

VARIATIONAL PROPERTIES OF VALUE FUNCTIONS

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Abstract. Regularization plays a key role in a variety of optimization formulations of inverse problems. A recurring question in regularization approaches is the selection of regularization parameters, and its effect on the solution and on the optimal value of the optimization problem. The sensitivity of the value function to the regularization parameter can be linked directly to the Lagrange multipliers. In this paper, we fully characterize the variational properties of the value functions for a broad class of convex formulations, which are not all covered by standard Lagrange multiplier theory. We also present an inverse function theorem that links the value functions of different regularization formulations (not necessarily convex). These results have implications for the selection of regularization parameters, and the development of specialized algorithms. We give numerical examples that illustrate the theoretical results.

1. Introduction. It is well known that there is a close connection between the sensitivity of the optimal value of a parametric optimization problem and its Lagrange multipliers. Consider the family of convex optimization problems

$$\begin{aligned} P(b, \tau) \quad & \underset{r, x}{\text{minimize}} \quad \rho(r) \\ & \text{subject to} \quad Ax + r = b : u, \\ & \quad \quad \quad \phi(x) \leq \tau : \mu, \end{aligned}$$

parameterized by $\tau > \inf \phi$ and $b \in \mathbb{R}^m$, where $A \in \mathbb{R}^{m \times n}$, and the functions $\phi : \mathbb{R}^n \rightarrow \mathbb{R} := (-\infty, \infty]$ and $\rho : \mathbb{R}^m \rightarrow \mathbb{R}$ are closed, proper and convex, and continuous relative to their domains. The value function

$$v(b, \tau) = \inf P(b, \tau)$$

gives the optimal objective value of $P(b, \tau)$ for fixed parameters b and τ . If $P(b, \tau)$ is a feasible *ordinary convex program* (cf. [28, Section 28]), then under standard hypotheses the subdifferential of v is

$$\partial v(b, \tau) = \{(u, \mu)\}, \tag{1.1}$$

where $u \in \mathbb{R}^m$ and $\mu \in \mathbb{R}$ are the Lagrange multipliers of $P(b, \tau)$, shown next to their constraints. This connection is extensively explored in Rockafeller's 1993 survey paper [27].

If we allow ϕ to take on infinite values on the domain of the objective—which can occur, for example, if ϕ is an arbitrary gauge—then $P(b, \tau)$ is no longer an ordinary convex program, and the standard Lagrange multiplier theory does not apply. While in some cases the problem can be remodeled to overcome this difficulty, we are interested in developing an extended Lagrange multiplier theory that avoids the need for such reformulation, and captures a wide range of data-fitting applications. Remarkably, even in this general setting, it is also possible to obtain explicit formulas of the subdifferential of the value function of $P(b, \tau)$, useful in many applications.

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1.1. Examples. We give two simple examples illustrating the need for the extended Lagrange multiplier theory. Both are of the form

$$\text{minimize } \frac{1}{2} \|Ax - b\|_2^2 \quad \text{subject to } \gamma(x | U) \leq 1, \quad (1.2)$$

where

$$\gamma(x | U) := \inf \{ \lambda \geq 0 \mid x \in \lambda U \}$$

is the *gauge function* for the closed nonempty convex set $U \subset \mathbb{R}^n$, which contains 0. Let $A = I$ and $b = (0, -1)^T$.

For our first example, we consider the set

$$U = \left\{ x \in \mathbb{R}^2 \mid \frac{1}{2}x_1^2 \leq x_2 \right\},$$

defined in [28, Section 10]. The gauge for this set is an example of a closed proper and convex function that is not continuous at a boundary point of its effective domain. It is straightforward to show that

$$\gamma(x | U) = \begin{cases} \frac{x_1^2}{2x_2}, & x_2 > 0 \\ 0, & x_1 = 0 = x_2 \\ +\infty, & \text{otherwise.} \end{cases}$$

The constraint region for (1.2) is the set U and the unique global solution is the point $x = 0$. However, since $0 = \gamma(0 | U) < 1$, the classical Lagrange multiplier theory fails: the solution is on the boundary of the feasible region, and yet no Lagrange multiplier exists. The problem is that the constraint is active at the solution, but not active in the functional sense, i.e., $\gamma(0 | U) < 1$. In contrast, the extended multiplier theory of Theorem 5.2 succeeds with the multiplier choice of 0.

For the second example, take $U = \mathbb{B}_2 \cap K$, where \mathbb{B}_2 is the unit ball associated with the Euclidean norm on \mathbb{R}^2 , and set $K = \{ (x_1, x_2) \mid x_2 \geq 0 \}$. Then $\gamma(x | \mathbb{B}_2 \cap K) = \|x\|_2 + \delta(x | K)$, and the constraint region for (1.2) is the set $\mathbb{B}_2 \cap K$. Again, the origin is the unique global solution to this optimization problem, and no classical Lagrange multiplier for this problem exists. The multiplier theory of Theorem 5.2 succeeds, again with the multiplier choice of 0.

