

# Iterative Hard Thresholding Methods for $l_0$ Regularized Convex Cone Programming

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## Abstract

In this paper we consider  $l_0$  regularized convex cone programming problems. In particular, we first propose an iterative hard thresholding (IHT) method and its variant for solving  $l_0$  regularized box constrained convex programming. We show that the sequence generated by these methods converges to a local minimizer. Also, we establish the iteration complexity of the IHT method for finding an  $\epsilon$ -local-optimal solution. We then propose a method for solving  $l_0$  regularized convex cone programming by applying the IHT method to its quadratic penalty relaxation and establish its iteration complexity for finding an  $\epsilon$ -approximate local minimizer. Finally, we propose a variant of this method in which the associated penalty parameter is dynamically updated, and show that every accumulation point is a local minimizer of the problem.

**Key words:** Sparse approximation, iterative hard thresholding method,  $l_0$  regularization, box constrained convex programming, convex cone programming

## 1 Introduction

Sparse approximations have over the last decade gained a great deal of popularity in numerous areas. For example, in compressed sensing, a large sparse signal is decoded by finding a sparse solution to a system of linear equalities and/or inequalities. Our particular interest of this paper is to find a sparse approximation to a convex cone programming problem in the form of

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & Ax - b \in \mathcal{K}^*, \\ & l \leq x \leq u \end{aligned} \tag{1}$$

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for some  $l \in \bar{\mathfrak{R}}_-^n$ ,  $u \in \bar{\mathfrak{R}}_+^n$ ,  $A \in \mathfrak{R}^{m \times n}$  and  $b \in \mathfrak{R}^m$ , where  $\mathcal{K}^*$  denotes the dual cone of a closed convex cone  $\mathcal{K} \subseteq \mathfrak{R}^m$ , i.e.,  $\mathcal{K}^* = \{s \in \mathfrak{R}^m : s^T x \geq 0, \forall x \in \mathcal{K}\}$ , and  $\bar{\mathfrak{R}}_-^n = \{x : -\infty \leq x_i \leq 0, 1 \leq i \leq n\}$  and  $\bar{\mathfrak{R}}_+^n = \{x \in \mathfrak{R}^n : 0 \leq x_i \leq \infty, 1 \leq i \leq n\}$ . A sparse solution to (1) can be sought by solving the following  $l_0$  regularized convex cone programming problem:

$$\begin{aligned} \min \quad & f(x) + \lambda \|x\|_0 \\ \text{s.t.} \quad & Ax - b \in \mathcal{K}^*, \\ & l \leq x \leq u \end{aligned} \tag{2}$$

for some  $\lambda > 0$ , where  $\|x\|_0$  denotes the cardinality of  $x$ . One special case of (2), that is, the  $l_0$ -regularized unconstrained least squares problem, has been well studied in the literature (e.g., [13, 10]), and some methods were developed for solving it. For example, the iterative hard thresholding (IHT) methods [6, 2, 3] and matching pursuit algorithms [11, 14] were proposed to solve this type of problems. Recently, Lu and Zhang [10] proposed a penalty decomposition method for solving a more general class of  $l_0$  minimization problems.

As shown by the extensive experiments in [2, 3], the IHT method performs very well in finding a sparse solution to unconstrained least squares problems. In addition, the similar type of methods [5, 8] were successfully applied to find low rank solutions in the context of matrix completion. Inspired by these works, in this paper we study IHT methods for solving  $l_0$  regularized convex cone programming problem (2). In particular, we first propose an IHT method and its variant for solving  $l_0$  regularized box constrained convex programming. We show that the sequence generated by these methods converges to a local minimizer. Also, we establish the iteration complexity of the IHT method for finding an  $\epsilon$ -local-optimal solution. We then propose a method for solving  $l_0$  regularized convex cone programming by applying the IHT method to its quadratic penalty relaxation and establish its iteration complexity for finding an  $\epsilon$ -approximate local minimizer of the problem. We also propose a variant of the method in which the associated penalty parameter is dynamically updated, and show that every accumulation point is a local minimizer of the problem.

The outline of this paper is as follows. In Subsection 1.1 we introduce some notations that are used in the paper. In Section 2 we present some technical results about a projected gradient method for convex programming. In Section 3 we propose IHT methods for solving  $l_0$  regularized box constrained convex programming and study their convergence. In section 4 we develop IHT methods for solving  $l_0$  regularized convex cone programming and study their convergence. Finally, in Section 5 we present some concluding remarks.

## 1.1 Notation

Given a nonempty closed convex  $\Omega \subseteq \mathfrak{R}^n$  and an arbitrary point  $x \in \Omega$ ,  $\mathcal{N}_\Omega(x)$  denotes the normal cone of  $\Omega$  at  $x$ . In addition,  $d_\Omega(y)$  denotes the Euclidean distance between  $y \in \mathfrak{R}^n$  and  $\Omega$ . All norms used in the paper are Euclidean norm denoted by  $\|\cdot\|$ . We use  $\mathcal{U}(r)$  to denote a ball centered at the origin with a radius  $r \geq 0$ , that is,  $\mathcal{U}(r) := \{x \in \mathfrak{R}^n : \|x\| \leq r\}$ .

## 2 Technical preliminaries

In this section we present some technical results about a projected gradient method for convex programming that will be subsequently used in this paper.

Consider the convex programming problem

$$\phi^* := \min_{x \in X} \phi(x), \quad (3)$$

where  $X \subseteq \mathfrak{R}^n$  is a closed convex set and  $\phi : X \rightarrow \mathfrak{R}$  is a smooth convex function whose gradient is Lipschitz continuous with constant  $L_\phi > 0$ . Assume that the set of optimal solutions of (3), denoted by  $X^*$ , is nonempty.

Let  $L \geq L_\phi$  be arbitrarily given. A projected gradient of  $\phi$  at any  $x \in X$  with respect to  $X$  is defined as

$$g(x) := L [x - \Pi_X (x - \nabla\phi(x)/L)], \quad (4)$$

where  $\Pi_X(\cdot)$  is the projection map onto  $X$  (see, for example, [12]).

The following properties of the projected gradient are essentially shown in Proposition 3 and Lemma 4 of [9] (see also [12]).

**Lemma 2.1** *Let  $x \in X$  be given and define  $x^+ := \Pi_X(x - \nabla\phi(x)/L)$ . Then, for any given  $\epsilon \geq 0$ , the following statements hold:*

- a)  $\|g(x)\| \leq \epsilon$  if and only if  $\nabla\phi(x) \in -\mathcal{N}_X(x^+) + \mathcal{U}(\epsilon)$ .
- b)  $\|g(x)\| \leq \epsilon$  implies that  $\nabla\phi(x^+) \in -\mathcal{N}_X(x^+) + \mathcal{U}(2\epsilon)$ .
- c)  $\phi(x^+) - \phi(x) \leq -\|g(x)\|^2/(2L)$ .
- d)  $\phi(x) - \phi(x^*) \geq \|g(x)\|^2/(2L)$ , where  $x^* \in \text{Arg min}\{\phi(y) : y \in X\}$ .

We next study a projected gradient method for solving (3).

### Projected gradient method for (3):

Choose an arbitrary  $x^0 \in X$ . Set  $k = 0$ .

- 1) Solve the subproblem

$$x^{k+1} = \arg \min_{x \in X} \{\phi(x^k) + \nabla\phi(x^k)^T(x - x^k) + \frac{L}{2}\|x - x^k\|^2\}. \quad (5)$$

- 2) Set  $k \leftarrow k + 1$  and go to step 1).

**end**

Some properties of the above projected gradient method are established in the following two theorems, which will be used in the subsequent sections of this paper.

**Theorem 2.2** Let  $\{x^k\}$  be generated by the above projected gradient method. Then the following statements hold:

(i) For every  $k \geq 0$  and  $l \geq 1$ ,

$$\phi(x^{k+l}) - \phi^* \leq \frac{L}{2l} \|x^k - x^*\|^2. \quad (6)$$

(ii)  $\{x^k\}$  converges to some optimal solution  $x^*$  of (3).

*Proof.* (i) Since the objective function of (5) is strongly convex with modulus  $L$ , it follows that for every  $x \in X$ ,

$$\phi(x^k) + \nabla \phi(x^k)^T (x - x^k) + \frac{L}{2} \|x - x^k\|^2 \geq \phi(x^k) + \nabla \phi(x^k)^T (x^{k+1} - x^k) + \frac{L}{2} \|x^{k+1} - x^k\|^2 + \frac{L}{2} \|x - x^{k+1}\|^2.$$

By the convexity of  $\phi$ , Lipschitz continuity of  $\nabla \phi$  and  $L \geq L_\phi$ , we have

$$\begin{aligned} \phi(x) &\geq \phi(x^k) + \nabla \phi(x^k)^T (x - x^k), \\ \phi(x^{k+1}) &\leq \phi(x^k) + \nabla \phi(x^k)^T (x^{k+1} - x^k) + \frac{L}{2} \|x^{k+1} - x^k\|^2, \end{aligned}$$

which together with the above inequality imply that

$$\phi(x) + \frac{L}{2} \|x - x^k\|^2 \geq \phi(x^{k+1}) + \frac{L}{2} \|x - x^{k+1}\|^2, \quad \forall x \in X. \quad (7)$$

Letting  $x = x^k$  in (7), we obtain that

$$\phi(x^k) - \phi(x^{k+1}) \geq L \|x^{k+1} - x^k\|^2 / 2.$$

Hence,  $\{\phi(x^k)\}$  is decreasing. Letting  $x = x^* \in X^*$  in (7), we have

$$\phi(x^{k+1}) - \phi^* \leq \frac{L}{2} (\|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2), \quad \forall k \geq 0.$$

Using this inequality and the monotonicity of  $\{\phi(x^k)\}$ , we obtain that

$$l(\phi(x^{k+l}) - \phi^*) \leq \sum_{i=k}^{k+l-1} [\phi(x^{i+1}) - \phi^*] \leq \frac{L}{2} (\|x^k - x^*\|^2 - \|x^{k+l} - x^*\|^2), \quad (8)$$

which immediately yields (6).

