau}{2}} d_0((u_n, v_{1n}, v_{2n}), (x_n, y_{1n}, y_{2n})),$$
$$d_0((u_n, v_{1n}, v_{2n}), (x_n, y_{1n}, y_{2n})) < \left(\rho + \frac{\tau}{2}\right)[d(z, R(x_n, y_{1n}, y_{2n})) - d(z, R(u_n, v_{1n}, v_{2n}))].$$

It follows that, for every n sufficiently large,

$$\begin{aligned} d_0((u_n, v_{1n}, v_{2n}), (x_n, y_{1n}, y_{2n})) &< \frac{\rho + \frac{\tau}{2}}{1-\theta}[\varphi_R((x, y_1, y_2), z) - \varphi_R((u_n, v_{1n}, v_{2n}), z)], \\ d_0((u_n, v_{1n}, v_{2n}), (x, y_1, y_2)) &\leq (1+\theta) d_0((u_n, v_{1n}, v_{2n}), (x_n, y_{1n}, y_{2n})) \\ &< \frac{1+\theta}{1-\theta}\left(\rho + \frac{\tau}{2}\right)[\varphi_R((x, y_1, y_2), z) - \varphi_R((u_n, v_{1n}, v_{2n}), z)] \\ &< (\rho + \tau)[\varphi_R((x, y_1, y_2), z) - \varphi_R((u_n, v_{1n}, v_{2n}), z)], \end{aligned}$$

i.e. (3.19) is proved.

Fix now $t \in (0, \min\{\rho, \tau\})$, denote

$$s := \min\left\{\delta, \frac{\delta}{6\rho}, \frac{\delta m}{6\rho}, \frac{\delta}{6l\rho}, \frac{\gamma}{4}\right\},$$

and take arbitrary $z \in B(\overline{z}, s) \subset B(\overline{z}, \delta)$. Then

$$\varphi_R((\overline{x}, \overline{y}_1, \overline{y}_2), z) \leq d(z, R(\overline{x}, \overline{y}_1, \overline{y}_2)) \leq d(z, \overline{z}) < s,$$

hence

$$\varphi_R((\overline{x}, \overline{y}_1, \overline{y}_2), z) < \inf_{(x, y_1, y_2) \in X \times Y_1 \times Y_2} \varphi_R((x, y_1, y_2), z) + s.$$

One can apply now the Ekeland Variational Principle to the lower semicontinuous function $(x, y_1, y_2) \mapsto \varphi_R((x, y_1, y_2), z)$ and for the distance d_0 to get the existence of $(u, v_1, v_2) \in X \times Y_1 \times Y_2$ such that

$$\varphi_R((u, v_1, v_2), z) \leq \varphi_R((\overline{x}, \overline{y}_1, \overline{y}_2), z) < s, \tag{3.25}$$
$$d_0((u, v_1, v_2), (\overline{x}, \overline{y}_1, \overline{y}_2)) \leq s(\rho + t), \tag{3.26}$$
$$\varphi_R((u, v_1, v_2), z) \leq \varphi_R((x', y_1', y_2'), z) + \frac{1}{\rho + t} d_0((u, v_1, v_2), (x', y_1', y_2')), \tag{3.27}$$

for every $(x', y_1', y_2') \in X \times Y_1 \times Y_2$.

It follows from (3.26) that $(u, v_1, v_2) \in B(\overline{x}, \delta) \times B(\overline{y}_1, \delta) \times B(\overline{y}_2, \delta)$. Suppose now that one has $\varphi_R((u, v_1, v_2), z) > 0$. By (3.25), $\varphi_R((u, v_1, v_2), z) < s < \gamma$, hence one can deduce the existence of $(u', v_1', v_2') \in X \times Y_1 \times Y_2$ such that, using relations (3.19) and (3.27),

$$\begin{aligned} 0 &< d_0((u, v_1, v_2), (u', v_1', v_2')) < (\rho + t)\left[\varphi_R((u, v_1, v_2), z) - \varphi_R((u', v_1', v_2'), z)\right] \\ &\leq d_0((u, v_1, v_2), (u', v_1', v_2')), \end{aligned}$$

a contradiction. Hence, $\varphi_R((u, v_1, v_2), z) = 0$, or, equivalently, $(u, v_1, v_2) \in R^{-1}(z)$.

Fix now arbitrary

$$\begin{aligned}(x, y_1, y_2) \in B_0((\overline{x}, \overline{y}_1, \overline{y}_2), 2s\rho) &:= \{(x, y_1, y_2) \mid d_0((x, y_1, y_2), (\overline{x}, \overline{y}_1, \overline{y}_2)) < 2s\rho\} \\ &\subset B(\overline{x}, \delta) \times B(\overline{y}_1, \delta) \times B(\overline{y}_2, \delta).\end{aligned}$$

If $\varphi_R((x, y_1, y_2), z) \geq \gamma$, then

$$\begin{aligned}d_0((x, y_1, y_2), R^{-1}(z)) &\leq d_0((x, y_1, y_2), (u, v_1, v_2)) \\ &\leq d_0((x, y_1, y_2), (\overline{x}, \overline{y}_1, \overline{y}_2)) + d_0((\overline{x}, \overline{y}_1, \overline{y}_2), (u, v_1, v_2)) \\ &< 4s\rho \leq \gamma\rho \leq \rho\varphi_R((x, y_1, y_2), z).\end{aligned}$$

The case $\varphi_R((x, y_1, y_2), z) = 0$ makes relation (3.13) to hold trivially. Consider now the case $\varphi_R((x, y_1, y_2), z) \in (0, \gamma)$. If there is no $(p, w_1, w_2) \in X \times Y_1 \times Y_2$ such that $d_0((x, y_1, y_2), (p, w_1, w_2)) < d_0((x, y_1, y_2), R^{-1}(z))$ and $\varphi_R((p, w_1, w_2), z) \leq \varphi_R((x, y_1, y_2), z)$, then

$$\begin{aligned}m(x, y_1, y_2) &:= \inf \left\{ \begin{array}{c} \sup_{\substack{(u', v_1', v_2') \in X \times Y_1 \times Y_2, \\ (u', v_1', v_2') \neq (x, y_1, y_2)}} \dfrac{\varphi_R((p, w_1, w_2), z) - \varphi_R((u', v_1', v_2'), z)}{d_0((p, w_1, w_2), (u', v_1', v_2'))} \mid \\ d_0((x, y_1, y_2), (p, w_1, w_2)) < d_0((x, y_1, y_2), R^{-1}(z)) \\ \varphi_R((p, w_1, w_2), z) \leq \varphi_R((x, y_1, y_2), z) \end{array} \right\} \\ &= +\infty > \frac{1}{\rho + \tau}.\end{aligned}$$

Take now any $(p, w_1, w_2) \in X \times Y_1 \times Y_2$ such that $d_0((x, y_1, y_2), (p, w_1, w_2)) < d_0((x, y_1, y_2), R^{-1}(z))$ and $\varphi_R((p, w_1, w_2), z) \leq \varphi_R((x, y_1, y_2), z)$. Then

$$\begin{aligned}d_0((p, w_1, w_2), (\overline{x}, \overline{y}_1, \overline{y}_2)) &\leq d_0((p, w_1, w_2), (x, y_1, y_2)) + d_0((x, y_1, y_2), (\overline{x}, \overline{y}_1, \overline{y}_2)) \\ &< d_0((x, y_1, y_2), R^{-1}(z)) + d_0((x, y_1, y_2), (\overline{x}, \overline{y}_1, \overline{y}_2)) \\ &\leq d_0((\overline{x}, \overline{y}_1, \overline{y}_2), R^{-1}(z)) + 2d_0((x, y_1, y_2), (\overline{x}, \overline{y}_1, \overline{y}_2)) \\ &\leq d_0((u, v_1, v_2), (\overline{x}, \overline{y}_1, \overline{y}_2)) + 2d_0((x, y_1, y_2), (\overline{x}, \overline{y}_1, \overline{y}_2)) \leq 6\rho s,\end{aligned}$$

hence $(p, w_1, w_2) \in B_0((\overline{x}, \overline{y}_1, \overline{y}_2), 6\rho s) \subset B(\overline{x}, \delta) \times B(\overline{y}_1, \delta) \times B(\overline{y}_2, \delta)$. According to (3.19), there exists $(u'', v_1'', v_2'') \in X \times Y_1 \times Y_2$, $(u'', v_1'', v_2'') \neq (p, w_1, w_2)$, such that

$$\frac{\varphi_R((p, w_1, w_2), z) - \varphi_R((u'', v_1'', v_2''), z)}{d_0((p, w_1, w_2), (u'', v_1'', v_2''))} > \frac{1}{\rho + \tau}.$$

