

# On the Relative Strength of Different Generalizations of Split Cuts

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## Abstract

Split cuts are among the most important and well-understood cuts for general mixed-integer programs. In this paper we consider some recent generalizations of split cuts and compare their relative strength. More precisely, we compare the elementary closures of split, cross, crooked cross and general multi-branch split cuts as well as cuts obtained from multi-row and basic relaxations.

We present a complete containment relationship between the closures of split, rank-2 split, cross, crooked cross and general multi-branch split cuts. More specifically, we show that 3-branch split cuts strictly dominate crooked cross cuts, which in turn strictly dominate cross cuts. We also show that multi-branch split cuts are incomparable to rank-2 split cuts. In addition, we also show that cross cuts, and hence crooked cross cuts, cannot always be obtained from 2-row relaxations or from basic relaxations. Together, these results settle some open questions raised in earlier papers.

## 1 Introduction

Cutting planes are crucial for solving mixed-integer programs (MIPs), and the Gomory mixed-integer (GMI) cut is currently among the most effective cutting planes for general MIPs. Cook, Kannan and Schrijver [CKS90] studied a class of disjunctive cuts called *split cuts* (formal definitions are presented in Section 2). Despite their simplicity, many known families of cutting planes (or *cuts* for short) such as the GMI, lift-and-project, and flow cover cuts can be viewed as split cuts from very simple disjunctions. Due to their importance, these cuts have been extensively studied, both theoretically [NW90, MW01, OM01, CL01, BP03, ACL05, BCM12] and computationally [BCCN96, BS08, DGL10, DG10, Bon12, FS11].

In the following, we refer to a *mixed-integer set* as the set of mixed-integer solutions of a given finite set of rational linear equations or inequalities (a fixed subset of variables are restricted to be integral), and refer to the polyhedron defined by these linear constraints as the *linear relaxation* of the mixed-integer set. Cuts for a mixed-integer set are linear inequalities valid for its convex hull. Split cuts (and other structured families of cuts) for a mixed-integer set are assumed to be derived using both the linear constraints and the integrality restrictions defining the set. The *elementary closure* of a family of cuts for a mixed-integer set is the set of (real) points in its linear relaxation satisfying all cuts in the family.

When the linear constraints defining a mixed-integer set are given in inequality form, Andersen, Cornuéjols and Li [ACL05] proved that any split cut can be obtained as a split cut from a *basic*

relaxation; also see [DGR11] for a simpler proof. A basic relaxation is the mixed-integer set defined by a maximal subset of linearly independent constraints of the linear relaxation and the original integrality restrictions (the remaining linear constraints are dropped). These relaxations generalize the corner relaxation introduced by Gomory [Gom69], as they also consider infeasible bases of the linear relaxation. When a mixed-integer set is defined by linear equations and nonnegativity constraints on some variables, any split cut can be obtained as a *mixed-integer rounding* (MIR) inequality, as described by Nemhauser and Wolsey; see [NW88, NW90]. MIR inequalities are obtained by using nonnegativity constraints together with a single equation obtained as a linear combination of (a linearly independent subset of) the constraints of the linear relaxation. Therefore, depending on how the linear relaxation of the set is defined, it is possible to view split cuts as valid inequalities obtained from basic relaxations, or, as cuts obtained from 1-row relaxations.

Recently, split cuts have been generalized in different ways to obtain more effective cutting planes. One such generalization is to use two or more split disjunctions simultaneously to obtain valid inequalities. This gives rise to multi-branch split cuts, or *t-branch split cuts* when  $t$  split disjunctions are used. These cuts were first studied by Li and Richard [LR08] and recently Dash and Günlük extended some of their results [DG13]. Dash, Dey and Günlük [DDG12, DDG11] study 2-branch split cuts (and call them *cross cuts*) and *crooked cross cuts* (which are derived using three linearly dependent split disjunctions). Crooked cross cuts subsume cross cuts and are implied by 3-branch split cuts.

A different generalization of split cuts is obtained by considering multi-row relaxations of the mixed-integer set instead of one-row relaxations. This approach was introduced by Andersen, Louveaux, Weismantel and Wolsey [ALWW] who study the so-called two-row continuous group relaxation and show that the convex hull of solutions of this relaxation is given by (two-dimensional) lattice-free cuts. This topic has received significant attention lately; see [CCZ11b] for a recent survey. More generally, a  $k$ -row relaxation of a mixed-integer set is constructed by aggregating the equations defining the linear relaxation into  $k$  equations.

In this paper we compare cuts obtained from different generalizations of split cuts. In particular, we compare the strength of split, cross, crooked cross and general  $t$ -branch split cuts as well as cuts obtained from multi-row and basic relaxations, by comparing their elementary closures. As mentioned earlier, some results comparing the strength of these closures are already present in the literature; we next review some of these results and highlight our contributions in this paper. We say that a family of cuts *dominates* another if for every mixed-integer set, the elementary closure of the first family of cuts for the set is contained in the elementary closure of the second family of cuts for the same set. We say that the dominance is *strict* if there are examples where the elementary closure of the first family is strictly contained in the elementary closure of the second family. Henceforth, we refer to the elementary closure of a family  $\mathcal{F}$  of cuts for a mixed-integer set as its  $\mathcal{F}$ -closure. Further, we define the *second*  $\mathcal{F}$ -closure as the elementary closure of the family  $\mathcal{F}$  of cuts for the mixed-integer set defined by the constraints in the  $\mathcal{F}$ -closure of the original mixed-integer set along with its integrality restrictions. A cut in the family  $\mathcal{F}$  that is valid for the second  $\mathcal{F}$ -closure is called a *rank-2*  $\mathcal{F}$ -cut.

We next give an overview of the main results in the paper. The statements of the theorems are stated more formally in later sections.

**Multi-branch split cuts.** Recall that split cuts are the same as 1-branch split cuts and cross cuts are the same as 2-branch split cuts. Cook, Kannan and Schrijver [CKS90, Example 2] presented

a simple mixed-integer set (with two integer variables and one continuous variable) such that its convex hull is strictly contained in its second split closure (actually in its  $n$ th split closure for any finite  $n$ ). The convex hull of this mixed-integer set equals its cross closure (by results in [ALWW] and [DDG12]). However, it is not known if in general cross cuts strictly dominate rank-2 split cuts. We answer this question and show that there is no such dominance relationship.

**Theorem 1.1.** *For every finite integer  $t > 0$ , there is a mixed-integer set whose second split closure is strictly contained in its  $t$ -branch split closure.*

Li and Richard [LR08] showed that for  $t > 2$ ,  $t$ -branch split cuts strictly dominate 2-branch split cuts. Subsequently, it was shown in [DG13] that  $t$ -branch split cuts strictly dominate  $k$ -branch split cuts for all  $t > k > 0$ . In addition, it is known that 3-branch split cuts dominate crooked cross cuts which, in turn, dominate cross cuts [DDG11, DDG12]. However, these two dominance relationships are not known to be strict. In [DDG11] the authors show that there is a crooked cross cut that is not implied by a *single* cross cut; however, this result does not rule out the possibility that the cross closure (which potentially contains infinitely many cuts) is *always equal* to the crooked cross closure. In this paper we establish that 3-branch split cuts strictly dominate crooked cross cuts which, in turn, strictly dominate 2-branch split cuts.

**Theorem 1.2.** *There is a mixed-integer set such that its crooked cross closure is strictly contained in its cross closure.*

**Theorem 1.3.** *There is a mixed-integer set such that its 3-branch closure is strictly contained in its crooked cross closure.*

**Observation 1.4.** *It was proved in [CKS90] that the split closure of a mixed-integer set  $P \cap (\mathbb{Z}^m \times \mathbb{R}^n)$  is given by*

$$\bigcap_{\pi \in \mathbb{Z}^m} \{(x, y) \in P : \pi x \in \mathbb{Z}\}.$$

*It is then natural to conjecture that the cross closure of this mixed-integer set is*

$$\bigcap_{\pi^1, \pi^2 \in \mathbb{Z}^m} \{(x, y) \in P : \pi^1 x, \pi^2 x \in \mathbb{Z}\}. \quad (1)$$

*However, Theorem 1.2 establishes that this is not the case, as it was proved in [DDG11, Theorem 3.1] that (1) equals the crooked cross closure of the mixed-integer set.*

In Figure 1 we summarize the containment relationships between these closures, or equivalently, the dominance relationships between corresponding cut families. A plain arrow from one closure to another means that the former closure is contained in the latter closure, and a crossed arrow from one closure to another means that the former closure is not always contained in the latter. Consequently when both types of arrows are present between a pair of closures, then the one family of cuts strictly dominates the other. Dashed arrows indicate results known prior to this paper and solid arrows indicate results obtained in this paper. In the figure, we denote the closure of  $t$ -branch split cuts with  $tBC$  for  $t = 1, 2, 3$  and we use  $4^+BC$  for all  $t > 3$ . We denote the crooked cross cut closure by  $CCC$  and use  $SC^2$  to denote the second split closure. Note that the displayed arrows can be used to infer the relationship between any pair of the closures considered.

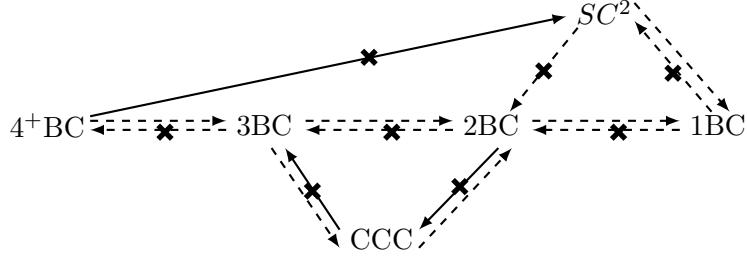


Figure 1: Comparing multi-branch split cuts with crooked cross cuts and rank 2 split cuts

**Cuts from relaxations.** Structured relaxations of mixed-integer sets have been widely used to generate cutting planes for the original sets. The literature on the theoretical aspects of different relaxations is extensive; for a small set of representative publications, see [MW99, CCZ11a, ALWW]. In this paper we focus on two relaxations: the *basic* and the *k-row* relaxations (see Section 2.2 for formal definitions). We show that, unlike split cuts, *t*-branch split cuts cannot always be obtained from basic relaxations.

**Theorem 1.5.** *There is a mixed-integer set such that its cross cut closure cannot be obtained by all cuts from its basic relaxations.*

When the linear relaxation of the mixed-integer set is given in equality form, Dash, Dey and Günlük [DDG12] showed that every cross cut (resp. crooked cross cut) can be obtained as a cross cut (resp. crooked cross cut) from a 3-row relaxation. However, they left as an open question whether these cuts can also be obtained from 2-row relaxations. They also note that if crooked cross cuts can be obtained as crooked cross cuts from 2-row relaxations, then crooked cross cuts would be equivalent to cuts from all 2-row continuous group relaxations of the set. In this paper we answer this question.

**Theorem 1.6.** *There is a mixed-integer set such that its cross cut closure cannot be obtained by all cuts from its 2-row relaxations.*

Since the crooked cross closure is contained in the cross closure, Theorems 1.5 and 1.6, respectively, imply following results.