1.2. Formulations. Appropriate definitions of the functions ρ and ϕ can be used to represent a range of practical problems. Choosing ρ to be the 2-norm and ϕ to be any norm yields the canonical regularized least-squares problem

$$\text{minimize}_x \|r\|_2 \quad \text{subject to } Ax + r = b, \|x\| \leq \tau, \quad (1.3)$$

which optimizes the misfit between the data b and the forward model Ax , subject to keeping x appropriately bounded in some norm. Interestingly, when the optimal residual r is nonzero, the value function for this family of problems is always differentiable in both b and τ —i.e., the subdifferential in (1.1) is a singleton. In particular,

$$\nabla v(b, \tau) = \left(\frac{r}{\|r\|_2}, \frac{\|A^T r\|_*}{\|r\|_2} \right), \quad (1.4)$$

where $\|\cdot\|_*$ is the norm dual to $\|\cdot\|$. The 2-norm constraint on x yields a Tikhonov regularization, popular in many inversion applications. A 1-norm constraint on x yields

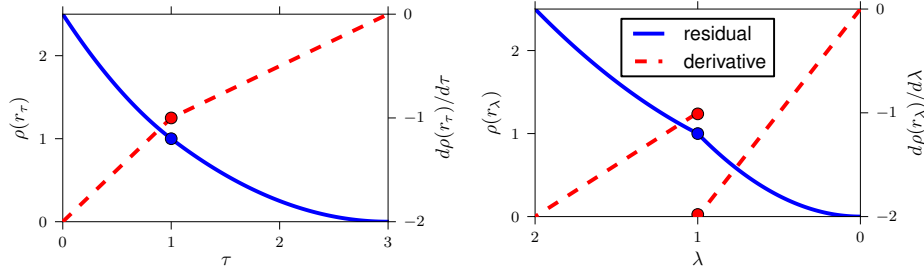


FIG. 1.1. The misfit $\rho(r)$ (solid line) and its derivative (dashed line) as a function of the regularization parameter for a 1-norm regularized example. The left panel shows the constrained formulation $P(b, \tau)$, which varies smoothly with τ ; the right panel shows that the penalized formulation does not vary smoothly with λ (note the reversed axis).

the Lasso problem [33], often used in sparse recovery and model-selection applications. The gradient (1.4) is derived by van den Berg and Friedlander [6]. The analysis of the sensitivity in τ of the value function for the Lasso problem led to the development of the SPGL1 solver [5], currently used in a variety of sparse inverse problems, with particular success in large-scale sparse inverse problems [21]. A subsequent analysis [8] that allows $\phi(x)$ to be a gauge paved the way for other applications, such as group-sparsity promotion [7].

An alternative to $P(b, \tau)$ is the class of penalized formulations

$$P_L(b, \lambda) \quad \underset{x}{\text{minimize}} \quad \rho(b - Ax) + \lambda\phi(x)$$

where the nonnegative parameter λ is used to control the tradeoff between the data misfit ρ and a convex regularization term ϕ . For example, taking $\rho(r) = \|r\|_2$ and $\phi(x) = \|x\|$ yields a formulation analogous to (1.3). This penalized formulation is commonly used in applications of Bayesian parametric regression [24, 25, 30, 34, 36], inference problems on dynamic linear systems [1, 11], feature selection, selective shrinkage, and compressed sensing [14, 15, 19], robust formulations [2, 17, 18, 23], support vector regression [20, 35], classification [16, 26, 32], and functional reconstruction [4, 12, 31].

From an algorithmic point of view, either formulation $P(b, \tau)$ or $P_L(b, \lambda)$ may be preferable. However, a key feature of $P(b, \tau)$ that sets it apart from $P_L(b, \lambda)$ and allows our comprehensive analysis is that its objective is a convex function of (b, τ) . This fact gives the convexity of the value function $v(b, \tau)$ (see section 1.3). In contrast, the optimal value of the penalized formulation $P_L(b, \lambda)$ is not in general a convex function of its parameters. The following example

$$\rho(r) = \frac{1}{2}\|r\|_2^2, \quad \phi(x) = \|x\|_1, \quad A = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

illustrates this situation. The optimal value of ρ in the two formulations, as functions of τ and λ , respectively, are given by

$$\rho(r_\tau) = \begin{cases} \frac{1}{4}(\tau - 3)^2 & \text{for } \tau \in [0, 1) \\ \frac{1}{2} + \frac{1}{2}(\tau - 2)^2 & \text{for } \tau \in [1, 3) \\ 0 & \text{otherwise;} \end{cases} \quad \rho(r_\lambda) = \begin{cases} \lambda^2 & \text{for } \lambda \in [0, 1) \\ \frac{1}{2} + \frac{1}{2}\lambda^2 & \text{for } \lambda \in [1, 2) \\ 5/2 & \text{otherwise.} \end{cases}$$