But this means that $m(x, y_1, y_2) \geq \dfrac{1}{\rho + \tau}$. As $\varphi_R((\cdot, \cdot, \cdot), z)$ is lower semicontinuous and $\{(a, b_1, b_2) \in X \times Y_1 \times Y_2 : \varphi_R((a, b_1, b_2), z) \leq 0\} = R^{-1}(z)$, apply Theorem 2.4 to get that

$$\frac{1}{\rho + \tau} \cdot d_0((x, y_1, y_2), R^{-1}(z)) \leq \varphi_R((x, y_1, y_2), z).$$

Take $\mathcal{U}_1 \times \mathcal{V}_1 \times \mathcal{V}_2 \subset B_0((\overline{x}, \overline{y}_1, \overline{y}_2), 2s\rho)$ a neighborhood of $(\overline{x}, \overline{y}_1, \overline{y}_2)$. As τ can be made arbitrary small, l, m, η, λ can be taken arbitrary close to the corresponding regularity moduli, and $(x, y_1, y_2) \in \mathcal{U}_1 \times \mathcal{V}_1 \times \mathcal{V}_2$, $z \in \mathcal{W} := B(\overline{z}, s)$ can be taken arbitrarily, we get the first conclusion.

For the second conclusion, take into account that $\varphi_R((x, y_1, y_2), z) \leq d(z, R(x, y_1, y_2))$ for every $(x, y_1, y_2, z) \in X \times Y_1 \times Y_2 \times Z$.

Now, observe that the implication $(i) \Rightarrow (ii)$ of Proposition 3.1 can be identically proven if one replaces the additive distance with d_0. In fact, it is essential to be able to write again relation (3.7) from (3.6) written for d_0, and this can be obviously done. Using Proposition 3.5, one gets the final conclusion. □

Remark 3.7 *This Theorem and Proposition 3.4 imply directly Theorem 4.6 given by Ioffe [32] for metric regularity of the composition multifunction of the form $G(x, F(x))$, where $F : X \rightrightarrows Y$ is a set-valued mapping and $G : X \times Y \to Y$ is a single-valued mapping.*

Consider now the case $Y_1 := Y_2 := Y$, where Y is a normed vector space, $G(y_1, y_2) := d(y_1, y_2)$. If $\overline{y}_1 \neq \overline{y}_2$, then G satisfies the conditions from the previous theorem (see [24, Section 4]). Observe now that

$$\begin{aligned} H^{-1}(0) &= \left\{ x \in X \mid 0 \in \bigcup_{(y_1,y_2)\in F_1(x)\times F_2(x)} d(y_1, y_2) \right\} \\ &= \{x \in X \mid \exists y \in Y \text{ such that } y \in F_1(x) \text{ and } y \in F_2(x)\} \\ &= \operatorname{Fix}(F_1^{-1} \circ F_2). \end{aligned}$$

Also,

$$\begin{aligned} d(0, G(F_1(x) \cap \mathcal{V}_1, F_2(x) \cap \mathcal{V}_2)) &= d\left(0, \bigcup_{(y_1,y_2)\in F_1(x)\cap\mathcal{V}_1\times F_2(x)\cap\mathcal{V}_2} d(y_1, y_2)\right) \\ &= \inf_{(y_1,y_2)\in F_1(x)\cap\mathcal{V}_1\times F_2(x)\cap\mathcal{V}_2} d(y_1, y_2) \\ &= d\left(F_1(x) \cap \mathcal{V}_1, F_2(x) \cap \mathcal{V}_2\right). \end{aligned}$$

As consequence, relation (3.13) from Theorem 3.6 implies, in the virtue of $(i) \Rightarrow (ii)$ from Proposition 3.1, that there exist a neighborhood $\mathcal{U} \times \mathcal{V}_1 \times \mathcal{V}_2 \subset X \times Y \times Y$ of $(\overline{x}, \overline{y}_1, \overline{y}_2)$ and $\rho > 0$ arbitrarily close to ρ_0 such that

$$d(x, \operatorname{Fix}(F_1^{-1} \circ F_2)) \leq \rho d\left(F_1(x) \cap \mathcal{V}_1, F_2(x) \cap \mathcal{V}_2\right) \quad \text{for all } x \in \mathcal{U}.$$

In this way, one can obtain a direct proof based on Ekeland Variational Principle, for fixed points/coincidence results of the type Arutyunov, Ioffe and Dontchev and Frankovska proved recently in [1], [2], [31], [14], [15].

4 Coderivative conditions

In this section, unless otherwise stated, we assume that all the spaces involved are Asplund, i.e., Banach spaces where every convex continuous function is generically Fréchet differentiable (in particular, any reflexive space is Asplund; see, e.g., [45] for more details). We recall next some standard notations and definitions we use in the sequel. If X is a normed vector space, we denote by B_X, $\overline{B}_X$, S_X the open unit ball, the closed unit ball and the unit sphere of X, respectively. As usual, by X^* we denote the topological dual of the normed vector space X, while the symbol $\langle \cdot, \cdot \rangle$ stands for the canonical duality pairing between X and X^*. The symbol $\xrightarrow{w^*}$ indicates the convergence in the weak-star topology of X^*. Given a set-valued mapping $F\colon X \rightrightarrows X^*$, recall that

$$\operatorname*{Limsup}_{x\to\overline{x}} F(x) := \left\{ x^* \in X^* \mid \exists x_n \to \overline{x}, x_n^* \xrightarrow{w^*} x^* \text{ with } x_n^* \in F(x_n),\ n \in \mathbb{N} \right\} \tag{4.1}$$

stands for the sequential Painlevé-Kuratowski outer/upper limit of F as $x \to \overline{x}$ with respect to the norm topology of X and the weak* topology of X^*.

If X is a normed vector space, S is a non-empty subset of X and $x \in S$, $\varepsilon \geq 0$, the set of $\varepsilon-$normals to S at x is

$$\widehat{N}_\varepsilon(S,x) := \left\{ x^* \in X^* \mid \limsup_{u \xrightarrow{S} x} \frac{\langle u-x, x^* \rangle}{\|u-x\|} \leq \varepsilon \right\}. \tag{4.2}$$

If $\varepsilon = 0$, the elements in the right-hand side of (4.2) are called Fréchet normals and their collection, denoted by $\widehat{N}(S,x)$, is the Fréchet normal cone to S at x.

Let $\overline{x} \in S$. The basic (or limiting, or Mordukhovich) normal cone to S at $\overline{x}$ is

$$N(S,\overline{x}) := \{x^* \in X^* \mid \exists \varepsilon_n \downarrow 0, x_n \xrightarrow{S} \overline{x}, x_n^* \xrightarrow{w^*} x^*, x_n^* \in \widehat{N}_{\varepsilon_n}(S,x_n), \forall n \in \mathbb{N}\},$$

where by $x_n \xrightarrow{S} \overline{x}$ we denote $x_n \xrightarrow{S} \overline{x}, x_n \in S$ for every n sufficiently large.