**Corollary 1.7.** *There is a mixed-integer set such that its crooked cross cut closure cannot be obtained by all cuts from its basic relaxations.*

**Corollary 1.8.** *There is a mixed-integer set such that its crooked cross cut closure cannot be obtained by all cuts from its 2-row relaxations.*

In Figure 2 we show some of the dominance relationships between these closures. We denote the closure of cuts from *k*-row relaxations by *kR* for *k* = 1, 2, 3 and we use *CCC* to denote crooked cross cuts, *CC* to denote cross cuts (2-branch split cuts) and *SC* to denote split cuts (1-branch split cuts). We use *BR* to denote cuts from basic relaxations. The fact that *2R* does not dominate *3R* follows from the fact that *2R* does not dominate *CC*. The fact that *1R* does not dominate *2R* can be proved using the example of Cook, Kannan and Schrijver [CKS90] where the integer hull has infinite split rank. It is shown in [ALWW] that the integer hull in this example can be

obtained from a 2-row relaxation. On the other hand, it is possible (and nontrivial) to show that all cuts from 1-row relaxations of this example are split cuts, and thus cannot yield the integer hull. As explained in the introduction, results in [NW90] imply that  $1R$  dominates  $SC$  and results in [ACL05] imply that  $BR$  dominates  $SC$ .

Some relationships between closures are missing in Figure 2. For example, though we expect that neither  $CCC$  nor  $BR$  dominates  $1R$ , we are not able to prove these relationships. Furthermore, we also cannot show that  $3R$  does not dominate  $BR$ .

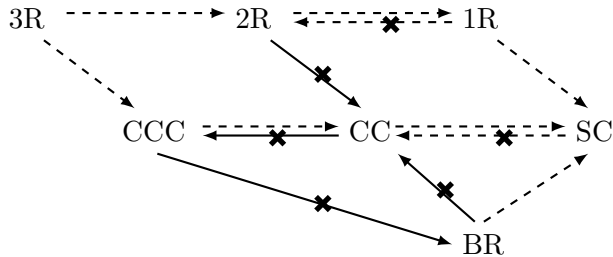


Figure 2: Comparing cuts from multi-row and basic relaxations with multi-branch split cuts

**Organization of the paper.** The outline of the paper is as follows. In the next section we formally define the families of cuts and relaxations studied in the paper. Unlike most previous related works, several of the sets we work with are not known to be polyhedral (e.g.  $t$ -branch split closures for  $t \geq 3$  and the intersection of  $k$ -row relaxations), creating the need of new tools for handling the interaction of potentially infinitely many cuts. Section 3 presents the main technical tool for this purpose, dubbed the “Height Lemma”.

In Sections 4, 5, and 6, we compare the closures of multi-branch split cuts and crooked cross cuts. In Sections 7 and 8 we compare the strength of cross cuts with cuts obtained from multi-row and basic relaxations.

## 2 Preliminaries

A mixed-integer set  $P^I$  is defined as the intersection of a rational polyhedron  $P \subseteq \mathbb{R}^\ell$  and a mixed-integer lattice  $I = \mathbb{Z}^m \times \mathbb{R}^n$ , where  $\ell = m + n$ , and the set  $P$  is called the *linear relaxation* of  $P^I$ . Clearly, different polyhedra, when intersected with  $I$ , can yield the same mixed-integer set. Throughout this paper, however, we associate the mixed-integer set  $P^I$  with the unique linear relaxation  $P$ . This association is necessary when considering elementary closures with respect to families of cuts, as these cuts are derived using the linear relaxation of the mixed-integer set in hand.

### 2.1 Split Cuts and More General Disjunctive Cuts

A *split disjunction* for a mixed-integer lattice  $\mathbb{Z}^m \times \mathbb{R}^n$  is a set of the form

$$D(\pi, \gamma) = \{(x, y) \in \mathbb{R}^{m+n} : \pi x \leq \gamma\} \cup \{(x, y) \in \mathbb{R}^{m+n} : \pi x \geq \gamma + 1\}$$

for some  $\pi \in \mathbb{Z}^m, \gamma \in \mathbb{Z}$ . Define the *split set* associated with the disjunction  $D(\pi, \gamma)$  as

$$S(\pi, \gamma) = \{(x, y) \in \mathbb{R}^{m+n} : \gamma < \pi x < \gamma + 1\} = \mathbb{R}^{m+n} \setminus D(\pi, \gamma).$$

**Split cuts.** We call a linear inequality a *split cut* for  $P$  with respect to the disjunction  $D(\pi, \gamma)$  if it is valid for  $P \cap D(\pi, \gamma)$  [CKS90]. Notice that multiple split cuts can be generated from the same split disjunction. The *split closure* of  $P$  with respect to the mixed-integer lattice  $I$  is denoted by  $SC(P)$  (the mixed-integer lattice  $I$  will be clear from the context).  $SC(P)$  is defined as the points in  $P$  satisfying all split cuts for  $P$  derived from split disjunctions for  $I$ . Equivalently, we have

$$SC(P) = \bigcap_{(\pi, \gamma) \in \mathbb{Z}^{m+1}} \text{conv}(P \cap D(\pi, \gamma)) = \bigcap_{(\pi, \gamma) \in \mathbb{Z}^{m+1}} \text{conv}(P \setminus S(\pi, \gamma)),$$

where  $\text{conv}(\cdot)$  denotes the convex hull operator (the first equation uses the fact that  $\text{conv}(P \cap D(\pi, \gamma))$  is always a closed set (see [DDV11, Lemma 2.3])).

As  $I$  is contained in  $D(\pi, \gamma)$  for all  $(\pi, \gamma) \in \mathbb{Z}^{m+1}$ , it follows that  $P^I \subseteq SC(P)$ , namely split cuts do not cut off any point in the set  $P^I$ . One can iterate the closure operator and define (for a positive integer  $k$ ) the  $k$ th split closure  $SC^k(P)$  recursively by setting  $SC^k(P) = SC(SC^{k-1}(P))$  and  $SC^1(P) = SC(P)$ .<sup>1</sup>

**$t$ -branch split cuts.** Consider an integer  $t$  together with  $\pi^i \in \mathbb{Z}^m$  and  $\gamma_i \in \mathbb{Z}$  for  $i = 1, \dots, t$ . The set  $D(\pi^1, \dots, \pi^t, \gamma_1, \dots, \gamma_t)$  given by

$$D(\pi^1, \dots, \pi^t, \gamma_1, \dots, \gamma_t) = \bigcap_{i=1}^t D(\pi^i, \gamma_i) = \mathbb{R}^{m+n} \setminus \bigcup_{i=1}^t S(\pi^i, \gamma_i) \quad (2)$$

is called a  *$t$ -branch split disjunction* for  $I$  [LR08]. The fact that  $I \subseteq D(\pi^i, \gamma_i)$  implies that  $P^I \subseteq D(\pi^1, \dots, \pi^t, \gamma_1, \dots, \gamma_t)$ . A linear inequality is a  *$t$ -branch split cut* for  $P$  with respect to a  $t$ -branch split disjunction  $D$  if it is valid for  $P \cap D$ . The  *$t$ -branch split closure* of  $P$  with respect to  $I$ , denoted by  $tBC(P)$ , is defined as the set of points in  $P$  which satisfy all  $t$ -branch split cuts; equivalently,

$$tBC(P) = \bigcap_{(\pi^1, \gamma_1), \dots, (\pi^t, \gamma_t) \in \mathbb{Z}^{m+1}} \text{conv}(P \cap D(\pi^1, \dots, \pi^t, \gamma_1, \dots, \gamma_t)).$$

Again this equivalence follows from the fact that  $\text{conv}(P \cap D(\pi^1, \dots, \pi^t, \gamma_1, \dots, \gamma_t))$  is a closed set ([DDV11, Lemma 2.3] can be easily extended to show this).

Similar to the split closure,  $tBC(P)$  depends on the mixed-integer lattice  $I$  and throughout the paper  $I$  will be clear from the context.

In [DDG12], 2-branch split disjunctions are called *cross disjunctions*, and 2-branch split cuts are called *cross cuts*. In this case, we have the equivalent definition of the *cross closure* as

$$CC(P) = \bigcap_{(\pi^1, \gamma_1), (\pi^2, \gamma_2) \in \mathbb{Z}^{m+1}} \text{conv}(P \setminus (S(\pi^1, \gamma_1) \cup S(\pi^2, \gamma_2))).$$

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<sup>1</sup>Recall from [CKS90] that the split closure of a rational polyhedron is again a rational polyhedron, which allow us to formally iterate the split closure operator with our current definitions.

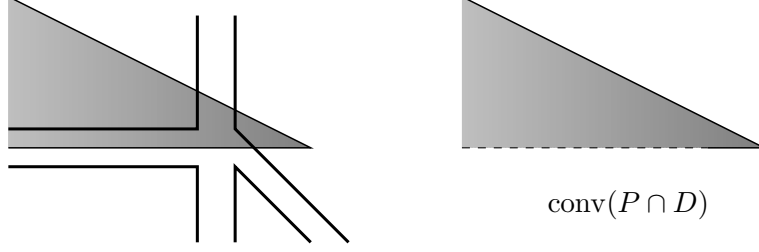


Figure 3: A crooked cross disjunction  $D$  and polyhedron  $P$  where  $\text{conv}(P \cap D)$  is not closed

**Crooked cross cuts.** Given  $\pi^1, \pi^2 \in \mathbb{Z}^m$  and  $\gamma_1, \gamma_2 \in \mathbb{Z}$  we define the sets

$$\begin{aligned} D_1^c(\pi^1, \pi^2, \gamma_1, \gamma_2) &= \{(x, y) \in \mathbb{R}^{m+n} : \pi^1 x \leq \gamma_1, (\pi^2 - \pi^1)x \leq \gamma_2 - \gamma_1\}, \\ D_2^c(\pi^1, \pi^2, \gamma_1, \gamma_2) &= \{(x, y) \in \mathbb{R}^{m+n} : \pi^1 x \leq \gamma_1, (\pi^2 - \pi^1)x \geq \gamma_2 - \gamma_1 + 1\}, \\ D_3^c(\pi^1, \pi^2, \gamma_1, \gamma_2) &= \{(x, y) \in \mathbb{R}^{m+n} : \pi^1 x \geq \gamma_1 + 1, \pi^2 x \leq \gamma_2\}, \\ D_4^c(\pi^1, \pi^2, \gamma_1, \gamma_2) &= \{(x, y) \in \mathbb{R}^{m+n} : \pi^1 x \geq \gamma_1 + 1, \pi^2 x \geq \gamma_2 + 1\}. \end{aligned}$$

We call the set  $D^c(\pi^1, \pi^2, \gamma_1, \gamma_2) = \bigcup_{i=1}^4 D_i^c(\pi^1, \pi^2, \gamma_1, \gamma_2)$  a *crooked cross disjunction* for  $I$ . A linear inequality is a *crooked cross cut* for  $P$  if it is valid for  $P \cap D^c$  for some crooked cross disjunction  $D^c$ . The *crooked cross closure* of  $P$ , which we denote by  $CCC(P)$ , is the set of points in  $P$  satisfying all the crooked cross cuts. Again we have the equivalence

$$CCC(P) = \bigcap_{(\pi^1, \gamma_1), (\pi^2, \gamma_2) \in \mathbb{Z}^{m+1}} \text{conv}(P \cap D^c(\pi^1, \pi^2, \gamma_1, \gamma_2)), \quad (3)$$

even though  $\text{conv}(P \cap D^c(\pi^1, \pi^2, \gamma_1, \gamma_2))$  may not be closed (see Figure 3 for an example); however, the right-hand side of (3) is indeed closed, which can be shown using [DDG11, Theorem 3.1], and thus gives the equivalence.