The optimal values and their derivatives are shown in Figure 1.1, where it is clear that $\rho(r_\tau)$ is convex (and in this case also smooth) in τ , but $\rho(r_\lambda)$ is not convex in λ .

The admissibility of variational analysis and convexity of the value function may convince some practitioners to explore formulations of type $P(b, \tau)$ rather than $P_L(b, \lambda)$. In fact, we give an example (in section 7) of how this variational information might be used for algorithm design in the context of large-scale inverse problems.

1.3. Approach. For many practical inverse problems, the formulation of primary interest is

$$P_R(b, \sigma) \quad \underset{x}{\text{minimize}} \quad \phi(x) \quad \text{subject to} \quad \rho(b - Ax) \leq \sigma,$$

in part because estimates of a tolerance level σ on a data fitting error $\rho(b - Ax)$ are more easily available than estimates of a bound on the regularization penalty $\phi(x)$. However, the formulation $P(b, \tau)$ can sometimes be easier to solve. The underlying numerical theme is to develop methods for solving $P_R(b, \sigma)$ using solutions to $P(b, \tau)$.

In section 2, we present an inverse function theorem for value functions that characterizes the relationship between $P(b, \tau)$ and $P_R(b, \sigma)$, and for more general nonconvex problems. One application of this result is to establish conditions under which it is possible to implement a root-finding approach for the nonlinear equation

$$\text{find } \tau \text{ such that } v(b, \tau) = \sigma, \tag{1.5}$$

where $P_R(b, \sigma)$ can be solved via a sequence of approximate solutions of $P(b, \tau)$. This generalizes the approach used by van den Berg and Friedlander [6, 8] for large-scale sparse optimization applications. The convex case is especially convenient, because both value functions are decreasing and convex. When the value function is differentiable, Newton's method is globally monotonic and locally quadratic. In section 5 we establish the variational properties (including conditions necessary for differentiability) of $P(b, \tau)$.

In section 4 we derive dual representations of $P(b, \tau)$ and their optimality conditions. These are used in section 5 to characterize the variational properties of the value function v . The conjugate, horizon, and perspective functions arise naturally as part of the analysis, and we present a calculus (section 3) for these functions that allows explicit computation of the subdifferential of v for large classes of misfit functions ρ and regularization functions ϕ (see section 6).

One of the motivating problems for the general analysis and methods we present is the treatment of a robust misfit function ρ (such as the popular Huber norm) in the context of sparsity promotion, which typically involves a nonsmooth regularizer ϕ . We demonstrate (section 7) how the sensitivity analysis can be applied to solve a sparse nonnegative denoising problem with robust misfit measures—both convex and nonconvex.

The proofs of all of the results are relegated to the appendix (section 8).

1.4. Notation. For a matrix $A \in \mathbb{R}^{m \times n}$, the image and inverse image of the sets E and F , respectively, are given by the sets

$$AE = \{ y \mid y = Ax, x \in E \} \quad \text{and} \quad A^{-1}F = \{ x \mid Ax \in F \}.$$

For a convex function p , its epigraph is denoted $\text{epi } p$, and the level set is denoted $\text{lev}_p(\tau) = \{ x \mid p(x) \leq \tau \}$. The function $\delta(x \mid X)$ is the indicator to a convex set X .

2. An inverse function theorem for optimal value functions. Let $\psi_i : X \subseteq \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$, $i \in \{1, 2\}$, be arbitrary scalar-valued functions, and consider the following pair of related problems, and their associated value functions:

$$\begin{aligned} v_1(\sigma) &:= \inf_{x \in X} \psi_1(x) + \delta((x, \sigma) \mid \text{epi } \psi_2), & \mathcal{P}_{1,2}(\sigma) \\ v_2(\tau) &:= \inf_{x \in X} \psi_2(x) + \delta((x, \tau) \mid \text{epi } \psi_1). & \mathcal{P}_{2,1}(\tau) \end{aligned}$$

This pair corresponds to the problems $P_{\mathbb{R}}(b, \sigma)$ and $P(b, \tau)$, defined in section 1 with the identifications

$$\psi_1(x) = \rho(b - Ax) \quad \text{and} \quad \psi_2(x) = \phi(x).$$

Our goal in this section is to establish general conditions under which the value functions v_1 and v_2 satisfy the inverse-function relationship

$$v_1 \circ v_2 = \text{id},$$

and for which the pair of problems $\mathcal{P}_{1,2}(\sigma)$ and $\mathcal{P}_{2,1}(\tau)$ have the same solution sets. The pair of problems $P(b, \tau)$ and $P_{\mathbb{R}}(b, \sigma)$ always satisfy the conditions of the next theorem, which applies to functions that are not necessarily convex.