If X is an Asplund space and S is locally closed around $\overline{x}$, the formula for the basic normal cone takes a simpler form:

$$N(S,\overline{x}) = \{x^* \in X^* \mid \exists x_n \xrightarrow{S} \overline{x}, x_n^* \xrightarrow{w^*} x^*, x_n^* \in \widehat{N}(S,x_n), \forall n \in \mathbb{N}\},$$

i.e.

$$N(S,\overline{x}) = \operatorname*{Limsup}_{x \xrightarrow{S} \overline{x}} \widehat{N}(S,x).$$

Consider now $f : X \to \overline{\mathbb{R}}$ such that is finite at $\overline{x} \in X$. The Fréchet subdifferential of f at $\overline{x}$ is the set

$$\widehat{\partial} f(\overline{x}) := \{x^* \in X^* \mid (x^*, -1) \in \widehat{N}(\operatorname{epi} f, (\overline{x}, f(\overline{x})))\},$$

and the basic (or limiting, or Mordukhovich) subdifferential of f at $\overline{x}$ is

$$\partial f(\overline{x}) := \{x^* \in X^* \mid (x^*, -1) \in N(\operatorname{epi} f, (\overline{x}, f(\overline{x})))\},$$

where $\operatorname{epi} f$ denotes the epigraph of f.

In general Banach spaces, for every lower semicontinuous function one has $\widehat{\partial} f(\overline{x}) \subset \partial f(\overline{x})$, and in Asplund spaces the next relation holds

$$\partial f(\overline{x}) = \operatorname*{Limsup}_{x \xrightarrow{f} \overline{x}} \widehat{\partial} f(x),$$

where by $x \xrightarrow{f} \overline{x}$ we mean $x \to \overline{x}, f(x) \to f(\overline{x})$.

It is well-known that the Fréchet subdifferential satisfies a fuzzy sum rule on Asplund spaces (see [45, Theorem 2.33]). More precisely, if X is an Asplund space and $\varphi_1, \varphi_2 : X \to \mathbb{R} \cup \{\infty\}$ are such that φ_1 is Lipschitz continuous around $\overline{x} \in \operatorname{dom} \varphi_1 \cap \operatorname{dom} \varphi_2$ and φ_2 is lower semicontinuous around $\overline{x}$, then for any $\gamma > 0$ one has

$$\widehat{\partial}(\varphi_1+\varphi_2)(\overline{x}) \subset \bigcup \{\widehat{\partial}\varphi_1(x_1)+\widehat{\partial}\varphi_2(x_2) \mid x_i \in \overline{x}+\gamma\overline{B}_X, |\varphi_i(x_i) - \varphi_i(\overline{x})| \leq \gamma, i = 1,2\}+\gamma\overline{B}_{X^*}. \tag{4.3}$$

If δ_Ω denotes the indicator function associated to a nonempty set $S \subset X$ (i.e. $\delta_S(x) = 0$ if $x \in S$, $\delta_S(x) = \infty$ otherwise), then for any $\overline{x} \in S$, $\widehat{\partial}\delta_S(\overline{x}) = \widehat{N}(S,\overline{x})$ and $\partial\delta_S(\overline{x}) = N(S,\overline{x})$.

Finally, let $F : X \rightrightarrows Y$ be a set-valued map and $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$. Then the Fréchet coderivative at $(\overline{x}, \overline{y})$ is the set-valued map $\widehat{D}^*F(\overline{x}, \overline{y}) : Y^* \rightrightarrows X^*$ given by

$$\widehat{D}^*F(\overline{x}, \overline{y})(y^*) := \{x^* \in X^* \mid (x^*, -y^*) \in \widehat{N}(\operatorname{Gr} F, (\overline{x}, \overline{y}))\}.$$

Similarly, the normal coderivative of F at $(\overline{x}, \overline{y})$ is the set-valued map $D^*F(\overline{x}, \overline{y}) : Y^* \rightrightarrows X^*$ given by

$$D^*F(\overline{x}, \overline{y})(y^*) := \{x^* \in X^* \mid (x^*, -y^*) \in N(\operatorname{Gr} F, (\overline{x}, \overline{y}))\}.$$

Note that, in fact, the concept of normal coderivative, independently of the normal cone used in its definition, was introduced in [44].

In order to obtain a result concerning coderivative conditions which assure the metric regularity of the composition multifunction, we need a calculus rule for the Fréchet normal cone of the intersection of a finite number of sets. We introduce next some concepts which are needed in order to formulate such a result.

Recall that the sets $S_1, ..., S_k$ satisfy the metric inequality at $\overline{x}$ if there are $\tau > 0$ and $r > 0$ such that

$$d(x, S_1 \cap ... \cap S_k) \leq \tau[d(x, S_1) + ... + d(x, S_k)] \text{ for every } x \in B(\overline{x}, r).$$

This inequality is shown to be very effective as assumption in order to infer generalized differentiation calculus rules for different kinds of operations with sets and multifunctions (see, e.g., [30], [55], [49]).

Given the closed subsets $S_1, ..., S_k$ of a normed vector space X, one says that they are allied at $\overline{x} \in S_1 \cap ... \cap S_k$ (for the Fréchet normal cones) whenever $(x_{in}) \xrightarrow{S_i} \overline{x}, x^*_{in} \in \widehat{N}(S_i, x_{in}), i = \overline{1,k}$, the relation $(x^*_{1n} + ... + x^*_{kn}) \to 0$ implies $(x^*_{in}) \to 0$ for every $i = \overline{1,k}$ (for more details, see [55], [42] and the references therein).

Theorem 4.1 *Suppose X is an Asplund space and let $S_1, ..., S_k$ be closed subsets such that $\overline{x} \in S_1 \cap ... \cap S_k$. Consider the assertions:*

(i) $S_2, ..., S_k$ are sequentially normally compact and

$$x^*_i \in N(S_i, \overline{x}), i = \overline{1,k} \text{ and } x^*_1 + ... + x^*_k = 0 \implies x^*_1 = ... = x^*_k = 0;$$

(ii) $S_1, ..., S_k$ are allied at $\overline{x}$;

(iii) there exist $a, r > 0$ such that for any $x_i \in S_i \cap B(\overline{x}, r), x^*_i \in \widehat{N}(S_i, x_i)$, one has

$$\max_{i=\overline{1,k}} \|x^*_i\| \geq 1 \implies \left\| \sum_{i=1}^{k} x^*_i \right\| \geq a^{-1};$$

(iv) $S_1, ..., S_k$ satisfy the metric inequality at $\overline{x}$;

(v) there exists $r > 0$ such that, for every $\varepsilon > 0$ and every $x \in [S_1 \cap ... \cap S_k] \cap B(\overline{x}, r)$, there exist $x_i \in S_i \cap B(x, \varepsilon), i = \overline{1,k}$ such that

$$\widehat{N}(S_1 \cap ... \cap S_k, x) \subset \widehat{N}(S_1, x_1) + ... + \widehat{N}(S_k, x_k) + \varepsilon \overline{B}_{X^*}.$$

Then (i) $\Rightarrow$ (iv) and (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v).

