Equivariant Perturbation in Gomory and Johnson's Infinite Group Problem. II. The Unimodular Two-Dimensional Case

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Abstract. We give an algorithm for testing the extremality of a large class of minimal valid functions for the two-dimensional infinite group problem.

1 Introduction

1.1 The group problem

Gomory's *group problem* [8] is a central object in the study of strong cutting planes for integer linear optimization problems. One considers an abelian (not necessarily finite) group G, written additively, and studies the set of functions $s\colon G \to \mathbb{R}$ satisfying the following constraints:

$$\sum_{\mathbf{r}\in G} \mathbf{r}\, s(\mathbf{r}) \in \mathbf{f} + S \qquad \text{(IR)}$$
$$s(\mathbf{r}) \in \mathbb{Z}_+ \quad \text{for all } \mathbf{r} \in G$$
$$s \text{ has finite support,}$$

where $\mathbf{f}$ is a given element in G, and S is a subgroup of G; so $\mathbf{f} + S$ is the coset containing the element $\mathbf{f}$. We will be concerned with the so-called *infinite group problem* [9,10], where $G = \mathbb{R}^k$ is taken to be the group of real k-vectors under addition, and $S = \mathbb{Z}^k$ is the subgroup of the integer vectors. We are interested in studying the convex hull $R_{\mathbf{f}}(G,S)$ of all functions satisfying the constraints in (IR). Observe that $R_{\mathbf{f}}(G,S)$ is a convex subset of the infinite-dimensional vector space $\mathcal{V}$ of functions $s\colon G \to \mathbb{R}$ with finite support.

Any linear inequality in $\mathcal{V}$ is given by a pair (π, α) where π is a function $\pi\colon G \to \mathbb{R}$ (not necessarily of finite support) and $\alpha \in \mathbb{R}$. The linear inequality is then given by $\sum_{\mathbf{r}\in G} \pi(\mathbf{r})s(\mathbf{r}) \geq \alpha$; the left-hand side is a finite sum because s has finite support. Such an inequality is called a *valid inequality* for $R_{\mathbf{f}}(G,S)$ if $\sum_{\mathbf{r}\in G} \pi(\mathbf{r})s(\mathbf{r}) \geq \alpha$ for all $s \in R_{\mathbf{f}}(G,S)$. It is customary to concentrate on those valid inequalities for which $\pi \geq 0$; then we can choose, after a scaling, $\alpha = 1$. Thus, we only focus on valid inequalities of the form $\sum_{\mathbf{r}\in G} \pi(\mathbf{r})s(\mathbf{r}) \geq 1$ with $\pi \geq 0$. Such functions π will be termed *valid functions* for $R_{\mathbf{f}}(G,S)$.

A valid function π for $R_{\mathbf{f}}(G,S)$ is said to be *minimal* for $R_{\mathbf{f}}(G,S)$ if there is no valid function $\pi' \neq \pi$ such that $\pi'(\mathbf{r}) \leq \pi(\mathbf{r})$ for all $\mathbf{r} \in G$. For every valid function π for $R_{\mathbf{f}}(G,S)$, there exists a minimal valid function π' such that $\pi' \leq \pi$ (cf. [3]), and thus non-minimal valid functions are redundant in the description of $R_{\mathbf{f}}(G,S)$. Minimal functions for $R_{\mathbf{f}}(G,S)$ were characterized by Gomory for finite groups G in [8], and later for $R_{\mathbf{f}}(\mathbb{R},\mathbb{Z})$ by Gomory and Johnson [9]. We state these results in a unified notation in the following theorem.

A function $\pi\colon G \to \mathbb{R}$ is *subadditive* if $\pi(\mathbf{x}+\mathbf{y}) \leq \pi(\mathbf{x})+\pi(\mathbf{y})$ for all $\mathbf{x},\mathbf{y} \in G$. We say that π is *symmetric* if $\pi(\mathbf{x}) + \pi(\mathbf{f}-\mathbf{x}) = 1$ for all $\mathbf{x} \in G$.

Theorem 1.1 (Gomory and Johnson [9]). *Let $\pi\colon G \to \mathbb{R}$ be a non-negative function. Then π is a minimal valid function for $R_{\mathbf{f}}(G,S)$ if and only if $\pi(\mathbf{r}) = 0$ for all $\mathbf{r} \in S$, π is subadditive, and π satisfies the symmetry condition. (The first two conditions imply that π is constant over any coset of S.)*

1.2 Characterization of extreme valid functions

A stronger notion is that of an *extreme function.* A valid function π is *extreme* for $R_{\mathbf{f}}(G,S)$ if it cannot be written as a convex combination of two other valid functions for $R_{\mathbf{f}}(G,S)$, i.e., $\pi = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2$ implies $\pi = \pi_1 = \pi_2$. Extreme functions are minimal. A tight characterization of extreme functions for $R_{\mathbf{f}}(\mathbb{R}^k,\mathbb{Z}^k)$ has eluded researchers for the past four decades now, however, various specific sufficient conditions for guaranteeing extremality [3,4,7,6,5,11] have been proposed. The standard technique for showing extremality is as follows. Suppose that $\pi = \frac{1}{2}\pi^1 + \frac{1}{2}\pi^2$, where π^1, π^2 are other (minimal) valid functions. All subadditivity relations that are tight for π are also tight for π^1, π^2. Then one uses a lemma of *real analysis*, the so-called Interval Lemma introduced by Gomory and Johnson in [11] or one of its variants. The Interval Lemma allows us to deduce certain affine linearity properties that π^1 and π^2 share with π. This is followed by a finite-dimensional *linear algebra* argument to establish uniqueness of π, implying $\pi = \pi^1 = \pi^2$, and thus the extremality of π.

Surprisingly, the *arithmetic* (number-theoretic) aspect of the problem has been largely overlooked, even though it is at the core of the theory of the closely related *finite* group problem. In [2], the authors showed that this aspect is the key for completing the classification of extreme functions. The authors studied the case $k = 1$ and gave a complete and algorithmic answer for the case of piecewise linear functions with rational breakpoints in the set $\frac{1}{q}\mathbb{Z}$. To capture the relevant arithmetics of the problem, the authors studied sets of additivity relations of the form $\pi(\mathbf{t}_i) + \pi(\mathbf{y}) = \pi(\mathbf{t}_i + \mathbf{y})$ and $\pi(\mathbf{x}) + \pi(\mathbf{r}_i - \mathbf{x}) = \pi(\mathbf{r}_i)$, where the points $\mathbf{t}_i$ and $\mathbf{r}_i$ are certain breakpoints of the function π. They give rise to the *reflection group* Γ generated by the reflections $\rho_{\mathbf{r}_i}\colon \mathbf{x} \mapsto \mathbf{r}_i - \mathbf{x}$ and translations $\tau_{\mathbf{t}_i}\colon \mathbf{y} \mapsto \mathbf{t}_i + \mathbf{y}$. The natural action of the reflection group Γ on the set of intervals delimited by the elements of $\frac{1}{q}\mathbb{Z}$ transfers the affine linearity established by the Interval Lemma on some interval I to a connected component of the orbit $\Gamma(I)$. When this establishes affine linearity of π^1, π^2 on all intervals where π is affinely linear, one proceeds with finite-dimensional linear algebra to decide extremality of π. Otherwise, there is a way to perturb π slightly to construct distinct minimal valid functions $\pi^1 = \pi + \bar{\pi}$ and $\pi^2 = \pi - \bar{\pi}$, using any sufficiently small, Γ-equivariant perturbation function (see subsection A.1), modified by restriction to a certain connected component.

1.3 Contributions of the paper

In the present paper, we continue the program of [2]. We study a remarkable class of minimal functions π of the two-dimensional infinite group problem ($k = 2$). Let q be a positive integer. Consider the arrangement $\mathcal{H}_q$ of all hyperplanes (lines) of the form $(0,1)\cdot\mathbf{x} = b$, $(1,0)\cdot\mathbf{x} = b$, and $(1,1)\cdot\mathbf{x} = b$, where $b \in \frac{1}{q}\mathbb{Z}$. The complement of the arrangement $\mathcal{H}_q$ consists of two-dimensional cells, whose closures are the triangles $T_0 = \frac{1}{q}\operatorname{conv}(\{\binom{0}{0}, \binom{1}{0}, \binom{0}{1}\})$ and $T_1 = \frac{1}{q}\operatorname{conv}(\{\binom{1}{0}, \binom{0}{1}, \binom{1}{1}\})$ and their translates by elements of the lattice $\frac{1}{q}\mathbb{Z}^2$. We denote by $\mathcal{P}_q$ the collection of these triangles and the vertices and edges that arise as intersections of the triangles. Thus $\mathcal{P}_q$ is a polyhedral complex that is a triangulation of the space $\mathbb{R}^2$. Within the polyhedral complex $\mathcal{P}_q$, let $\mathcal{P}_{q,0}$ be the set of 0-faces (vertices), $\mathcal{P}_{q,1}$ be the set of 1-faces (edges), and $\mathcal{P}_{q,2}$ be the set of 2-faces (triangles). The sets of diagonal, vertical, and horizontal edges will be denoted by $\mathcal{P}_{q,\diagdown}$, $\mathcal{P}_{q,|}$, and $\mathcal{P}_{q,-}$, respectively. We will use $\oplus$ and $\ominus$ to denote vector addition and subtraction modulo 1, respectively. We use the same notation for pointwise sums and differences of sets. By quotienting out by $\mathbb{Z}^2$, we obtain a finite complex that triangulates $\mathbb{R}^2/\mathbb{Z}^2$; we still denote it by $\mathcal{P}_q$.

We call a function $\pi\colon \mathbb{R}^2 \to \mathbb{R}$ *continuous piecewise linear over* $\mathcal{P}_q$ if it is an affine linear function on each of the triangles of $\mathcal{P}_q$. We introduce the following notation. For every $I \in \mathcal{P}_q$, the restriction $\pi|_I$ is an affine function, that is $\pi|_I(\mathbf{x}) = \mathbf{m}_I \cdot \mathbf{x} + b_I$ for some $\mathbf{m}_I \in \mathbb{R}^2$, $b_I \in \mathbb{R}$. We abbreviate $\pi|_I$ as π_I.

For a valid function π, we consider the set $E(\pi) = \{\, (\mathbf{x},\mathbf{y}) \mid \pi(\mathbf{x}) + \pi(\mathbf{y}) = \pi(\mathbf{x} \oplus \mathbf{y}) \,\}$ of pairs $(\mathbf{x},\mathbf{y})$, for which the subadditivity relations are tight. Because $\mathcal{P}_q$ enjoys a strong unimodularity property (Lemma A.6), we can give a finite combinatorial representation of the set $E(\pi)$ using the faces of $\mathcal{P}_q$; this extends a technique in [2]. For faces $I, J, K \in \mathcal{P}_q$, let

$$F(I,J,K) = \{\, (\mathbf{x},\mathbf{y}) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid \mathbf{x} \in I,\ \mathbf{y} \in J,\ \mathbf{x} \oplus \mathbf{y} \in K \,\}.$$

A triple (I,J,K) of faces is called a *valid triple* (Definition A.5) if none of the sets I, J, K can individually decreased without changing the resulting set $F(I,J,K)$. Let $E(\pi,\mathcal{P}_q)$ denote the set of valid triples (I,J,K) such that

$$\pi(\mathbf{x}) + \pi(\mathbf{y}) = \pi(\mathbf{x} \oplus \mathbf{y}) \quad \text{for all} \quad (\mathbf{x},\mathbf{y}) \in F(I,J,K).$$

$E(\pi,\mathcal{P}_q)$ is partially ordered by letting $(I,J,K) \leq (I',J',K')$ if and only if $I \subseteq I'$, $J \subseteq J'$, and $K \subseteq K'$. Let $E_{\max}(\pi,\mathcal{P}_q)$ be the set of all maximal valid triples of the poset $E(\pi,\mathcal{P}_q)$. Then $E(\pi)$ is exactly covered by the sets $F(I,J,K)$ for the maximal valid triples $(I,J,K) \in E_{\max}(\pi,\mathcal{P}_q)$ (Lemma A.7).

In the present paper, we will restrict ourselves to a setting without maximal valid triples that include horizontal or vertical edges.

Definition 1.2. *A continuous piecewise linear function π on $\mathcal{P}_q$ is called* diagonally constrained *if whenever $(I, J, K) \in E_{\max}(\pi, \mathcal{P}_q)$, then $I, J, K \in \mathcal{P}_{q,0} \cup \mathcal{P}_{q,\diagdown} \cup \mathcal{P}_{q,2}$.*

Remark 1.3. Given a piecewise linear continuous valid function $\zeta\colon \mathbb{R} \to \mathbb{R}$ for the one-dimensional infinite group problem, Dey–Richard [5, Construction 6.1] consider the function $\kappa\colon \mathbb{R}^2 \to \mathbb{R}$, $\kappa(\mathbf{x}) = \zeta(\mathbf{1}\cdot\mathbf{x})$, where $\mathbf{1} = (1, 1)$, and show that κ is minimal and extreme if and only if ζ is minimal and extreme, respectively. If ζ has rational breakpoints in $\frac{1}{q}\mathbb{Z}$, then κ belongs to our class of diagonally constrained continuous piecewise linear functions over $\mathcal{P}_q$.

We prove the following main theorem.

Theorem 1.4. *Consider the following problem.*

> *Given a minimal valid function π for $R_f(\mathbb{R}^2, \mathbb{Z}^2)$ that is piecewise linear continuous on $\mathcal{P}_q$ and diagonally constrained, decide if π is extreme.*

There exists an algorithm for this problem that takes a number of elementary operations over the reals that is bounded by a polynomial in q.

As a direct corollary of the proof of the theorem, we obtain the following result relating the finite and infinite group problems.