(ii) It follows from (8) that

$$\|x^{k+l} - x^*\| \leq \|x^k - x^*\|, \quad \forall k \geq 0, l \geq 1. \quad (9)$$

Hence,  $\|x^k - x^*\| \leq \|x^0 - x^*\|$  for every  $k$ . It implies that  $\{x^k\}$  is bounded. Then, there exists a subsequence  $K$  such that  $\{x^k\}_K \rightarrow \hat{x}^* \in X$ . It can be seen from (6) that  $\{\phi(x^k)\}_K \rightarrow \phi^*$ . Hence,  $\phi(\hat{x}^*) = \lim_{k \in K \rightarrow \infty} \phi(x^k) = \phi^*$ , which implies that  $\hat{x}^* \in X^*$ . Since (9) holds for any  $x^* \in X^*$ , we also have  $\|x_{k+l} - \hat{x}^*\| \leq \|x^k - \hat{x}^*\|$  for every  $k \geq 0$  and  $l \geq 1$ . This together with the fact  $\{x^k\}_K \rightarrow \hat{x}^*$  implies that  $\{x^k\} \rightarrow \hat{x}^*$  and hence statement (ii) holds.  $\blacksquare$

**Theorem 2.3** *Suppose that  $\phi$  is strongly convex with modulus  $\sigma > 0$ . Let  $\{x^k\}$  be generated by the above projected gradient method. Then, for any given  $\epsilon > 0$ , the following statements hold:*

(i)  $\phi(x^k) - \phi^* \leq \epsilon$  whenever

$$k \geq 2\lceil L/\sigma \rceil \left\lceil \log \frac{\phi(x^0) - \phi^*}{\epsilon} \right\rceil.$$

(ii)  $\phi(x^k) - \phi^* < \epsilon$  whenever

$$k \geq 2\lceil L/\sigma \rceil \left\lceil \log \frac{\phi(x^0) - \phi^*}{\epsilon} \right\rceil + 1.$$

*Proof.* (i) Let  $M = \lceil L/\sigma \rceil$ . It follows from Theorem 2.2 that

$$\phi(x^{k+l}) - \phi^* \leq \frac{L}{2l} \|x^k - x^*\|^2 \leq \frac{L}{\sigma l} (\phi(x^k) - \phi^*),$$

where  $x^*$  is the optimal solution of (3). Hence, we have

$$\phi(x^{k+2M}) - \phi^* \leq \frac{L}{2\sigma M} (\phi(x^k) - \phi^*) \leq \frac{1}{2} (\phi(x^k) - \phi^*),$$

which implies that

$$\phi(x^{2^j M}) - \phi^* \leq \frac{1}{2^j} (\phi(x^0) - \phi^*).$$

Let  $K = \lceil \log((\phi(x^0) - \phi^*)/\epsilon) \rceil$ . Hence, when  $k \geq 2KM$ , we have

$$\phi(x^k) - \phi^* \leq \phi(x^{2KM}) - \phi^* \leq \frac{1}{2^K} (\phi(x^0) - \phi^*) \leq \epsilon,$$

which immediately implies that statement (i) holds.

(ii) Let  $K$  and  $M$  be defined as above. If  $\phi(x^{2KM}) = \phi^*$ , by monotonicity of  $\{\phi(x^k)\}$  we have  $\phi(x^k) = \phi^*$  when  $k > 2KM$ , and hence the conclusion holds. We now suppose that  $\phi(x^{2KM}) > \phi^*$ . It implies that  $g(x^{2KM}) \neq 0$ , where  $g$  is defined in (4). Using this relation, Lemma 2.1 (c) and statement (i), we obtain that  $\phi(x^{2KM+1}) < \phi(x^{2KM}) \leq \epsilon$ , which together with the monotonicity of  $\{\phi(x^k)\}$  implies that the conclusion holds. ■

Finally, we consider the convex programming problem

$$f^* := \min\{f(x) : Ax - b \in \mathcal{K}^*, x \in X\}, \quad (10)$$

for some  $A \in \mathfrak{R}^{m \times n}$  and  $b \in \mathfrak{R}^m$ , where  $f : X \rightarrow \mathfrak{R}$  is a smooth convex function whose gradient is Lipschitz continuous with constant  $L_f > 0$ ,  $X \subseteq \mathfrak{R}^n$  is a closed convex set, and  $\mathcal{K}^*$  is the dual cone of a closed convex cone  $\mathcal{K}$ .

The Lagrangian dual function associated with (10) is given by

$$d(\mu) := \inf\{f(x) + \mu^T(Ax - b) : x \in X\}, \quad \forall \mu \in -\mathcal{K}.$$

Assume that there exists a Lagrange multiplier for (10), that is, a vector  $\mu^* \in -\mathcal{K}$  such that  $d(\mu^*) = f^*$ . Under this assumption, the following results are established in Corollary 2 and Proposition 10 of [9], respectively.

**Lemma 2.4** *Let  $\mu^*$  be a Lagrange multiplier for (10). There holds:*

$$f(x) - f^* \geq -\|\mu^*\|d_{\mathcal{K}^*}(Ax - b), \quad \forall x \in X.$$

**Lemma 2.5** *Let  $\rho > 0$  be given and  $L_\rho = L_f + \rho\|A\|^2$ . Consider the problem*

$$\Phi_\rho^* := \min_{x \in X} \{\Phi_\rho(x) := f(x) + \frac{\rho}{2}[d_{\mathcal{K}^*}(Ax - b)]^2\}. \quad (11)$$

*If  $x \in X$  is a  $\xi$ -approximate solution of (11), i.e.,  $\Phi_\rho(x) - \Phi_\rho^* \leq \xi$ , then the pair  $(x^+, \mu)$  defined as*

$$\begin{aligned} x^+ &:= \Pi_X(x - \nabla\Phi_\rho(x)/L_\rho), \\ \mu &:= \rho[Ax^+ - b - \Pi_{\mathcal{K}^*}(Ax^+ - b)] \end{aligned}$$

*is in  $X \times (-\mathcal{K})$  and satisfies  $\mu^T \Pi_{\mathcal{K}^*}(Ax^+ - b) = 0$  and the relations*

$$\begin{aligned} d_{\mathcal{K}^*}(Ax^+ - b) &\leq \frac{1}{\rho}\|\mu^*\| + \sqrt{\frac{\xi}{\rho}}, \\ \nabla f(x^+) + A^T \mu &\in -\mathcal{N}_X(x^+) + \mathcal{U}(2\sqrt{2L_\rho\xi}), \end{aligned}$$

*where  $\mu^*$  is an arbitrary Lagrange multiplier for (10).*

### 3 $l_0$ regularized box constrained convex programming

In this section we consider a special case of (2), that is,  $l_0$  regularized box constrained convex programming problem in the form of:

$$\begin{aligned} \underline{F}^* &:= \min F(x) := f(x) + \lambda\|x\|_0 \\ &\text{s.t. } l \leq x \leq u \end{aligned} \quad (12)$$

for some  $\lambda > 0$ ,  $l \in \bar{\mathfrak{R}}_-^n$  and  $u \in \bar{\mathfrak{R}}_+^n$ . Recently, Blumensath and Davies [2, 3] proposed an iterative hard thresholding (IHT) method for solving a special case of (12) with  $f(x) = \|Ax - b\|^2$ ,  $l_i = -\infty$  and  $u_i = \infty$  for all  $i$ . Our aim is to extend their IHT method to solve (12) and study its convergence. In addition, we establish its iteration complexity for finding an  $\epsilon$ -local-optimal solution of (12). Finally, we propose a variant of the IHT method in which only ‘‘local’’ Lipschitz constant of  $\nabla f$  is used.

Throughout this section we assume that  $f$  is a smooth convex function in  $\mathcal{B}$  whose gradient is Lipschitz continuous with constant  $L_f > 0$ , and also that  $f$  is bounded below on the set  $\mathcal{B}$ , where

$$\mathcal{B} := \{x \in \mathfrak{R}^n : l \leq x \leq u\}. \quad (13)$$

We now present an IHT method for solving problem (12).

**Iterative hard thresholding method for (12):**

Choose an arbitrary  $x^0 \in \mathcal{B}$ . Set  $k = 0$ .

1) Solve the subproblem

$$x^{k+1} \in \text{Arg min}_{x \in \mathcal{B}} \{f(x^k) + \nabla f(x^k)^T(x - x^k) + \frac{L}{2}\|x - x^k\|^2 + \lambda\|x\|_0\}. \quad (14)$$

2) Set  $k \leftarrow k + 1$  and go to step 1).

**end**

*Remark.* The subproblem (14) has a closed form solution given in (21). ■

In what follows, we study the convergence of the above IHT method for (12). Before proceeding, we introduce some notations that will be used subsequently. Define

$$\mathcal{B}_I := \{x \in \mathcal{B} : x_I = 0\}, \quad \forall I \subseteq \{1, \dots, n\}, \quad (15)$$

$$\Pi_{\mathcal{B}}(x) := \arg \min\{\|y - x\| : y \in \mathcal{B}\}, \quad \forall x \in \mathfrak{R}^n,$$

$$s_L(x) := x - \frac{1}{L}\nabla f(x), \quad \forall x \in \mathcal{B}, \quad (16)$$

$$I(x) := \{i : x_i = 0\}, \quad \forall x \in \mathfrak{R}^n \quad (17)$$

for some constant  $L > L_f$ .

The following lemma establishes some properties of the operators  $s_L(\cdot)$  and  $\Pi_{\mathcal{B}}(s_L(\cdot))$ , which will be used subsequently.