Proof. For the proof of $(i) \Rightarrow (iv)$, see, e.g., [49, Theorem 3.8], [55, Corollary 3.10]. For the rest of the implications, one may find the proofs (with obvious minor modifications) in [55, Theorem 3.7, Propositions 3.8, 3.9], [49, Theorem 3.8]. □

Let X, Y_1, Y_2, Z be Asplund spaces. Suppose that $F_1 : X \rightrightarrows Y_1$, $F_2 : X \rightrightarrows Y_2$ and $G : Y_1 \times Y_2 \rightrightarrows Z$ are multifunctions and $(\overline{x}, \overline{y}_1, \overline{y}_2, \overline{z}) \in X \times Y_1 \times Y_2 \times Z$ is such that $(\overline{x}, \overline{y}_1) \in \operatorname{Gr} F_1$, $(\overline{x}, \overline{y}_2) \in \operatorname{Gr} F_2$, $((\overline{y}_1, \overline{y}_2), \overline{z}) \in \operatorname{Gr} G$. Setting

$$\begin{aligned} C_1 &:= \{(x, y_1, y_2, z) \in X \times Y_1 \times Y_2 \times Z : y_1 \in F_1(x)\}, \\ C_2 &:= \{(x, y_1, y_2, z) \in X \times Y_1 \times Y_2 \times Z : y_2 \in F_2(x)\}, \\ C_3 &:= \{(x, y_1, y_2, z) \in X \times Y_1 \times Y_2 \times Z : z \in G(y_1, y_2)\}, \end{aligned} \tag{4.4}$$

observe that $(\overline{x}, \overline{y}_1, \overline{y}_2, \overline{z}) \in C_1 \cap C_2 \cap C_3$.

Consider now the alliedness property of C_1, C_2, C_3 at $(\overline{x}, \overline{y}_1, \overline{y}_2, \overline{z})$: for any sequences $(x_n, y_{1n}) \overset{\operatorname{Gr} F_1}{\longrightarrow} (\overline{x}, \overline{y}_1)$, $(u_n, v_{2n}) \overset{\operatorname{Gr} F_2}{\longrightarrow} (\overline{x}, \overline{y}_2)$, $(b_{1n}, b_{2n}, c_n) \overset{\operatorname{Gr} G}{\longrightarrow} (\overline{y}_1, \overline{y}_2, \overline{z})$ and every $x_n^* \in \widehat{D}^* F_1(x_n, y_{1n})(y_{1n}^*)$, $u_n^* \in \widehat{D}^* F_2(u_n, v_{2n})(v_{2n}^*)$, $(b_{1n}^*, b_{2n}^*) \in \widehat{D}^* G(b_{1n}, b_{2n}, c_n)(c_n^*)$, the relations $(x_n^* + u_n^*) \to 0$, $(y_{1n}^* + b_{1n}^*) \to 0$, $(v_{2n}^* + b_{2n}^*) \to 0$, $(c_n^*) \to 0$ imply

$$(x_n^*) \to 0, (u_n^*) \to 0, (y_{1n}^*) \to 0, (b_{1n}^*) \to 0, (v_{2n}^*) \to 0, (b_{2n}^*) \to 0.$$

The next result is twofold. On one hand, it provides a sufficient Fréchet coderivative condition for linear openness/metric regularity on Asplund spaces. On the other hand, it serves in the sequel as the basis for getting again the conclusion of the main result from the previous section, i.e. Theorem 3.6.

Theorem 4.2 *Let X, Y_1, Y_2 and Z be Asplund spaces, $F_1 : X \rightrightarrows Y_1$, $F_2 : X \rightrightarrows Y_2$, $G : Y_1 \times Y_2 \rightrightarrows Z$ be closed-graph multifunctions, and $(\overline{x}, \overline{y}_1, \overline{y}_2, \overline{z}) \in X \times Y_1 \times Y_2 \times Z$ be such that $\overline{z} \in G(\overline{y}_1, \overline{y}_2), (\overline{y}_1, \overline{y}_2) \in F_1(\overline{x}) \times F_2(\overline{x})$. Assume that the sets C_1, C_2, C_3 defined by (4.4) are allied at $(\overline{x}, \overline{y}_1, \overline{y}_2, \overline{z})$ and that there exists $c > 0$ such that*

$$c < \liminf_{\substack{(u_1,v_1) \overset{F_1}{\to} (\overline{x}, \overline{y}_1),\ (u_2,v_2) \overset{F_2}{\to} (\overline{x}, \overline{y}_2) \\ (t_1,t_2,w) \overset{G}{\to} (\overline{y}_1, \overline{y}_2, \overline{z}),\ \delta \downarrow 0}} \left\{ \|x_1^* + x_2^*\| : \begin{array}{l} x_1^* \in \widehat{D}^* F_1(u_1, v_1)(t_1^*), \\ x_2^* \in \widehat{D}^* F_2(u_2, v_2)(t_2^*), \\ (z_1^* + t_1^*, z_2^* + t_2^*) \in \widehat{D}^* G(t_1, t_2, w)(w^*), \\ \|w^*\| = 1, \|z_1^*\| < \delta, \|z_2^*\| < \delta, \end{array} \right\}. \tag{C}$$

Then for every $a \in (0, c)$, H is $a-$open at $(\overline{x}, \overline{z})$. If, moreover, $(F_1, F_2), G$ are locally composition-stable around $(\overline{x}, (\overline{y}_1, \overline{y}_2), \overline{z})$, then for every $a \in (0, c)$, H is $a-$open around $(\overline{x}, \overline{z})$.

Proof. Fix $a \in (0, c)$. Remark that the condition (C) assures the existence of $r > 0$ such that for every $\delta \in (0, r)$, every $(u_1, v_1) \in \operatorname{Gr} F_1 \cap [B(\overline{x}, r) \times B(\overline{y}_1, r)]$, $(u_2, v_2) \in \operatorname{Gr} F_2 \cap [B(\overline{x}, r) \times B(\overline{y}_2, r)]$, $(t_1, t_2, w) \in \operatorname{Gr} G \cap [B(\overline{y}_1, r) \times B(\overline{y}_2, r) \times B(\overline{z}, r)]$, every $w^* \in S_{Z^*}$, $t_1^* \in Y_1^*$, $t_2^* \in Y_2^*$, $z_1^* \in \delta B_{Y_1^*}$, $z_2^* \in \delta B_{Y_2^*}$ such that $(z_1^* + t_1^*, z_2^* + t_2^*) \in \widehat{D}^* G((t_1, t_2), w)(w^*)$ and every $x_1^* \in \widehat{D}^* F_1(u_1, v_1)(t_1^*)$, $x_2^* \in \widehat{D}^* F_2(u_2, v_2)(t_2^*)$,

$$c \leq \|x_1^* + x_2^*\|. \tag{4.5}$$

We will prove the existence of $\varepsilon > 0$ such that, for every $\rho \in (0, \varepsilon)$ and every $(\widehat{x}, \widehat{y}_1, \widehat{y}_2, \widehat{z}) \in [B(\overline{x}, \varepsilon) \times B(\overline{y}_1, \varepsilon) \times B(\overline{y}_2, \varepsilon) \times B(\overline{z}, \varepsilon)] \cap [C_1 \cap C_2 \cap C_3]$,

$$B(\widehat{z}, a\rho) \subset H(B(\widehat{x}, \rho)). \tag{4.6}$$

This will show in particular that H is $a-$open at $(\overline{x},\overline{z})$. Moreover, if $(F_1,F_2),G$ are locally composition-stable around $(\overline{x},(\overline{y}_1,\overline{y}_2),\overline{z})$, the definition of this concept ensures that if (x,z) is close to $(\overline{x},\overline{z})$, one can find (y_1,y_2) close to $(\overline{y}_1,\overline{y}_2)$ such that $(x,y_1,y_2,z)\in C_1\cap C_2\cap C_3$, hence one may apply (4.6) in order to conclude that H is $a-$open around $(\overline{x},\overline{z})$.