Theorem 1.5. *Let π be a minimal continuous piecewise linear function over $\mathcal{P}_q$ that is diagonally constrained. Then π is extreme for $R_{\mathbf{f}}(\mathbb{R}^2, \mathbb{Z}^2)$ if and only if the restriction $\pi\big|_{\frac{1}{4q}\mathbb{Z}^2}$ is extreme for $R_{\mathbf{f}}(\frac{1}{4q}\mathbb{Z}^2, \mathbb{Z}^2)$.*

We conjecture that the hypothesis on π being diagonally constrained can be removed.

2 Real analysis lemmas

For any element $\mathbf{x} \in \mathbb{R}^k$, $k \geq 1$, $|\mathbf{x}|$ will denote the standard Euclidean norm. The proof of the following theorem appears in appendix A.3.

Theorem 2.1. *If $\pi\colon \mathbb{R}^k \to \mathbb{R}$ is a minimal valid function, and $\pi = \frac{1}{2}\pi^1 + \frac{1}{2}\pi^2$, where π^1, π^2 are valid functions, then π^1, π^2 are both minimal. Moreover, if $\limsup_{\mathbf{h}\to 0} \frac{|\pi(\mathbf{h})|}{|\mathbf{h}|} < \infty$, then this condition also holds for π^1 and π^2. This implies that π, π^1 and π^2 are all Lipschitz continuous.*

The following lemmas are corollaries of a general version of the interval lemma or similar real analysis arguments. Proofs appear in appendix A.7.

Lemma 2.2. *Suppose π is a continuous function and let $(I, J, K) \in E(\pi, \mathcal{P}_q)$ be a valid triple of triangles, i.e., $I, J, K \in \mathcal{P}_{q,2}$. Then π is affine in I, J, K with the same gradient.*

Lemma 2.3. *Suppose π is a continuous function and let $(I,J,K) \in E(\pi,\mathcal{P}_q)$ where $I \in \mathcal{P}_{q,\diagdown}$, $J,K \in \mathcal{P}_{q,\diagdown} \cup \mathcal{P}_{q,2}$. Then π is affine in the diagonal direction in I,J,K, i.e., there exists $c \in \mathbb{R}$ such that such that $\pi(\mathbf{v} + \lambda \binom{-1}{1}) = \pi(\mathbf{v}) + c \cdot \lambda$ for all $\mathbf{v} \in I$ (resp., $\mathbf{v} \in J$, $\mathbf{v} \in K$) and $\lambda \in \mathbb{R}$ such that $\mathbf{v} + \lambda \binom{-1}{1} \in I$ (resp., $\mathbf{v} + \lambda \binom{-1}{1} \in J$, $\mathbf{v} + \lambda \binom{-1}{1} \in K$).*

Lemma 2.4. *Let $I,J \in \mathcal{P}_{q,2}$ be triangles such that $I \cap J \in \mathcal{P}_{q,|} \cup \mathcal{P}_{q,-}$. Let π be a continuous function defined on $I \cup J$ satisfying the following properties:*

(i) π is affine on I.
(ii) There exists $c \in \mathbb{R}$ such that $\pi(\mathbf{v} + \lambda \binom{-1}{1}) = \pi(\mathbf{v}) + c \cdot \lambda$ for all $\mathbf{v} \in J$ and $\lambda \in \mathbb{R}$ such that $\mathbf{v} + \lambda \binom{-1}{1} \in J$.

Then π is affine on J.

3 Proof of the main results

Let $\partial_{\mathbf{v}}$ denote the directional derivative in the direction of $\mathbf{v}$.

Definition 3.1. *Let π be a minimal valid function.*

(a) *For any $I \in \mathcal{P}_q$, if π is affine in I and if for all valid functions π^1, π^2 such that $\pi = \frac{1}{2}\pi^1 + \frac{1}{2}\pi^2$ we have that π^1, π^2 are affine in I, then we say that π is* affine imposing in I.
(b) *For any $I \in \mathcal{P}_q$, if $\partial_{(-1,1)}\pi$ is constant in I and if for all valid functions π^1, π^2 such that $\pi = \frac{1}{2}\pi^1 + \frac{1}{2}\pi^2$ we have that $\partial_{(-1,1)}\pi^1$, $\partial_{(-1,1)}\pi^2$ are constant in I, then we say that π is* diagonally affine imposing in I.
(c) *For a collection $\mathcal{P} \subseteq \mathcal{P}_q$, if for all $I \in \mathcal{P}$, π is affine imposing (or diagonally affine imposing) in I, then we say that π is* affine imposing (diagonally affine imposing) in $\mathcal{P}$.

We either show that π is affine imposing in $\mathcal{P}_q$ (subsection 3.1) or construct a continuous piecewise linear Γ-equivariant perturbation over $\mathcal{P}_{4q}$ that proves π is not extreme (subsections 3.2 and 3.3). If π is affine imposing in $\mathcal{P}_q$, we set up a system of linear equations to decide if π is extreme or not (subsection 3.4). This implies the main theorem stated in the introduction.

3.1 Imposing affine linearity on faces of $\mathcal{P}_q$

For the remainder of this paper, we will use reflections and translations modulo 1 to compensate for the fact that our function is periodic with period 1. Working modulo 1 is accounted for by applying the translations $\tau_{(1,0)}$ and $\tau_{(0,1)}$ whenever needed. Hence, we define the reflection $\bar{\rho}_{\mathbf{v}}(\mathbf{x}) = \mathbf{v} \ominus \mathbf{x}$ and the translation $\bar{\tau}_{\mathbf{v}}(\mathbf{x}) = \mathbf{v} \oplus \mathbf{x}$. The reflections and translations arise from certain valid triples as follows.

Lemma 3.2. *Suppose (I,J,K) is a valid triple.*

1. *If* $K = \{\mathbf{a}\} \in \mathcal{P}_{q,0}$, *then* $J = \bar{\rho}_{\mathbf{a}}(I)$,
2. *If* $J = \{\mathbf{a}\} \in \mathcal{P}_{q,0}$, *then* $K = \bar{\tau}_{\mathbf{a}}(I)$.

Proof. Part 1. Since $(I, J, \{\mathbf{a}\})$ is a valid triple, then for all $\mathbf{x} \in I$, there exists a $\mathbf{y} \in J$ such that $x \oplus \mathbf{y} = \mathbf{a}$, i.e., $\mathbf{y} = \mathbf{a} \ominus \mathbf{x} \in J$, and therefore $J \supseteq \bar{\rho}_{\mathbf{a}}(I)$. Also, for all $\mathbf{y} \in J$, there exists a $\mathbf{x} \in I$ such that $\mathbf{x} \oplus \mathbf{y} = \mathbf{a}$. Again, $\mathbf{y} = \mathbf{a} \ominus \mathbf{x}$, i.e., $J \subseteq \bar{\rho}_{\mathbf{a}}(I)$. Hence, $J = \bar{\rho}_{\mathbf{a}}(I)$.

Part 2. Since $(I, \{\mathbf{a}\}, K)$ is a valid triple and J is a singleton, then for all $\mathbf{x} \in I$, we have $\mathbf{x} \oplus \mathbf{a} \in K$, i.e., $K \supseteq \bar{\tau}_{\mathbf{a}}(I)$. Also, for all $\mathbf{z} \in K$, there exists a $\mathbf{x} \in I$ such that $\mathbf{x} \oplus \mathbf{a} = \mathbf{z}$, i.e., $K \subseteq \bar{\tau}_{\mathbf{a}}(I)$. Hence, $K = \bar{\tau}_{\mathbf{a}}(I)$. □

Let $\mathcal{G} = \mathcal{G}(\mathcal{P}_{q,2}, \mathcal{E})$ be an undirected graph with node set $\mathcal{P}_{q,2}$ and edge set $\mathcal{E} = \mathcal{E}_0 \cup \mathcal{E}_{\diagdown}$ where $\{I, J\} \in \mathcal{E}_0$ ($\{I, J\} \in \mathcal{E}_{\diagdown}$) if and only if for some $K \in \mathcal{P}_{q,0}$ ($K \in \mathcal{P}_{q,\diagdown}$), we have $(I, J, K) \in E(\pi, \mathcal{P}_q)$ or $(I, K, J) \in E(\pi, \mathcal{P}_q)$. For each $I \in \mathcal{P}_{q,2}$, let $\mathcal{G}_I$ be the connected component of $\mathcal{G}$ containing I.

We now consider faces of $\mathcal{P}_{q,2}$ on which we will apply lemmas from section 2.

$$\mathcal{P}^1_{q,2} = \{\, I, J \in \mathcal{P}_{q,2} \mid \exists K \in \mathcal{P}_{q,\diagdown} \text{ with } (I, J, K) \in E(\pi, \mathcal{P}_q) \text{ or } (I, K, J) \in E(\pi, \mathcal{P}_q) \,\},$$

$$\mathcal{P}^2_{q,2} = \{\, I, J, K \in \mathcal{P}_{q,2} \mid (I, J, K) \in E(\pi, \mathcal{P}_q) \,\}.$$

It follows from Lemma 2.2 that π is affine imposing in $\mathcal{P}^2_{q,2}$ and from Lemma 2.3 that π is diagonally affine imposing in $\mathcal{P}^1_{q,2}$.

Faces connected in the graph have related slopes.

Lemma 3.3. *Let* $\mathbf{v} \in \mathbb{R}^2$. *For* $\theta = \pi, \pi^1$, *or* π^2, *if* θ *is affine in the* $\mathbf{v}$ *direction in* I, *i.e., there exists* $c \in \mathbb{R}$ *such that such that* $\pi(\mathbf{x} + \lambda\mathbf{v}) = \pi(\mathbf{x}) + c \cdot \lambda$ *for all* $\mathbf{x} \in I$ *and* $\lambda \in \mathbb{R}$ *such that* $\mathbf{x} + \lambda\mathbf{v} \in I$, *and* $\{I, J\} \in \mathcal{E}$, *then* θ *is affine in the* $\mathbf{v}$ *direction in* J *as well.*

The proof appears in appendix A.8.

With this in mind, we define the two sets of faces and any faces connected to them in the graph $\mathcal{G}$,

$$\mathcal{S}^1_{q,2} = \{\, J \in \mathcal{P}_{q,2} \mid J \in \mathcal{G}_I \text{ for some } I \in \mathcal{P}^1_{q,2} \,\},$$

$$\mathcal{S}^2_{q,2} = \{\, J \in \mathcal{P}_{q,2} \mid J \in \mathcal{G}_I \text{ for some } I \in \mathcal{P}^2_{q,2} \,\}.$$

It follows from Lemma 3.3 that π is affine imposing in $\mathcal{S}^2_{q,2}$ and diagonally affine imposing in $\mathcal{S}^1_{q,2}$.

From Lemma 2.4, it follows that if $I \in \mathcal{S}^2_{q,2}$, $J \in \mathcal{S}^1_{q,2}$ and $I \cap J \in \mathcal{P}_{q,|} \cup \mathcal{P}_{q,\diagdown}$, then π is affine imposing in J. Let

$$\bar{\mathcal{S}}_{q,2} = \{K \in \mathcal{G}_I \mid I \in \mathcal{S}^1_{q,2} \text{ and there exists a } J \in \mathcal{S}^2_{q,2} \text{ such that } I \cap J \in \mathcal{P}_{q,|} \cup \mathcal{P}_{q,-}\}.$$

Now set $\bar{\mathcal{S}}^2_{q,2} = \mathcal{S}^2_{q,2} \cup \bar{\mathcal{S}}_{q,2}$ and $\bar{\mathcal{S}}^1_{q,2} = \mathcal{S}^1_{q,2} \setminus \bar{\mathcal{S}}_{q,2}$. The following theorem is a consequence of Lemmas 2.2, 2.4, and 3.3.

Theorem 3.4. *If* $\bar{\mathcal{S}}^2_{q,2} = \mathcal{P}_{q,2}$, *then* π *is affine imposing in* $\mathcal{P}_{q,2}$, *and therefore* θ *is continuous piecewise linear over* $\mathcal{P}_q$ *for* $\theta = \pi^1, \pi^2$.

3.2 Non-extremality by two-dimensional equivariant perturbation

In this and the following subsection, we will prove the following result.

Lemma 3.5. *Let π be a minimal, continuous piecewise linear function over $\mathcal{P}_q$ that is diagonally constrained. If $\bar{\mathcal{S}}^2_{q,2} \neq \mathcal{P}_{q,2}$, then π is not extreme.*

In the proof, we will need two different equivariant perturbations that we construct as follows (see subsection A.1). Let $\Gamma_0 = \langle\, \rho_{\mathbf{g}}, \tau_{\mathbf{g}} \mid \mathbf{g} \in \frac{1}{q}\mathbb{Z}^2 \,\rangle$ be the group generated by reflections and translations corresponding to all possible vertices of $\mathcal{P}_q$. We define the function $\psi\colon \mathbb{R}^2 \to \mathbb{R}$ as a continuous piecewise linear function over $\mathcal{P}_{4q}$ in the following way: let $T_0 = \frac{1}{q}\operatorname{conv}(\{\binom{0}{0}, \binom{1}{0}, \binom{0}{1}\})$, and at all vertices of $\mathcal{P}_{4q}$ that lie in T_0, let ψ take the value 0, except at the interior vertices $\frac{1}{4q}\binom{1}{1}, \frac{1}{4q}\binom{2}{1}, \frac{1}{4q}\binom{1}{2}$, where we assign ψ to have the value 1. Since ψ is continuous piecewise linear over $\mathcal{P}_{4q}$, this uniquely determines the function on T_0. We then extend ψ to all of $\mathbb{R}^2$ using the equivariance formula (2).

Lemma 3.6. *The function $\psi\colon \mathbb{R}^2 \to \mathbb{R}$ constructed above is well-defined and has the following properties:*

(i) $\psi(\mathbf{g}) = 0$ *for all* $\mathbf{g} \in \frac{1}{q}\mathbb{Z}^2$,
(ii) $\psi(\mathbf{x}) = -\psi(\rho_{\mathbf{g}}(\mathbf{x})) = -\psi(\mathbf{g} - \mathbf{x})$ *for all* $\mathbf{g} \in \frac{1}{q}\mathbb{Z}^2, \mathbf{x} \in [0,1]^2$,
(iii) $\psi(\mathbf{x}) = \psi(\tau_{\mathbf{g}}(\mathbf{x})) = \psi(\mathbf{g} + \mathbf{x})$ *for all* $\mathbf{g} \in \frac{1}{q}\mathbb{Z}^2, \mathbf{x} \in [0,1]^2$,
(iv) ψ *is continuous piecewise linear over* $\mathcal{P}_{4q}$.