**Lemma 3.1** *For any  $x, y \in \mathfrak{R}^n$ , there hold:*

- (1)  $|[s_L(x)]_i^2 - [s_L(y)]_i^2| \leq 4(\|x - y\| + |[s_L(y)]_i|)\|x - y\|;$
- (2)  $|[\Pi_{\mathcal{B}}(s_L(x)) - s_L(x)]_i^2 - [\Pi_{\mathcal{B}}(s_L(y)) - s_L(y)]_i^2| \leq 4(\|x - y\| + |[\Pi_{\mathcal{B}}(s_L(y)) - s_L(y)]_i|)\|x - y\|.$

*Proof.* (1) We observe that

$$\begin{aligned} \|s_L(x) - s_L(y)\| &= \|x - y - \frac{1}{L}(\nabla f(x) - \nabla f(y))\| \leq \|x - y\| + \frac{1}{L}\|\nabla f(x) - \nabla f(y)\|, \\ &\leq (1 + \frac{L_f}{L})\|x - y\| \leq 2\|x - y\|. \end{aligned} \quad (18)$$

It follows from (18) that

$$\begin{aligned}
|[s_L(x)]_i^2 - [s_L(y)]_i^2| &= |[s_L(x)]_i + [s_L(y)]_i| \cdot |[s_L(x)]_i - [s_L(y)]_i|, \\
&\leq (|[s_L(x)]_i - [s_L(y)]_i| + 2|[s_L(y)]_i|) \cdot |[s_L(x)]_i - [s_L(y)]_i|, \\
&\leq 4(\|x - y\| + |[s_L(y)]_i|)\|x - y\|.
\end{aligned}$$

(2) It can be shown that

$$\|\Pi_{\mathcal{B}}(x) - x + y - \Pi_{\mathcal{B}}(y)\| \leq \|x - y\|.$$

Using this inequality and (18), we then have

$$\begin{aligned}
&|[\Pi_{\mathcal{B}}(s_L(x)) - s_L(x)]_i^2 - [\Pi_{\mathcal{B}}(s_L(y)) - s_L(y)]_i^2| \\
&\leq (|[\Pi_{\mathcal{B}}(s_L(x)) - s_L(x)]_i - [\Pi_{\mathcal{B}}(s_L(y)) - s_L(y)]_i| + 2|[\Pi_{\mathcal{B}}(s_L(y)) - s_L(y)]_i|) \\
&\quad \cdot |[\Pi_{\mathcal{B}}(s_L(x)) - s_L(x)]_i - [\Pi_{\mathcal{B}}(s_L(y)) - s_L(y)]_i|, \\
&\leq (\|s_L(x) - s_L(y)\| + 2|[\Pi_{\mathcal{B}}(s_L(y)) - s_L(y)]_i|) \cdot \|s_L(x) - s_L(y)\|, \\
&\leq 4(\|x - y\| + |[\Pi_{\mathcal{B}}(s_L(y)) - s_L(y)]_i|)\|x - y\|.
\end{aligned}$$

■

The following lemma shows that for the sequence  $\{x^k\}$ , the magnitude of any nonzero component  $x_i^k$  cannot be too small for  $k \geq 1$ .

**Lemma 3.2** *Let  $\{x^k\}$  be generated by the above IHT method. Then, for all  $k \geq 0$ ,*

$$|x_j^{k+1}| \geq \delta := \min_{i \notin I_0} \delta_i > 0, \quad \text{if } x_j^{k+1} \neq 0, \quad (19)$$

where  $I_0 = \{i : l_i = u_i = 0\}$  and

$$\delta_i = \begin{cases} \min(u_i, \sqrt{2\lambda/L}), & \text{if } l_i = 0, \\ \min(-l_i, \sqrt{2\lambda/L}), & \text{if } u_i = 0, \\ \min(-l_i, u_i, \sqrt{2\lambda/L}), & \text{otherwise,} \end{cases} \quad \forall i \in I_0. \quad (20)$$

*Proof.* One can observe from (14) that for  $i = 1, \dots, n$ ,

$$x_i^{k+1} = \begin{cases} [\Pi_{\mathcal{B}}(s_L(x^k))]_i, & \text{if } [s_L(x^k)]_i^2 - [\Pi_{\mathcal{B}}(s_L(x^k)) - s_L(x^k)]_i^2 > \frac{2\lambda}{L}, \\ 0, & \text{if } [s_L(x^k)]_i^2 - [\Pi_{\mathcal{B}}(s_L(x^k)) - s_L(x^k)]_i^2 < \frac{2\lambda}{L}, \\ [\Pi_{\mathcal{B}}(s_L(x^k))]_i \text{ or } 0, & \text{otherwise} \end{cases} \quad (21)$$

(see, for example, [10]). Suppose that  $j$  is an index such that  $x_j^{k+1} \neq 0$ . Clearly,  $j \notin I_0$ , where  $I_0$  is define above. It follows from (21) that

$$x_j^{k+1} = [\Pi_{\mathcal{B}}(s_L(x^k))]_j \neq 0, \quad [s_L(x^k)]_j^2 - [\Pi_{\mathcal{B}}(s_L(x^k)) - s_L(x^k)]_j^2 \geq \frac{2\lambda}{L}. \quad (22)$$



The second relation of (22) implies that  $|[s_L(x^k)]_j| \geq \sqrt{2\lambda/L}$ . In addition, by the first relation of (22) and the definition of  $\Pi_{\mathcal{B}}$ , we have

$$x_j^{k+1} = [\Pi_{\mathcal{B}}(s_L(x^k))]_j = \min(\max([s_L(x^k)]_j, l_j), u_j) \neq 0. \quad (23)$$

Recall that  $j \notin I_0$ . We next show that  $|x_j^{k+1}| \geq \delta_j$  by considering three separate cases: i)  $l_j = 0$ ; ii)  $u_j = 0$ ; and iii)  $l_j u_j \neq 0$ . For case i), it follows from (23) that  $[s_L(x^k)]_j \geq 0$  and  $x_j^{k+1} = \min([s_L(x^k)]_j, u_j)$ . This together with the relation  $|[s_L(x^k)]_j| \geq \sqrt{2\lambda/L}$  and the definition of  $\delta_j$  implies that  $|x_j^{k+1}| \geq \delta_j$ . By the similar arguments, we can show that  $|x_j^{k+1}| \geq \delta_j$  also holds for the other two cases. Then, it is easy to see that the conclusion of this lemma holds.  $\blacksquare$

We next establish that the sequence  $\{x^k\}$  converges to a local minimizer of (12), and moreover,  $F(x^k)$  converges to a local minimum value of (12).

**Theorem 3.3** *Let  $\{x^k\}$  be generated by the above IHT method. Then,  $x^k$  converges to a local minimizer  $x^*$  of problem (12) and moreover,  $I(x^k) \rightarrow I(x^*)$ ,  $\|x^k\|_0 \rightarrow \|x^*\|_0$  and  $F(x^k) \rightarrow F(x^*)$ .*

*Proof.* Since  $\nabla f$  is Lipschitz continuous with constant  $L_f$ , we have

$$f(x^{k+1}) \leq f(x^k) + \nabla f(x^k)^T(x - x^k) + \frac{L_f}{2}\|x^{k+1} - x^k\|^2.$$

Using this inequality, the fact that  $L > L_f$ , and (14), we obtain that

$$\begin{aligned} F(x^{k+1}) &= f(x^{k+1}) + \lambda\|x^{k+1}\|_0 \leq \overbrace{f(x^k) + \nabla f(x^k)^T(x^{k+1} - x^k) + \frac{L_f}{2}\|x^{k+1} - x^k\|^2 + \lambda\|x^{k+1}\|_0}^a \\ &\leq \underbrace{f(x^k) + \nabla f(x^k)^T(x^{k+1} - x^k) + \frac{L}{2}\|x^{k+1} - x^k\|^2 + \lambda\|x^{k+1}\|_0}_b \\ &\leq f(x^k) + \lambda\|x^k\|_0 = F(x^k), \end{aligned}$$

where the last inequality follows from (14). The above inequality implies that  $\{F(x^k)\}$  is nonincreasing and moreover,

$$F(x^k) - F(x^{k+1}) \geq b - a = \frac{L - L_f}{2}\|x^{k+1} - x^k\|^2. \quad (24)$$

By the assumption, we know that  $f$  is bounded below in  $\mathcal{B}$ . It then follows that  $\{F(x^k)\}$  is bounded below. Hence,  $\{F(x^k)\}$  converges to a finite value as  $k \rightarrow \infty$ , which together with (24) implies that

$$\lim_{k \rightarrow \infty} \|x^{k+1} - x^k\| = 0. \quad (25)$$

Let  $I_k = I(x^k)$ , where  $I(\cdot)$  is defined in (17). In view of (19), we observe that

$$\|x^{k+1} - x^k\| \geq \delta \text{ if } I_k \neq I_{k+1}. \quad (26)$$

This together with (25) implies that  $I_k$  does not change when  $k$  is sufficient large. Hence, there exist some  $K \geq 0$  and  $I \subseteq \{1, \dots, n\}$  such that  $I_k = I$  for all  $k \geq K$ . Then one can observe from (14) that

$$x^{k+1} = \arg \min_{x \in \mathcal{B}_I} \{f(x^k) + \nabla f(x^k)^T(x - x^k) + \frac{L}{2}\|x - x^k\|^2\}, \quad \forall k > K,$$

where  $\mathcal{B}_I$  is defined in (15). It follows from Lemma 2.2 that  $x^k \rightarrow x^*$ , where

$$x^* \in \text{Arg min}\{f(x) : x \in \mathcal{B}_I\}. \quad (27)$$

It is not hard to see from (27) that  $x^*$  is a local minimizer of (12). In addition, we know from (19) that  $|x_i^k| \geq \delta$  for  $k > K$  and  $i \notin I$ . It yields  $|x_i^*| \geq \delta$  for  $i \notin I$  and  $x_i^* = 0$  for  $i \in I$ . Hence,  $I(x^k) = I(x^*) = I$  for all  $k > K$ , which clearly implies that  $\|x^k\|_0 = \|x^*\|_0$  for every  $k > K$ . By continuity of  $f$ , we have  $f(x^k) \rightarrow f(x^*)$ . It then follows that

$$F(x^k) = f(x^k) + \lambda\|x^k\|_0 \rightarrow f(x^*) + \lambda\|x^*\|_0 = F(x^*).$$

■

As shown in Theorem 3.3,  $x^k \rightarrow x^*$  for some local minimizer  $x^*$  of (12) and  $F(x^k) \rightarrow F(x^*)$ . Our next aim is to establish the iteration complexity of the IHT method for finding an  $\epsilon$ -local-optimal solution  $x_\epsilon \in \mathcal{B}$  of (12) satisfying  $F(x_\epsilon) \leq F(x^*) + \epsilon$  and  $I(x_\epsilon) = I(x^*)$ . Before proceeding, we define

$$\alpha = \min_{I \subseteq \{1, \dots, n\}} \left\{ \min_i \left| [s_L(x^*)]_i^2 - [\Pi_{\mathcal{B}}(s_L(x^*)) - s_L(x^*)]_i^2 - \frac{2\lambda}{L} \right| : x^* \in \text{Arg min}\{f(x) : x \in \mathcal{B}_I\} \right\}, \quad (28)$$

$$\beta = \max_{I \subseteq \{1, \dots, n\}} \left\{ \max_i (|s_L(x^*)|_i + |\Pi_{\mathcal{B}}(s_L(x^*)) - s_L(x^*)|_i) : x^* \in \text{Arg min}\{f(x) : x \in \mathcal{B}_I\} \right\}. \quad (29)$$