Let us proceed to the proof of (4.6). Choose $b\in(0,1)$ such that $\dfrac{a}{a+1}<b<\dfrac{c}{c+1}$ and fix $\delta\in(0,r)$ as above. Then one can find $\alpha,\varepsilon>0$ such that the following are satisfied:

- $b^{-1}a\varepsilon<4^{-1}r$;
- $\alpha^{-1}a\varepsilon<4^{-1}r$;
- $\varepsilon<2^{-1}r$;
- $\dfrac{a}{a+1}<b+2\varepsilon<\dfrac{c}{c+1}$;
- $b+2\varepsilon<1$;
- $\dfrac{\alpha(1+2\varepsilon)}{1-(b+2\varepsilon)}<\delta$.

Fix $\rho\in(0,\varepsilon)$ and $(\widehat{x},\widehat{y}_1,\widehat{y}_2,\widehat{z})\in B(\overline{x},\varepsilon)\times B(\overline{y}_1,\varepsilon)\times B(\overline{y}_2,\varepsilon)\times B(\overline{z},\varepsilon)]\cap[C_1\cap C_2\cap C_3]$. Choose now $v\in B(\widehat{z},\rho a)$. Observe that the set $C_1\cap C_2\cap C_3$ is closed because of the closedness of the graphs of F_1,F_2 and G. Endow the space $X\times Y_1\times Y_2\times Z$ with the distance

$$d((x,y_1,y_2,z),(x',y_1',y_2',z')):=b\left\|x-x'\right\|+\alpha\left\|y_1-y_1'\right\|+\alpha\left\|y_2-y_2'\right\|+b\left\|z-z'\right\|$$

and apply the Ekeland Variational Principle for the function $f:C_1\cap C_2\cap C_3\to\mathbb{R}$, $f(x,y_1,y_2,z):=\|v-z\|$ and $(\widehat{x},\widehat{y}_1,\widehat{y}_2,\widehat{z})\in\operatorname{dom}f$ in order to find $(x_0,y_{10},y_{20},z_0)\in C_1\cap C_2\cap C_3$ such that

$$\|v-z_0\|\leq\|v-\widehat{z}\|-(b\left\|\widehat{x}-x_0\right\|+\alpha\left\|\widehat{y}_1-y_{10}\right\|+\alpha\left\|\widehat{y}_2-y_{20}\right\|+b\left\|\widehat{z}-z_0\right\|) \tag{4.7}$$

and

$$\|v-z_0\|\leq\|v-z\|+(b\left\|x-x_0\right\|+\alpha\left\|y_1-y_{10}\right\|+\alpha\left\|y_2-y_{20}\right\|+b\left\|z-z_0\right\|), \tag{4.8}$$

for every $(x,y_1,y_2,z)\in C_1\cap C_2\cap C_3$.

We have

$$\begin{aligned}\|\widehat{x}-x_0\|+\|\widehat{z}-z_0\|&\leq b^{-1}\left\|v-\widehat{z}\right\|<b^{-1}a\rho\leq b^{-1}a\varepsilon<4^{-1}r,\\ \|\widehat{y}_1-y_{10}\|+\|\widehat{y}_2-y_{20}\|&\leq\alpha^{-1}\left\|v-\widehat{z}\right\|<\alpha^{-1}a\rho\leq\alpha^{-1}a\varepsilon<4^{-1}r,\end{aligned}$$

hence

$$\begin{aligned}(x_0,y_{10},y_{20},z_0)&\in B(\widehat{x},4^{-1}r)\times B(\widehat{y}_1,4^{-1}r)\times B(\widehat{y}_2,4^{-1}r)\times B(\widehat{z},4^{-1}r).\\ &\subset B(\overline{x},2^{-1}r)\times B(\overline{y}_1,2^{-1}r)\times B(\overline{y}_2,2^{-1}r)\times B(\overline{z},2^{-1}r).\end{aligned}$$

If $v=z_0$, then

$$\begin{aligned}b\left\|\widehat{x}-x_0\right\|&\leq(1-b)\left\|v-\widehat{z}\right\|\\ &<(1-b)a\rho<b\rho,\end{aligned}$$

hence $x_0 \in B(\widehat{x}, \rho)$ and $v = z_0 \in H(x_0) \subset H(B(\widehat{x}, \rho))$, which proves the desired conclusion.

We want to prove now that $v = z_0$ is the sole possible situation. Suppose then by contradiction that $v \neq z_0$ and consider the function $h : X \times Y_1 \times Y_2 \times Z \to \mathbb{R}$,

$$h(x, y_1, y_2, z) := \|v - z\| + (b\,\|x - x_0\| + \alpha\,\|y_1 - y_{10}\| + \alpha\,\|y_2 - y_{20}\| + b\,\|z - z_0\|).$$

From (4.8), we have that the point $(x_0, y_{10}, y_{20}, z_0)$ is a minimum point for h on the set $C_1 \cap C_2 \cap C_3$, or, equivalently, $(x_0, y_{10}, y_{20}, z_0)$ is a global minimum point for the function $h + \delta_{C_1 \cap C_2 \cap C_3}$. Applying the generalized Fermat rule, we have that

$$(0, 0, 0, 0) \in \widehat{\partial}(h(\cdot, \cdot, \cdot, \cdot) + \delta_{C_1 \cap C_2 \cap C_3}(\cdot, \cdot, \cdot, \cdot))(x_0, y_{10}, y_{20}, z_0).$$

Observe now that h is Lipschitz and $\delta_{C_1 \cap C_2 \cap C_3}$ is lower semicontinuous, so one can apply the fuzzy calculus rule for the Fréchet subdifferential. Choose $\gamma \in (0, \min\{\rho, \alpha\rho, 4^{-1}r\})$ such that $v \notin \overline{B}(z_0, \gamma)$ and obtain that there exist

$$\begin{aligned}
&(u_1, v_{11}, v_{21}, w_1) \in \overline{B}(x_0, \gamma) \times \overline{B}(y_{10}, \gamma) \times \overline{B}(y_{20}, \gamma) \times \overline{B}(z_0, \gamma),\\
&(u_2, v_{12}, v_{22}, w_2) \in [\overline{B}(x_0, \gamma) \times \overline{B}(y_{10}, \gamma) \times \overline{B}(y_{20}, \gamma) \times \overline{B}(z_0, \gamma)] \cap [C_1 \cap C_2 \cap C_3]
\end{aligned}$$

such that

$$(0, 0, 0, 0) \in \widehat{\partial} h(u_1, v_{11}, v_{21}, w_1) + \widehat{\partial}\delta_{C_1 \cap C_2 \cap C_3}(u_2, v_{12}, v_{22}, w_2) + \gamma(\overline{B}_{X^*} \times \overline{B}_{Y_1^*} \times \overline{B}_{Y_2^*} \times \overline{B}_{Z^*}). \quad (4.9)$$

Observe that h is the sum of five convex functions, Lipschitz on $X \times Y_1 \times Y_2 \times Z$, hence $\widehat{\partial} h$ coincides with the sum of the convex subdifferentials. Also, because $v \neq w_1 \in \overline{B}(z_0, \gamma)$, we get

$$\widehat{\partial}\,\|v - \cdot\|\,(w_1) = \{z^* \mid z^* \in S_{Z^*},\ z^*(v - w_1) = \|v - w_1\|\}.$$