Proof. The properties follow directly from the equivariance formula (2). □

It is now convenient to introduce the function $\Delta\pi(\mathbf{x},\mathbf{y}) = \pi(\mathbf{x}) + \pi(\mathbf{y}) - \pi(\mathbf{x} \oplus \mathbf{y})$, which measures the slack in the subadditivity constraints. Let $\Delta\mathcal{P}_q$ be the polyhedral complex containing all polytopes $F = F(I,J,K)$ where $I,J,K \in \mathcal{P}_q$. Observe that $\Delta\pi|_F$ is affine; if we introduce the function $\Delta\pi_F(\mathbf{x},\mathbf{y}) = \pi_I(\mathbf{x}) + \pi_J(\mathbf{y}) - \pi_K(\mathbf{x} \oplus \mathbf{y})$ for all $\mathbf{x},\mathbf{y} \in \mathbb{R}^2$, then $\Delta\pi(\mathbf{x},\mathbf{y}) = \Delta\pi_F(\mathbf{x},\mathbf{y})$ for all $(\mathbf{x},\mathbf{y}) \in F$. Furthermore, if (I,J,K) is a valid triple, then $(I,J,K) \in E(\pi,\mathcal{P}_q)$ if and only if $\Delta\pi|_{F(I,J,K)} = 0$. We will use $\operatorname{vert}(F)$ to denote the set of vertices of the polytope F.

Lemma 3.7. *Let $F \in \Delta\mathcal{P}_q$ and let $(\mathbf{x},\mathbf{y})$ be a vertex of F. Then $\mathbf{x},\mathbf{y}$ are vertices of the complex $\mathcal{P}_q$, i.e., $\mathbf{x},\mathbf{y} \in \frac{1}{q}\mathbb{Z}^2$.*

The proof again uses the strong unimodularity properties of $\mathcal{P}_q$ and appears in appendix A.2.

Lemma 3.8. *Let π be a minimal, continuous piecewise linear function over $\mathcal{P}_q$ that is diagonally constrained. Suppose there exists $I^* \in \mathcal{P}_{q,2} \setminus (\bar{\mathcal{S}}^2_{q,2} \cup \bar{\mathcal{S}}^1_{q,2})$. Then π is not extreme.*

Proof. Let $R = \bigcup_{J \in \mathcal{G}_{I^*}} \operatorname{int}(J) \subseteq [0,1]^2$. Since R is a union of interiors, it does not contain any points in $\frac{1}{2q}\mathbb{Z}^2$. Let ψ be the Γ_0-equivariant function of Lemma 3.6. Let

$$\epsilon = \min\{\, \Delta\pi_{\hat{F}}(\mathbf{x},\mathbf{y}) \neq 0 \mid \hat{F} \in \Delta\mathcal{P}_{4q},\ (\mathbf{x},\mathbf{y}) \in \operatorname{vert}(\hat{F}) \,\},$$

and let $\bar{\pi} = \delta_R \cdot \psi$ where δ_R is the indicator function for the set R. We will show that for

$$\pi^1 = \pi + \tfrac{\epsilon}{3}\bar{\pi}, \quad \pi^2 = \pi - \tfrac{\epsilon}{3}\bar{\pi},$$

that π^1, π^2 are minimal, and therefore valid functions, and hence π is not extreme. We will show this just for π^1 as the proof for π^2 is the same.

Since $\psi(\mathbf{0}) = 0$ and $\psi(\mathbf{f}) = 0$, we see that $\pi^1(\mathbf{0}) = 0$ and $\pi^1(\mathbf{f}) = 1$.

We want to show that π^1 is symmetric and subadditive. We will do this by analyzing the function $\Delta\pi^1(\mathbf{x}, \mathbf{y}) = \pi^1(\mathbf{x}) + \pi^1(\mathbf{y}) - \pi^1(\mathbf{x} \oplus \mathbf{y})$. Since ψ is piecewise linear over $\mathcal{P}_{4q}$, π^1 is also piecewise linear over $\mathcal{P}_{4q}$, and thus we only need to focus on vertices of $\Delta\mathcal{P}_{4q}$, which, by Lemma 3.7, are contained in $\frac{1}{4q}\mathbb{Z}^2$.

Let $\mathbf{u}, \mathbf{v} \in \frac{1}{4q}\mathbb{Z}^2$. First, if $\Delta\pi_(\mathbf{u}, \mathbf{v}) > 0$, then

$$\Delta\pi^1(\mathbf{u}, \mathbf{v}) \geq \pi(\mathbf{u}) - \epsilon/3 + \pi(\mathbf{v}) - \epsilon/3 - \pi(\mathbf{u} \oplus \mathbf{v}) - \epsilon/3 = \Delta\pi(\mathbf{u}, \mathbf{v}) - \epsilon \geq 0.$$

Next, we will show that if $\Delta\pi(\mathbf{u}, \mathbf{v}) = 0$, then $\Delta\pi^1(\mathbf{u}, \mathbf{v}) = 0$. This will prove two things. First, $\Delta\pi^1(\mathbf{x}, \mathbf{y}) \geq 0$ for all $\mathbf{x}, \mathbf{y} \in [0,1]^2$, and therefore π^1 is subadditive. Second, since π is symmetric, $\Delta\pi(\mathbf{x}, \mathbf{f} \ominus \mathbf{x}) = 0$ for all $\mathbf{x} \in \frac{1}{4q}\mathbb{Z}^2$, which would imply that $\Delta\pi^1(\mathbf{x}, \mathbf{f} \ominus \mathbf{x}) = 0$ for all $\mathbf{x} \in \frac{1}{4q}\mathbb{Z}^2$, proving π^1 is symmetric via Lemma A.9.

Suppose that $\Delta\pi(\mathbf{u}, \mathbf{v}) = 0$. We will proceed by cases.

Case 1. Suppose $\mathbf{u}, \mathbf{v}, \mathbf{u} \oplus \mathbf{v} \notin R$. Then $\delta_R(\mathbf{u}) = \delta_R(\mathbf{v}) = \delta_R(\mathbf{u} \oplus \mathbf{v}) = 0$, and $\Delta\pi^1(\mathbf{u}, \mathbf{v}) = \Delta\pi(\mathbf{u}, \mathbf{v}) \geq 0$.

Case 2. Suppose we are not in Cases 1. That is, suppose $\Delta\pi(\mathbf{u}, \mathbf{v}) = 0$, and at least one of $\mathbf{u}, \mathbf{v}, \mathbf{u} \oplus \mathbf{v}$ is in R. Since $R \cap \frac{1}{2q}\mathbb{Z}^2 = \emptyset$, at least one of $\mathbf{u}, \mathbf{v}\mathbf{u} \oplus \mathbf{v} \notin \frac{1}{2q}\mathbb{Z}^2$. This implies that at least one of $\mathbf{u}, \mathbf{v} \notin \frac{1}{2q}\mathbb{Z}^2$. Since $\Delta\pi^1(\mathbf{x}, \mathbf{y})$ is symmetric in $\mathbf{x}$ and $\mathbf{y}$, without loss of generality, we will assume that $\mathbf{u} \notin \frac{1}{2q}\mathbb{Z}^2$.

Since $\mathbf{u} \notin \frac{1}{2q}\mathbb{Z}^2$, $(\mathbf{u}, \mathbf{v}) \notin \operatorname{vert}(\Delta\mathcal{P}_q)$. Therefore, there exists a face $F \in \Delta\mathcal{P}_q$ such that $(\mathbf{u}, \mathbf{v}) \in \operatorname{rel\,int}(F)$. Since $\Delta\pi_F \geq 0$ (π is subadditive) and $\Delta\pi_F(\mathbf{u}, \mathbf{v}) = 0$, it follows that $\Delta\pi_F = 0$. Now let $(I, J, K) \in E_{\max}(\pi, \mathcal{P}_q)$ such that $F(I, J, K) \supseteq F$. We discuss the possible cases for I, J, K from Lemma A.11.

1. If $I, J, K \notin \mathcal{P}_{q,2}$, then $I, J, K \in \mathcal{P}_{q,0} \cup \mathcal{P}_{q,\diagdown}$ are all vertices or edges of $\mathcal{P}_q$, which are all not contained in R since R is the union of interiors of sets from $\mathcal{P}_{q,2}$. Therefore, $\mathbf{u}, \mathbf{v}, \mathbf{u} \oplus \mathbf{v} \notin R$, which means we are in Case 1.
2. If $I, J, K \in \mathcal{P}_{q,2}$, then $I, J, K \in \mathcal{S}^2_{q,2}$. Therefore, $\mathbf{u}, \mathbf{v}, \mathbf{u} \oplus \mathbf{v} \notin R$, which means we are in Case 1.
3. One of I, J, K is a diagonal edge in $\mathcal{P}_{q,1}$, while the other two are in $\mathcal{P}_{q,2}$, which means these sets are in $\mathcal{S}^1_{q,2}$. Since edges are not in R, and $R \cap \mathcal{S}^1_{q,2} = \emptyset$, and again, $\mathbf{u}, \mathbf{v}, \mathbf{u} \oplus \mathbf{v} \notin R$, which means we are in Case 1.
4. This leaves us with the case where two of I, J, K are in $\mathcal{P}_{q,2}$ and the third is a vertex, i.e., is in $\mathcal{P}_{q,0}$. Since $\mathbf{u} \notin \frac{1}{q}\mathbb{Z}^2$, I cannot be a vertex. Therefore, $I \in \mathcal{P}_{q,2}$. We proceed with this knowledge.

There are two possible cases.

Case 2a. $J \in \mathcal{P}_{q,0}$, $I, K \in \mathcal{P}_{q,2}$ and hence $\mathbf{v} \in \frac{1}{q}\mathbb{Z}^2$.

Therefore $\{I, K\} \in \mathcal{E}_0$ and $\delta_R(\mathbf{u}) = \delta_R(\mathbf{u} \oplus \mathbf{v})$. Since $\mathbf{v} \in \frac{1}{q}\mathbb{Z}^2$, we have $\psi(\mathbf{v}) = 0$ and $\psi(\mathbf{u}) = \psi(\bar\tau_{\mathbf{v}}(\mathbf{u})) = \psi(\mathbf{u} \oplus \mathbf{v})$ by Lemma 3.6 (iii). It follows that $\bar\pi(\mathbf{u}) + \bar\pi(\mathbf{v}) - \bar\pi(\mathbf{u} \oplus \mathbf{v}) = 0$, and therefore $\Delta\pi^1(\mathbf{u}, \mathbf{v}) = \Delta\pi(\mathbf{u}, \mathbf{v}) = 0$.

Case 2b. $I, J \in \mathcal{P}_{q,2}$, $K \in \mathcal{P}_{q,0}$ and hence $\mathbf{u} \oplus \mathbf{v} \in \frac{1}{q}\mathbb{Z}^2$. Therefore $\{I, J\} \in \mathcal{E}_0$ and $\delta_R(\mathbf{u}) = \delta_R(\mathbf{v})$. Since $\mathbf{u} \oplus \mathbf{v} \in \frac{1}{q}\mathbb{Z}^2$, $\psi(\mathbf{u}) = -\psi(\bar\rho_{\mathbf{u}\oplus\mathbf{v}}(\mathbf{u})) = -\psi(\mathbf{v})$ by Lemma 3.6 (ii). It follows that $\bar\pi(\mathbf{u}) + \bar\pi(\mathbf{v}) - \bar\pi(\mathbf{u} \oplus \mathbf{v}) = 0$, and therefore $\Delta\pi^1(\mathbf{u}, \mathbf{v}) = \Delta\pi(\mathbf{u}, \mathbf{v}) = 0$.

We conclude that π^1 (and similarly π^2) is subadditive and symmetric, and therefore minimal and hence valid. Therefore π is not extreme. ☐

3.3 Non-extremality by diagonal equivariant perturbation

We next construct a different equivariant perturbation function. Let $\Gamma_{\diagdown} = \langle\, \rho_{\mathbf{g}}, \tau_{\mathbf{g}} \mid \mathbf{1} \cdot \mathbf{g} \equiv 0 \pmod{\frac{1}{q}} \,\rangle$, where $\mathbf{1} = (1, 1)$, be the group generated by reflections and translations corresponding to all points on diagonal edges of $\mathcal{P}_q$. We define the function $\varphi \colon \mathbb{R}^2 \to \mathbb{R}$ as a continuous piecewise linear function over $\mathcal{P}_{4q}$ in the following way:

$$\varphi(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{1} \cdot \mathbf{x} \equiv \frac{1}{4q} \pmod{\frac{1}{q}}, \\ -1 & \text{if } \mathbf{1} \cdot \mathbf{x} \equiv \frac{3}{4q} \pmod{\frac{1}{q}}, \\ 0 & \text{if } \mathbf{1} \cdot \mathbf{x} \equiv 0 \text{ or } \frac{2}{4q} \pmod{\frac{1}{q}}. \end{cases}$$

This function satisfies all properties of Lemma 3.6, but is also $\Gamma_{\diagdown}$-equivariant.

Lemma 3.9. *Suppose there exists $I^* \in \bar{\mathcal{S}}^1_{q,2}$ and π is diagonally constrained. Then π is not extreme.*

Proof. Let $R = (\bigcup_{J \in \mathcal{G}_{I^*}} J) \setminus \{\, \mathbf{x} \mid \mathbf{1} \cdot \mathbf{x} \equiv 0 \text{ or } \frac{2}{4q} \pmod{\frac{1}{q}} \,\}$.