Finally, notice that we have  $P^I \subseteq CCC(P)$ .

## 2.2 Relaxations of Mixed-integer Sets

**$k$ -row relaxation.** Now consider a rational polyhedron of the form

$$P = \{(x, y) \in \mathbb{R}^m \times \mathbb{R}^n : Ax + Gy = b, y \geq 0\},$$

where  $A, G$  and  $b$  have  $r$  rows, and the mixed-integer lattice  $I = \mathbb{Z}^n \times \mathbb{R}^m$ , giving rise to the mixed-integer set  $P^I$ . Notice that any rational mixed-integer program can be modeled by such mixed-integer set [DDG12].

A  *$k$ -row relaxation* of  $P$  is obtained by combining the  $r$  equality constraints defining the set into  $k \leq r$  equalities. More precisely, it is  $P(M) \triangleq \{(x, y) \in \mathbb{R}^m \times \mathbb{R}^n : MAx + MGy = Mb, y \geq 0\}$ , where  $M$  is a  $k \times r$  matrix. Any inequality valid for  $P(M)^I \triangleq P(M) \cap I$  is called a *cut from a  $k$ -row relaxation*, see [ALWW, AWW09, DDG12].

**Basic relaxation.** We now consider a rational polyhedron defined in inequality form. Let  $P = \{(x, y) \in \mathbb{R}^m \times \mathbb{R}^n : Ax + Gy \leq b\}$  where  $A, G$  and  $b$  have  $r \geq m + n$  rows, and let  $I = \mathbb{Z}^m \times \mathbb{R}^n$ . For a subset  $J \subseteq \{1, \dots, r\}$  of row indices, we use  $A_J$  to denote the submatrix of  $A$  consisting of the rows of  $A$  corresponding to the indices in  $J$ . We define  $G_J$  and  $b_J$  similarly. Then a *basic relaxation* of  $P$  is obtained by keeping in the linear relaxation only linearly independent constraints, namely it is a set of the form  $P_{[J]} = \{(x, y) \in \mathbb{R}^m \times \mathbb{R}^n : A_J x + G_J y \leq b_J\}$  for some  $J \subseteq \{1, 2, \dots, r\}$  such that the matrix  $[A_J \ G_J]$  has full-row rank. A basic relaxation of the mixed-integer set  $P^I$  is obtained as follows:  $P_{[J]}^I = P_{[J]} \cap I$  and any inequality valid for  $P_{[J]}^I$  is called *a cut from a basic relaxation*, see [ACL05, DGR11].

### 2.3 Notation

We use  $\|\cdot\|$  to denote the  $\ell_2$  norm. Given a point  $x \in \mathbb{R}^n$  and a positive real  $r > 0$ , we use  $B(x, r) = \{y \in \mathbb{R}^n : \|x - y\| < r\}$  to denote the open ball centered at  $x$  with radius  $r$ . For a set  $S \subseteq \mathbb{R}^n$  we use  $\text{conv}(S)$  to denote the convex hull of  $S$ , and  $\text{aff}(S)$  to denote the affine hull of  $S$ . Given a set of vectors  $V \subseteq \mathbb{R}^n$  we use  $\text{span}(V)$  to denote the subspace spanned by  $V$ . Given a matrix  $M \in \mathbb{R}^{n \times m}$ , we use  $\text{rowspan}(M)$  to denote the subspace spanned by the rows of  $M$ .

## 3 Height Lemma

In preparation for the proof of our results we present the main technical tool used, called *Height Lemma* (this generalizes a similar result in [DG13]). Intuitively this lemma states the following: consider a collection (of arbitrary cardinality) of full dimensional pyramids, all sharing the same base. If we have a uniform lower bound on the height of the pyramids, plus the property that their apexes are not arbitrarily far from each other, then the intersection of all these pyramids contains a point outside of the common base. The motivation is that these pyramids will later represent what is ‘left over’ of  $P$  when we employ a subset of the cuts of interest, so this result allow us to talk about the left over of  $P$  when we add all these cuts together. In the formal statement below, the points  $s^1, s^2, \dots, s^n$  form the base of the pyramids and the points in  $Q$  are the apexes.

**Lemma 3.1** (Height Lemma). *Consider  $(a, b) \in \mathbb{R}^n \times \mathbb{R}$  with  $a \neq 0$  and let  $s^1, s^2, \dots, s^n$  be affinely independent points in the hyperplane  $\{x \in \mathbb{R}^n : ax = b\}$ . Consider  $b' > b$  and let  $Q$  be a bounded and non-empty subset of  $\{x \in \mathbb{R}^n : ax \geq b'\}$ . Then there exists a point  $x$  in  $\bigcap_{q \in Q} \text{conv}(s^1, s^2, \dots, s^n, q)$  satisfying the strict inequality  $ax > b$ .*

*Proof.* Let  $H = \{x \in \mathbb{R}^n : ax = b\}$  and  $S = \text{conv}(s^1, s^2, \dots, s^n)$ . We say that a point  $x$  that satisfies  $ax > b$  has positive *height*; so our goal is to find a point in  $\bigcap_{q \in Q} \text{conv}(s^1, s^2, \dots, s^n, q)$  with positive height. To simplify the notation, we assume without loss of generality that  $\|a\| = 1$ .

Clearly  $S$  is an  $(n - 1)$ -dimensional simplex contained in  $H$  and, by comparing dimensions, the affine hull of  $S$  equals  $H$ . Consider a point  $x^*$  in the relative interior of  $S$ , and let  $r > 0$  be such that the ball  $B(x^*, r) \cap H$  is contained in  $S$ . Let  $U \in \mathbb{R}$  be large enough so that it upper bounds the the norm of the points in  $S$  and the norm of the points in the projection of  $Q$  into  $H$ ; notice that such finite upper bound exist since  $S$  and  $Q$  are bounded. We show that the point  $x^* + (b' - b) \frac{r}{2U} a$  belongs to  $\bigcap_{q \in Q} \text{conv}(s^1, s^2, \dots, s^n, q)$ , which gives the desired result.

Consider  $q \in Q$  and let  $q^*$  denote its orthogonal projection into  $H$ , so  $q^* = q - b'' a$  for some  $b'' \geq b' - b$ . The idea is to show that  $x^*$  can be written as a convex combination  $\alpha q^* + (1 - \alpha) y^*$



for some point  $y^*$  in  $S$  (see Figure 4). Then replacing  $q^*$  by  $q - b''a$  in this expression, we get by convexity that  $\alpha q + (1 - \alpha)y^* = x^* + \alpha b''a$  belongs to  $\text{conv}(s^1, s^2, \dots, s^n, q)$  and has positive height. Importantly, our construction will guarantee that we can lower bound  $\alpha$  independently of the choice of  $q$ .

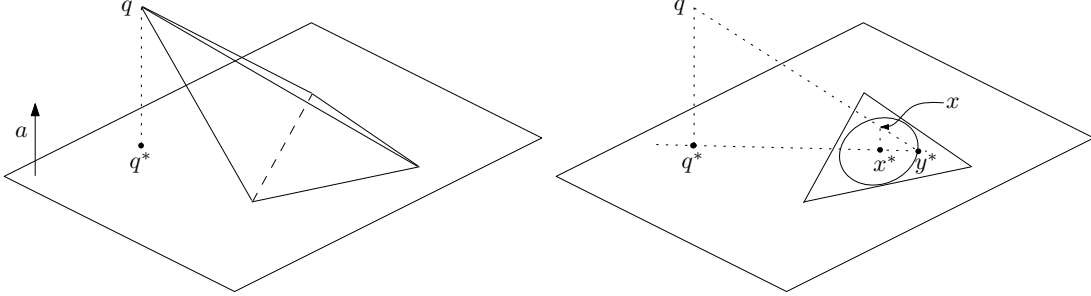


Figure 4: The left picture shows  $\text{conv}(s_1, s_2, \dots, s_n, q)$  and the hyperplane  $H = \{x \in \mathbb{R}^n : ax = b\}$ . The right picture shows the construction of  $y^*$  and the point  $x$  which belongs to  $\text{conv}(s_1, s_2, \dots, s_n, q)$  and has positive height, namely it satisfies  $ax > b$ .

To make this construction, consider the line  $\{q^* + \lambda(x^* - q^*) : \lambda \in \mathbb{R}\}$  passing through the points  $q^*$  and  $x^*$ , and notice that it lies in the hyperplane  $H$ . This line intersects the boundary of the closed ball  $\bar{B}(x^*, r) \cap H$  in two points, so let  $y^*$  denote such point which is farthest from  $q^*$  (notice that this point belongs to  $S$ ); specifically, we have  $y^* = q^* + \lambda^*(x^* - q^*)$  for  $\lambda^* = 1 + \frac{r}{\|x^* - q^*\|}$ , and notice that  $\|y^* - q^*\| = \lambda^*\|x^* - q^*\| = r + \|x^* - q^*\|$ . Rearranging, we can write explicitly  $x^*$  as a convex combination of  $q^*$  and  $y^*$ :  $x^* = \alpha q^* + (1 - \alpha)y^*$  for  $\alpha = \frac{r}{\|y^* - q^*\|} \in [0, 1]$ . As mentioned previously, we get that the point  $\alpha q + (1 - \alpha)y^* = x^* + \frac{r}{\|y^* - q^*\|}b''a$  belongs to  $\text{conv}(s^1, s^2, \dots, s^n, q)$ . Using the triangle inequality, we get that

$$\frac{r}{\|y^* - q^*\|}b'' \geq \frac{r}{\|y^*\| + \|q^*\|}b'' \geq \frac{r}{2U}b'' \geq \frac{r}{2U}(b' - b).$$

Using convexity we conclude that the point  $x^* + (b' - b)\frac{r}{2U}a$  belongs to  $\text{conv}(s^1, s^2, \dots, s^n, q)$ . Since the point is independent of  $q$ , it belongs to  $\bigcap_{q \in Q} \text{conv}(s^1, s^2, \dots, s^n, q)$  and the result follows.  $\square$

Also note that in the proof we do not use the property that the norms of the points in  $Q$  are bounded, we only use the fact that their projection on  $H$  has bounded norm. It is therefore possible to generalize this result slightly to unbounded  $Q$  that has bounded projection on  $H$ .