**Theorem 3.4** *Assume that  $f$  is a smooth strongly convex function with modulus  $\sigma > 0$ . Suppose that  $L > L_f$  is chosen such that  $\alpha > 0$ . Let  $\{x^k\}$  be generated by the above IHT method,  $I_k = I(x^k)$  for all  $k$ ,  $x^* = \lim_{k \rightarrow \infty} x^k$ , and  $F^* = F(x^*)$ . Then, for any given  $\epsilon > 0$ , the following statements hold:*

(i) *The number changes of  $I_k$  is at most  $\left\lceil \frac{2(F(x^0) - F^*)}{(L - L_f)\delta^2} \right\rceil$ .*

(ii) *The total number of iterations by the IHT method for finding an  $\epsilon$ -local-optimal solution  $x_\epsilon \in \mathcal{B}$  satisfying  $I(x_\epsilon) = I(x^*)$  and  $F(x_\epsilon) \leq F^* + \epsilon$  is at most  $2\lceil L/\sigma \rceil \log \frac{\theta}{\epsilon}$ , where*

$$\theta = (F(x^0) - F^*)2^{\frac{\omega+3}{2}}, \quad \omega = \max_t \left\{ (d - 2c)t - ct^2 : 0 \leq t \leq \left\lfloor \frac{2(F(x^0) - F^*)}{(L - L_f)\delta^2} \right\rfloor \right\}, \quad (30)$$

$$c = \frac{(L - L_f)\delta^2}{2(F(x^0) - F^*)}, \quad \gamma = \sigma(\sqrt{2\alpha + \beta^2} - \beta)^2/32, \quad (31)$$

$$d = 2 \log(F(x^0) - F^*) + 4 - 2 \log \gamma + c.$$

*Proof.* (i) As shown in Theorem 3.3,  $I_k$  only changes for a finite number of times. Assume that  $I_k$  only changes at  $k = n_1 + 1, \dots, n_J + 1$ , that is,

$$I_{n_{j-1}+1} = \dots = I_{n_j} \neq I_{n_{j+1}} = \dots = I_{n_{j+1}}, \quad j = 1, \dots, J-1, \quad (32)$$

where  $n_0 = 0$ .

We next bound  $J$ , i.e., the total number of changes of  $I_k$ . In view of (26) and (32), one can observe that

$$\|x^{n_{j+1}} - x^{n_j}\| \geq \delta, \quad j = 1, \dots, J,$$

which together with (24) implies that

$$F(x^{n_j}) - F(x^{n_{j+1}}) \geq \frac{1}{2}(L - L_f)\delta^2, \quad j = 1, \dots, J. \quad (33)$$

Summing up these inequalities and using the monotonicity of  $\{F(x^k)\}$ , we have

$$\frac{1}{2}(L - L_f)\delta^2 J \leq F(x^{n_1}) - F(x^{n_{J+1}}) \leq F(x^0) - F^*, \quad (34)$$

and hence

$$J \leq \left\lfloor \frac{2(F(x^0) - F^*)}{(L - L_f)\delta^2} \right\rfloor. \quad (35)$$

(ii) Let  $n_j$  be defined as above for  $j = 1, \dots, J$ . We first show that

$$n_j - n_{j-1} \leq 2 + 2\lceil L/\sigma \rceil \left\lceil \log \left( F(x^0) - (j-1)(L - L_f)\delta^2/2 - \underline{F}^* \right) - \log \gamma \right\rceil, \quad j = 1, \dots, J, \quad (36)$$

where  $\underline{F}^*$  and  $\gamma$  are defined in (12) and (31), respectively. Indeed, one can observe from (14) that

$$x^{k+1} = \arg \min_{x \in \mathcal{B}} \{f(x^k) + \nabla f(x^k)^T(x - x^k) + \frac{L}{2}\|x - x^k\|^2 : x_{I_{k+1}} = 0\}.$$

Therefore, for  $j = 1, \dots, J$  and  $k = n_{j-1}, \dots, n_j - 1$ ,

$$x^{k+1} = \arg \min_{x \in \mathcal{B}} \{f(x^k) + \nabla f(x^k)^T(x - x^k) + \frac{L}{2}\|x - x^k\|^2 : x_{I_{n_j}} = 0\}.$$

We arbitrarily choose  $1 \leq j \leq J$ . Let  $\bar{x}^*$  (depending on  $j$ ) denote the optimal solution of

$$\min_{x \in \mathcal{B}} \{f(x) : x_{I_{n_j}} = 0\}. \quad (37)$$

One can observe that

$$\|\bar{x}^*\|_0 \leq \|x^{n_{j-1}+1}\|_0.$$

Also, it follows from (33) and the monotonicity of  $\{F(x^k)\}$  that

$$F(x^{n_{j+1}}) \leq F(x^0) - \frac{j}{2}(L - L_f)\delta^2, \quad j = 1, \dots, J. \quad (38)$$

Using these relations and the fact that  $F(\bar{x}^*) \geq \underline{F}^*$ , we have

$$\begin{aligned} f(x^{n_{j-1}+1}) - f(\bar{x}^*) &= F(x^{n_{j-1}+1}) - \lambda \|x^{n_{j-1}+1}\|_0 - F(\bar{x}^*) + \lambda \|\bar{x}^*\|_0, \\ &\leq F(x^0) - \frac{j-1}{2}(L - L_f)\delta^2 - \underline{F}^*. \end{aligned} \quad (39)$$

Suppose for a contradiction that (36) does not hold for some  $1 \leq j \leq J$ . Hence, we have

$$n_j - n_{j-1} > 2 + 2\lceil L/\sigma \rceil \lceil \log(F(x^0) - (j-1)(L - L_f)\delta^2/2 - \underline{F}^*) - \log \gamma \rceil.$$

This inequality and (39) yields

$$n_j - n_{j-1} > 2 + 2\lceil L/\sigma \rceil \left\lceil \log \frac{f(x^{n_{j-1}+1}) - f(\bar{x}^*)}{\gamma} \right\rceil.$$

Using the strong convexity of  $f$  and applying Theorem 2.3 (ii) to (37) with  $\epsilon = \gamma$ , we obtain that

$$\frac{\sigma}{2} \|x^{n_j} - \bar{x}^*\|^2 \leq f(x^{n_j}) - f(\bar{x}^*) < \frac{\sigma}{32} (\sqrt{2\alpha + \beta^2} - \beta)^2.$$

It implies that

$$\|x^{n_j} - \bar{x}^*\| < \frac{\sqrt{2\alpha + \beta^2} - \beta}{4}. \quad (40)$$

Using (40), Lemma 3.1 and the definition of  $\beta$ , we have

$$\begin{aligned} &|[s_L(x^{n_j})]_i^2 - [s_L(\bar{x}^*)]_i^2 - [\Pi_{\mathcal{B}}(s_L(x^{n_j})) - s_L(x^{n_j})]_i^2 + [\Pi_{\mathcal{B}}(s_L(\bar{x}^*)) - s_L(\bar{x}^*)]_i^2| \\ &\leq |[s_L(x^{n_j})]_i^2 - [s_L(\bar{x}^*)]_i^2| + |[\Pi_{\mathcal{B}}(s_L(x^{n_j})) - s_L(x^{n_j})]_i^2 - [\Pi_{\mathcal{B}}(s_L(\bar{x}^*)) - s_L(\bar{x}^*)]_i^2| \\ &\leq 4(2\|x^{n_j} - \bar{x}^*\| + \beta)\|x^{n_j} - \bar{x}^*\| < \alpha, \end{aligned} \quad (41)$$

where the last inequality is due to (40). Let

$$I^* = \left\{ i : [s_L(\bar{x}^*)]_i^2 - [\Pi_{\mathcal{B}}(s_L(\bar{x}^*)) - s_L(\bar{x}^*)]_i^2 < \frac{2\lambda}{L} \right\}$$

and let  $\bar{I}^* = \{1, \dots, n\} \setminus I^*$ . Since  $\alpha > 0$ , we know that

$$[s_L(\bar{x}^*)]_i^2 - [\Pi_{\mathcal{B}}(s_L(\bar{x}^*)) - s_L(\bar{x}^*)]_i^2 > \frac{2\lambda}{L}, \quad \forall i \in \bar{I}^*.$$

It then follows from (41) and the definition of  $\alpha$  that

$$\begin{aligned} [s_L(x^{n_j})]_i^2 - [\Pi_{\mathcal{B}}(s_L(x^{n_j})) - s_L(x^{n_j})]_i^2 &< \frac{2\lambda}{L}, \quad \forall i \in I^*, \\ [s_L(x^{n_j})]_i^2 - [\Pi_{\mathcal{B}}(s_L(x^{n_j})) - s_L(x^{n_j})]_i^2 &> \frac{2\lambda}{L}, \quad \forall i \in \bar{I}^*. \end{aligned}$$

Observe that  $[\Pi_{\mathcal{B}}(s_L(x^{n_j}))]_i \neq 0$  for all  $i \in \bar{I}^*$ . This fact together with (21) implies that

$$x_i^{n_j+1} = 0, \quad i \in I^* \quad \text{and} \quad x_i^{n_j+1} \neq 0, \quad i \in \bar{I}^*.$$

By a similar argument, one can show that

$$x_i^{n_j} = 0, \quad i \in I^* \quad \text{and} \quad x_i^{n_j} \neq 0, \quad i \in \bar{I}^*.$$

Hence,  $I_{n_j} = I_{n_{j+1}} = I^*$ , which is a contradiction to (32). We thus conclude that (36) holds.