Let
$$\epsilon = \min\{\, \Delta\pi_F(\mathbf{x}, \mathbf{y}) \neq 0 \mid F \in \Delta\mathcal{P}_{4q},\ (\mathbf{x}, \mathbf{y}) \in \operatorname{vert}(F) \,\},$$
and let $\bar\pi$ be the unique continuous piecewise linear function over $\mathcal{P}_{4q}$ such that for any vertex $\mathbf{x}$ of $\mathcal{P}_{4q}$, we have $\bar\pi(\mathbf{x}) = \delta_R(\mathbf{x}) \cdot \varphi(\mathbf{x})$ where δ_R is the indicator function for the set R. By construction, $\bar\pi$ is a continuous function that vanishes on all diagonal hyperplanes in the complex $\mathcal{P}_q$. We will show that for

$$\pi^1 = \pi + \tfrac{\epsilon}{3}\bar\pi, \quad \pi^2 = \pi - \tfrac{\epsilon}{3}\bar\pi,$$

that π^1, π^2 are minimal, and therefore valid functions, and hence π is not extreme. We will show this just for π^1 as the proof for π^2 is the same.

Since, $\varphi(\mathbf{0}) = 0$ and $\varphi(\mathbf{f}) = 0$, we see that $\pi^1(\mathbf{0}) = 0$ and $\pi^1(\mathbf{f}) = 1$.

We want to show that π^1 is symmetric and subadditive. We will do this by analyzing the function $\Delta\pi^1(\mathbf{x}, \mathbf{y}) = \pi^1(\mathbf{x}) + \pi^1(\mathbf{y}) - \pi^1(\mathbf{x} \oplus \mathbf{y})$. Since $\bar\pi$ is continuous piecewise linear over $\mathcal{P}_{4q}$, π^1 is also continuous piecewise linear over

$\mathcal{P}_{4q}$, and thus we only need to focus on vertices of $\Delta\mathcal{P}_{4q}$, which, by Lemma 3.7, are contained in $\frac{1}{4q}\mathbb{Z}^2$.

Let $\mathbf{u}, \mathbf{v} \in \frac{1}{4q}\mathbb{Z}^2$.

First, if $\Delta\pi_(\mathbf{u}, \mathbf{v}) > 0$, then $\Delta\pi_(\mathbf{u}, \mathbf{v}) \geq \epsilon$ and therefore

$$\Delta\pi^1(\mathbf{u}, \mathbf{v}) \geq \pi(\mathbf{u}) - \epsilon/3 + \pi(\mathbf{v}) - \epsilon/3 - \pi(\mathbf{u} \oplus \mathbf{v}) - \epsilon/3 = \Delta\pi(\mathbf{u}, \mathbf{v}) - \epsilon \geq 0.$$

Next, we will show that if $\Delta\pi(\mathbf{u}, \mathbf{v}) = 0$, then $\Delta\pi^1(\mathbf{u}, \mathbf{v}) = 0$. This will prove two things. First, $\Delta\pi^1(\mathbf{x}, \mathbf{y}) \geq 0$ for all $\mathbf{x}, \mathbf{y} \in [0,1]^2$, and therefore π^1 is subadditive. Second, since π is symmetric, $\Delta\pi(\mathbf{x}, \mathbf{f} \ominus \mathbf{x}) = 0$ for all $\mathbf{x} \in \frac{1}{4q}\mathbb{Z}^2$, which would imply that $\Delta\pi^1(\mathbf{x}, \mathbf{f} \ominus \mathbf{x}) = 0$ for all $\mathbf{x} \in \frac{1}{4q}\mathbb{Z}^2$, proving π^1 is symmetric via Lemma A.9.

Suppose that $\Delta\pi(\mathbf{u}, \mathbf{v}) = 0$. We will proceed by cases.

Case 1. Suppose $\mathbf{u}, \mathbf{v}, \mathbf{u} \oplus \mathbf{v} \notin R$. Then $\delta_R(\mathbf{u}) = \delta_R(\mathbf{v}) = \delta_R(\mathbf{u} \oplus \mathbf{v}) = 0$, and $\Delta\pi^1(\mathbf{u}, \mathbf{v}) = \Delta\pi(\mathbf{u}, \mathbf{v}) \geq 0$.

Case 2. Suppose $\mathbf{u}, \mathbf{v} \in \frac{1}{2q}\mathbb{Z}^2$. Then $\mathbf{1} \cdot (\mathbf{u} \oplus \mathbf{v}) \equiv 0 \pmod{\frac{1}{q}}$ and, by definition of R, $\mathbf{u}, \mathbf{v}, \mathbf{u} \oplus \mathbf{v} \notin R$, and we are actually in Case 1.

Case 3. Suppose we are not in Cases 1 or 2. That is, suppose $\Delta\pi(\mathbf{u}, \mathbf{v}) = 0$, not both $\mathbf{u}, \mathbf{v}$ are in $\frac{1}{2q}\mathbb{Z}^2$, and at least one of $\mathbf{u}, \mathbf{v}, \mathbf{u} \oplus \mathbf{v}$ is in R. Since $\Delta\pi^1(\mathbf{x}, \mathbf{y})$ is symmetric in $\mathbf{x}$ and $\mathbf{y}$, without loss of generality, since not both $\mathbf{u}, \mathbf{v}$ are in $\frac{1}{2q}\mathbb{Z}^2$, we will assume that $\mathbf{u} \notin \frac{1}{2q}\mathbb{Z}^2$.

Since $\mathbf{u} \notin \frac{1}{2q}\mathbb{Z}^2$, $(\mathbf{u}, \mathbf{v}) \notin \operatorname{vert}(\Delta\mathcal{P}_q)$. Therefore, there exists a face $F \in \Delta\mathcal{P}_q$ such that $(\mathbf{u}, \mathbf{v}) \in \operatorname{rel\,int}(F)$. Since $\Delta\pi_F \geq 0$ (π is subadditive) and $\Delta\pi_F(\mathbf{u}, \mathbf{v}) = 0$, it follows that $\Delta\pi_F = 0$. Now let $(I, J, K) \in E_{\max}(\pi, \mathcal{P}_q)$ such that $F(I, J, K) \supseteq F$. Since π is diagonally constrained, by definition, I, J, K are each either a vertex, diagonal edge, or triangle in $\mathcal{P}_q$. We discuss the possible cases for I, J, K according to Lemma A.11.

1. If $I, J, K \notin \mathcal{P}_{q,2}$, then I, J, K are all vertices or diagonal edges of $\mathcal{P}_q$, which are all not contained in R since all vertices and diagonal edges are subsets of $\{\, \mathbf{x} \mid \mathbf{x}_1 + \mathbf{x}_2 \equiv 0 \pmod{\frac{1}{q}} \,\}$. Therefore, $\mathbf{u}, \mathbf{v}, \mathbf{u} \oplus \mathbf{v} \notin R$, which means we are in Case 1.
2. If $I, J, K \in \mathcal{P}_{q,2}$, then $I, J, K \in \mathcal{S}^2_{q,2}$. By definition of $\bar{\mathcal{S}}^1_{q,2}$, for any $I' \in \mathcal{S}^2_{q,2}$ and $J' \in \bar{\mathcal{S}}^1_{q,2}$, either $I' \cap J' = \emptyset$, or $I' \cap J' \in \mathcal{P}_{q,\diagdown}$. Therefore, $\mathbf{u}, \mathbf{v}, \mathbf{u} \oplus \mathbf{v} \notin R$, which means we are in Case 1.
3. If two of I, J, K are in $\mathcal{P}_{q,2}$ and the third is a vertex, i.e., is in $\mathcal{P}_{q,0}$. Since $\mathbf{u} \notin \frac{1}{q}\mathbb{Z}^2$, I cannot be a vertex. Therefore, $I \in \mathcal{P}_{q,2}$. For this case, the proof is exactly the same as Case 2a and Case 2b in the proof of Lemma 3.8 because $\bar{\pi}(\mathbf{x}) = 0$ for all vertices $\mathbf{x} \in \mathcal{P}_{q,0}$. For brevity, we will not repeat it here.
4. If one of I, J, K is in $\mathcal{P}_{q,\diagdown}$, call it I', and the other two are in $\mathcal{P}_{q,2}$, call them J',K', then $J', K' \in \mathcal{S}^1_{q,2}$ and $\{J', K'\} \in \mathcal{E}_\diagdown$. Since $I' \in \mathcal{P}_{q,\diagdown}$, $I' \cap R = \emptyset$. Recall that $\mathcal{S}^1_{q,2} \subseteq \bar{\mathcal{S}}^1_{q,2} \cup \bar{\mathcal{S}}^2_{q,2}$. If either J' or K' is in $\bar{\mathcal{S}}^2_{q,2}$, then they both are in $\bar{\mathcal{S}}^2_{q,2}$, i.e., $J' \cup K' \cap R = \emptyset$ and therefore $\mathbf{u}, \mathbf{v}, \mathbf{u} \oplus \mathbf{v} \notin R$, which is Case 1. We proceed to consider the case where $I' \in \mathcal{P}_{q,\diagdown}$ and $J', K' \in \bar{\mathcal{S}}^1_{q,2}$ with $\{J', K'\} \in \mathcal{E}_\diagdown$ of which there are three possible cases.

Case 3a. $I \in \mathcal{P}_{q,\diagdown}$, $J, K \in \mathcal{P}_{q,2}$. Since $\{J, K\} \in \mathcal{E}_{\diagdown}$, $\delta_R(\mathbf{v}) = \delta_R(\mathbf{u} \oplus \mathbf{v})$. Since $I \in \mathcal{P}_{q,\diagdown}$ and $\mathbf{u} \in I$, $\mathbf{1} \cdot \mathbf{u} \equiv 0 \pmod{\frac{1}{q}}$. It follows that $\varphi(\mathbf{u}) = 0$ and $\mathbf{1} \cdot \mathbf{v} \equiv \mathbf{1} \cdot (\mathbf{u} \oplus \mathbf{v}) \pmod{\frac{1}{q}}$. Therefore, $\varphi(\mathbf{v}) = \varphi(\mathbf{u} \oplus \mathbf{v})$. Combining these, we have $\bar\pi(\mathbf{u}) + \bar\pi(\mathbf{v}) - \bar\pi(\mathbf{u} \oplus \mathbf{v}) = 0$, and therefore $\Delta\pi^1(\mathbf{u}, \mathbf{v}) = \Delta\pi(\mathbf{u}, \mathbf{v}) = 0$.

Case 3b. $J \in \mathcal{P}_{q,\diagdown}$, $I, K \in \mathcal{P}_{q,2}$. This is similar to Case 3a and the proof need not be repeated.

Case 3c. $I, J \in \mathcal{P}_{q,2}$, $K \in \mathcal{P}_{q,\diagdown}$ and hence $\mathbf{1} \cdot (\mathbf{u} \oplus \mathbf{v}) \equiv 0 \pmod{\frac{1}{q}}$. Since $\{I, J\} \in \mathcal{E}_{\diagdown}$, we have $\delta_R(\mathbf{u}) = \delta_R(\mathbf{v})$. Since $\mathbf{1} \cdot (\mathbf{u} \oplus \mathbf{v}) \equiv 0 \pmod{\frac{1}{q}}$, we have $\mathbf{1} \cdot \mathbf{u} \equiv -\mathbf{1} \cdot \mathbf{v} \pmod{\frac{1}{q}}$, and hence $\varphi(\mathbf{u}) = -\varphi(\mathbf{v})$. It follows that $\bar\pi(\mathbf{u}) + \bar\pi(\mathbf{v}) - \bar\pi(\mathbf{u} \oplus \mathbf{v}) = 0$, and therefore $\Delta\pi^1(\mathbf{u}, \mathbf{v}) = \Delta\pi(\mathbf{u}, \mathbf{v}) = 0$.

We conclude that π^1 (and similarly π^2) is subadditive and symmetric, and therefore minimal and hence valid. Therefore π is not extreme. □

Proof (of Lemma 3.5). This follows directly from Lemmas 3.8 and 3.9. □

The specific form of our perturbations as continuous piecewise linear function over $\mathcal{P}_{4q}$ implies the following corollary.

Corollary 3.10. *Suppose π is a continuous piecewise linear function over $\mathcal{P}_q$ and is diagonally constrained. If π is not affine imposing over $\mathcal{P}_{q,2}$, then there exist distinct minimal π^1, π^2 that are continuous piecewise linear over $\mathcal{P}_{4q}$ such that $\pi = \frac{1}{2}\pi^1 + \frac{1}{2}\pi^2$.*

3.4 Extremality and non-extremality by linear algebra

In this section we suppose π is a minimal continuous piecewise linear function over $\mathcal{P}_q$ that is affine imposing in $\mathcal{P}_{q,2}$. Therefore, π^1 and π^2 must also be continuous piecewise linear functions over $\mathcal{P}_q$. It is clear that whenever $\pi(\mathbf{x}) + \pi(\mathbf{y}) = \pi(\mathbf{x} \oplus \mathbf{y})$, the functions π^1 and π^2 must also satisfy this equality relation, that is, $\pi^i(\mathbf{x}) + \pi^i(\mathbf{y}) = \pi^i(\mathbf{x} \oplus \mathbf{y})$. We now set up a system of linear equations that π satisfies and that π_1 and π_2 must also satisfy. Let $\varphi\colon \frac{1}{q}\mathbb{Z}^2 \to \mathbb{R}$. Suppose φ satisfies the following system of linear equations:

$$\begin{cases} \varphi(\mathbf{0}) = 0,\ \varphi(\mathbf{f}) = 1,\ \varphi(\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)) = 0,\ \varphi(\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)) = 0,\ \varphi(\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right))) = 0, \\ \varphi(\mathbf{u}) + \varphi(\mathbf{v}) = \varphi(\mathbf{u} \oplus \mathbf{v}) \text{ if } \mathbf{u}, \mathbf{v} \in \frac{1}{q}\mathbb{Z}^2,\ \pi(\mathbf{u}) + \pi(\mathbf{v}) = \pi(\mathbf{u} \oplus \mathbf{v}) \end{cases} \tag{1}$$

Since π exists and satisfies (1), we know that the system has a solution.