By employing an affine transformation, this lemma also carries over to affine subspaces of  $\mathbb{R}^n$ .

**Corollary 3.2.** *Let  $A \subseteq \mathbb{R}^n$  be an affine subspace of dimension  $k$ . Consider  $(a, b) \in \mathbb{R}^n \times \mathbb{R}$  with  $a \neq 0$  and let  $s^1, s^2, \dots, s^k$  be affinely independent points in  $A \cap \{x \in \mathbb{R}^n : ax = b\}$ . Consider  $b' > b$  and let  $Q$  be a bounded and non-empty subset of  $\{x \in A : ax \geq b'\}$ . Then there exists a point  $x$  in  $\bigcap_{q \in Q} \text{conv}(s^1, s^2, \dots, s^k, q)$  satisfying the strict inequality  $ax > b$ .*

For a vector  $v \in \mathbb{R}^n$ , a  $n \times n$  matrix  $M$  and a set  $S \subseteq \mathbb{R}^n$ , let  $S - v = \{s - v : s \in S\}$  and  $MS = \{Ms : s \in S\}$ . To see that the corollary follows from Lemma 3.1, let  $M$  be an  $n \times n$  matrix with determinant one such that  $M(A - s^1) = \mathbb{R}^k \times \{0\}^{n-k}$ . Applying this affine transformation and subsequently removing the last  $k$  coordinates, the corollary reduces to the previous lemma applied to objects in  $\mathbb{R}^k$  (points in  $Q$  are mapped to points in  $\mathbb{R}^k \times \{0\}^{n-k}$  with bounded norm).

## 4 $t$ -branch Split Closure Versus Second Split Closure

In this section we prove Theorem 1.1, which states that there is a polyhedral set  $P$  whose  $t$ -branch split closure is not contained in its second split closure. More specifically, we will work with the integer lattice  $I = \mathbb{Z}^{n+1}$  and the distorted simplex  $P_\epsilon$  defined by

$$P_\epsilon = \left\{ x \in \mathbb{R}^{n+1} : \sum_{i=1}^n x_i + 2x_{n+1} \leq n + 2 - \epsilon, \quad x_i \geq \epsilon, \quad i = 1, \dots, n \right\},$$

for  $\epsilon > 0$  (see Figure 5). We show that for every  $t < n$ , there is an  $\epsilon^* > 0$  such that  $tBC(P_{\epsilon^*}) \not\subseteq SC^2(P_{\epsilon^*})$ .

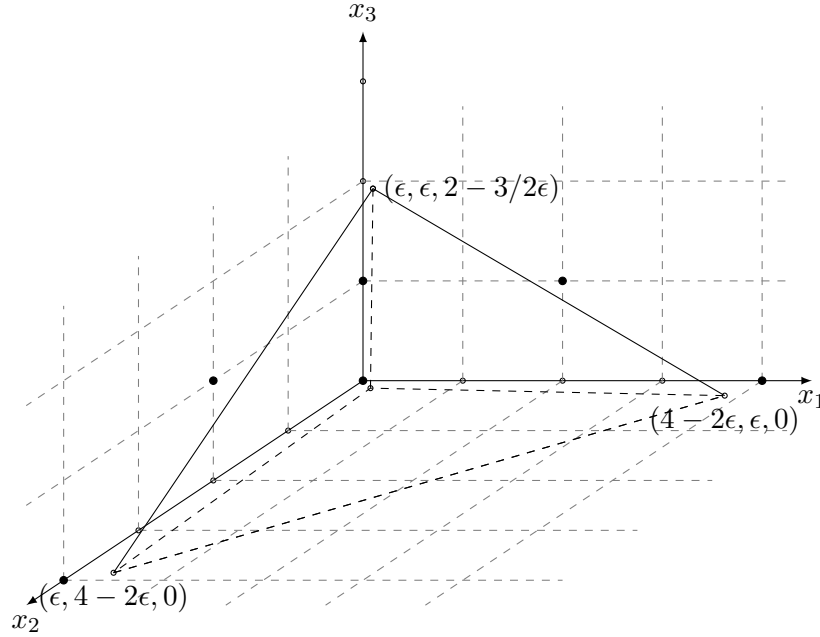


Figure 5: The set  $P$  when  $n = 2$

First we claim that the cut  $x_{n+1} \leq 0$  is valid for  $SC^2(P_\epsilon)$  for all  $\epsilon > 0$ . First notice that Chvátal-Gomory cuts for  $P_\epsilon$  can be obtained by rounding the right-hand sides of the constraints above. Since every Chvátal-Gomory cut is also a split cut, we observe that

$$\begin{aligned} SC(P_\epsilon) &\subseteq \left\{ x \in \mathbb{R}^{n+1} : \sum_{i=1}^n x_i + 2x_{n+1} \leq n + 1, \quad x_i \geq 1, \quad i = 1, \dots, n \right\} \\ &\subseteq \left\{ x \in \mathbb{R}^{n+1} : x_{n+1} \leq \frac{1}{2} \right\}. \end{aligned}$$

Again by using Chvátal-Gomory cuts, we get that

$$SC^2(P_\epsilon) \subseteq \{x \in \mathbb{R}^{n+1} : x_{n+1} \leq 0\},$$

which proves the claim. Since  $P^I \subseteq SC^2(P_\epsilon)$ , we get the following.

**Lemma 4.1.** *For all  $\epsilon > 0$  the inequality  $x_{n+1} \leq 0$  is valid for  $SC^2(P_\epsilon)$ . Furthermore, it is facet defining as it contains the following  $n + 1$  affinely independent points in  $P_\epsilon^I$ :*

$$s_1 = (2, 1, \dots, 1, 0), s_2 = (1, 2, \dots, 1, 0), \dots, s_n = (1, 1, \dots, 2, 0), s_{n+1} = (1, 1, \dots, 1, 0). \quad (4)$$

We next argue that when  $t < n$ , there is a sufficiently small  $\epsilon > 0$  such that the inequality  $x_{n+1} \leq 0$  is not valid for the  $t$ -branch split closure of  $P_\epsilon$ . First we show that a single  $t$ -branch split cut cannot imply the cut  $x_{n+1} \leq 0$ . The main tool used is the fact that simplices cannot be covered by a small collection of split sets. More precisely, define the simplex

$$\Delta_n = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^n x_i \leq n, \quad x_i \geq 0, \quad i = 1, \dots, n \right\}.$$

**Theorem 4.2** ([DG13]). *For every integer  $n > 0$ , there exists a constant  $\delta > 0$  such that for every collection of  $n - 1$  split sets  $S_1, \dots, S_{n-1}$  the volume of  $\Delta_n \setminus \bigcup_{i=1}^{n-1} S_i$  is at least  $\delta$ .*

**Lemma 4.3.** *Consider  $t < n$ . Then there is  $\epsilon^* > 0$  such that for every collection  $S_1, \dots, S_t$  of split sets for  $P_{\epsilon^*}$ , the set  $P_{\epsilon^*} \setminus \bigcup_{j=1}^t S_j$  contains a point  $x$  such that  $x_{n+1} = 1$ .*

*Proof.* To simplify the notation, define the union of the splits  $S \triangleq \bigcup_{i=1}^t S_j$ . Consider the slice of  $P_\epsilon$  with  $x_{n+1} = 1$ , namely

$$\begin{aligned} T_\epsilon &\triangleq P \cap \{x \in \mathbb{R}^{n+1} : x_{n+1} = 1\} \\ &= \left\{ x \in \mathbb{R}^{n+1} : \sum_{i=1}^n x_i \leq n - \epsilon, \quad x_{n+1} = 1, \quad x_i \geq \epsilon, \quad i = 1, \dots, n \right\}. \end{aligned}$$

We will show that there is  $\epsilon^* > 0$  independent of  $S_1, \dots, S_t$  such that  $T_{\epsilon^*} \setminus S \neq \emptyset$ , which proves the lemma. Let  $\pi^j \in \mathbb{Z}^{n+1}$  and  $\gamma_j \in \mathbb{Z}$  be such that  $S_j = \{x \in \mathbb{R}^{n+1} : \gamma_j < \pi^j x < \gamma_j + 1\}$ . Notice that

$$T_\epsilon \cap S_j = T_\epsilon \cap \left\{ x \in \mathbb{R}^{n+1} : \gamma_j - \pi_{n+1}^j < \sum_{i=1}^n \pi_i^j x_i < \gamma_j - \pi_{n+1}^j + 1 \right\}$$

and therefore  $T_\epsilon \cap S_j = T_\epsilon \cap (S_j^* \times \mathbb{R})$ , where  $S_j^*$  is the split set  $S(\pi^j, \gamma_j - \pi_{n+1}^j)$  contained in  $\mathbb{R}^n$ . Let  $S^* = \bigcup_{j=1}^t S_j^*$  and observe that  $T_\epsilon \cap S = T_\epsilon \cap (S^* \times \mathbb{R})$ . Let  $T_\epsilon^*$  denote the projection of  $T_\epsilon$  onto the first  $n$  coordinates, namely

$$T_\epsilon^* = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^n x_i \leq n - \epsilon, \quad x_i \geq \epsilon \quad i = 1, \dots, n \right\},$$

and notice that  $T_\epsilon \setminus S \neq \emptyset$  if and only if  $T_\epsilon^* \setminus S^* \neq \emptyset$ , so it suffices to work with the latter.

Let  $\delta > 0$  be the volume lower bound given by Theorem 4.2 (which is independent of  $S_1, \dots, S_t$ ); in particular we have  $\text{vol}(\Delta_n \setminus S^*) \geq \delta$ . Since  $T_\epsilon^*$  is a perturbation of  $\Delta_n$ , we claim that for a small enough  $\epsilon > 0$  we have  $T_\epsilon^* \setminus S^*$  non-empty. First notice that the volume of  $T_\epsilon^*$  is a continuous function of  $\epsilon$ , achieving values  $\text{vol}(T_0^*) = \text{vol}(\Delta_n) > \text{vol}(\Delta_n \setminus S^*)$  and  $\text{vol}(T_{n/(n+1)}^*) = 0 < \text{vol}(\Delta_n \setminus S^*)$ . Then by the Intermediate Value Theorem there is a value  $\epsilon^* > 0$  (independent of  $S_1, \dots, S_t$ ) such that  $\text{vol}(T_{\epsilon^*}^*) \geq \text{vol}(\Delta_n) - \frac{\delta}{2}$ . Since both  $T_{\epsilon^*}^*$  and  $\Delta_n \setminus S^*$  are contained in  $\Delta_n$ , it follows that the intersection  $T_{\epsilon^*}^* \cap (\Delta_n \setminus S^*) = T_{\epsilon^*}^* \setminus S^*$  is non-empty, otherwise  $\text{vol}(\Delta_n) \geq \text{vol}(T_{\epsilon^*}^*) + \text{vol}(\Delta_n \setminus S) \geq \text{vol}(\Delta_n) + \frac{\delta}{2}$ . This concludes the proof.  $\square$

Applying the height lemma, we can make a statement about the simultaneous effect of every possible collection of  $t < n$  split sets  $S_1, S_2, \dots, S_t$  on  $P_{\epsilon^*}$ .