Let  $N_\epsilon$  denote the total number of iterations for finding an  $\epsilon$ -local-optimal solution  $x_\epsilon \in \mathcal{B}$  by the IHT method satisfying  $I(x_\epsilon) = I(x^*)$  and  $F(x_\epsilon) \leq F^* + \epsilon$ . We next establish an upper bound for  $N_\epsilon$ . Summing up the inequality (36) for  $j = 1, \dots, J$ , we obtain that

$$n_J \leq \sum_{j=1}^J \left\{ 2 + 2\lceil L/\sigma \rceil \left[ \log(F(x^0) - \frac{j-1}{2}(L - L_f)\delta^2 - \underline{F}^*) - \log \gamma \right] \right\}.$$

Using this inequality, (34), and the facts that  $L \geq \sigma$  and  $\log(1-t) \leq -t$  for all  $t \in (0, 1)$ , we have

$$\begin{aligned} n_J &\leq \sum_{j=1}^J \left[ 2 + 2\lceil L/\sigma \rceil \left( \log(F(x^0) - \frac{j-1}{2}(L - L_f)\delta^2 - \underline{F}^*) - \log \gamma + 1 \right) \right], \\ &\leq \sum_{j=1}^J \left[ 2 + 2\lceil L/\sigma \rceil \left( \log(F(x^0) - \underline{F}^*) - \frac{(L - L_f)\delta^2}{2(F(x^0) - \underline{F}^*)}(j-1) - \log \gamma + 1 \right) \right], \\ &\leq \lceil L/\sigma \rceil \left[ \underbrace{\left( 2\log(F(x^0) - \underline{F}^*) + 4 - 2\log \gamma + \frac{(L - L_f)\delta^2}{2(F(x^0) - \underline{F}^*)} \right)}_d J - \underbrace{\frac{(L - L_f)\delta^2}{2(F(x^0) - \underline{F}^*)}}_c J^2 \right] \end{aligned} \tag{42}$$

By the definition of  $n_J$ , we observe that after  $n_J + 1$  iterations, the IHT method becomes the projected gradient method applied to the problem

$$x^* = \arg \min_{x \in \mathcal{B}} \{f(x) : x_{I_{n_J+1}} = 0\}.$$

In addition, we know from Theorem 3.3 that  $I(x^k) = I(x^*)$  for all  $k > n_J$ . Hence,  $f(x^k) - f(x^*) = F(x^k) - F^*$  when  $k > n_J$ . Using these facts and Theorem 2.3 (ii), we have

$$N_\epsilon \leq n_J + 1 + 2\lceil L/\sigma \rceil \left\lceil \log \frac{F(x^{n_J+1}) - F^*}{\epsilon} \right\rceil.$$

Using this inequality, (38), (42) and the facts that  $F^* \geq \underline{F}^*$ ,  $L \geq \sigma$  and  $\log(1-t) \leq -t$  for all  $t \in (0, 1)$ , we obtain that

$$\begin{aligned} N_\epsilon &\leq n_J + 1 + 2\lceil L/\sigma \rceil \left( \log(F(x^0) - \frac{J}{2}(L - L_f)\delta^2 - F^*) + 1 - \log \epsilon \right), \\ &\leq n_J + \lceil L/\sigma \rceil \left( 2\log(F(x^0) - F^*) - \frac{(L - L_f)\delta^2 J}{F(x^0) - F^*} + 3 - 2\log \epsilon \right) \\ &\leq \lceil L/\sigma \rceil \left[ (d - 2c)J - cJ^2 + 2\log(F(x^0) - F^*) + 3 - 2\log \epsilon \right], \end{aligned}$$

which together with (35) and (30) implies that

$$N_\epsilon \leq 2\lceil L/\sigma \rceil \log \frac{\theta}{\epsilon}.$$

■

The iteration complexity given in Theorem 3.4 is based on the assumption that  $f$  is strongly convex in  $\mathcal{B}$ . We next consider a case where  $\mathcal{B}$  is bounded and  $f$  is convex but not strongly convex. We will establish the iteration complexity of finding an  $\epsilon$ -local-optimal solution of (12) by the IHT method applied to a perturbation of (12) obtained from adding a “small” strongly convex regularization term to  $f$ .

Consider a perturbation of (12) in the form of

$$\underline{F}_\nu^* := \min_{x \in \mathcal{B}} \{F_\nu(x) := f_\nu(x) + \lambda \|x\|_0\}, \quad (43)$$

where  $\nu > 0$  and

$$f_\nu(x) := f(x) + \frac{\nu}{2} \|x\|^2.$$

One can easily see that  $f_\nu$  is strongly convex in  $\mathcal{B}$  with modulus  $\nu$  and moreover  $\nabla f_\nu$  is Lipschitz continuous with constant  $L_\nu$ , where

$$L_\nu = L_f + \nu. \quad (44)$$

We next establish the iteration complexity of finding an  $\epsilon$ -local-optimal solution of (12) by the IHT method applied to (43). Given any  $L > 0$ , let  $s_L$ ,  $\alpha$  and  $\beta$  be defined according to (16), (28) and (29), respectively, by replacing  $f$  by  $f_\nu$ , and let  $\delta$  be defined in (19).

**Theorem 3.5** *Suppose that  $\mathcal{B}$  is bounded and  $f$  is convex but not strongly convex. Let  $\epsilon > 0$  be arbitrarily given,  $D = \max\{\|x\| : x \in \mathcal{B}\}$ ,  $\nu = \epsilon/D^2$ , and  $L > L_\nu$  be chosen such that  $\alpha > 0$ . Let  $\{x^k\}$  be generated by the IHT method applied to (43), and let  $x^* = \lim_{k \rightarrow \infty} x^k$ ,  $F_\nu^* = F_\nu(x^*)$  and  $F^* = \min\{F(x) : x \in \mathcal{B}_{I^*}\}$ , where  $I^* = \{i : x_i^* = 0\}$ . Then, the total number of iterations by the IHT method for finding an  $\epsilon$ -local-optimal solution  $x_\epsilon \in \mathcal{B}$  satisfying  $F(x_\epsilon) \leq F^* + \epsilon$  is at most  $2 \left\lceil \frac{D^2 L_f}{\epsilon} + 1 \right\rceil \log \frac{2\theta}{\epsilon}$ , where*

$$\begin{aligned} \theta &= (F_\nu(x^0) - F_\nu^*) 2^{\frac{\omega+3}{2}}, \quad \omega = \max_t \left\{ (d-2c)t - ct^2 : 0 \leq t \leq \left\lfloor \frac{2(F_\nu(x^0) - F_\nu^*)}{(L-L_\nu)\delta^2} \right\rfloor \right\}, \\ c &= \frac{(L-L_\nu)\delta^2}{2(F_\nu(x^0) - F_\nu^*)}, \quad \gamma = \nu(\sqrt{2\alpha + \beta^2} - \beta)^2/32, \\ d &= 2 \log(F_\nu(x^0) - F_\nu^*) + 4 - 2 \log \gamma + c. \end{aligned}$$

*Proof.* By Theorem 3.4 (ii), we see that the IHT method applied to (43) finds an  $\epsilon/2$ -local-optimal solution  $x_\epsilon \in \mathcal{B}$  of (43) satisfying  $I(x_\epsilon) = I(x^*)$  and  $F_\nu(x_\epsilon) \leq F_\nu^* + \epsilon/2$  within  $2\lceil L_\nu/\nu \rceil \log \frac{2\theta}{\epsilon}$  iterations. From the proof of Theorem 3.3, we observe that

$$F_\nu(x^*) = \min\{F_\nu(x) : x \in \mathcal{B}_{I^*}\}.$$

Hence, we have

$$F_\nu^* = F_\nu(x^*) \leq \min_{x \in \mathcal{B}_{F^*}} f(x) + \frac{\nu D^2}{2} \leq F^* + \frac{\epsilon}{2}.$$

In addition, we observe that  $F(x_\epsilon) \leq F_\nu(x_\epsilon)$ . Hence, it follows that

$$F(x_\epsilon) \leq F_\nu(x_\epsilon) \leq F_\nu^* + \frac{\epsilon}{2} \leq F^* + \epsilon.$$

Note that  $F^*$  is a local optimal value of (12). Hence,  $x_\epsilon$  is an  $\epsilon$ -local-optimal solution of (12). The conclusion of this theorem then follows from (44) and  $\nu = \epsilon/D^2$ .  $\blacksquare$

For the above IHT method, a fixed  $L$  is used through all iterations, which may be too conservative. To improve its practical performance, we can use “local”  $L$  that is update dynamically. The resulting variant of the method is presented as follows.

**A variant of IHT method for (12):**

Let  $0 < L_{\min} < L_{\max}$ ,  $\tau > 1$  and  $\eta > 0$  be given. Choose an arbitrary  $x^0 \in \mathcal{B}$  and set  $k = 0$ .

1) Choose  $L_k^0 \in [L_{\min}, L_{\max}]$  arbitrarily. Set  $L_k = L_k^0$ .

1a) Solve the subproblem

$$x^{k+1} \in \text{Arg min}_{x \in \mathcal{B}} \{f(x^k) + \nabla f(x^k)^T(x - x^k) + \frac{L_k}{2}\|x - x^k\|^2 + \lambda\|x\|_0\}. \quad (45)$$

1b) If

$$F(x^k) - F(x^{k+1}) \geq \frac{\eta}{2}\|x^{k+1} - x^k\|^2 \quad (46)$$

is satisfied, then go to step 2).

1c) Set  $L_k \leftarrow \tau L_k$  and go to step 1a).

2) Set  $k \leftarrow k + 1$  and go to step 1).

**end**

*Remark.*  $L_k^0$  can be chosen by the similar scheme as used in [1, 4], that is,

$$L_k^0 = \max \left\{ L_{\min}, \min \left\{ L_{\max}, \frac{\Delta f^T \Delta x}{\|\Delta x\|^2} \right\} \right\},$$

where  $\Delta x = x^k - x^{k-1}$  and  $\Delta f = \nabla f(x^k) - \nabla f(x^{k-1})$ .  $\blacksquare$

At each iteration, the IHT method solves a single subproblem in step 1). Nevertheless, its variant needs to solve a sequence of subproblems. We next show that for each outer iteration, its number of inner iterations is finite.

**Theorem 3.6** For each  $k \geq 0$ , the inner termination criterion (46) is satisfied after at most  $\left\lceil \frac{\log(L_f + \eta) - \log(L_{\min})}{\log \tau} + 2 \right\rceil$  inner iterations.