Consequently, we have from (4.9) that

$$\begin{aligned}
(0, 0, 0, 0) \in\ & \{0\} \times \{0\} \times \{0\} \times S_{Z^*} + b\overline{B}_{X^*} \times \{0\} \times \{0\} \times \{0\}\\
&+ \{0\} \times \alpha\overline{B}_{Y_1^*} \times \{0\} \times \{0\} + \{0\} \times \{0\} \times \alpha\overline{B}_{Y_2^*} \times \{0\} + \{0\} \times \{0\} \times \{0\} \times b\overline{B}_{Z^*}\\
&+ \widehat{N}(C_1 \cap C_2 \cap C_3, (u_2, v_{12}, v_{22}, w_2)) + \rho\overline{B}_{X^*} \times \alpha\rho\overline{B}_{Y_1^*} \times \alpha\rho\overline{B}_{Y_2^*} \times \rho\overline{B}_{Z^*}\\
\subset\ & \{0\} \times \{0\} \times \{0\} \times S_{Z^*} + \widehat{N}(C_1 \cap C_2 \cap C_3, (u_2, v_{12}, v_{22}, w_2))\\
&+ (b + \rho)\overline{B}_{X^*} \times \alpha(1 + \rho)\overline{B}_{Y_1^*} \times \alpha(1 + \rho)\overline{B}_{Y_2^*} \times (b + \rho)\overline{B}_{Z^*}.
\end{aligned}$$

Now, use the alliedness of C_1, C_2, C_3 at $(\overline{x}, \overline{y}_1, \overline{y}_2, \overline{z})$ to get that

$$\begin{aligned}
\widehat{N}(C_1 \cap C_2 \cap C_3, (u_2, v_{12}, v_{22}, w_2)) \subset\ & \widehat{N}(C_1, (u_3, v_{13}, v_{23}, w_3)) + \widehat{N}(C_2, (u_4, v_{14}, v_{24}, w_4))\\
&+ \widehat{N}(C_3, (u_5, v_{15}, v_{25}, w_5)) + \rho\overline{B}_{X^*} \times \alpha\rho\overline{B}_{Y_1^*} \times \alpha\rho\overline{B}_{Y_2^*} \times \rho\overline{B}_{Z^*},
\end{aligned}$$

where

$$\begin{aligned}
&(u_3, v_{13}, v_{23}, w_3) \in [\overline{B}(u_2, \gamma) \times \overline{B}(v_{12}, \gamma) \times \overline{B}(v_{22}, \gamma) \times \overline{B}(w_2, \gamma)] \cap C_1,\\
&(u_4, v_{14}, v_{24}, w_4) \in [\overline{B}(u_2, \gamma) \times \overline{B}(v_{12}, \gamma) \times \overline{B}(v_{22}, \gamma) \times \overline{B}(w_2, \gamma)] \cap C_2,\\
&(u_5, v_{15}, v_{25}, w_5) \in [\overline{B}(u_2, \gamma) \times \overline{B}(v_{12}, \gamma) \times \overline{B}(v_{22}, \gamma) \times \overline{B}(w_2, \gamma)] \cap C_3.
\end{aligned}$$

In conclusion, there exist

$$
\begin{aligned}
w_0^* &\in S_{Z^*},\\
(u_3^*, v_{13}^*, 0, 0) &\in \widehat{N}(C_1, (u_3, v_{13}, v_{23}, w_3)) \iff u_3^* \in \widehat{D}^* F_1(u_3, v_{13})(-v_{13}^*),\\
(u_4^*, 0, v_{24}^*, 0) &\in \widehat{N}(C_2, (u_4, v_{14}, v_{24}, w_4)) \iff u_4^* \in \widehat{D}^* F_2(u_4, v_{14})(-v_{24}^*),\\
(u_5^*, v_{15}^*, v_{25}^*, w_5^*) &\in \overline{B}_{X^*} \times \overline{B}_{Y_1^*} \times \overline{B}_{Y_2^*} \times \overline{B}_{Z^*}
\end{aligned}
$$

such that

$$
\begin{aligned}
&(-u_3^* - u_4^* - (b+2\rho)u_5^*, -v_{13}^* - \alpha(1+2\rho)v_{15}^*, -v_{24}^* - \alpha(1+2\rho)v_{25}^*, -w_0^* - (b+2\rho)w_5^*)\\
&\in \widehat{N}(C_3, (u_5, v_{15}, v_{25}, w_5)),
\end{aligned}
$$

i.e.

$$
\begin{aligned}
-u_3^* - u_4^* - (b+2\rho)u_5^* = 0,\\
(-v_{13}^* - \alpha(1+2\rho)v_{15}^*, -v_{24}^* - \alpha(1+2\rho)v_{25}^*) \in \widehat{D}^* G(v_{15}, v_{25}, w_5)(w_0^* + (b+2\rho)w_5^*).
\end{aligned}
$$

Observe that

$$
\|w_0^* + (b+2\rho)w_5^*\| \geq 1 - (b+2\rho) > 1 - (b+2\varepsilon) > 0,
$$

and denote

$$
\begin{aligned}
x_1^* &:= \|w_0^* + (b+2\rho)w_5^*\|^{-1} u_3^*,\\
x_2^* &:= \|w_0^* + (b+2\rho)w_5^*\|^{-1} u_4^*,\\
t_1^* &:= -\|w_0^* + (b+2\rho)w_5^*\|^{-1} v_{13}^*,\\
t_2^* &:= -\|w_0^* + (b+2\rho)w_5^*\|^{-1} v_{24}^*,\\
w^* &:= \|w_0^* + (b+2\rho)w_5^*\|^{-1} (w_0^* + (b+2\rho)w_5^*),\\
z_1^* &:= -\|w_0^* + (b+2\rho)w_5^*\|^{-1} \alpha(1+2\rho)v_{15}^*\\
z_2^* &:= -\|w_0^* + (b+2\rho)w_5^*\|^{-1} \alpha(1+2\rho)v_{25}^*.
\end{aligned}
$$

In conclusion, one has

$$
\begin{aligned}
x_1^* &\in \widehat{D}^* F_1(u_3, v_{13})(t_1^*),\\
x_2^* &\in \widehat{D}^* F_2(u_4, v_{24})(t_2^*),\\
(t_1^* + z_1^*, t_2^* + z_2^*) &\in \widehat{D}^* G(v_{15}, v_{25}, w_5)(w^*),
\end{aligned}
$$

where

$$
\|w^*\| = 1, \tag{4.10}
$$

$$
\|x_1^* + x_2^*\| = \frac{\|u_3^* + u_4^*\|}{\|w_0^* + (b+2\rho)w_5^*\|} \leq \frac{b+2\rho}{1-(b+2\rho)} < c \tag{4.11}
$$

$$
\|z_1^*\| \leq \frac{\alpha(1+2\rho)}{1-(b+2\rho)} < \delta, \tag{4.12}
$$

$$
\|z_2^*\| \leq \frac{\alpha(1+2\rho)}{1-(b+2\rho)} < \delta. \tag{4.13}
$$

Remark that $(u_3, v_{13}) \in \operatorname{Gr} F_1$ and

$$(u_3, v_{13}) \in \overline{B}(u_2, \gamma) \times \overline{B}(y_{12}, \gamma) \subset \overline{B}(x_0, 2\gamma) \times \overline{B}(y_{10}, 2\gamma)$$
$$\subset B(\widehat{x}, 2^{-1}r) \times B(\widehat{y}_1, 2^{-1}r) \subset B(\overline{x}, r) \times B(\overline{y}_2, r).$$

Analogously, $(u_4, v_{24}) \in \operatorname{Gr} F_2 \cap [B(\overline{x}, r) \times B(\overline{y}_2, r)]$ and $(v_{15}, v_{25}, w_5) \in \operatorname{Gr} G \cap [B(\overline{y}_1, r) \times B(\overline{y}_2, r) \times B(\overline{z}, r)]$. Using now (4.5) and (4.11), we get that

$$c \leq \|x_1^* + x_2^*\| < c,$$

a contradiction. It follows that $v = z_0$ is the sole possible situation and the conclusion follows. □

We remark that the conclusion of Theorem 4.2 could be deduced (with a different technique) using as an intermediate step some estimations for the strong slope of the lower semicontinuous envelope φ_R, as done in [52], [20]. We prefered here this direct aproach for clarity and for the sake of including our result in the framework opened by Mordukhovich and Shao [46] and Penot [56].