Theorem 3.11. *Let $\pi\colon \mathbb{R}^2 \to \mathbb{R}$ be a continuous piecewise linear valid function over $\mathcal{P}_q$.*

i. *If the system* (1) *does not have a unique solution, then π is not extreme.*
ii. *Suppose π is minimal and affine imposing in $\mathcal{P}_{q,2}$. Then π is extreme if and only if the system of equations* (1) *has a unique solution.*

The proof, similar to one in [2], appears in appendix A.9.

3.5 Connection to a finite group problem

Theorem 3.12. *Let π be a minimal continuous piecewise linear function over $\mathcal{P}_q$ that is diagonally constrained. Then π is extreme if and only if the system of equations* (1) *with $\frac{1}{4q}\mathbb{Z}^2$ has a unique solution.*

Proof. Since π is continuous piecewise linear over $\mathcal{P}_q$, it is also continuous piecewise linear over $\mathcal{P}_{4q}$. The forward direction is the contrapositive of Theorem 3.11 (i), applied when we view π piecewise linear over $\mathcal{P}_{4q}$. For the reverse direction, observe that if the system of equations (1) with $\frac{1}{4q}\mathbb{Z}^2$ has a unique solution, then there cannot exist distinct minimal π^1, π^2 that are continuous piecewise linear over $\mathcal{P}_{4q}$ such that $\pi = \frac{1}{2}\pi^1 + \frac{1}{2}\pi^2$. By the contrapositive of Corollary 3.10, π is affine imposing in $\mathcal{P}_{q,2}$. Then π is also affine imposing on $\mathcal{P}_{4q,2}$ since it is a finer set. By Theorem 3.11 (ii), since π is affine imposing in $\mathcal{P}_{4q,2}$ and the system of equations (1) on $\mathcal{P}_{4q}$ has a unique solution, π is extreme. □

Theorem 1.4 is proved by testing for minimality using Lemma A.10 and then testing for extremality using Theorem 3.12. Theorem 1.5 is a direct consequence of Theorem 3.12.

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A Appendix

A.1 Equivariant perturbations

In this section we outline the theory of equivariant perturbations for the infinite group problem, used first in [2] for the case $k = 1$.

We consider a subgroup of the group $\mathrm{Aff}(\mathbb{R}^k)$ of invertible affine linear transformations of $\mathbb{R}^k$ as follows.

Definition A.1. *For a point* $\mathbf{r} \in \mathbb{R}^k$, *define the* reflection $\rho_{\mathbf{r}} \colon \mathbb{R}^k \to \mathbb{R}^k$, $\mathbf{x} \mapsto \mathbf{r} - \mathbf{x}$. *For a vector* $t \in \mathbb{R}^k$, *define the* translation $\tau_{\mathbf{t}} \colon \mathbb{R}^k \to \mathbb{R}$, $\mathbf{x} \mapsto \mathbf{x} + \mathbf{t}$.

Given a set R of points and a set U of vectors, we will define the subgroup

$$\Gamma = \langle\, \rho_{\mathbf{r}}, \tau_{\mathbf{t}} \mid \mathbf{r} \in R,\, \mathbf{t} \in U \,\rangle.$$

Let $\mathbf{r}, \mathbf{s}, \mathbf{w}, \mathbf{t} \in \mathbb{R}^k$. Each reflection is an involution: $\rho_{\mathbf{r}} \circ \rho_{\mathbf{r}} = id$, two reflections give one translation: $\rho_{\mathbf{r}} \circ \rho_{\mathbf{s}} = \tau_{\mathbf{r}-\mathbf{s}}$. Thus, if we assign a *character* $\chi(\rho_{\mathbf{r}}) = -1$ to every reflection and $\chi(\tau_{\mathbf{t}}) = +1$ to every translation, then this extends to a *group character* of Γ, that is, a group homomorphism $\chi \colon \Gamma \to \mathbb{C}^\times$.

On the other hand, not all pairs of reflections need to be considered: $\rho_{\mathbf{s}} \circ \rho_{\mathbf{w}} = (\rho_{\mathbf{s}} \circ \rho_{\mathbf{r}}) \circ (\rho_{\mathbf{r}} \circ \rho_{\mathbf{w}}) = (\rho_{\mathbf{r}} \circ \rho_{\mathbf{s}})^{-1} \circ (\rho_{\mathbf{r}} \circ \rho_{\mathbf{w}})$. Thus the subgroup $T = \ker \chi$ of translations in Γ is generated as follows. Let $\mathbf{r}_1 \in R$ be any of the reflection points; then

$$T = \langle\, \tau_{\mathbf{r}-\mathbf{r}_1}, \tau_{\mathbf{t}} \mid \mathbf{r} \in R,\, \mathbf{t} \in U \,\rangle.$$

It is *normal* in Γ, as it is stable by conjugation by any reflection: $\rho_{\mathbf{r}} \circ \tau_{\mathbf{t}} \circ \rho_{\mathbf{r}}^{-1} = \tau_{-\mathbf{t}}$. If $\gamma \in \Gamma$ is not a translation, i.e., $\chi(\gamma) = -1$, then it is generated by an odd number of reflections, and thus can be written as $\gamma = \tau \rho_{\mathbf{r}_1}$ with $\tau \in T$. Thus $\Gamma/T = \langle \rho_{\mathbf{r}_1} \rangle$ is of order 2. In short, we have the following lemma.

Lemma A.2. *The group* Γ *is the semidirect product* $T \rtimes \langle \rho_{\mathbf{r}_1} \rangle$*, where the (normal) subgroup of translations can be written as*

$$T = \{\, \tau_{\mathbf{t}} \mid \mathbf{t} \in \Lambda \,\},$$

where Λ *is the additive subgroup*

$$\Lambda = \langle\, \mathbf{r} - \mathbf{r}_1, \mathbf{t} \mid \mathbf{r} \in R,\, \mathbf{t} \in U \,\rangle_{\mathbb{Z}} \subseteq \mathbb{R}^k.$$

Definition A.3. *A function* $\psi \colon \mathbb{R}^k \to \mathbb{R}$ *is called* Γ-equivariant *if it satisfies the* equivariance formula

$$\psi(\gamma(\mathbf{x})) = \chi(\gamma)\psi(\mathbf{x}) \quad \textit{for } \mathbf{x} \in \mathbb{R} \textit{ and } \gamma \in \Gamma. \tag{2}$$

We note that if Λ is discrete, i.e., a lattice, then there is a way to construct continuous Γ-equivariant functions by defining them on a fundamental domain and extending them to all of $\mathbb{R}^k$ via the equivariance formula (2). The same is true for the case where Λ is a mixed lattice, i.e., a direct sum of a lattice in a subspace and another subspace. We omit the details.

A.2 Polyhedral complexes $\mathcal{P}_q$, $\Delta\mathcal{P}_q$, and unimodularity

We first comment that $\mathbf{f}$ must be a vertex of $\mathcal{P}_q$ of any minimal valid function. We omit the proof here as it is very similar to ([2], Lemma 2.1).

Lemma A.4. *If π is a minimal function, then $\mathbf{f} \in \frac{1}{q}\mathbb{Z}^2$.*

Definition A.5. *For $I, J, K \in \mathcal{P}_q \setminus \{\emptyset\}$, we say (I, J, K) is a* valid triple *provided that the following occur:*

i. $K \subseteq I \oplus J$,
ii. *For all $\mathbf{u} \in I$ there exists a $\mathbf{v} \in J$ such that $\mathbf{u} \oplus \mathbf{v} \in K$,*
iii. *For all $\mathbf{v} \in J$ there exists a $\mathbf{u} \in I$ such that $\mathbf{u} \oplus \mathbf{v} \in K$,*

Equivalently, a valid triple (I, J, K) is characterized by the following property.

iv. Whenever I', J', K' are sets such that $I' \subseteq I$, $J' \subseteq J$, $K' \subseteq K$ and $F(I, J, K) = F(I', J', K')$ we have that $I' = I$, $J' = J$, $K' = K$.

The construction of $\mathcal{P}_q$ has convenient properties such as the following.

Lemma A.6. *Let $I, J \in \mathcal{P}_q$. Then $I \oplus J$ and $I \ominus J$ are both unions of faces in $\mathcal{P}_q$.*

Proof. By construction, for any face $K \in \mathcal{P}_q$, the set $\{\, \mathbf{x} \bmod \mathbf{1} \mid \mathbf{x} \in K \,\}$ is also a face in $\mathcal{P}_q$. Therefore we only need to show that the Minkowski sums $I + J$ and $I - J$ are unions of faces in $\mathcal{P}_q$. Let

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{bmatrix}^T$$

Let $\mathbf{a}^i$ be the i^{th} row vector of A. Then there exists vectors $\mathbf{b}^1, \mathbf{b}^2$ such that $I = \{\, \mathbf{x} \mid A\mathbf{x} \le \mathbf{b}^1 \,\}$, $J = \{\, \mathbf{y} \mid A\mathbf{y} \le \mathbf{b}^2 \,\}$. Moreover, due to the total unimodularity of the matrix A, the right-hand side vectors $\mathbf{b}^1, \mathbf{b}^2$ can be chosen so that $\mathbf{b}^1, \mathbf{b}^2$ are tight, i.e.,

$$\max_{\mathbf{x} \in I} \mathbf{a}^i \cdot \mathbf{x} = \mathbf{b}^1_i, \quad \max_{\mathbf{y} \in J} \mathbf{a}^i \cdot \mathbf{y} = \mathbf{b}^2_i, \tag{3}$$

and $\mathbf{b}^1, \mathbf{b}^2 \in \frac{1}{q}\mathbb{Z}^2$.

We claim that $I + J = \{\, \mathbf{x} \mid A\mathbf{x} \le \mathbf{b}^1 + \mathbf{b}^2 \,\}$. Clearly $I + J \subseteq \{\, \mathbf{x} \mid A\mathbf{x} \le \mathbf{b}^1 + \mathbf{b}^2 \,\}$. We show the reverse direction. Let K' be a facet (edge) of $I + J$. Then $K' = I' + J'$, where I' is a face of I and J' is a face of J. Without loss of generality, assume that I' is an edge; then J' is either a vertex or an edge. By well-known properties of Minkowski sums, the normal cone of K' is the intersection of the normal cones of I' in I and J' in J. Thus K' has the same normal direction as the facet (edge) I'. This proves that $I + J = \{\, \mathbf{x} \mid A\mathbf{x} \le \mathbf{b} \,\}$ for some vector $\mathbf{b}$.

Let $\mathbf{x}^*$, $\mathbf{y}^*$ be maximizers in 3. Then $\mathbf{x}^* + \mathbf{y}^* \in I + J$. Then

$$\mathbf{b}^1_i + \mathbf{b}^2_i = \mathbf{a}^i \cdot \mathbf{x}^* + \mathbf{a}^i \cdot \mathbf{y}^* \le \max_{\mathbf{z} \in I+J} \mathbf{a}^i \cdot \mathbf{z} \le \max_{\mathbf{x} \in I} \mathbf{a}^i \cdot \mathbf{x} + \max_{\mathbf{y} \in J} \mathbf{a}^i \cdot \mathbf{y} = \mathbf{b}^1_i + \mathbf{b}^2_i.$$

Therefore, $\max_{\mathbf{z}\in I+J} \mathbf{a}^i\cdot\mathbf{z} = \mathbf{b}_i^1+\mathbf{b}_i^2$, which shows that every constraint $a_i\cdot\mathbf{z} \le \mathbf{b}_i^1$ is met at equality, and therefore $I + J = \{\, \mathbf{x} \mid A\mathbf{x} \le \mathbf{b}^1 + \mathbf{b}^2 \,\}$ and we conclude that $I + J$ must be a union of subsets in $\mathcal{P}_q$.

The case $I - J = \{\, \mathbf{z} - \mathbf{y} \mid \mathbf{z} \in K, \mathbf{y} \in J \,\}$ is shown similarly. □

Lemma A.7. $E(\pi) = \bigcup\{F(I, J, K) \mid (I, J, K) \in E_{\max}(\pi, \mathcal{P}_q)\}$.

Proof. Clearly the right hand side is a subset of $E(\pi)$. We show $E(\pi)$ is a subset of the right hand side. Suppose $(\mathbf{x}, \mathbf{y}) \in E(\pi)$. Let $I, J, K \in \mathcal{P}_q$ be minimal faces by set inclusion containing $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{x} \oplus \mathbf{y}$, respectively. We show that (I, J, K) is a valid triple. By Lemma A.6, $I \oplus J$ is a union of faces in $\mathcal{P}_q$. Since $\mathbf{x} \oplus \mathbf{y} \in I \oplus J$ and $\mathbf{x} \oplus \mathbf{y} \in K$, we have that $K \cap (I + J) \neq \emptyset$, and in particular, is a union of faces of $\mathcal{P}_q$ containing $\mathbf{x} + \mathbf{y}$. Since K was chosen to be a minimal such face in $\mathcal{P}_q$ containing $\mathbf{x} \oplus \mathbf{y}$, we have that $K \subseteq I \oplus J$.

Similarly, by Lemma A.6, $K \ominus J$ is also a union of sets in $\mathcal{P}_q$ containing $\mathbf{x}$. Since I is a minimal set containing $\mathbf{x}$, it must be that $I \subseteq K \ominus J$. Therefore, for any $\mathbf{u} \in I$, there exists a $\mathbf{v} \in J$ such that $\mathbf{u} \oplus \mathbf{v} \in K$.
Similarly, we find that for any $\mathbf{v} \in J$, there exists a $\mathbf{u} \in I$ such that $\mathbf{u} \oplus \mathbf{v} \in K$.

Since I, J, K were chosen to be minimal in $\mathcal{P}_q$, the triple satisfies criterion (iv) of being a valid triple. Hence, (I, J, K) is a valid triple.