**Lemma 4.4.** *Let  $\epsilon^*$  be given by Lemma 4.3. Then for  $t < n$ , the inequality  $x_{n+1} \leq 0$  is not valid for the  $t$ -branch split closure of  $P_{\epsilon^*}$ .*

*Proof.* To simplify the notation, let  $P \triangleq P_{\epsilon^*}$ . Let  $\mathcal{S}_t$  denote the family of  $t$ -branch split sets for  $I = \mathbb{Z}^{n+1}$ , namely sets of the form  $\bigcup_{i=1}^t S_i$  where each  $S_i$  is a split set for  $I$ . To prove the lemma, we show that  $\bigcap_{S \in \mathcal{S}_t} \text{conv}(P \setminus S)$  contains a point  $x$  with  $x_{n+1} > 0$ .

For each  $S \in \mathcal{S}_t$ , let  $x^S$  be the point given by Lemma 4.3. Since  $\|x^S\| \leq n+1$ , we can apply the Height Lemma with parameters  $a = (0, 0, \dots, 0, 1)$ ,  $b = 0$ ,  $b' = 1$ ,  $Q = \{x \in \mathbb{R}^{n+1} : ax \geq b', \|x\| \leq n+1\}$ , and  $s_1, s_2, \dots, s_{n+1}$  defined in (4) to get that  $\bigcap_{S \in \mathcal{S}_t} \text{conv}(s_1, s_2, \dots, s_{n+1}, x^S)$  contains a point  $x$  with  $x_{n+1} > 0$ . Notice that for each  $S \in \mathcal{S}_t$  we have  $\text{conv}(s_1, s_2, \dots, s_{n+1}, x^S) \subseteq \text{conv}(P \setminus S)$  (since the integer points  $s_1, s_2, \dots, s_{n+1}$  belong to  $P^I \subseteq \text{conv}(P \setminus S)$ ), which implies that  $\bigcap_{S \in \mathcal{S}_t} \text{conv}(P \setminus S)$  contains a point  $x$  with  $x_{n+1} > 0$ . This concludes the proof.  $\square$

Using Lemmas 4.1 and 4.4 we now prove Theorem 1.1.

**Theorem 1.1 (restated).** *Let  $P_\epsilon$  and  $I$  be defined as above, and let  $\epsilon^*$  be given by Lemma 4.3. Then for any positive integer  $t < n$ ,  $SC^2(P_{\epsilon^*})$  is strictly contained in  $tBC(P_{\epsilon^*})$ .*

*Proof.* Again define  $P \triangleq P_{\epsilon^*}$ . We have already argued that

$$SC^2(P) \subseteq \left\{ x \in \mathbb{R}^{n+1} : \sum_{i=1}^n x_i + 2x_{n+1} \leq n+1, \quad x_{n+1} \leq 0, \quad x_i \geq 1, \quad i = 1, \dots, n \right\} \subseteq P. \quad (5)$$

We claim that the second polyhedron in this expression is integral, and hence equal to  $\text{conv}(P^I)$ . To see this, notice that the polyhedron is defined by  $n+2$  inequalities in  $\mathbb{R}^{n+1}$  and hence has at most  $n+2$  extreme points, which are obtained by intersecting all but one of the defining hyperplanes. It can be checked that the only fractional point that can be obtained by intersecting  $n+1$  of these hyperplanes is obtained by excluding the inequality  $x_{n+1} \leq 0$ . The corresponding point, however, violates  $x_{n+1} \leq 0$  and therefore is not an extreme point of the polyhedron; this proves its integrality.

Then using the fact that  $SC^2(P) \supseteq \text{conv}(P^I)$ , the first inclusion in (5) gives that  $SC^2(P) = \text{conv}(P^I)$ , and also that the inequality  $x_{n+1} \leq 0$  is valid for  $SC^2(P)$ . Since  $tBC(P) \supseteq \text{conv}(P^I) = SC^2(P)$  and  $x_{n+1} \leq 0$  is not valid for  $tBC(P)$  (Lemma 4.4), we have that  $SC^2(P)$  is strictly contained in  $tBC(P)$ .  $\square$

Notice that as  $P_\epsilon \subseteq \mathbb{R}^{n+1}$  is defined by  $n+1$  linearly independent linear inequalities, the mixed-integer set  $P_\epsilon^I$  is actually a basic relaxation of itself. Therefore cuts from basic relaxations of  $P_\epsilon^I$  give  $\text{conv}(P_\epsilon^I)$  whereas  $\text{conv}(P_\epsilon^I)$  does not equal  $tBC(P)$  for  $t < n$ . This yields the following corollary of the previous theorem.

**Corollary 4.5.** *For any integer  $t > 0$ , there exists a mixed-integer set in  $\mathbb{Z}^{t+1} \times \mathbb{R}$  such that its  $t$ -branch split closure strictly contains the set of points satisfying all cuts from its basic relaxations.*

## 5 Crooked cross Closure Versus Cross Closure

In this section we prove Theorem 1.2 by constructing a polyhedral set  $P$  whose crooked cross closure  $CCC(P)$  is strictly contained in its cross closure  $CC(P)$ . One important component of the construction is a triangle that cannot be covered by a cross set.

**Theorem 5.1** ([DDG<sup>+</sup>14]). *There exists a rational triangle  $T^* \subseteq \mathbb{R}^2$  satisfying the following: (i)  $T^*$  does not contain integer points in its interior; (ii)  $T^*$  contains the points  $(0, 0)$ ,  $(1, 0)$ ,  $(0, 1)$  in its boundary; (iii) there is  $\delta > 0$  such that for any pair of split sets  $S_1, S_2$  for  $\mathbb{Z}^2$ , the set  $T^* \setminus (S_1 \cup S_2)$  has area at least  $\delta$ .*

Let  $T^*$  be such a triangle and let  $x^*$  be a point in the interior of  $T^*$ , say, its centroid (which has rational coordinates). In this section we work with the polyhedron  $P$  defined as

$$P = \{(x, y) \in \mathbb{R}^2 \times \mathbb{R} : (x, y) \in \text{conv}((T^* \times \{0\}) \cup (x^* \times \{1\}))\},$$

and the mixed integer lattice  $I = \mathbb{Z}^2 \times \mathbb{R}$ . The associated mixed-integer set  $P^I = P \cap (\mathbb{Z}^2 \times \mathbb{R})$ . We also define  $T_\epsilon \triangleq P \cap \{x \in \mathbb{R}^3 : x_3 = \epsilon\}$  for  $\epsilon \geq 0$  and define  $T_\epsilon^*$  to be the projection of  $T_\epsilon$  onto the first 2 coordinates. We next obtain the following result.

**Lemma 5.2.** *The inequality  $x_3 \leq 0$  is valid for  $CCC(P)$ .*

*Proof.* Notice that  $T_0^* = T^*$  and  $T_1^* = x^*$ , and as the latter belongs to the interior of  $T^*$ , we conclude that  $T_\epsilon$  is contained in the interior of  $T^*$  for all  $\epsilon > 0$ . As  $T^*$  does not contain any integer points in its interior,  $T_\epsilon^* \cap \mathbb{Z}^2 = \emptyset$  for all  $\epsilon > 0$  and therefore  $\text{conv}(P) = T^* \times \{0\}$ . Consequently, the inequality  $x_3 \leq 0$  is valid for  $\text{conv}(P)$ .

To conclude the proof, we recall the fact that the convex hull of any polyhedral mixed-integer set in  $\mathbb{Z}^2 \times \mathbb{R}$  is given by crooked cross cuts. In particular,  $\text{conv}(P^I) = CCC(P)$  and consequently  $x_3 \leq 0$  is valid for  $CCC(P)$ , concluding the proof.  $\square$

We next show that the inequality  $x_3 \leq 0$  is not valid for  $CC(P)$ ; since  $CC(P)$  always contains  $CCC(P)$  – which equals  $\text{conv}(P^I)$  in this section – such a result would imply that  $CC(P)$  strictly contains  $CCC(P)$ , as desired. We start by showing that a single cross disjunction cannot imply the cut  $x_3 \geq 0$ .

**Lemma 5.3.** *There exists  $\epsilon^* > 0$  such that for any pair of split sets  $S_1, S_2$  for  $P$ , the set  $T_{\epsilon^*} \setminus (S_1 \cup S_2)$  is non-empty.*

*Proof.* Notice that  $\text{area}(T_0^*) > \text{area}(T_1^*) = 0$  and  $\text{area}(T_\epsilon^*)$  is continuous as a function of  $\epsilon$ . Let  $\delta > 0$  be given by Theorem 5.1 and take  $\epsilon^* > 0$  such that  $\text{area}(T_{\epsilon^*}^*) \geq \text{area}(T_0^*) - \delta/2$ ; the existence of  $\epsilon^*$  is guaranteed by the Intermediate Value Theorem.

Let  $S_1^*$  denote the projection of the split set  $S_1$  onto the first 2 (integer) coordinates, and notice that  $S_1 = S_1^* \times \mathbb{R}$  and that  $S_1^*$  is a split set for  $(T^*, \mathbb{Z}^2)$ . Define  $S_2^*$  similarly. It then follows that  $T_{\epsilon^*} \setminus (S_1 \cup S_2)$  is non-empty if and only if  $T_{\epsilon^*}^* \setminus (S_1^* \cup S_2^*)$  is non-empty; we prove the latter. Theorem 5.1 guarantees that the set  $T^* \setminus (S_1^* \cup S_2^*)$  has area at least  $\delta$ , and so  $T_{\epsilon^*}^* \setminus (S_1^* \cup S_2^*)$  has area at least  $\delta/2$ . Therefore  $T_{\epsilon^*} \setminus (S_1 \cup S_2)$  is non-empty.  $\square$

Together with the previous lemma, the Height Lemma directly implies that the cut  $x_3 \leq 0$  is not valid for the cross closure of  $P$ ; the proof is exactly the same as in Lemma 4.4 and is omitted.

**Lemma 5.4.** *The inequality  $x_3 \leq 0$  is not valid for  $CC(P)$ .*

Employing Lemmas 5.2 and 5.4 we obtain Theorem 1.2:

**Theorem 1.2 (restated).**  $CCC(P) \subsetneq CC(P)$ .

## 6 Crooked Cross Cuts Versus 3-branch Split Cuts

In this section we prove Theorem 1.3 by constructing an integer set  $P^I = P \cap I$  where  $I = \mathbb{Z}^3$  such that  $3BC(P) = \text{conv}(P^I) = \emptyset$  but  $CCC(P) \neq \emptyset$ . We define the polyhedron  $P$  to be the intersection of a specific octahedron with the unit cube, i.e.,

$$P = \left\{ x \in [0, 1]^3 : \sum_{i \in S} (1 - x_i) + \sum_{i \notin S} x_i \geq \frac{1}{2}, \forall S \subseteq \{1, 2, 3\} \right\}.$$

The inequalities defining  $P$  are defined for all choices of  $S$  including the empty set. Notice that for any  $\bar{x} \in \{0, 1\}^3$ , letting  $\bar{S} = \{i : \bar{x}_i = 1\}$ , we have  $\sum_{i \in \bar{S}} (1 - \bar{x}_i) + \sum_{i \notin \bar{S}} \bar{x}_i = 0$  which implies that  $\bar{x}$  violates an inequality defining  $P$ . Therefore  $P^I$  is the empty set.