We want to discuss in the sequel the possible relations between the coderivative conditions which assure the linear openness/metric regularity of H and the main result, i.e. Theorem 3.6. To this, let us formulate first an auxiliary result.

Proposition 4.3 *(A) Let X, Y be Banach spaces, $\Gamma : X \rightrightarrows Y$ be a multifunction and $(\overline{x}, \overline{y}) \in \operatorname{Gr}\Gamma$.*

(i) If Γ has the Aubin property around $(\overline{x}, \overline{y})$, then there exists $\alpha > 0$ arbitrarily close to $\operatorname{lip}\Gamma(\overline{x}, \overline{y})$ and $r > 0$ such that, for every $(x, y) \in \operatorname{Gr}\Gamma \cap [B(\overline{x}, r) \times B(\overline{y}, r)]$, $y^* \in Y^*$ and $x^* \in \widehat{D}^*\Gamma(x, y)(y^*)$,

$$\|x^*\| \leq \alpha \|y^*\|.$$

(ii) If Γ is open at linear rate around $(\overline{x}, \overline{y})$, then there exists $\alpha > 0$ arbitrarily close to $\operatorname{lop}\Gamma(\overline{x}, \overline{y})$ and $r > 0$ such that, for every $(x, y) \in \operatorname{Gr}\Gamma \cap [B(\overline{x}, r) \times B(\overline{y}, r)]$, $y^* \in Y^*$ and $x^* \in \widehat{D}^*\Gamma(x, y)(y^*)$,

$$\|x^*\| \geq \alpha \|y^*\|.$$

(B) Let X, Y, Z be Banach spaces, $\Phi : X \times Y \rightrightarrows Z$ be a multifunction and $((\overline{x}, \overline{y}), \overline{z}) \in \operatorname{Gr}\Phi$.

(i) If Φ has the Aubin property with respect to x, uniformly in y around $((\overline{x}, \overline{y}), \overline{z})$, then there exists $\alpha > 0$ arbitrarily close to $\widehat{\operatorname{lip}}_x\Phi((\overline{x}, \overline{y}), \overline{z})$ and $r > 0$ such that, for every $y \in B(\overline{y}, r)$, every $(x, z) \in \operatorname{Gr}\Phi_y \cap [B(\overline{x}, r) \times B(\overline{z}, r)]$, $z^* \in Z^*$ and $x^* \in \widehat{D}^*\Phi_y(x, z)(z^*)$,

$$\|x^*\| \leq \alpha \|z^*\|.$$

(ii) If Φ is open at linear rate with respect to x, uniformly in y around $((\overline{x}, \overline{y}), \overline{z})$, then there exists $\alpha > 0$ arbitrarily close to $\widehat{\operatorname{lop}}_x\Phi((\overline{x}, \overline{y}), \overline{z})$ and $r > 0$ such that, for every $y \in B(\overline{y}, r)$, every $(x, z) \in \operatorname{Gr}\Phi_y \cap [B(\overline{x}, r) \times B(\overline{z}, r)]$, $z^* \in Z^*$ and $x^* \in \widehat{D}^*\Phi_y(x, z)(z^*)$,

$$\|x^*\| \geq \alpha \|z^*\|.$$

Proof. The assertion from (A) are well-known. See, for instance, [45, Theorems 1.43, 1.54]. Concerning (B), the proof is very similar to the one of (A) and it mimics the proofs of the mentioned theorems. For the readers convenience, let us prove the item (i) from (B).

Suppose that Φ has the Aubin property with respect to x, uniformly in y around $((\overline{x},\overline{y}),\overline{z})$, hence there exists $\alpha > 0$ arbitrarily close to $\widehat{\operatorname{lip}}_x \Phi((\overline{x},\overline{y}),\overline{z})$ and $\varepsilon > 0$ such that, for every $y \in B(\overline{y},\varepsilon)$ and every $x,u \in B(\overline{x},\varepsilon)$,

$$\Phi_y(x) \cap B(\overline{z},\varepsilon) \subset \Phi_y(u) + \alpha \|x-u\| B_Z. \tag{4.14}$$

Consider $r := 2^{-1}\varepsilon$ and take arbitrary $y \in B(\overline{y},r)$, $(x,z) \in \operatorname{Gr}\Phi_y \cap [B(\overline{x},r) \times B(\overline{z},r)]$, $z^* \in Z^*$, $x^* \in \widehat{D}^*\Phi_y(x,z)(z^*)$ and $\delta > 0$. Employing the definition of the Fréchet coderivatives, one may find $\gamma < \min\{2^{-1}\varepsilon, 2^{-1}\alpha\varepsilon\}$ such that

$$\langle x^*, u-x\rangle - \langle z^*, w-z\rangle \le \delta(\|u-x\| + \|w-z\|) \tag{4.15}$$

for every $(u,w) \in \operatorname{Gr}\Phi_y \cap [B(x,\alpha^{-1}\gamma) \times B(z,\gamma)]$.

Take now arbitrary $u \in B(x,\alpha^{-1}\gamma)$. Then

$$\|u-\overline{x}\| \le \|u-x\| + \|x-\overline{x}\| \le \alpha^{-1}\gamma + r < \varepsilon,$$

hence one may apply (4.14) for this u and $z \in \Phi_y(x) \cap B(\overline{z},\varepsilon)$ in order to find $w \in \Phi_y(u)$ such that

$$\|w-u\| \le \alpha \|x-u\| \le \gamma.$$

Consequently, using (4.15), one obtains

$$\alpha^{-1}\gamma \|x^*\| \le \gamma \|z^*\| + \delta\gamma(1+\alpha^{-1}).$$

As δ is arbitrary positive, the conclusion follows.

The proof of the item (ii) from (B) takes into account two facts. The first one is the obvious equivalence

$$x^* \in \widehat{D}^*\Phi_y(x,z)(z^*) \iff -z^* \in \widehat{D}^*\Phi_y^{-1}(z,x)(-x^*).$$

The second one is more technical, and it concerns a link between the regularity notions in the parametric case. More precisely, following the lines of the proof of Theorem 2.2, if Φ is open at linear rate with respect to x, uniformly in y around $((\overline{x},\overline{y}),\overline{z})$, then Φ_y^{-1} is Aubin around $(\overline{z},\overline{x})$, for every y in a neighborhood which can be taken exactly the same as in the definition of the uniform openness in y. Then (A), item (i), can be applied and the conclusion follows. □

Let us make the final step of our plan to derive again Theorem 3.6 on Asplund spaces, by means of coderivative conditions in Theorem 4.2.

Theorem 4.4 *Let X, Y_1, Y_2 and Z be Asplund spaces, $F_1 : X \rightrightarrows Y_1$, $F_2 : X \rightrightarrows Y_2$, $G : Y_1 \times Y_2 \rightrightarrows Z$ be closed-graph multifunctions, and $(\overline{x},\overline{y}_1,\overline{y}_2,\overline{z}) \in X \times Y_1 \times Y_2 \times Z$ be such that $\overline{z} \in G(\overline{y}_1,\overline{y}_2), (\overline{y}_1,\overline{y}_2) \in F_1(\overline{x}) \times F_2(\overline{x})$. Suppose that the assumptions (i)-(v) of Theorem 3.6 are satisfied. Then the sets C_1, C_2, C_3 given by (4.4) are allied at $(\overline{x},\overline{y}_1,\overline{y}_2,\overline{z})$. Moreover, there exist $L, M, C, D > 0$ arbitrarily close to $\operatorname{reg} F_1(\overline{x},\overline{y}_1)$, $\operatorname{lip} F_2(\overline{x},\overline{y}_2)$, $\widehat{\operatorname{reg}}_{y_1} G((\overline{y}_1,\overline{y}_2),\overline{z})$, $\widehat{\operatorname{lip}}_{y_2} G((\overline{y}_1,\overline{y}_2),\overline{z})$ such that $LMCD < 1$ and for every $\varepsilon > 0$ arbitrary small, the relation (C) is satisfied with*

$$c := \frac{1}{LC} - MD - \varepsilon > 0.$$

In conclusion, for every $a \in (0,c)$, H is metrically regular at $(\overline{x},\overline{z})$ with constant a. If, moreover, $(F_1,F_2), G$ are locally composition-stable around $(\overline{x},(\overline{y}_1,\overline{y}_2),\overline{z})$, then for every $a \in (0,c)$, H is metrically regular around $(\overline{x},\overline{z})$ with constant a. As consequence,

$$\operatorname{reg} H(\overline{x},\overline{z}) \le \rho_0,$$

where ρ_0 is given by (3.12).