Next we argue that $(I, J, K) \in E(\pi, \mathcal{P}_q)$. This is because $\Delta\pi$ is affine in $F(I, J, K)$, $\Delta\pi \ge 0$, $(\mathbf{x}, \mathbf{y}) \in \operatorname{rel\,int}(F(I, J, K))$, $\Delta\pi(\mathbf{x}, \mathbf{y}) = 0$ and therefore $\Delta\pi|_{F(I,J,K)} = 0$, i.e., $(I, J, K) \in E(\pi, \mathcal{P}_q)$.

Lastly, if (I, J, K) is not maximal in $E(\pi, \mathcal{P}_q)$, then there exists a maximal (I', J', K') such that $F(I', J', K') \supset (I, J, K)$, namely, $(\mathbf{x}, \mathbf{y}) \in F(I', J', K')$. □

Next we study the complex $\Delta\mathcal{P}_q$.

Proof (of Lemma 3.7). Since $F \in \Delta\mathcal{P}_q$, we can write F using the system of inequalities $F = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^4 \colon \hat{A}(\mathbf{x}, \mathbf{y}) \le \mathbf{b}\}$ where $\mathbf{b} \in \frac{1}{q}\mathbb{Z}^9$, the matrix A is given by

$$A = \begin{bmatrix} 1\,0\,1\,0\,0\,0\,1\,0\,1 \\ 0\,1\,1\,0\,0\,0\,0\,1\,1 \\ 0\,0\,0\,1\,0\,1\,1\,0\,1 \\ 0\,0\,0\,0\,1\,1\,0\,1\,1 \end{bmatrix}^T$$

and the matrix $\hat{A}$ differs from A only by scaling each row individually by ± 1. (This inequality representation of F will usually be redundant.) By checking every subdeterminant of the matrix A, it can be verified that A is totally unimodular, and therefore $\hat{A}$ is also totally unimodular. Therefore, the polytope $qF = \{\, (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^4 \colon \hat{A}(\mathbf{x}, \mathbf{y}) \le q\mathbf{b} \,\}$ has integral vertices in $\mathbb{Z}^4$.

It follows that P has vertices in $\frac{1}{q}\mathbb{Z}^4$. Therefore, $\mathbf{x}, \mathbf{y} \in \frac{1}{q}\mathbb{Z}^2$ and therefore are vertices of $\mathcal{P}_q$. □

A.3 Continuity results

In this section we prove Theorem 2.1 on continuity. Although similar results appear in [10], we provide proofs of these facts to keep this paper more self-contained. We first prove the following lemma.

Lemma A.8. *If $\theta\colon \mathbb{R}^k \to \mathbb{R}$ is a subadditive function and* $\limsup_{h\to 0} \frac{|\theta(\mathbf{h})|}{|\mathbf{h}|} = L < \infty$*, then θ is Lipschitz continuous with Lipschitz constant L.*

Proof. Fix any $\delta > 0$. Since $\limsup_{\mathbf{h}\to 0} \frac{|\theta(\mathbf{h})|}{|\mathbf{h}|} = L$, there exists $\epsilon > 0$ such that for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$ satisfying $|\mathbf{x} - \mathbf{y}| < \epsilon$, $\frac{|\theta(\mathbf{x}-\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|} < L + \delta$. By subadditivity, $|\theta(\mathbf{x} - \mathbf{y})| \geq |\theta(\mathbf{x}) - \theta(\mathbf{y})|$ and so $\frac{|\theta(\mathbf{x})-\theta(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|} < L + \delta$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$ satisfying $|\mathbf{x}-\mathbf{y}| < \epsilon$. This immediately implies that for *all* $\mathbf{x}, \mathbf{y} \in \mathbb{R}$, $\frac{|\theta(\mathbf{x})-\theta(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|} < L+\delta$, by simply breaking the interval $[\mathbf{x}, \mathbf{y}]$ into equal subintervals of size at most ϵ. Since the choice of δ was arbitrary, this shows that for every $\delta > 0$, $\frac{|\theta(\mathbf{x})-\theta(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|} < L + \delta$ and therefore, $\frac{|\theta(\mathbf{x})-\theta(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|} \leq L$. Therefore, θ is Lipschitz continuous with Lipschitz constant L. □

Proof (of Theorem 2.1). The minimality of π^1, π^2 is clear. Since we assume $\pi^1, \pi^2 \geq 0$, $\pi = \frac{1}{2}\pi^1 + \frac{1}{2}\pi^2$ implies that $\pi^i \leq 2\pi$ for $i = 1, 2$. Therefore if $\limsup_{\mathbf{h}\to 0} \frac{|\pi(\mathbf{h})|}{|\mathbf{h}|} = L < \infty$, then $\limsup_{\mathbf{h}\to 0} \frac{|\pi^i(\mathbf{h})|}{|\mathbf{h}|} \leq 2L < \infty$ for $i = 1, 2$. Applying Lemma A.8, we get Lipschitz continuity for all three functions. □

The following is a slight generalization of the Interval Lemma that appears in [1]. The proof is a minor modification of the original proof.

A.4 Finite test for minimality of piecewise linear functions

In this subsection, we show that there is an easy test to see if a continuous piecewise linear function over $\mathcal{P}_q$ is minimal.

Lemma A.9. *Suppose that π is a continuous piecewise linear function over $\mathcal{P}_q$ and $\pi(\mathbf{0}) = 0$.*

1. *π is subadditive if and only if $\pi(\mathbf{x}) + \pi(\mathbf{y}) \geq \pi(\mathbf{x} \oplus \mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \frac{1}{q}\mathbb{Z}^2$,*
2. *π is symmetric if and only if $\pi(\mathbf{x}) + \pi(\mathbf{f} \ominus \mathbf{x}) = 1$ for all $\mathbf{x} \in \frac{1}{q}\mathbb{Z}^2$.*

Proof. Clearly the forward direction of both statements is true. We will show the reverse of each. For subadditivity, we need to show that $\Delta\pi \geq 0$. Since $\Delta\pi$ is piecewise linear over $\Delta\mathcal{P}_q$, we just need to show that $\Delta\pi(\mathbf{x}, \mathbf{y}) \geq 0$ for any $(\mathbf{x}, \mathbf{y}) \in \operatorname{vert}(\Delta\mathcal{P}_q)$. By Lemma 3.7, $\operatorname{vert}(\Delta\mathcal{P}_q) \subseteq \frac{1}{q}\mathbb{Z}^4$, and the result follows.
Next, we show symmetry. Since $\mathbf{0}, \mathbf{f} \in \frac{1}{q}\mathbb{Z}^2$ and $\pi(\mathbf{0}) = 0$, we have that $\pi(\mathbf{f}) = 1$. Let $\mathbf{x} \in [0, 1]^2$ and let $F \in \Delta\mathcal{P}_q$ such that $(\mathbf{x}, \mathbf{f} \ominus \mathbf{x}) \in F$.

Similarly, to show symmetry, we need to show that $\Delta\pi(\mathbf{x}, \mathbf{y}) = 0$ for all $\mathbf{x}, \mathbf{y} \in [0, 1]^2$ such that $\mathbf{x} \oplus \mathbf{y} = \mathbf{f}$. Let $\mathbf{x}, \mathbf{y} \in [0, 1]^2$ such that $\mathbf{x} \oplus \mathbf{y} = \mathbf{f}$. Since

$\mathbf{f} \in \frac{1}{q}\mathbb{Z}^2$ by Lemma A.4, $(\mathbf{x}, \mathbf{y}) \in \operatorname{rel\,int}(\hat{F})$ for some face $\hat{F}$ of some $F \in \Delta\mathcal{P}_q$ and $\hat{F} \subseteq \{\, (\mathbf{x}, \mathbf{y}) \mid \mathbf{x} \oplus \mathbf{y} = f \,\}$. Since $\Delta\pi_F(\mathbf{u}, \mathbf{v}) = 0$ for all $(\mathbf{u}, \mathbf{v}) \in \operatorname{vert}(F) \subset \frac{1}{q}\mathbb{Z}^2$ when $\mathbf{u} \oplus \mathbf{v} = \mathbf{f}$, and $\Delta\pi_F$ is affine, it follows that $\Delta\pi(\mathbf{x}, \mathbf{y}) = \Delta\pi_F(\mathbf{x}, \mathbf{y}) = 0$. □

The following theorem is a direct corollary of Lemma A.9 and Theorem 1.1.

Theorem A.10 (Minimality test). *A function $\pi\colon \mathbb{R}^2 \to \mathbb{R}$ that is continuous piecewise linear over $\mathcal{P}_q$ is minimal if and only if*

1. $\pi(\mathbf{0}) = 0$,
2. $\pi(\mathbf{x}) + \pi(\mathbf{y}) \geq \pi(\mathbf{x} \oplus \mathbf{y})$ *for all* $\mathbf{x}, \mathbf{y} \in \frac{1}{q}\mathbb{Z}^2$,
3. $\pi(\mathbf{x}) + \pi(\mathbf{f} \ominus \mathbf{x}) = 1$ *for all* $\mathbf{x} \in \frac{1}{q}\mathbb{Z}^2$.

A.5 Properties of valid triples

Lemma A.11. *Suppose π is continuous piecewise linear over $\mathcal{P}_q$ and is diagonally constrained. Suppose that $(I, J, K) \in E(\pi, \mathcal{P}_q)$. Then one of the following is true.*

1. $I, J, K \in \mathcal{P}_{q,0} \cup \mathcal{P}_{q,\diagdown}$,
2. $I, J, K \in \mathcal{P}_{q,2}$,
3. *One of I, J, K is in $\mathcal{P}_{q,0}$, while the other two are in $\mathcal{P}_{q,2}$,*
4. *One of I, J, K is in $\mathcal{P}_{q,\diagdown}$, while the other two are in $\mathcal{P}_{q,2}$*

Proof. By definition of diagonally constrained, $I, J, K \in \mathcal{P}_{q,0} \cup \mathcal{P}_{q,\diagdown} \cup \mathcal{P}_{q,2}$. There are 27 possible ways to put I, J, K into those three sets. Above, 15 possibilities are described. We will show that the 12 remaining cases not list above are not possible because (I, J, K) is assumed to be a valid triple.

1. Suppose $I, J \in \mathcal{P}_{q,0} \cup \mathcal{P}_{q,\diagdown}$, $K \in I_{q,2}$. Then $K' = I \oplus J \subsetneq K$, and therefore $F(I, J, K) = F(I, J, K')$, and therefore (I, J, K) is not a valid triple.
2. Suppose $I, K \in \mathcal{P}_{q,0} \cup \mathcal{P}_{q,\diagdown}$, $J \in I_{q,2}$. Then $K \ominus I \subsetneq J$, and therefore, there exists a $J' \subsetneq J$ such that $F(I, J, K) = F(I, J', K)$, and therefore (I, J, K) is not a valid triple.
3. Suppose $J, K \in \mathcal{P}_{q,0} \cup \mathcal{P}_{q,\diagdown}$, $I \in I_{q,2}$. This is similar to the last case.

□

Lemma A.12. *Suppose (I, J, K) is a valid triple. The following are true.*

i. *Suppose $I, J \in \mathcal{P}_{q,2}$. Then for every point $\mathbf{u} \in \operatorname{int}(I)$ there exists a point $\mathbf{v} \in \operatorname{int}(J)$ such that $\mathbf{u} \oplus \mathbf{v} \in \operatorname{rel\,int}(K)$.*
ii. *Suppose $I, K \in \mathcal{P}_{q,2}$. Then for every point $\mathbf{w} \in \operatorname{int}(K)$ there exists a point $\mathbf{u} \in \operatorname{int}(I)$ such that $\mathbf{w} \ominus \mathbf{u} \in \operatorname{rel\,int}(J)$.*

Proof. Part (i). Let $\mathbf{u} \in \operatorname{int}(I)$ and so $(1, 0)^T\mathbf{u}$, $(0, 1)^T\mathbf{u}$ and $(1, 1)^T\mathbf{u}$ are all nonzero modulo $\frac{1}{q}$. Since (I, J, K) is a valid triple, there exist $\mathbf{v} \in J$ and $\mathbf{w} \in K$ such that $\mathbf{u} \oplus \mathbf{v} = \mathbf{w}$. Thus, $(1, 0)^T\mathbf{v}$ and $(1, 0)^T\mathbf{w}$ are different modulo $\frac{1}{q}$ (resp. for $(0, 1)^T\mathbf{v}$, $(0, 1)^T\mathbf{w}$ and $(1, 1)^T\mathbf{v}$, $(1, 1)^T\mathbf{w}$). Note that for any point $\mathbf{x} \in \mathbb{R}^2$, either $(1,0)^T\mathbf{x}$, $(0,1)^T\mathbf{x}$ and $(1,1)^T\mathbf{x}$ are all 0 modulo $\frac{1}{q}$, or exactly one of these numbers is 0 modulo $\frac{1}{q}$, or none of them are 0. Thus, we consider these cases :

Case 1: $(1,0)^T\mathbf{w}$, $(0,1)^T\mathbf{w}$ and $(1,1)^T\mathbf{w}$ are all 0 modulo $\frac{1}{q}$. Then $\mathbf{v} \in \operatorname{int}(J)$ since $J \in \mathcal{P}_{q,2}$. Then one can choose a vector $\mathbf{d}$ such that $\mathbf{w}' = \mathbf{w}+\mathbf{d} \in \operatorname{rel\,int}(K)$ and $\mathbf{v}' = \mathbf{v} + \mathbf{d} \in \operatorname{int}(J)$. Then $\mathbf{u} \oplus \mathbf{v}' = \mathbf{w}'$ and we are done.

Case 2: $(1,0)^T\mathbf{v}$, $(0,1)^T\mathbf{v}$ and $(1,1)^T\mathbf{v}$ are all 0 modulo $\frac{1}{q}$. Then $\mathbf{w} \in \operatorname{int}(K)$ and one can choose again a vector $\mathbf{d}$ such that $\mathbf{w}' = \mathbf{w} + \mathbf{d} \in \operatorname{int}(K)$ and $\mathbf{v}' = \mathbf{v} + \mathbf{d} \in \operatorname{int}(J)$. Then $\mathbf{u} \oplus \mathbf{v}' = \mathbf{w}'$ and we are done.