Also note that for any  $x \in [0, 1]^3$  with  $x_j = 1/2$ , we have  $(1 - x_i) \geq 0$  and  $x_i \geq 0$  for all  $i \neq j$  and  $(1 - x_j) = x_j = 1/2$ . Therefore all such points are contained in  $P$ .

We first claim that  $3BC(P) = \emptyset$ . To see this, consider the 3-branch split disjunction  $D = D(e^1, e^2, e^3, 0, 0, 0)$ , where  $e^i$  is the  $i$ th unit vector in  $\mathbb{R}^3$ . Notice that  $x$  belongs to  $D$  iff  $x_i \notin (0, 1)$  for all  $i = 1, 2, 3$ , and therefore  $x \in P \cap D$  if and only if  $x$  is a 0-1 vector. Therefore  $P \cap D = \emptyset$ . Since  $3BC(P) \subseteq P \cap D$ , the claim follows.

Now we need to show that  $CCC(P) \neq \emptyset$ ; in particular, we show that  $(1/2, 1/2, 1/2)$  belongs to  $CCC(P)$ . For that, we need the following characterization of the crooked cross closure.

**Theorem 6.1.** (*[DDG11, Theorem 3.1]*) *For any polyhedron  $\tilde{P} \subseteq \mathbb{R}^m \times \mathbb{R}^n$  and mixed-integer lattice  $I = \mathbb{Z}^m \times \mathbb{R}^n$ ,*

$$CCC(\tilde{P}) = \bigcap_{\pi^1, \pi^2 \in \mathbb{Z}^m} \text{conv} \left( \tilde{P} \cap \{(x, y) : \pi^1 x \in \mathbb{Z}, \pi^2 x \in \mathbb{Z}\} \right).$$

**Lemma 6.2.** *The point  $(1/2, 1/2, 1/2)$  belongs to  $CCC(P)$ .*

*Proof.* Consider an arbitrary pair of vectors  $\pi^1, \pi^2 \in \mathbb{Z}^3$  and define  $P_{\pi^1, \pi^2} = \text{conv}(P \cap \{x \in \mathbb{R}^3 : \pi^1 x \in \mathbb{Z}, \pi^2 x \in \mathbb{Z}\})$ . Given Theorem 6.1, it suffices to show that

$$(1/2, 1/2, 1/2) \in P_{\pi^1, \pi^2}. \tag{6}$$

For that, let  $v \in \mathbb{R}^3$  be a non-zero vector orthogonal to  $\pi^1$  and  $\pi^2$ . We assume that  $v_1 \neq 0$ ; the argument is identical for the cases  $v_2 \neq 0$  and  $v_3 \neq 0$ . Then without loss of generality we can rescale  $v$  and assume  $v_1 = 1$ . The idea in the analysis is that the set  $\{x \in \mathbb{R}^3 : \pi^1 x \in \mathbb{Z}, \pi^2 x \in \mathbb{Z}\}$  contains all lines in the direction of  $v$  that pass through an integer point. We are interested in the lines that cross the intersection of  $P$  with the plane  $x_1 = 1/2$ ; therefore, it suffices to project  $\mathbb{Z}^3$  onto this plane along  $v$  and analyze the obtained set of points  $\Lambda$ , and show that  $\text{conv}(P \cap \Lambda)$  contains the point  $(1/2, 1/2, 1/2)$ .

Define the integer points  $w^1 = (0, -\lfloor \frac{v_2}{2} \rfloor, -\lfloor \frac{v_3}{2} \rfloor)$  and  $w^2 = (1, 1 + \lfloor \frac{v_2}{2} \rfloor, 1 + \lfloor \frac{v_3}{2} \rfloor)$ ; clearly  $w^j \pi^i \in \mathbb{Z}$  for  $i, j \in \{1, 2\}$ . Now consider the points  $u^1 = w^1 + v/2$  and  $u^2 = w^2 - v/2$ , which lie in the plane  $x_1 = 1/2$ . We can use the fact that  $v$  is orthogonal to  $\pi^1, \pi^2$  to deduce that  $u^j \pi^i \in \mathbb{Z}$  for  $i, j \in \{1, 2\}$ . Also, notice that  $u_2^j$  and  $u_3^j$  belong to the interval  $[0, 1]$  for  $j \in \{1, 2\}$ . Now any point in  $[0, 1]^3$  with one component equal to  $1/2$  is contained in  $P$ , and therefore so are  $u^1, u^2$ . Therefore, these points belong to  $P_{\pi^1, \pi^2}$ . By convexity of  $P_{\pi^1, \pi^2}$ , the point  $(u^1 + u^2)/2 = (1/2, 1/2, 1/2)$  also belongs to it, which concludes the proof of the lemma.  $\square$

The fact that  $3BC(P) = \emptyset \neq CCC(P)$  then concludes the proof of Theorem 1.3:

**Theorem 1.3 (restated).**  $3BC(P) \subsetneq CCC(P)$ .

## 7 Cross Cuts from Basic Relaxations

In this section we prove Theorem 1.5 by constructing a polyhedral set  $P$  with the property that the intersection of all cuts from its basic relaxations does not dominate its cross closure (see Figure 6)

$$P = \left\{ (x, w) \in \mathbb{R}^2 \times \mathbb{R} : \begin{aligned} -x_1 - x_2 + w &\leq 0, \\ x_1 + x_2 + w &\leq 2, \\ -x_1 + x_2 + w &\leq 1, \\ x_1 - x_2 + w &\leq 1 \end{aligned} \right\}, \quad (7)$$

and the mixed-integer lattice  $I = \mathbb{Z}^2 \times \mathbb{R}$ . For  $j = 1, 2, 3, 4$ , let  $P_j$  denote the relaxation of  $P$  obtained by dropping the  $j$ th constraint in (7); also let  $P_j^I = P_j \cap I$ . We remark that  $P^I$  is a convenient modification of the construction presented in Section 2.3 of [ACL05] to show that not every cut can be obtained as a cut from a basic relaxation.

As  $P \subseteq \mathbb{R}^3$  is defined by 4 constraints, the sets  $P_j$  for  $j = 1, 2, 3, 4$  give all the basic relaxations of  $P$ . Thus we want to show that  $\bigcap_{j=1}^4 \text{conv}(P_j^I) \not\subseteq CC(P)$ . For that, we show that  $w \leq 0$  is a cross cut for  $P$  but it is not valid for  $\bigcap_{j=1}^4 \text{conv}(P_j^I)$ .

**Lemma 7.1.** *The inequality  $w \leq 0$  is a valid cross cut for  $P$ .*

*Proof.* We will show that  $w \leq 0$  is a cross cut for  $P$  derived from the cross disjunction  $D(e^1, e^2, 0, 0) = \mathbb{R}^3 \setminus (S_1 \cup S_2)$  where  $e^i$  is the  $i$ th unit vector in  $\mathbb{R}^3$  and  $S_1$  is the split set  $\{(x, w) \in \mathbb{R}^2 \times \mathbb{R} : 0 < x_1 < 1\}$  and  $S_2$  is the split set  $\{(x, w) \in \mathbb{R}^2 \times \mathbb{R} : 0 < x_2 < 1\}$ .

This statement would be false only if there exists some point  $(x, w)$  belonging to both  $P$  and  $D(e^1, e^2, 0, 0)$  with  $w > 0$ . But if  $(x, w)$  belongs to  $P$  and  $w > 0$ , the inequalities in (7) immediately imply that  $0 < x_1 + x_2 < 2$  and  $-1 < x_1 - x_2 < 1$ . Therefore (see Figure 6)

$$\left\{ (x, w) \in P : w > 0 \right\} \subseteq S_1 \cup S_2 = \mathbb{R}^3 \setminus D(e^1, e^2, 0, 0),$$

and hence  $(x, w)$  does not belong to  $D(e^1, e^2, 0, 0)$ . The result then follows.  $\square$

Next we show that this cut cannot be obtained from basic relaxations.

**Lemma 7.2.** *The inequality  $w \leq 0$  is not valid for  $\bigcap_{j=1}^4 \text{conv}(P_j^I)$ .*

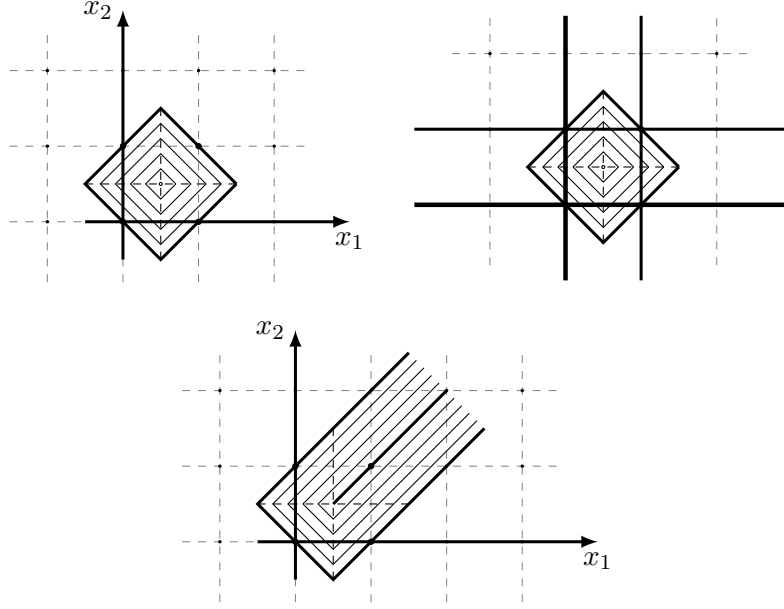


Figure 6: The top left picture shows  $P$  along the  $x_1, x_2$  axis, with the boundary of the intersection of  $P$  with the plane  $w = 0$  in bold. The top right picture shows  $P$  together with the cross disjunction  $D(e^1, e^2, 0, 0)$  in bold. The bottom picture shows the basic relaxation  $P_1$ , which gives rise to the set  $P_1^I = P_1 \cap I$ .

*Proof.* Observe that the points

$$p_1 = (0, 0, 0), p_2 = (1, 1, 0), p_3 = (0, 1, 0), p_4 = (1, 0, 0)$$

all belong to  $P^I$  and therefore to  $\text{conv}(P_j^I)$  for  $j = 1, \dots, 4$ . Also, the points

$$q_1 = (0, 0, 1), q_2 = (1, 1, 1), q_3 = (0, 1, 1), q_4 = (1, 0, 1)$$

belong to, respectively,  $P_1^I, \dots, P_4^I$  ( $q_j$  violates only the  $j$ th constraint defining  $P$ ). But  $(p_j + q_j)/2 = (1/2, 1/2, 1/2)$  for  $j = 1, \dots, 4$ , and therefore the point  $(1/2, 1/2, 1/2)$  belongs to  $\bigcap_{j=1}^4 \text{conv}(P_j^I)$  but violates  $w \leq 0$ .  $\square$

Theorem 1.5 follows from the previous two lemmas:

**Theorem 1.5 (restated).** *Let  $P_j$  for  $j \in J$  denote the set of basic relaxations of  $P$ , then*

$$P \cap \left( \bigcap_{j \in J} \text{conv}(P_j^I) \right) \not\subseteq CC(P).$$

As the crooked cross closure of  $P$  is contained in the cross closure of  $P$ , the above result implies that the crooked cross closure cannot be obtained by all cuts from the basic relaxations of  $P$ .