*Proof.* Let  $\bar{L}_k$  denote the final value of  $L_k$  at the  $k$ th outer iteration. By (45) and the similar arguments as for deriving (24), one can show that

$$F(x^k) - F(x^{k+1}) \geq \frac{L_k - L_f}{2} \|x^{k+1} - x^k\|^2.$$

Hence, (46) holds whenever  $L_k \geq L_f + \eta$ , which together with the definition of  $\bar{L}_k$  implies that  $\bar{L}_k / \tau < L_f + \eta$ , that is,  $\bar{L}_k < \tau(L_f + \eta)$ . Let  $n_k$  denote the number of inner iterations for the  $k$ th outer iteration. Then, we have

$$L_{\min} \tau^{n_k - 1} \leq L_k^0 \tau^{n_k - 1} = \bar{L}_k < \tau(L_f + \eta).$$

Hence,  $n_k \leq \left\lceil \frac{\log(L_f + \eta) - \log(L_{\min})}{\log \tau} + 2 \right\rceil$  and the conclusion holds.  $\blacksquare$

We next establish that the sequence  $\{x^k\}$  generated by the above variant of IHT method converges to a local minimizer of (12) and moreover,  $F(x^k)$  converges to a local minimum value of (12).

**Theorem 3.7** Let  $\{x^k\}$  be generated by the above variant of IHT method. Then,  $x^k$  converges to a local minimizer  $x^*$  of problem (12), and moreover,  $I(x^k) \rightarrow I(x^*)$ ,  $\|x^k\|_0 \rightarrow \|x^*\|_0$  and  $F(x^k) \rightarrow F(x^*)$ .

*Proof.* Let  $\bar{L}_k$  denote the final value of  $L_k$  at the  $k$ th outer iteration. From the proof of Theorem 3.6, we know that  $\bar{L}_k \in [L_{\min}, \tau(L_f + \eta))$ . Using this fact and a similar argument as used to prove (19), one can obtain that

$$|x_i^{k+1}| \geq \bar{\delta} := \min_{i \notin I_0} \bar{\delta}_i > 0, \quad \text{if } x_j^{k+1} \neq 0,$$

where  $I_0 = \{i : l_i = u_i = 0\}$  and  $\bar{\delta}_i$  is defined according to (20) by replacing  $L$  by  $\tau(L_f + \eta)$  for all  $i \in I_0$ . It implies that

$$\|x^{k+1} - x^k\| \geq \bar{\delta} \quad \text{if } I(x^k) \neq I(x^{k+1}).$$

The conclusion then follows from this inequality and the similar arguments as used in the proof of Theorem 3.3.  $\blacksquare$

## 4 $l_0$ -regularized convex cone programming

In this section we consider  $l_0$ -regularized convex cone programming problem (2) and propose IHT methods for solving it. In particular, we apply the IHT method proposed in Section



3 to a quadratic penalty relaxation of (2) and establish the iteration complexity for finding an  $\epsilon$ -approximate local minimizer of (2). We also propose a variant of the method in which the associated penalty parameter is dynamically updated, and show that every accumulation point is a local minimizer of (2).

Let  $\mathcal{B}$  be defined in (13). We assume that  $f$  is a smooth convex function in  $\mathcal{B}$ ,  $\nabla f$  is Lipschitz continuous with constant  $L_f$  and that  $f$  is bounded below on  $\mathcal{B}$ . In addition, we make the following assumption throughout this section.

**Assumption 1** *For each  $I \subseteq \{1, \dots, n\}$ , there exists a Lagrange multiplier for*

$$f_I^* = \min\{f(x) : Ax - b \in \mathcal{K}^*, x \in \mathcal{B}_I\}, \quad (47)$$

*provided that (47) is feasible, that is, there exists  $\mu^* \in -\mathcal{K}$  such that  $f_I^* = d_I(\mu^*)$ , where*

$$d_I(\mu) := \inf\{f(x) + \mu^T(Ax - b) : x \in \mathcal{B}_I\}, \quad \forall \mu \in -\mathcal{K}.$$

Let  $x^*$  be a point in  $\mathcal{B}$ , and let  $I^* = \{i : x_i^* = 0\}$ . One can observe that  $x^*$  is a local minimizer of (2) if and only if  $x^*$  is a minimizer of (47) with  $I = I^*$ . Then, in view of Assumption 1, we see that  $x^*$  is a local minimizer of (2) if and only if  $x^* \in \mathcal{B}$  and there exists  $\mu^* \in -\mathcal{K}$  such that

$$\begin{aligned} Ax^* - b &\in \mathcal{K}^*, \quad (\mu^*)^T(Ax^* - b) = 0, \\ \nabla f(x^*) + A^T \mu^* &\in -\mathcal{N}_{\mathcal{B}_{I^*}}(x^*). \end{aligned} \quad (48)$$

Based on the above observation, we can define an approximate local minimizer of (2) to be the one that nearly satisfies (48).

**Definition 1** *Let  $x^*$  be a point in  $\mathcal{B}$ , and let  $I^* = \{i : x_i^* = 0\}$ .  $x^*$  is an  $\epsilon$ -approximate local minimizer of (2) if there exists  $\mu^* \in -\mathcal{K}$  such that*

$$\begin{aligned} d_{\mathcal{K}^*}(Ax^* - b) &\leq \epsilon, \quad (\mu^*)^T \Pi_{\mathcal{K}^*}(Ax^* - b) = 0, \\ \nabla f(x^*) + A^T \mu^* &\in -\mathcal{N}_{\mathcal{B}_{I^*}}(x^*) + \mathcal{U}(\epsilon). \end{aligned}$$

In what follows, we propose an IHT method for finding an approximate local minimizer of (2). In particular, we apply the IHT method or its variant to a quadratic penalty relaxation of (2) which is in the form of

$$\underline{\Psi}_\rho^* := \min_{x \in \mathcal{B}} \{\Psi_\rho(x) := \Phi_\rho(x) + \lambda \|x\|_0\}, \quad (49)$$

where

$$\Phi_\rho(x) := f(x) + \frac{\rho}{2} [d_{\mathcal{K}^*}(Ax - b)]^2 \quad (50)$$

It is not hard to show that the function  $\Phi_\rho$  is convex differentiable and moreover  $\nabla \Phi_\rho$  is Lipschitz continuous with constant

$$L_\rho = L_f + \rho \|A\|^2 \quad (51)$$

(see, for example, Proposition 8 and Corollary 9 of [9]). Therefore, problem (49) can be suitably solved by the IHT method or its variant proposed in Section 3.

Under the assumption that  $f$  is strongly convex in  $\mathcal{B}$ , we next establish the iteration complexity of the IHT method applied to (49) for finding an approximate local minimizer of (2). Given any  $L > 0$ , let  $s_L$ ,  $\alpha$  and  $\beta$  be defined according to (16), (28) and (29), respectively, by replacing  $f$  by  $\Phi_\rho$ , and let  $\delta$  be defined in (19).

**Theorem 4.1** *Assume that  $f$  is a smooth strongly convex function with modulus  $\sigma > 0$ . Given any  $\epsilon > 0$ , let*

$$\rho = \frac{t}{\epsilon} + \frac{1}{\sqrt{8}\|A\|} \quad (52)$$

for any  $t \geq \max_{I \subseteq \{1, \dots, n\}} \min_{\mu \in \Lambda_I} \|\mu\|$ , where  $\Lambda_I$  is the set of Lagrange multipliers of (47). Let  $L > L_\rho$  be chosen such that  $\alpha > 0$ . Let  $\{x^k\}$  be generated by the IHT method applied to (49), and let  $x^* = \lim_{k \rightarrow \infty} x^k$  and  $\Psi_\rho^* = \Psi_\rho(x^*)$ . Then the IHT method finds an  $\epsilon$ -approximate local minimizer of (2) in at most

$$N := 2 \left\lceil \frac{L_\rho}{\sigma} \right\rceil \log \frac{8L_\rho\theta}{\epsilon^2}$$

iterations, where

$$\theta = (\Psi_\rho(x^0) - \Psi_\rho^*)2^{\frac{\omega+3}{2}}, \quad \omega = \max_t \left\{ (d-2c)t - ct^2 : 0 \leq t \leq \left\lfloor \frac{2(\Psi_\rho(x^0) - \Psi_\rho^*)}{(L-L_\rho)\delta^2} \right\rfloor \right\},$$

$$c = \frac{(L-L_\rho)\delta^2}{2(\Psi_\rho(x^0) - \Psi_\rho^*)}, \quad \gamma = \sigma(\sqrt{2\alpha + \beta^2} - \beta)^2/32,$$

$$d = 2 \log(\Psi_\rho(x^0) - \Psi_\rho^*) + 4 - 2 \log \gamma + c.$$

*Proof.* We know from Theorem 3.3 that  $x^k \rightarrow x^*$  for some local minimizer  $x^*$  of (49),  $I(x^k) \rightarrow I(x^*)$  and  $\Psi_\rho(x^k) \rightarrow \Psi_\rho(x^*) = \Psi_\rho^*$ . By Theorem 3.4, after at most  $N$  iterations, the IHT method generates  $\tilde{x} \in \mathcal{B}$  such that  $I(\tilde{x}) = I(x^*)$  and  $\Psi_\rho(\tilde{x}) - \Psi_\rho(x^*) \leq \xi := \epsilon^2/(8L_\rho)$ . It then follows that  $\Phi_\rho(\tilde{x}) - \Phi_\rho(x^*) \leq \xi$ . Since  $x^*$  is a local minimizer of (49), we observe that

$$x^* = \arg \min_{x \in \mathcal{B}_{I^*}} \Phi_\rho(x), \quad (53)$$

where  $I^* = I(x^*)$ . Hence,  $\tilde{x}$  is a  $\xi$ -approximate solution of (53). Let  $\mu^* \in \text{Arg min}\{\|\mu\| : \mu \in \Lambda_{I^*}\}$ , where  $\Lambda_{I^*}$  is the set of Lagrange multipliers of (47) with  $I = I^*$ . In view of Lemma 2.5, we see that the pair  $(\tilde{x}^+, \mu)$  defined as  $\tilde{x}^+ := \Pi_{\mathcal{B}_{I^*}}(\tilde{x} - \nabla \Phi_\rho(\tilde{x})/L_\rho)$  and  $\mu := \rho[A\tilde{x}^+ - b - \Pi_{\mathcal{K}^*}(A\tilde{x}^+ - b)]$  satisfies

$$\nabla f(\tilde{x}^+) + A^T \mu \in -\mathcal{N}_{\mathcal{B}_{I^*}}(\tilde{x}^+) + \mathcal{U}(2\sqrt{2L_\rho\xi}) = \mathcal{N}_{\mathcal{B}_I}(\tilde{x}^+) + \mathcal{U}(\epsilon),$$

$$d_{\mathcal{K}^*}(A\tilde{x}^+ - b) \leq \frac{1}{\rho}\|\mu^*\| + \sqrt{\frac{\xi}{\rho}} \leq \frac{1}{\rho} \left( \|\mu^*\| + \frac{\epsilon}{\sqrt{8}\|A\|} \right) \leq \epsilon,$$

where the last inequality is due to (52) and the assumption  $t \geq \hat{t} \geq \|\mu^*\|$ . Hence,  $\tilde{x}^+$  is an  $\epsilon$ -approximate local minimizer of (2).  $\blacksquare$