Case 3: Exactly one of $(1,0)^T\mathbf{w}$, $(0,1)^T\mathbf{w}$ and $(1,1)^T\mathbf{w}$ is 0 modulo $\frac{1}{q}$ and the same holds for $\mathbf{v}$. This means $\mathbf{w}$ and $\mathbf{v}$ lie on different hyperplanes in the arrangement $\mathcal{H}_q$. But then one can again choose a vector $\mathbf{d}$ such that $\mathbf{w}' = \mathbf{w} + \mathbf{d} \in \operatorname{rel\,int}(K)$ and $\mathbf{v}' = \mathbf{v} + \mathbf{d} \in \operatorname{int}(J)$. Then $\mathbf{u} \oplus \mathbf{v}' = \mathbf{w}'$ and we are done.

Case 4: None of $(1,0)^T\mathbf{w}$, $(0,1)^T\mathbf{w}$ and $(1,1)^T\mathbf{w}$ is 0 modulo $\frac{1}{q}$ and the same holds for $\mathbf{v}$. This means $\mathbf{v} \in \operatorname{int}(J)$ and $\mathbf{w} \in \operatorname{int}(K)$ already and we are done.

Part (ii) can be proved in a similar way.

□

A.6 Interval lemma

The so-called Interval Lemma was introduced by Gomory and Johnson in [11]. We prove this in a more general setting with three functions by a modifying a proof from [1].

Lemma A.13 (Interval Lemma). *Given real numbers $u_1 < u_2$ and $v_1 < v_2$, let $U = [u_1, u_2]$, $V = [v_1, v_2]$, and $U + V = [u_1 + v_1, u_2 + v_2]$. Let $f : U \to \mathbb{R}$, $g : V \to \mathbb{R}$, $h : U + V \to \mathbb{R}$ be bounded functions.*
If $f(u) + g(v) = h(u + v)$ for every $u \in U$ and $v \in V$, then there exists $c \in \mathbb{R}$ such that $f(u) = f(u_1) + c(u - u_1)$ for every $u \in U$, $g(v) = g(v_1) + c(v - v_1)$ for every $v \in V$, $h(w) = h(u_1 + v_1) + c(w - u_1 - v_1)$ for every $w \in U + V$.

Proof. We first show the following.

Claim 1. Let $u \in U$, and let $\varepsilon > 0$ such that $v_1 + \varepsilon \in V$. For every nonnegative integer p such that $u + p\varepsilon \in U$, we have $f(u + p\varepsilon) - f(u) = p(g(v_1 + \varepsilon) - g(v_1))$.

For $h = 1, \dots, p$, by hypothesis $f(u + h\varepsilon) + g(v_1) = h(u + h\varepsilon + v_1) = f(u + (h-1)\varepsilon) + g(v_1 + \varepsilon)$. Thus $f(u + h\varepsilon) - f(u + (h-1)\varepsilon) = g(v_1 + \varepsilon) - g(v_1)$, for $h = 1, \dots, p$. By summing the above p equations, we obtain $f(u + p\varepsilon) - f(u) = p(g(v_1 + \varepsilon) - g(v_1))$. This concludes the proof of Claim 1.

Let $\bar{u}, \bar{u}' \in U$ such that $\bar{u} - \bar{u}' \in \mathbb{Q}$ and $\bar{u} > \bar{u}'$. Define $c := \frac{f(\bar{u}) - f(\bar{u}')}{\bar{u} - \bar{u}'}$.

Claim 2. For every $u, u' \in U$ such that $u - u' \in \mathbb{Q}$, we have $f(u) - f(u') = c(u - u')$.

We only need to show that, given $u, u' \in U$ such that $u - u' \in \mathbb{Q}$, we have $f(u) - f(u') = c(u - u')$. We may assume $u > u'$. Choose a positive rational ε

such that $\bar{u} - \bar{u}' = \bar{p}\varepsilon$ for some integer $\bar{p}$, $u - u' = p\varepsilon$ for some integer p, and $v_1 + \varepsilon \in V$. By Claim 1,

$$f(\bar{u}) - f(\bar{u}') = \bar{p}(g(v_1 + \varepsilon) - g(v_1)) \quad \text{and} \quad f(u) - f(u') = p(g(v_1 + \varepsilon) - g(v_1)).$$

Dividing the last equality by $u - u'$ and the second to last by $\bar{u} - \bar{u}'$, we get

$$\frac{g(v_1 + \varepsilon) - g(v_1)}{\varepsilon} = \frac{f(\bar{u}) - f(\bar{u}')}{\bar{u} - \bar{u}'} = \frac{f(u) - f(u')}{u - u'} = c.$$

Thus $f(u) - f(u') = c(u - u')$. This concludes the proof of Claim 2.

Claim 3. For every $u \in U$, $f(u) = f(u_1) + c(u - u_1)$.

Let $\delta(x) = f(x) - cx$ for all $x \in U$. We show that $\delta(u) = \delta(u_1)$ for all $u \in U$ and this proves the claim. Since f is bounded on U, δ is bounded over U. Let M be a number such that $|\delta(x)| \leq M$ for all $x \in U$.

Suppose by contradiction that, for some $u^* \in U$, $\delta(u^*) \neq \delta(u_1)$. Let N be a positive integer such that $|N(\delta(u^*) - \delta(u_1))| > 2M$.
By Claim 2, $\delta(u^*) = \delta(u)$ for every $u \in U$ such that $u^* - u$ is rational. Thus there exists $\bar{u}$ such that $\delta(\bar{u}) = \delta(u^*)$, $u_1 + N(\bar{u} - u_1) \in U$ and $v_1 + \bar{u} - u_1 \in V$. Let $\bar{u} - u_1 = \varepsilon$. By Claim 1,

$$\begin{aligned}
\delta(u_1 + N\varepsilon) - \delta(u_1) &= (f(u_1 + N\varepsilon) - c(u_1 + N\varepsilon)) - (f(u_1) - cu_1) \\
&= N(g(v_1 + \varepsilon) - g(v_1)) - c(N\varepsilon) \\
&= N(f(u_1 + \varepsilon) - f(u_1)) - c(N\varepsilon) \\
&= N(f(u_1 + \varepsilon) - f(u_1) - c\varepsilon) \\
&= N(\delta(u_1 + \varepsilon) - \delta(u_1)) \\
&= N(\delta(\bar{u}) - \delta(u_1))
\end{aligned}$$

Thus $|\delta(u_1 + N\varepsilon) - \delta(u_1)| = |N(\delta(\bar{u}) - \delta(u_1))| = |N(\delta(u^*) - \delta(u_1))| > 2M$, which implies $|\delta(u_1 + N\varepsilon)| + |\delta(u_1)| > 2M$, a contradiction. This concludes the proof of Claim 3.

By symmetry between U and V, Claim 3 implies that there exists some constant c' such that, for every $v \in V$, $g(v) = g(v_1) + c'(v - v_1)$. We show $c' = c$. Indeed, given $\varepsilon > 0$ such that $u_1 + \varepsilon \in U$ and $v_1 + \varepsilon \in V$, $c\varepsilon = f(u_1 + \varepsilon) - f(u_1) = g(v_1 + \varepsilon) - g(v_1) = c'\varepsilon$, where the second equality follows from Claim 1.
Therefore, for every $v \in V$, $g(v) = g(v_1) + cg(v - v_1)$. Finally, since $f(u) + g(v) = h(u + v)$ for every $u \in U$ and $v \in V$, we have that for every $w \in U + V$, $h(w) = h(u_1 + v_1) + c(w - u_1 - v_1)$. □

A.7 Generalized interval lemma and corollaries

The following lemma is a generalization to higher dimensions of the interval lemma that appears in the literature for the infinite group problem.

Lemma A.14 (Higher Dimensional Interval Lemma). *Let $\pi : \mathbb{R}^k \to \mathbb{R}$ be a bounded function. Let U and V be compact convex subsets of $\mathbb{R}^k$ such that $\pi(\mathbf{u}) + \pi(\mathbf{v}) = \pi(\mathbf{u}+\mathbf{v})$ for all $\mathbf{u} \in U$ and $\mathbf{v} \in V$. Corresponding to every linear subspace L of $\mathbb{R}^k$, there exists a vector $\mathbf{g}$ in the dual space L' of L with the following property. For any $\mathbf{u}^0 \in U$ and $\mathbf{v}^0 \in V$ such that $\mathbf{u}^0$ (resp. $\mathbf{v}^0$) is in the interior of $(\mathbf{u}^0 + L) \cap U$ (resp. $(\mathbf{v}^0 + L) \cap V$) in the relative topology of L, the following conditions hold:*

(i) $\pi(\mathbf{u}^0 + \mathbf{p}) = \pi(\mathbf{u}^0) + \langle \mathbf{g}, \mathbf{p}\rangle$ *for all* $\mathbf{p} \in L$ *such that* $\mathbf{u}^0 + \mathbf{p} \in U$.
(ii) $\pi(\mathbf{v}^0 + \mathbf{p}) = \pi(\mathbf{v}^0) + \langle \mathbf{g}, \mathbf{p}\rangle$ *for all* $\mathbf{p} \in L$ *such that* $\mathbf{v}^0 + \mathbf{p} \in V$.
(iii) $\pi(\mathbf{u}^0 + \mathbf{v}^0 + \mathbf{p}) = \pi(\mathbf{u}^0 + \mathbf{v}^0) + \langle \mathbf{g}, \mathbf{p}\rangle$ *for all* $\mathbf{p} \in L$ *such that* $\mathbf{v}^0 + \mathbf{p} \in U + V$.

Proof. We fix an arbitrary linear subspace L and exhibit a vector $\mathbf{g} \in L'$ with the stated property. Let $\mathbf{p}^1, \dots, \mathbf{p}^m$ be a basis for L (we obviously have $m \leq k$). Now consider any $\mathbf{u}^0 \in U$ and $\mathbf{v}^0 \in V$ such that $\mathbf{u}^0$ (resp. $\mathbf{v}^0$) is in the interior of $(\mathbf{u}^0 + L) \cap U$ (resp. $(\mathbf{v}^0 + L) \cap V$) in the relative topology of L. Let $u_1^i < u_2^i \in \mathbb{R}$, $i = 1, \dots, m$ be such that the intersection of the line $\mathbf{u}^0 + \lambda \mathbf{p}^i$ with U is given by $\{u^0 + \lambda p \colon u_1^i \leq \lambda \leq u_2^i\}$ (these numbers exist since U is assumed to be compact and convex), similarly, $v_1^i < v_2^i \in \mathbb{R}$ are defined with respect to V, $\mathbf{v}^0$ and $\mathbf{p}^i$.

Let $f^i : [u_1^i, u_2^i] \to \mathbb{R}$ be defined by $f^i(\lambda) = \pi(\mathbf{u}^0 + \lambda \mathbf{p}^i)$, $g^i : [v_1^i, v_2^i] \to \mathbb{R}$ be defined by $g^i(\lambda) = \pi(\mathbf{v}^0 + \lambda \mathbf{p}^i)$ and $h^i : [u_1^i + v_1^i, u_2^i + v_2^i] \to \mathbb{R}$ be defined by $h^i(\lambda) = \pi(\mathbf{u}^0 + \mathbf{v}^0 + \lambda \mathbf{p}^i)$. Applying Lemma A.13, there exists a constant $c_i \in \mathbb{R}$ such that

$$\begin{aligned}
&\pi(\mathbf{u}^0 + \lambda \mathbf{p}^i) = \pi(\mathbf{u}^0) + c_i \cdot \lambda \text{ for all } \lambda \in [u_1^i, u_2^i],\\
&\pi(\mathbf{v}^0 + \lambda \mathbf{p}^i) = \pi(\mathbf{u}^0) + c_i \cdot \lambda \text{ for all } \lambda \in [v_1^i, v_2^i] \text{ and}\\
&\pi(\mathbf{u}^0 + \mathbf{v}^0 + \lambda \mathbf{p}^i) = \pi(\mathbf{u}^0 + \mathbf{v}^0) + c_i \cdot \lambda \text{ for all } \lambda \in [u_1^i + v_1^i, u_2^i + v_2^i].
\end{aligned} \tag{4}$$

Notice that this argument could be made with $\mathbf{u}^0$ and *any other* $\mathbf{v} \in V$ with the property that $\mathbf{v}$ is in the interior of $(\mathbf{v} + L) \cap V$. Thus, c_i is independent of $\mathbf{v}^0$. Applying a symmetric argument by fixing $\mathbf{v}^0$ and considering different $\mathbf{u} \in U$, we see that c_i is also independent of $\mathbf{u}^0$. In other words, c_i, $i = 1, \dots, m$ only depend on π, L and the two sets U and V, and (4) holds for any $\mathbf{u} \in U$ and $\mathbf{v} \in V$ with the property that $\mathbf{u}$ (resp. $\mathbf{v}$) is in the interior of $(\mathbf{u} + L) \cap V$ (resp. $(\mathbf{u} + L) \cap V$) in the relative topology of L.