## 8 Cross Cuts that Cannot be Obtained from 2-row Relaxations

In this section we prove Theorem 1.6, namely we exhibit a polyhedral set such that the intersection of all cuts from its 2-row relaxations does not dominate its cross closure. The polyhedron we work with in this section is

$$P = \left\{ (x, y) \in \mathbb{R}^2 \times \mathbb{R}^4 : \begin{aligned} x_1 &= \frac{1}{2} + \frac{1}{2}y_1 - \frac{1}{2}y_4, \\ x_2 &= \frac{1}{2} + \frac{1}{2}y_1 - \frac{1}{2}y_3, \\ -y_1 - y_2 + y_3 + y_4 &= 0, \quad y \geq 0 \end{aligned} \right\}, \quad (8)$$

and the associated mixed-integer set is  $P^I = P \cap I$  with the mixed-integer lattice  $I = \mathbb{Z}^2 \times \mathbb{R}^4$ .

**Observation 8.1.** *The set  $P^I$  contains the points  $p^k = (x^k, y^k)$  for  $k = 1, \dots, 4$  given by*

$$\begin{aligned} x^1 &= (0, 0), y^1 = (0, 2, 1, 1) \\ x^2 &= (1, 1), y^2 = (2, 0, 1, 1) \\ x^3 &= (0, 1), y^3 = (1, 1, 0, 2) \\ x^4 &= (1, 0), y^4 = (1, 1, 2, 0). \end{aligned} \quad (9)$$

Moreover, the points  $p^1, p^2$ , and  $p^3$  are affinely independent.

For convenience, we define

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1/2 & 0 & 0 & -1/2 \\ 1/2 & 0 & -1/2 & 0 \\ -1 & -1 & 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix}$$

so that  $P = \{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^4 : Dx - Ay = b, y \geq 0\}$ . We use  $A_i$  to denote the  $i$ th row of  $A$ . Note that  $\text{rank}(A) = 3$ , and so  $\dim(P) \leq 3$ . On the other hand,  $P$  contains the affinely independent points  $p^1, p^2, p^3$  and  $(1/2, 1/2, 0, 0, 0, 0)$ , and so  $\dim(P) = 3$ .

$P$  can be obtained from the polyhedron in (7) by: (i) introducing slack variable  $y_i$  to convert the  $i$ th ( $i = 1, \dots, 4$ ) inequality to an equation, e.g.,  $-x_1 - x_2 + w + y_1 = 0$ ; (ii) Replacing  $w$  in the second to the fourth equations by  $x_1 + x_2 - y_1$  (obtained from the first equation) and, (iii) subtracting the third and fourth equations from the second equation, and then dividing the third and fourth equations by 2. It follows from the above operations that there is a one-to-one correspondence between the solutions of (7) and (8). For any solution  $(x_1, x_2, w)$  of (7), one gets a solution  $(x_1, x_2, y_1, \dots, y_4)$  of (8) by keeping  $x_1, x_2$  unchanged and letting  $y_1, \dots, y_4$  stand for the slacks of the inequalities in (8). Conversely, for any solution  $(x_1, x_2, y_1, \dots, y_4)$  of (8),  $(x_1, x_2, x_1 + x_2 - y_1)$  or  $(x_1, x_2, 1 - (y_1 + \dots + y_4)/4)$  is a solution of (7). The latter claim follows from the fact that adding up the four constraints in (7) (after introducing the slack variables) yields  $4w + y_1 + y_2 + y_3 + y_4 = 4$ .

Any 2-row relaxation of  $P$  is of the form

$$P(M) = \left\{ (x, y) \in \mathbb{R}^2 \times \mathbb{R}^4 : Mx - MAy = Mb, y \geq 0 \right\}$$

for a  $2 \times 3$  matrix  $M$ . To prove Theorem 1.6, we will show that

$$P \cap \left( \bigcap_{M \in \mathbb{R}^{2 \times 3}} \text{conv}(P(M)^I) \right) \not\subseteq CC(P).$$

Before starting, we observe that it is sufficient to consider matrices  $M$  that have full row rank.

**Lemma 8.2.** *For any  $M \in \mathbb{R}^{2 \times 3}$ , there is a rank 2 matrix  $M' \in \mathbb{R}^{2 \times 3}$  such that  $\text{conv}(P(M')^I) \subseteq \text{conv}(P(M)^I)$ .*

*Proof.* Clearly there exists a rank 2 matrix  $M' \in \mathbb{R}^{2 \times 3}$  such that  $\text{rowspan}(M') \supseteq \text{rowspan}(M)$ . It is easy to verify that such  $M'$  satisfies  $P(M') \subseteq P(M)$ , and hence  $\text{conv}(P(M')^I) \subseteq \text{conv}(P(M)^I)$ .  $\square$

We start by showing that the inequality  $cy \geq 4$ , where  $c = (1, 1, 1, 1)$ , is a cross cut for  $P$ . Notice that the inequality  $cy \geq 4$  translates to the inequality  $w \leq 0$  for the polyhedron (7).

**Lemma 8.3.** *The inequality  $cy \geq 4$  is a cross cut for  $P$ .*

*Proof.* We will show that  $cy \geq 4$  is a cross cut for  $P$  derived from the cross disjunction  $D(e^1, e^2, 0, 0) = \mathbb{R}^6 \setminus (S_1 \cup S_2)$ , where  $e^i$  is the  $i$ th unit vector in  $\mathbb{R}^6$ ,  $S_1 = \{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^4 : 0 < x_1 < 1\}$  and  $S_2 = \{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^4 : 0 < x_2 < 1\}$ .

This statement would be false only if there exists some point  $(\bar{x}, \bar{y})$  belonging to both  $P$  and  $D(e^1, e^2, 0, 0)$  with  $c\bar{y} < 4$ . But if  $(\bar{x}, \bar{y})$  belongs to  $P$  and  $c\bar{y} < 4$ , then  $(\bar{x}, \bar{w})$  with  $\bar{w} = 1 - cy/4$  is a solution of (7) with  $\bar{w} > 0$ . As in the proof of Lemma 7.1, we then infer that  $0 < \bar{x}_1 + \bar{x}_2 < 2$  and  $-1 < \bar{x}_1 - \bar{x}_2 < 1$ , hence  $(\bar{x}, \bar{y}) \in S_1 \cup S_2 = \mathbb{R}^6 \setminus D(e^1, e^2, 0, 0)$  and thus  $(\bar{x}, \bar{y})$  does not belong to  $D(e^1, e^2, 0, 0)$ . The result then follows.  $\square$

We will next show that there exists a point  $(\bar{x}, \bar{y}) \in P \cap \left( \bigcap_{M \in \mathbb{R}^{2 \times 3}} \text{conv}(P(M)^I) \right)$  such that  $c\bar{y} < 4$ , and hence the cross cut  $cy \geq 4$  is not valid for this set. To this end, we will show that for any  $M \in \mathbb{R}^{2 \times 3}$  we can construct a point  $(x(M), y(M)) \in P \cap \text{conv}(P(M)^I)$  such that  $cy(M) \leq 3$ . We will then apply the Height Lemma using these points and a common base formed by points  $p^1, p^2$  and  $p^3$  presented in Observation 8.1. The following lemma, whose proof is deferred to Section 8.1, shows the existence of the points mentioned above.

**Lemma 8.4.** *Consider a matrix  $M \in \mathbb{R}^{2 \times 3}$  of rank 2. Then, there is a point  $(x, y)$ , depending on  $M$ , with the following properties: (i)  $(x, y) \in P \cap \text{conv}(P(M)^I)$ ; (ii)  $cy \leq 3$ ; (iii)  $\|(x, y)\| \leq 6$ .*

Using Lemma 8.4 we next prove Theorem 1.6:

**Theorem 1.6 (restated).** *The cross cut closure of  $P$  cannot be obtained by all cuts from its 2-row relaxations. More precisely,*

$$P \cap \left( \bigcap_{M \in \mathbb{R}^{2 \times 3}} \text{conv}(P(M)^I) \right) \not\subseteq CC(P).$$

*Proof.* Consider a matrix  $M \in \mathbb{R}^{2 \times 3}$ . Using Lemma 8.4 (and Lemma 8.2 if necessary), find a point  $(x(M), y(M))$  in  $P \cap \text{conv}(P(M)^I)$  such that  $cy(M) \leq 3$  and  $\|(x(M), y(M))\| \leq 6$ . Also, for  $i = 1, 2, 3$ , the affinely independent points  $p^i$  in Observation 8.1 belong to  $P \cap \text{conv}(P(M)^I)$  and

satisfy  $cy^i = 4$ . Then applying Corollary 3.2 (with  $A = \text{aff}(P)$ ,  $a = (0, 0, -c)$ ,  $b = -4$  and  $b' = -3$ ), we conclude that the set

$$Q = \bigcap_{M \in \mathbb{R}^{2 \times 3}} \text{conv}(p^1, p^2, p^3, (x(M), y(M)))$$

contains a point  $(x^*, y^*)$  satisfying  $cy^* < 4$ . Note that it is possible to apply Corollary 3.2 because the dimension of  $\text{aff}(P)$  is 3.

Since  $P \cap (\bigcap_{M \in \mathbb{R}^{2 \times 3}} \text{conv}(P(M)^I))$  contains  $Q$ , it also contains  $(x^*, y^*)$ . This shows that the cut  $cy \geq 4$  is not valid for this set; together with Lemma 8.3, this concludes the proof of Theorem 1.6.  $\square$

As the crooked cross closure of  $P$  is contained in the cross closure of  $P$ , the above result implies that the crooked cross closure cannot be obtained by all cuts from 2-row relaxations of  $P$ .

### 8.1 Proof of Lemma 8.4

Let  $M \in \mathbb{R}^{2 \times 3}$  be a rank 2 matrix. Since  $\text{rank}(A) = 3$ , this implies that  $\text{rank}(MA) = 2$ . We will construct the points  $(x(M), y(M))$  satisfying the properties of the lemma in three steps. In the first step, we will construct points in  $\text{aff}(P)$ , which violate  $cy \geq 4$ , but do not belong to  $P \cap \text{conv}(P(M)^I)$ ; informally, these points almost belong to  $P \cap \text{conv}(P(M)^I)$ , except that they do not satisfy the required non-negativity condition. In the second step, we create two directions  $d^1$  and  $d^2$  in order to ‘correct’ the points constructed in the first step. In the final step, we use these directions to correct the points created in the first step, obtaining the desired point  $(x(M), y(M))$  in  $P \cap \text{conv}(P(M)^I)$  but still violating  $cy \geq 4$ .