We next consider finding an  $\epsilon$ -approximate local minimizer of (2) for the case where  $\mathcal{B}$  is bounded and  $f$  is convex but not strongly convex. In particular, we apply the IHT method or its variant to a quadratic penalty relaxation of a perturbation of (2) obtained from adding a ‘small’ strongly convex regularization term to  $f$ .

Consider a perturbation of (2) in the form of

$$\min_{x \in \mathcal{B}} \left\{ f(x) + \frac{\nu}{2} \|x\|^2 + \lambda \|x\|_0 : Ax - b \in \mathcal{K}^* \right\}. \quad (54)$$

The associated quadratic penalty problem for (54) is given by

$$\underline{\Psi}_{\rho, \nu}^* := \min_{x \in \mathcal{B}} \{ \Psi_{\rho, \nu}(x) := \Phi_{\rho, \nu}(x) + \lambda \|x\|_0 \}, \quad (55)$$

where

$$\Phi_{\rho, \nu}(x) := f(x) + \frac{\nu}{2} \|x\|^2 + \frac{\rho}{2} [d_{\mathcal{K}^*}(Ax - b)]^2.$$

One can easily see that  $\Phi_{\rho, \nu}$  is strongly convex in  $\mathcal{B}$  with modulus  $\nu$  and moreover  $\nabla \Phi_{\rho, \nu}$  is Lipschitz continuous with constant

$$L_{\rho, \nu} := L_f + \rho \|A\|^2 + \nu.$$

Clearly, the IHT method or its variant can be suitably applied to (55). We next establish the iteration complexity of the IHT method applied to (55) for finding an approximate local minimizer of (2). Given any  $L > 0$ , let  $s_L$ ,  $\alpha$  and  $\beta$  be defined according to (16), (28) and (29), respectively, by replacing  $f$  by  $\Phi_{\rho, \nu}$ , and let  $\delta$  be defined in (19).

**Theorem 4.2** *Suppose that  $\mathcal{B}$  is bounded and  $f$  is convex but not strongly convex. Let  $\epsilon > 0$  be arbitrarily given,  $D = \max\{\|x\| : x \in \mathcal{B}\}$ ,*

$$\rho = \frac{\left( \sqrt{D} + \sqrt{D + 16t + \frac{2\sqrt{2}\epsilon}{\|A\|}} \right)^2}{16\epsilon}, \quad \nu = \frac{\epsilon}{2D} \quad (56)$$

for any  $t \geq \max_{I \subseteq \{1, \dots, n\}} \min_{\mu \in \Lambda_I} \|\mu\|$ , where  $\Lambda_I$  is the set of Lagrange multipliers of (47). Let  $L > L_{\rho, \nu}$

be chosen such that  $\alpha > 0$ . Let  $\{x^k\}$  be generated by the IHT method applied to (55), and let  $x^* = \lim_{k \rightarrow \infty} x^k$  and  $\Psi_{\rho, \nu}^* = \Psi_{\rho, \nu}(x^*)$ . Then the IHT method finds an  $\epsilon$ -approximate local minimizer of (2) in at most

$$N := 2 \left\lceil \frac{2DL_{\rho, \nu}}{\epsilon} \right\rceil \log \frac{32L_{\rho, \nu}\theta}{\epsilon^2}$$

iterations, where

$$\theta = (\Psi_{\rho, \nu}(x^0) - \Psi_{\rho, \nu}^*) 2^{\frac{\omega+3}{2}}, \quad \omega = \max_t \left\{ (d - 2c)t - ct^2 : 0 \leq t \leq \left\lfloor \frac{2(\Psi_{\rho, \nu}(x^0) - \Psi_{\rho, \nu}^*)}{(L - L_{\rho, \nu})\delta^2} \right\rfloor \right\},$$

$$c = \frac{(L - L_{\rho, \nu})\delta^2}{2(\Psi_{\rho, \nu}(x^0) - \Psi_{\rho, \nu}^*)}, \quad \gamma = \nu(\sqrt{2\alpha_{\rho, \nu} + \beta_{\rho, \nu}^2} - \beta_{\rho, \nu})^2 / 32,$$

$$d = 2 \log(\Psi_{\rho, \nu}(x^0) - \underline{\Psi}_{\rho, \nu}^*) + 4 - 2 \log \gamma + c.$$

*Proof.* From Theorem 3.3, we know that  $x^k \rightarrow x^*$  for some local minimizer  $x^*$  of (55),  $I(x^k) \rightarrow I(x^*)$  and  $\Psi_{\rho,\nu}(x^k) \rightarrow \Psi_{\rho,\nu}(x^*) = \Psi_{\rho,\nu}^*$ . By Theorem 3.4, after at most  $N$  iterations, the IHT method applied to (55) generates  $\tilde{x} \in \mathcal{B}$  such that  $I(\tilde{x}) = I(x^*)$  and  $\Psi_{\rho,\nu}(\tilde{x}) - \Psi_{\rho,\nu}(x^*) \leq \xi := \epsilon^2/(32L_{\rho,\nu})$ . It then follows that  $\Phi_{\rho,\nu}(\tilde{x}) - \Phi_{\rho,\nu}(x^*) \leq \xi$ . Since  $x^*$  is a local minimizer of (55), we see that

$$x^* = \arg \min_{x \in \mathcal{B}_{I^*}} \Phi_{\rho,\nu}(x), \quad (57)$$

where  $I^* = I(x^*)$ . Hence,  $\tilde{x}$  is a  $\xi$ -approximate solution of (57). In view of Lemma 2.5, we see that the pair  $(\tilde{x}^+, \mu)$  defined as  $\tilde{x}^+ := \Pi_{\mathcal{B}_I}(\tilde{x} - \nabla \Phi_{\rho,\nu}(\tilde{x})/L_{\rho,\nu})$  and  $\mu := \rho[A\tilde{x}^+ - b - \Pi_{\mathcal{K}^*}(A\tilde{x}^+ - b)]$  satisfies

$$\nabla f(\tilde{x}^+) + \nu \tilde{x}^+ + A^T \mu \in -\mathcal{N}_{\mathcal{B}_{I^*}}(\tilde{x}^+) + \mathcal{U}(2\sqrt{2L_{\rho,\nu}\xi}) = -\mathcal{N}_{\mathcal{B}_{I^*}}(\tilde{x}^+) + \mathcal{U}(\epsilon/2),$$

which together with the fact that  $\nu \|\tilde{x}^+\| \leq \nu D \leq \epsilon/2$  implies that

$$\nabla f(\tilde{x}^+) + A^T \mu \in -\nu \tilde{x}^+ - \mathcal{N}_{\mathcal{B}_{I^*}}(\tilde{x}^+) + \mathcal{U}(\epsilon/2) \subseteq -\mathcal{N}_{\mathcal{B}_{I^*}}(\tilde{x}^+) + \mathcal{U}(\epsilon).$$

In addition, it follows from Lemma 2.1 (c) that  $\Phi_{\rho,\nu}(\tilde{x}^+) \leq \Phi_{\rho,\nu}(\tilde{x})$ , and hence

$$\Phi_{\rho,\nu}(\tilde{x}^+) - \Phi_{\rho,\nu}(x^*) \leq \Phi_{\rho,\nu}(\tilde{x}) - \Phi_{\rho,\nu}(x^*) \leq \xi.$$

Let  $\Phi_\rho^* = \min\{\Phi_\rho(x) : x \in \mathcal{B}_{I^*}\}$ , where  $\Phi_\rho$  is defined in (50). Notice that  $\Phi_{\rho,\nu}(x^*) \leq \Phi_\rho^* + \nu D^2/2$ . It then follows that

$$\Phi_\rho(\tilde{x}^+) - \Phi_\rho^* \leq \Phi_{\rho,\nu}(\tilde{x}^+) - \Phi_{\rho,\nu}(x^*) + \frac{\nu D^2}{2} \leq \xi + \frac{\epsilon D}{4} \leq \frac{\epsilon^2}{32\rho\|A\|^2} + \frac{\epsilon D}{4}.$$

Let  $\mu^* \in \text{Arg min}\{\|\mu\| : \mu \in \Lambda_{I^*}\}$ , where  $\Lambda_{I^*}$  is the set of Lagrange multipliers of (47) with  $I = I^*$ . In view of Lemma 2.5 and the assumption  $t \geq \hat{t} \geq \|\mu^*\|$ , we obtain that

$$d_{\mathcal{K}^*}(A\tilde{x}^+ - b) \leq \frac{1}{\rho}\|\mu^*\| + \sqrt{\frac{\epsilon^2}{32\rho^2\|A\|^2} + \frac{\epsilon D}{4\rho}} \leq \frac{1}{\rho} \left( t + \frac{\epsilon}{\sqrt{32}\|A\|} \right) + \sqrt{\frac{\epsilon D}{4\rho}} = \epsilon,$$

where the last inequality is due to (56). Hence,  $\tilde{x}^+$  is an  $\epsilon$ -approximate local minimizer of (2).  $\blacksquare$

For the above method, the fixed penalty parameter  $\rho$  is used through all iterations, which may be too conservative. To improve its practical performance, we can update  $\rho$  dynamically. The resulting variant of the method is presented as follows. Before proceeding, we define the projected gradient of  $\Phi_\rho$  at  $x \in \mathcal{B}_I$  with respect to  $\mathcal{B}_I$  as

$$g(x; \rho, I) = L_\rho[x - \Pi_{\mathcal{B}_I}(x - \frac{1}{L_\rho}\nabla\Phi_\rho(x))], \quad (58)$$

where  $I \subseteq \{1, \dots, n\}$ , and  $\Phi_\rho$  and  $L_\rho$  are defined in (50) and (51), respectively.