We choose $\mathbf{g} \in L'$ as the unique dual vector satisfying $\langle \mathbf{g}, \mathbf{p}^i \rangle = c_i$. Now for any $\mathbf{p} \in L$ such that $\mathbf{u}^0 + \mathbf{p} \in U$. We can then represent $\mathbf{p} = \sum_{i=1}^m \lambda_i \mathbf{p}^i$ such that $u_1^i \leq \lambda_i \leq u_2^i$. Thus, $\pi(\mathbf{u}^0 + \mathbf{p}) = \pi(\mathbf{u}^0 + \sum_{i=1}^m \lambda_i \mathbf{p}^i)$. Now using (4) with $i = m$ we have

$$\begin{aligned}
\pi(\mathbf{u}^0 + \textstyle\sum_{i=1}^m \lambda_i \mathbf{p}^i) &= \pi(\mathbf{u}^0 + \textstyle\sum_{i=1}^{m-1} \lambda_i \mathbf{p}^i + \lambda_m \mathbf{p}^m)\\
&= \pi(\mathbf{u}^0 + \textstyle\sum_{i=1}^{m-1} \lambda_i \mathbf{p}^i) + c_m \cdot \lambda_m
\end{aligned}$$

which follows because the c_i's do not depend on the particular point $\mathbf{u}^0$ and in the case above we apply it on the point $\mathbf{u}^0 + \sum_{i=1}^{m-1} \lambda_i \mathbf{p}^i$. By applying this argument iteratively, we find that

$$\begin{aligned}\pi(\mathbf{u}^0 + \textstyle\sum_{i=1}^m \lambda_i \mathbf{p}^i) &= \pi(\mathbf{u}^0) + \textstyle\sum_{i=1}^m c_i \cdot \lambda_i \\ &= \pi(\mathbf{u}^0) + \textstyle\sum_{i=1}^m \langle \mathbf{g}, \mathbf{p}^i \rangle \cdot \lambda_i \\ &= \pi(\mathbf{u}^0) + \langle \mathbf{g}, \textstyle\sum_{i=1}^m \lambda_i \mathbf{p}^i \rangle \\ &= \pi(\mathbf{u}^0) + \langle \mathbf{g}, \mathbf{p} \rangle\end{aligned}$$

This proves condition (i) in the statement of the lemma. The same argument applies for proving conditions (ii) and (iii). □

Now the lemmas stated in section 2 follow as corollaries.

Proof (of Lemma 2.2). Let $U(\mathbf{x}, r) \subseteq \mathbb{R}^2$ denote the ℓ_∞ ball of radius r around $\mathbf{x} \in \mathbb{R}^2$. Define

$$r(\mathbf{u}) = \sup\{r \in \mathbb{R} \colon \exists \mathbf{v}, \mathbf{w} \text{ such that } U(\mathbf{u}, r) \subseteq I, U(\mathbf{v}, r) \subseteq J, U(\mathbf{w}, 2r) \subseteq K\}.$$

Since (I, J, K) is a valid triple, by Lemma A.12 (i), for any $\mathbf{u} \in \operatorname{int}(I)$, there exist $\mathbf{v} \in \operatorname{int}(J)$ and $\mathbf{w} \in \operatorname{int}(K)$ such that $\mathbf{u} \oplus \mathbf{v} = \mathbf{w}$. Thus, $r(\mathbf{u}) > 0$ for every $\mathbf{u} \in I$.

Claim. $r(\mathbf{u})$ is a continuous function of $\mathbf{u}$.

Proof. $r(\mathbf{u})$ is the optimal value of the linear program with variables $r, \mathbf{v}, \mathbf{w}$ given by

$$\max r \text{ subject to } \mathbf{u} \oplus \mathbf{v} = \mathbf{w}, U(\mathbf{u}, r) \subseteq I, U(\mathbf{v}, r) \subseteq J, U(\mathbf{w}, 2r) \subseteq K.$$

All the constraints can be written as linear constraints. Since the value of a parametric linear program is continuous in the parameter (in this case the parameter is $\mathbf{u}$) we are done. □

We will now show that for any two points $\mathbf{x}_1, \mathbf{x}_2 \in \operatorname{int}(I)$, there exist finitely many full-dimensional parallelotopes $U_1, \dots, U_k$ in $\mathbb{R}^2$ such that $\mathbf{x}_1 \in U_1$, $\mathbf{x}_2 \in U_k$ and $\operatorname{int}(U_i) \cap \operatorname{int}(U_{i+1}) \neq \emptyset$ for all $i = 1, \dots, k-1$. Moreover, we will show that π is affine over each U_i, $i = 1, \dots, k$. This will imply that in fact π is affine over $\operatorname{int}(I)$ and therefore, by continuity, over I. By a symmetric argument, one can show that π is affine over J. This will then show that π is affine over K.

Given $\mathbf{x}_1, \mathbf{x}_2 \in \operatorname{int}(I)$, consider the minimum value ϵ of $r(\mathbf{u})$ as $\mathbf{u}$ varies over the line segment $[\mathbf{x}_1, \mathbf{x}_2]$. Note that ϵ is strictly greater than 0 as it is the minimum of a strictly positive function over a compact set. This implies that we can find a set of points $\mathbf{u}_1 = \mathbf{x}_1, \dots, \mathbf{u}_k = \mathbf{x}_2$ on the line segment $[\mathbf{x}_1, \mathbf{x}_2]$ such that if we let $U_i = U(\mathbf{u}_i, \epsilon)$ we have the property that $\mathbf{x}_1 \in U_1$, $\mathbf{x}_2 \in U_k$ and $\operatorname{int}(U_i) \cap \operatorname{int}(U_{i+1}) \neq \emptyset$ for all $i = 1, \dots, k-1$. Now, by the definition of $r(\mathbf{u}_i)$ which is greater than or equal to ϵ, there exist $\mathbf{v}_i$ and $\mathbf{w}_i$, $i = 1, \dots, k$ such that $U(\mathbf{u}_i, r(u_i)) \in I, U(\mathbf{v}_i, r(\mathbf{u}_i)) \in J, U(\mathbf{w}_i, 2r(\mathbf{u}_i)) \in K$. Applying Lemma A.14, with $L = \mathbb{R}^2$, $U = U(\mathbf{u}_i, r(\mathbf{u}_i))$, $V = U(\mathbf{v}_i, r(\mathbf{u}_i))$ and $\mathbf{u}_0 = \mathbf{u}_i$ and $\mathbf{v}_0 \in v_i$, we obtain that π is affine over $U(\mathbf{u}_i, r(\mathbf{u}_i))$ and hence over $U_i \subseteq U(\mathbf{u}_i, r(\mathbf{u}_i))$.

The fact that the gradient over I and J (and hence over K) are the same follows from the observation that Lemma A.14 gives the same gradient over the parallelotopes $U = U(\mathbf{u}_i, r(\mathbf{u}_i))$ and $V = U(\mathbf{v}_i, r(\mathbf{u}_i))$ in the above argument.

□

Similar arguments can be used to show Lemma 2.3. We omit the proof.

A.8 Transferring affine linearity

Proof (of Lemma 2.4). Let $e \in \mathcal{P}_{q,1}$ be the common edge for I and J. We assume that e is horizontal (the argument for vertical edges is exactly the same) and let $\mathbf{v}^0 \in \mathbb{R}^2$ be the vertex of e such that the other vertex is $\mathbf{v}^0 + (0,1)^T$. Since π is affine on I, there exists $c' \in \mathbb{R}$ such that $\pi(\mathbf{v}^0 + \lambda(0,1)^T) = \pi(\mathbf{v}^0) + c' \cdot \lambda$ for all $0 \leq \lambda \leq 1$. Now observe that any point in J can be written as $\mathbf{v}^0 + \mu_1(0,1)^T + \mu_2(-1,1)^T$ with $0 \leq \mu_1, \mu_2 \leq 1$ and therefore, $\pi(\mathbf{v}^0 + \mu_1(0,1)^T + \mu_2(-1,1)^T) = \pi(\mathbf{v}^0 + \mu_1(0,1)^T) + c \cdot \mu_2$ (using (ii) in the hypothesis) and $\pi(\mathbf{v}^0 + \mu_1(0,1)^T) + c \cdot \mu_2 = \pi(\mathbf{v}^0) + c' \cdot \mu_1 + c \cdot \mu_2$. Thus, π is affine on J. ☐

Proof (of Lemma 3.3). Case (i). Suppose $\{I, J\} \in \mathcal{E}_0$. Since π, π^1, π^2 are all continuous, we just prove that $\partial_{\mathbf{v}}$ is constant on $\operatorname{int}(J)$. If $\{I, J\} \in \mathcal{E}$, $\exists \mathbf{a} \in \frac{1}{q}\mathbb{Z}^2$ such that, setting $K = \{\mathbf{a}\} \in \mathcal{P}_{q,0}$, one of the following two cases occurs.

Case 1. $(I, J, K) \in E(\pi, \mathcal{P}_q)$. Then $\pi_I(\mathbf{x}) + \pi_J(\mathbf{y}) = \pi_K(\mathbf{a})$ for all $\mathbf{x} \in I, \mathbf{y} \in J, \mathbf{x} \oplus \mathbf{y} = \mathbf{a}$, or rewriting this, we have $\pi_J(\mathbf{x}) = \pi_K(\mathbf{a}) - \pi_I(\mathbf{a} \ominus \mathbf{x})$. For any $\mathbf{u} \in \operatorname{int}(J)$, it follows from Lemma 3.2 that $a \ominus \mathbf{u} \in \operatorname{int}(I)$. Since the right hand side is differentiable in the direction of $\mathbf{v}$ at $\mathbf{a} \ominus \mathbf{u}$, the left hand side is as well. The result in this case follows by the chain rule.

Case 2. $(I, K, J) \in E(\pi, \mathcal{P}_q)$. Then $\pi_I(\mathbf{x}) + \pi_K(\mathbf{a}) = \pi_J(\mathbf{y})$ for all $\mathbf{x} \in I, \mathbf{y} \in J, \mathbf{x} \oplus \mathbf{a} = \mathbf{y}$, or rewriting this, we have $\pi_J(\mathbf{x} \oplus \mathbf{a}) = \pi_I(\mathbf{x}) + \pi_K(\mathbf{a})$. For any $\mathbf{u} \in \operatorname{int}(J)$, it follows from Lemma 3.2 that $\mathbf{u} \ominus \mathbf{a} \in \operatorname{int}(I)$. Since the right hand side is differentiable in the direction of $\mathbf{v}$, the left hand side is as well. The result again follows by the chain rule.

Case (ii). Suppose $\{I, J\} \in \mathcal{E}_{\diagdown}$. Using Lemma A.12, the proof follows similar to Case (i). ☐

A.9 Proof of Theorem 3.11

Proof (of Theorem 3.11). Part (i). Suppose (1) does not have a unique solution. Let $\bar{\varphi}\colon \frac{1}{q}\mathbb{Z}^2 \to \mathbb{R}$ be a non-trivial element in the kernel of the system above. Then for any ϵ, $\pi|_{\frac{1}{q}\mathbb{Z}^2} + \epsilon\bar{\varphi}$ also satisfies the system of equations. Let

$$\epsilon = \min\{\, \Delta\pi_F(\mathbf{x}, \mathbf{y}) \neq 0 \mid F \in \Delta\mathcal{P}_q,\ (\mathbf{x}, \mathbf{y}) \in \operatorname{vert}(F) \,\}.$$

Let $\bar{\pi}\colon \mathbb{R}^2 \to \mathbb{R}$ be the continuous piecewise linear extension of φ over $\mathcal{P}_q$ and set $\pi^1 = \pi + \frac{\epsilon}{3||\bar{\pi}||_\infty}\bar{\pi}$, $\pi^2 = \pi - \frac{\epsilon}{3||\bar{\pi}||_\infty}\bar{\pi}$. Note that $0 < ||\bar{\pi}||_\infty < \infty$ since $\bar{\varphi}$ comes from a non-trivial element in the kernel. We claim that π^1, π^2 are both minimal. As before, we show this for π^1, and the proof for π^2 is similar. Since $\pi|_{\frac{1}{q}\mathbb{Z}^2}$ satisfies the system (1) and $\bar{\varphi}$ is an element of the kernel, π^1 satisfies the system (1) as well. In particular, we have $\pi^1(\mathbf{0}) = 0, \pi^1(\mathbf{f}) = 1, \pi^1((0,1)) = 0, \pi^1((0,1)) = 0, \pi^1((1,1)) = 0$.

Next, π^1 is symmetric because the symmetry conditions are implied here, that is, since we require that $\varphi(\mathbf{f}) = 1$, and since π is minimal, $\Delta\pi(\mathbf{u}, \mathbf{f} - \mathbf{u}) = 0$ whenever $\mathbf{u} \in \frac{1}{q}\mathbb{Z}^2$, hence, by Theorem A.10, π^1 is symmetric.

Lastly, we show that π^1 is subadditive. Let $\mathbf{u}, \mathbf{v} \in \frac{1}{q}\mathbb{Z}^2$. If $\Delta\pi(\mathbf{u}, \mathbf{v}) = 0$, then $\Delta\varphi(\mathbf{u}, \mathbf{v}) = 0$, as implied by the system of equations. Otherwise, if $\Delta\pi(\mathbf{u}, \mathbf{v}) > 0$, then

$$\begin{aligned}\Delta\pi^1(\mathbf{u}, \mathbf{v}) &= \Delta\pi(\mathbf{u}, \mathbf{v}) + \frac{\epsilon}{3||\bar{\varphi}||_\infty}\bar{\varphi}(\mathbf{u}) + \frac{\epsilon}{3||\bar{\varphi}||_\infty}\bar{\varphi}(\mathbf{v}) - \frac{\epsilon}{3||\bar{\varphi}||_\infty}\bar{\varphi}(\mathbf{u} \oplus \mathbf{v}) \\ &\geq \Delta\pi_F(\mathbf{u}, \mathbf{v}) - \tfrac{\epsilon}{3} - \tfrac{\epsilon}{3} - \tfrac{\epsilon}{3} \geq 0\end{aligned}$$

Therefore, by Theorem A.10, π^1 (and π^2) is subadditive and therefore minimal and valid. Therefore π is not extreme.

Part (ii). Suppose there exist distinct, valid functions π^1, π^2 such that $\pi = \frac{1}{2}\pi^1 + \frac{1}{2}\pi^2$. Since π is minimal and affine imposing in $\mathcal{P}_{q,2}$, π^1, π^2 are minimal continuous piecewise linear functions over $\mathcal{P}_q$. Furthermore, $\pi|_{\frac{1}{q}\mathbb{Z}^2}$ and, also $\pi^1|_{\frac{1}{q}\mathbb{Z}^2}, \pi^2|_{\frac{1}{q}\mathbb{Z}^2}$ satisfy the system of equations (1). If this system has a unique solution, then $\pi = \pi^1 = \pi^2$, which is a contradiction since π^1, π^2 were assumed distinct. Therefore π is extreme.

On the other hand, if the system (1) does not have a unique solution, then by Theorem 3.11, π is not extreme. □