**Step 1.** Consider the points  $(x^i, y^i) \in P^I$  for  $i = 1, \dots, 4$  from Observation 8.1, and recall that they all satisfy  $cy^i = 4$ . Since they belong to  $P^I$ , we have  $Dx^i - Ay^i = b$  for  $i = 1, \dots, 4$ . Moreover, since  $Ac = 0$ , we have  $Dx^i - A(y^i - c/2) = b$  for all  $i$ , which then implies  $MDx^i - MA(y^i - c/2) = Mb$  for all  $i$ . In other words, the points  $(x^i, \bar{y}^i) = (x^i, y^i - c/2)$  ( $i = 1, \dots, 4$ ) satisfy the equations defining both  $P$  and  $P(M)^I$  but violate one non-negativity inequality each, as

$$\begin{aligned} \bar{y}^1 &= (0, 2, 1, 1) - c/2 = (-1, 3, 1, 1) / 2 \\ \bar{y}^2 &= (2, 0, 1, 1) - c/2 = (3, -1, 1, 1) / 2 \\ \bar{y}^3 &= (1, 1, 0, 2) - c/2 = (1, 1, -1, 3) / 2 \\ \bar{y}^4 &= (1, 1, 2, 0) - c/2 = (1, 1, 3, -1) / 2. \end{aligned} \tag{10}$$

(Notice we have not changed the  $x^i$ 's, and hence the points  $(x^i, \bar{y}^i)$  still satisfy the integrality constraints in  $P(M)^I$ .) Note that each point above has exactly one negative coefficient which equals  $-1/2$ , and the remaining coefficients are strictly positive and at least  $1/2$ . These four points also violate the inequality  $cy \geq 4$ , as  $c \cdot c = 4$  and therefore,  $(x^i, y^i - c/2)$  satisfies  $c(y^i - c/2) = 2$ .

**Step 2.** We now define the ‘correcting’ directions  $d^1, d^2 \in \mathbb{R}^4$  in the  $y$ -space. To do so, recall that  $\text{rowspan}(A)$  has dimension 3 and by assumption  $\text{rowspan}(MA)$  is a 2-dimensional subspace of  $\text{rowspan}(A)$ . If  $A_3 \notin \text{rowspan}(MA)$ , let  $i^* = 3$ , and if  $A_3 \in \text{rowspan}(MA)$ , let  $i^* \in \{1, 2\}$  be the index such that  $A_{i^*}$  does not belong to  $\text{rowspan}(MA)$ . Notice that the rows of  $MA$  together with  $A_{i^*}$  span exactly  $\text{rowspan}(A)$ .

Now define  $d^1, d^2 \in \mathbb{R}^4$  to be solutions of the following two systems of four equations each (the coefficient  $\eta$  is specified later):

$$\begin{bmatrix} MA \\ A_{i^*} \\ c \end{bmatrix} d^1 = \begin{bmatrix} \mathbf{0} \\ 1 \\ \eta \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} MA \\ A_{i^*} \\ c \end{bmatrix} d^2 = \begin{bmatrix} \mathbf{0} \\ -1 \\ \eta \end{bmatrix}. \quad (11)$$

As the rows of  $MA$  together with  $A_{i^*}$  span exactly  $\text{rowspan}(A)$  and the vector  $c$  is orthogonal to the rows of  $A$  and hence to  $\text{rowspan}(A)$ , the matrix in the left-hand side of equations in (11) (which is the same) is invertible. Therefore, these systems have unique nonzero solutions.

We will show that for some  $\eta$ , there are scalars  $\lambda^1, \lambda^2 > 0$  such that  $\lambda^1 d^1$  and  $\lambda^2 d^2$  are nonzero vectors satisfying the following properties:

1.  $MA(\lambda^1 d^1) = MA(\lambda^2 d^2) = 0$ .
2. There exists an  $\alpha \in [0, 1]$  such that  $A(\alpha \lambda^1 d^1 + (1 - \alpha) \lambda^2 d^2) = 0$ .
3.  $\max_i \lambda^1 d_i^1 = 1/2 = \max_i |\lambda^1 d_i^1|$  and  $\max_i \lambda^2 d_i^2 = 1/2 = \max_i |\lambda^2 d_i^2|$ .
4.  $c \lambda^1 d^1 \leq 1$  and  $c \lambda^2 d^2 \leq 1$ .

The motivation for these properties is the following: (i) the first and second properties will ensure that the ‘corrected’ vectors  $(x^i, \bar{y}^i + (\alpha d^1 + (1 - \alpha) d^2))$  still satisfy all the constraints of  $P \cap \text{conv}(P(M)^I)$ , except for the non-negativity conditions; (ii) we will use the third property to argue that there is an index  $i$  such that the corresponding corrected vector satisfies the non-negativity conditions, and hence belongs to  $P \cap \text{conv}(P(M)^I)$ ; (iii) the fourth property will ensure that the corrected vector does not satisfy the inequality  $cy \geq 4$ .

Note that Properties 1 and 2 hold for all  $\lambda^1, \lambda^2 > 0$ , independent of the choice of  $\eta$ . Property 1 follows directly from the first two equations in both systems in (11). For Property 2, since  $A_{i^*} d^1 = 1$  and  $A_{i^*} d^2 = -1$ , we have that  $A_{i^*} (d^1 + d^2) = 0$ ; then for a given  $\lambda^1, \lambda^2$ , we can set  $\alpha \in [0, 1]$  such that  $\alpha \lambda^1 = (1 - \alpha) \lambda^2$  and have  $A_{i^*} (\alpha \lambda^1 d^1 + (1 - \alpha) \lambda^2 d^2) = \alpha \lambda^1 A_{i^*} (d^1 + d^2) = 0$ . Therefore,  $\alpha \lambda^1 d^1 + (1 - \alpha) \lambda^2 d^2$  is orthogonal to the rows of  $MA$  and to  $A_{i^*}$ , and hence to the rows of  $A$  (as rows of  $MA$  and  $A_{i^*}$  span  $\text{rowspan}(A)$ ).

In order to obtain Properties 3 and 4 we consider two cases depending on whether  $A_3$  belongs to  $\text{rowspan}(MA)$  or not, and set  $\lambda^1, \lambda^2$ , and  $\eta$  appropriately.

**Case 1:**  $A_3 \in \text{rowspan}(MA)$ . Set  $\eta = 0$ . In this case, the last constraint in both systems in (11) (which are identical) guarantee that  $\lambda^1 d^1$  and  $\lambda^2 d^2$  satisfy Property 4 for all  $\lambda^1, \lambda^2$ .

We now consider Property 3 for a rescaling of  $d^1$ ; the proof for  $d^2$  is identical. Since  $A_3$  belongs to  $\text{rowspan}(MA)$ , the first two constraints in the first system in (11) guarantee that  $A_3 d^1 = 0$ , and therefore  $d_1^1 + d_2^1 = d_3^1 + d_4^1$ . The last constraint implies that  $d_1^1 + d_2^1 + d_3^1 + d_4^1 = 0$ . In addition,  $d^1 \neq 0$  as  $A_{i^*} d^1 \neq 0$ . Therefore,  $d_1^1 + d_2^1 = d_3^1 + d_4^1 = 0$  and hence  $\max_i d_i^1 = \max_i |d_i^1|$ , so we can multiply  $d^1$  by an appropriate positive scalar  $\lambda^1$  so that  $\max_i \lambda^1 d_i^1 = 1/2$ . The vector  $\lambda^1 d^1$  then satisfies Properties 1,2,3, and 4.

**Case 2:**  $A_3 \notin \text{rowspan}(MA)$ . Set  $\eta = 1$ . In this case  $i^* = 3$ , namely both systems in (11) contain a constraint of the form  $A_3 d = \pm 1$  (instead of the implied constraint  $A_3 d = 0$  in the previous case). Adding the third and fourth constraints in the first system in (11), we get

$d_3^1 + d_4^1 = 1$ . Subtracting the third constraint from the fourth constraint, we get  $d_1^1 + d_2^1 = 0$ . Therefore  $\max_i d_i^1 = \max_i |d_i^1| \geq 1/2$ . We can then rescale  $d^1$  by  $\lambda^1 \in (0, 1]$  so that  $\lambda^1 d^1$  satisfies Property 3. Further,  $\lambda^1 d^1$  satisfies Property 4, since  $c d^1 \leq 1$ . Therefore,  $\lambda^1 d^1$  satisfies Properties 1, 2, 3 and 4.

As for  $d^2$ , adding and subtracting constraints as in the case of  $d^1$ , we see that  $d_3^2 + d_4^2 = 0$  and  $d_1^2 + d_2^2 = 1$ . Once again we can scale  $d^2$  so that it satisfies all properties.

**Step 3.** Consider the vectors  $\lambda^1 d^1$  and  $\lambda^2 d^2$  from the previous step. Let  $i = \operatorname{argmax}_k d_k^1$  and  $j = \operatorname{argmax}_k d_k^2$ . As  $\lambda^1 d^1$  is nonzero, and because of Property 3, we have  $\lambda^1 d_i^1 = 1/2$  and  $\bar{y}^i + \lambda^1 d^1 \geq 0$ . Property 1 implies that  $MDx - MA(\bar{y}^i + \lambda^1 d^1) = MDx - MA\bar{y}^i = Mb$ , so using the integrality of  $x^i$  we have that  $(x^i, \bar{y}^i + \lambda^1 d^1)$  belongs to  $P(M)^I$  (but not necessarily to  $P$ , since we can still have  $Dx^i - A(\bar{y}^i + \lambda^1 d^1) \neq b$ ). Also, Property 4 implies that  $c(\bar{y}^i + \lambda^1 d^1) \leq 3$ , and hence the point does not satisfy the inequality  $cy \geq 4$ . Similarly, Properties 1 and 3 imply that  $(x^j, \bar{y}^j + \lambda^2 d^2) \in P^I(M)$ , and  $c(\bar{y}^j + \lambda^2 d^2) \leq 3$ .

Finally, by Property 2 there is an  $\alpha \in [0, 1]$  such that the convex combination

$$(x(M), y(M)) \triangleq \alpha(x^i, \bar{y}^i + \lambda^1 d^1) + (1 - \alpha)(x^j, \bar{y}^j + \lambda^2 d^2)$$

satisfies  $Dx(M) - Ay(M) = \alpha(Dx^i - A\bar{y}^i) + (1 - \alpha)(Dx^j - A\bar{y}^j)$ ; but by construction of our points,  $Dx^i - A\bar{y}^i = Dx^j - A\bar{y}^j = b$ , and hence  $Dx(M) - Ay(M) = b$ . Therefore, the point  $(x(M), y(M))$  belongs to  $P \cap \operatorname{conv}(P(M)^I)$ . In addition, we clearly have  $cy(M) \leq 3$ , and it is easy to verify that  $\|(x(M), y(M))\| \leq 6$ . This concludes the proof of Lemma 8.4.

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