Proof. Suppose that the multifunctions F_1, F_2, G satisfy the assumptions (ii)-(iv) of Theorem 3.6 and consider arbitrary sequences $(x_n, y_{1n}) \stackrel{\operatorname{Gr} F_1}{\longrightarrow} (\overline{x}, \overline{y}_1)$, $(u_n, v_{2n}) \stackrel{\operatorname{Gr} F_2}{\longrightarrow} (\overline{x}, \overline{y}_2)$, $(b_{1n}, b_{2n}, c_n) \stackrel{\operatorname{Gr} G}{\longrightarrow} (\overline{y}_1, \overline{y}_2, \overline{z})$ and $x_n^* \in \widehat{D}^* F_1(x_n, y_{1n})(y_{1n}^*)$, $u_n^* \in \widehat{D}^* F_2(u_n, v_{2n})(v_{2n}^*)$, $(b_{1n}^*, b_{2n}^*) \in \widehat{D}^* G(b_{1n}, b_{2n}, c_n)(c_n^*)$ such that the relations $(x_n^* + u_n^*) \to 0$, $(y_{1n}^* + b_{1n}^*) \to 0, (v_{2n}^* + b_{2n}^*) \to 0, (c_n^*) \to 0$ hold. Then for every n, $b_{1n}^* \in \widehat{D}^* G_{b_{2n}}(b_{1n}, c_n)(c_n^*)$ and $b_{2n}^* \in \widehat{D}^* G_{b_{1n}}(b_{2n}, c_n)(c_n^*)$. Using now Theorem 4.3, there exist $L, M, C, D > 0$ arbitrarily close to $\operatorname{reg} F_1(\overline{x}, \overline{y}_1)$, $\operatorname{lip} F_2(\overline{x}, \overline{y}_2)$, $\widehat{\operatorname{reg}}_{y_1} G((\overline{y}_1, \overline{y}_2), \overline{z})$, $\widehat{\operatorname{lip}}_{y_2} G((\overline{y}_1, \overline{y}_2), \overline{z})$ such that $LMCD < 1$ and, for every n, one has

$$\begin{aligned} \|x_n^*\| &\geq L^{-1} \|y_{1n}^*\|, \\ \|u_n^*\| &\leq M \|v_{2n}^*\|, \\ \|b_{1n}^*\| &\geq C^{-1} \|c_n^*\|, \\ \|b_{2n}^*\| &\leq D \|c_n^*\|. \end{aligned} \tag{4.16}$$

As $(c_n^*) \to 0$, it follows successively from (4.16) and the assumptions made that $(b_{2n}^*) \to 0$, $(v_{2n}^*) \to 0$, $(u_n^*) \to 0$, $(x_n^*) \to 0$, $(y_{1n}^*) \to 0$, $(b_{1n}^*) \to 0$. In conclusion, the sets C_1, C_2, C_3 are allied at $(\overline{x}, \overline{y}_1, \overline{y}_2, \overline{z})$.

Using again the properties (i)-(iv) and Theorem 4.3, one gets the existence of $r > 0$ and of $L, M, C, D > 0$ as above such that:

1. for every $(x, y_1) \in \operatorname{Gr} F_1 \cap [B(\overline{x}, r) \times B(\overline{y}_1, r)]$, $y_1^* \in Y^*$ and $x^* \in \widehat{D}^* F_1(x, y_1)(y_1^*)$,
$$\|x^*\| \geq L^{-1} \|y_1^*\|; \tag{4.17}$$
2. for every $(x, y_2) \in \operatorname{Gr} F_2 \cap [B(\overline{x}, r) \times B(\overline{y}_2, r)]$, $y_2^* \in Y^*$ and $x^* \in \widehat{D}^* F_2(x, y_2)(y_2^*)$,
$$\|x^*\| \leq M \|y_2^*\|; \tag{4.18}$$
3. for every $y_2 \in B(\overline{y}_2, r)$, every $(y_1, z) \in \operatorname{Gr} G_{y_2} \cap [B(\overline{y}_1, r) \times B(\overline{z}, r)]$, $z^* \in Z^*$ and $y_1^* \in \widehat{D}^* G_{y_2}(y_1, z)(z^*)$,
$$\|y_1^*\| \geq C^{-1} \|z^*\|; \tag{4.19}$$
4. for every $y_1 \in B(\overline{y}_1, r)$, every $(y_2, z) \in \operatorname{Gr} G_{y_1} \cap [B(\overline{y}_2, r) \times B(\overline{z}, r)]$, $z^* \in Z^*$ and $y_2^* \in \widehat{D}^* G_{y_1}(y_2, z)(z^*)$,
$$\|y_2^*\| \leq D \|z^*\|. \tag{4.20}$$

Choose next arbitrary $\varepsilon > 0$ and $\delta \in (0, r)$ such that $(L^{-1} + M)\delta < \varepsilon$. Take $(u_1, v_1) \in \operatorname{Gr} F_1 \cap [B(\overline{x}, r) \times B(\overline{y}_1, r)]$, $(u_2, v_2) \in \operatorname{Gr} F_2 \cap [B(\overline{x}, r) \times B(\overline{y}_2, r)]$, $(t_1, t_2, w) \in \operatorname{Gr} G \cap [B(\overline{y}_1, r) \times B(\overline{y}_2, r) \times B(\overline{z}, r)]$, $w^* \in S_{Z^*}$, $t_1^* \in Y_1^*$, $t_2^* \in Y_2^*$, $z_1^* \in \delta B_{Y_1^*}$, $z_2^* \in \delta B_{Y_2^*}$ such that $(z_1^* + t_1^*, z_2^* + t_2^*) \in \widehat{D}^* G((t_1, t_2), w)(w^*)$ and $x_1^* \in \widehat{D}^* F_1(u_1, v_1)(t_1^*)$, $x_2^* \in \widehat{D}^* F_2(u_2, v_2)(t_2^*)$. Then $z_1^* + t_1^* \in \widehat{D}^* G_{t_2}(t_1, w)(w^*)$ and $z_2^* + t_2^* \in \widehat{D}^* G_{t_1}(t_2, w)(w^*)$, and using (4.17)-(4.20), one gets

$$\begin{aligned} \|x_1^* + x_2^*\| &\geq \|x_1^*\| - \|x_2^*\| \geq L^{-1} \|t_1^*\| - M \|t_2^*\| \\ &\geq L^{-1} (\|z_1^* + t_1^*\| - \|z_1^*\|) - M(\|z_2^* + t_2^*\| + \|z_2^*\|) \\ &\geq L^{-1} C^{-1} \|w^*\| - MD \|w^*\| - (L^{-1} + M)\delta \\ &> L^{-1} C^{-1} - MD - \varepsilon. \end{aligned}$$

It follows that relation (C) is satisfied for $c = L^{-1} C^{-1} - MD - \varepsilon > 0$, i.e. the conclusion follows. □

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