**A variant of IHT method for (2):**

Let  $\{\epsilon_k\}$  be a positive decreasing sequence. Let  $\rho_0 > 0$ ,  $\tau > 1$ ,  $t > \max_{I \subseteq \{1, \dots, n\}} \min_{\mu \in \Lambda_I} \|\mu\|$ , where  $\Lambda_I$  is the set of Lagrange multipliers of (47). Choose an arbitrary  $x^0 \in \mathcal{B}$ . Set  $k = 0$ .

- 1) Start from  $x^{k-1}$  and apply the IHT method or its variant to problem (49) with  $\rho = \rho_k$  until finding some  $x^k \in \mathcal{B}$  such that

$$d_{\mathcal{K}^*}(Ax^k - b) \leq \frac{t}{\rho_k}, \quad \|g(x^k; \rho_k, I_k)\| \leq \min\{1, L_{\rho_k}\}\epsilon_k, \quad (59)$$

where  $I_k = I(x^k)$ .

- 2) Set  $\rho_{k+1} := \tau\rho_k$ .
- 3) Set  $k \leftarrow k + 1$  and go to step 1).

**end**

The following theorem shows that  $x^k$  satisfying (59) can be found within a finite number of iterations by the IHT method or its variant applied to problem (49) with  $\rho = \rho_k$ . Without loss of generality, we consider the IHT method or its variant applied to problem (49) with any given  $\rho > 0$ .

**Theorem 4.3** *Let  $x_0 \in \mathcal{B}$  be an arbitrary point and the sequence  $\{x_l\}$  be generated by the IHT method or its variant applied to problem (49). Then, the following statements hold:*

- (i)  $\lim_{l \rightarrow \infty} g(x_l; \rho, I_l) = 0$ , where  $I_l = I(x_l)$  for all  $l$ .
- (ii)  $\lim_{l \rightarrow \infty} d_{\mathcal{K}^*}(Ax_l - b) \leq \frac{\hat{t}}{\rho}$ , where  $\hat{t} := \max_{I \subseteq \{1, \dots, n\}} \min_{\mu \in \Lambda_I} \|\mu\|$  and  $\Lambda_I$  is the set of Lagrange multipliers of (47).

*Proof.* (i) It follows from Theorems 3.3 and 3.7 that  $x_l \rightarrow x^*$  for some local minimizer  $x^*$  of (49) and moreover,  $\Phi_\rho(x_l) \rightarrow \Phi_\rho(x^*)$  and  $I_l \rightarrow I^*$ , where  $I_l = I(x_l)$  and  $I^* = I(x^*)$ . We also know that

$$x^* \in \text{Arg} \min_{x \in \mathcal{B}_{I^*}} \Phi_\rho(x).$$

It then follows from Lemma 2.1 (d) that

$$\Phi_\rho(x_l) - \Phi_\rho(x^*) \geq \frac{1}{2L_\rho} \|g(x_l; \rho, I^*)\|^2, \quad l \geq N.$$

Using this inequality and  $\Phi_\rho(x_l) \rightarrow \Phi_\rho(x^*)$ , we thus have  $g(x_l; \rho, I^*) \rightarrow 0$ . Since  $I_l = I^*$  for  $l \geq N$ , we also have  $g(x_l; \rho, I_l) \rightarrow 0$ .

(ii) Let  $f_I^*$  be defined in (47). Applying Lemma 2.4 to problem (47), we know that

$$f(x_l) - f_{I(l)}^* \geq -\hat{t}d_{\mathcal{K}^*}(Ax_l - b), \quad \forall l, \quad (60)$$

where  $\hat{t}$  is defined above. Let  $x^*$  and  $I^*$  be defined in the proof of statement (i). We observe that  $f_{I^*}^* \geq \Phi_\rho(x^*)$ . Using this relation and (60), we have that for sufficiently large  $l$ ,

$$\begin{aligned} \Phi_\rho(x_l) - \Phi_\rho(x^*) &= f(x_l) + \frac{\rho}{2}[d_{\mathcal{K}^*}(Ax_l - b)]^2 - \Phi_\rho(x^*) \geq f(x_l) - f_{I^*}^* + \frac{\rho}{2}[d_{\mathcal{K}^*}(Ax_l - b)]^2 \\ &= f(x_l) - f_{I(l)}^* + \frac{\rho}{2}[d_{\mathcal{K}^*}(Ax_l - b)]^2 \geq -\hat{t}d_{\mathcal{K}^*}(Ax_l - b) + \frac{\rho}{2}[d_{\mathcal{K}^*}(Ax_l - b)]^2, \end{aligned}$$

which implies that

$$d_{\mathcal{K}^*}(Ax_l - b) \leq \frac{\hat{t}}{\rho} + \sqrt{\frac{\Phi_\rho(x_l) - \Phi_\rho(x^*)}{\rho}}.$$

This inequality together with the fact  $\lim_{l \rightarrow \infty} \Phi_\rho(x_l) = \Phi_\rho(x^*)$  yields statement (ii).  $\blacksquare$

*Remark.* From Theorem 4.3, we can see that the inner iterations of the above method terminates finitely.  $\blacksquare$

We next establish convergence of the outer iterations of the above variant of the IHT method for (2). In particular, we show that every accumulation point of  $\{x^k\}$  is a local minimizer of (2).

**Theorem 4.4** *Let  $\{x^k\}$  be the sequence generated by the above variant of the IHT method for solving (2). Then any accumulation point of  $\{x^k\}$  is a local minimizer of (2).*

*Proof.* Let

$$\tilde{x}^k = \Pi_{\mathcal{B}_{I_k}}(x^k - \frac{1}{L_{\rho_k}} \nabla \Phi_{\rho_k}(x^k)).$$

Since  $\{x^k\}$  satisfies (59), it follows from Lemma 2.1 (a) that

$$\nabla \Phi_{\rho_k}(x^k) \in -\mathcal{N}_{\mathcal{B}_{I_k}}(\tilde{x}^k) + \mathcal{U}(\epsilon_k), \quad (61)$$

where  $I_k = I(x^k)$ . Let  $x^*$  be any accumulation point of  $\{x^k\}$ . Then there exists a subsequence  $K$  such that  $\{x^k\}_K \rightarrow x^*$ . By passing to a subsequence if necessary, we can assume that  $I_k = I$  for all  $k \in K$ . Let

$$\mu^k = \rho_k[Ax^k - b - \Pi_{\mathcal{K}^*}(Ax^k - b)].$$

We clearly see that

$$(\mu^k)^T \Pi_{\mathcal{K}^*}(Ax^k - b) = 0. \quad (62)$$

Using (61) and the definitions of  $\Phi_\rho$  and  $\mu^k$ , we have

$$\nabla f(x^k) + A^T \mu^k \in -\mathcal{N}_{\mathcal{B}_I}(\tilde{x}^k) + \mathcal{U}(\epsilon_k), \quad \forall k \in \mathcal{K}. \quad (63)$$

By (58), (59) and the definition of  $\tilde{x}^k$ , one can observe that

$$\|\tilde{x}^k - x^k\| = \frac{1}{L_{\rho_k}} \|g(x^k; \rho_k, I_k)\| \leq \epsilon_k. \quad (64)$$

In addition, notice that  $\|\mu^k\| = \rho_k d_{\mathcal{K}^*}(Ax^k - b)$ , which together with (59) implies that  $\|\mu^k\| \leq t$  for all  $k$ . Hence,  $\{\mu^k\}$  is bounded. By passing to a subsequence if necessary, we can assume that  $\{\mu^k\}_K \rightarrow \mu^*$ . Using (64) and upon taking limits on both sides of (62) and (63) as  $k \in K \rightarrow \infty$ , we have

$$(\mu^*)^T \Pi_{\mathcal{K}^*}(Ax^* - b) = 0, \quad \nabla f(x^*) + A^T \mu^* \in -\mathcal{N}_{\mathcal{B}_I}(x^*)$$

In addition, since  $x_I^k = 0$  for  $k \in K$ , we know that  $x_I^* = 0$ . Also, it follows from (59) that  $d_{\mathcal{K}^*}(Ax^* - b) = 0$ , which implies that  $Ax^* - b \in \mathcal{K}^*$ . These relations yield that

$$x^* \in \text{Arg min}_{x \in \mathcal{B}_I} \{f(x) : Ax - b \in \mathcal{K}^*\},$$

and hence,  $x^*$  is a local minimizer of (2). ■

## 5 Concluding remarks

In this paper we studied iterative hard thresholding (IHT) methods for solving  $l_0$  regularized convex cone programming problems. In particular, we first proposed an IHT method and its variant for solving  $l_0$  regularized box constrained convex programming. We showed that the sequence generated by these methods converges to a local minimizer. Also, we established the iteration complexity of the IHT method for finding an  $\epsilon$ -local-optimal solution. We then proposed a method for solving  $l_0$  regularized convex cone programming by applying the IHT method to its quadratic penalty relaxation and established its iteration complexity for finding an  $\epsilon$ -approximate local minimizer. Finally, we proposed a variant of this method in which the associated penalty parameter is dynamically updated, and showed that every accumulation point is a local minimizer of the problem.

Some of the methods studied in this paper can be extended to solve some  $l_0$  regularized nonconvex optimization problems. For example, the IHT method and its variant can be applied to problem (12) in which  $f$  is nonconvex and  $\nabla f$  is Lipschitz continuous. In addition, the numerical study of the IHT methods will be presented in the working paper [7]. Finally, it would be interesting to extend the methods of this paper to solve rank minimization problems and compare them with the methods studied in [5, 8]. This is left as a future research.

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