THEOREM 2.1. *Let $\psi_i : X \subseteq \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$, $i \in \{1, 2\}$, be as defined in $\mathcal{P}_{1,2}(\sigma)$, and define*

$$\mathcal{S}_{1,2} := \{ \sigma \in \overline{\mathbb{R}} \mid \emptyset \neq \arg \min \mathcal{P}_{1,2}(\sigma) \subset \{ x \in X \mid \psi_2(x) = \sigma \} \}.$$

Let $\mathcal{S}_{2,1}$ be defined symmetrically to $\mathcal{S}_{1,2}$ by interchanging the roles of the indices. Then, for every $\sigma \in \mathcal{S}_{1,2}$,

- (a) $v_2(v_1(\sigma)) = \sigma$, and
- (b) $\arg \min \mathcal{P}_{1,2}(\sigma) = \arg \min \mathcal{P}_{2,1}(v_1(\sigma)) \subset \{ x \in X \mid \psi_1(x) = v_1(\sigma) \}$.

Moreover, $\mathcal{S}_{2,1} = \{ v_1(\sigma) \mid \sigma \in \mathcal{S}_{1,2} \}$, and so

$$\{ (\sigma, v_1(\sigma)) \mid \sigma \in \mathcal{S}_{1,2} \} = \{ (v_2(\tau), \tau) \mid \tau \in \mathcal{S}_{2,1} \}.$$

3. Convex analysis. In order to present the duality results of section 4, we require a few basic tools from convex analysis. There are many excellent references for the necessary background material in convex analysis, with several appearing within the past 10 years. In this study we make use of Rockafellar [28] and Rockafellar and Wets [29], although similar results can be found elsewhere [9, 10, 22]. We review the necessary results here.

3.1. Functional operations. The convex function $h : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ generates the following convex functions:

1. *Legendre-Fenchel conjugate* of h :

$$h^*(y) := \sup_x [\langle y, x \rangle - h(x)].$$

2. *Horizon function* of h :

$$h^\infty(z) := \sup_{x \in \text{dom } h} [h(x+z) - h(x)].$$

3. *Perspective function of h :*

$$\tilde{h}(z, \lambda) := \begin{cases} \lambda h(\lambda^{-1}z) & \text{if } \lambda > 0, \\ \delta(x \mid 0) & \text{if } \lambda = 0, \\ +\infty & \text{if } \lambda < 0. \end{cases}$$

4. *Closure of the perspective function of h :*

$$h^\pi(z, \lambda) := \begin{cases} \lambda h(\lambda^{-1}z) & \text{if } \lambda > 0, \\ h^\infty(z) & \text{if } \lambda = 0, \\ +\infty & \text{if } \lambda < 0. \end{cases}$$

If h is additionally closed and proper, then so are h^* (Theorem 12.2), h^∞ (Theorem 8.5), and h^π (p.35, Corollaries 8.5.2 and 13.5.1), where the results are all from Rockafellar [28]. Note that these functions can also be defined by considering the epigraphical perspective and properties of convex sets. The conjugate function can be derived from the support function of the epigraph. The epigraph of the horizon function of h is the recession cone of the epigraph of h . The perspective function of h is the positively homogeneous function generated by the convex function $\hat{h}(x, \lambda) := h(x) + \delta(\lambda \mid \{1\})$ [28, pp. 35 and 67].

Note that for every closed proper and convex function h , the associated horizon and perspective function, h^∞ and h^π , are positively homogeneous and so can be represented as the support functional for some convex set [28, Theorem 13.2]. Moreover, if h is a support functional, then $h^\infty = h^\pi = h$.

3.2. Cones. We associate the following cones with a convex set C and a convex function h .

1. *Polar cone:* The polar cone of C is denoted by C° :

$$C^\circ := \{ x^* \mid \langle x^*, x \rangle \leq 0 \ \forall x \in C \}.$$

2. *Recession cone:* The recession cone of C is denoted by C^∞ :

$$C^\infty := \{ x \mid C + x \subset C \} = \{ x \mid y + \lambda x \in C \ \forall \lambda \geq 0, \forall y \in C \}.$$

3. *Barrier cone:* The barrier cone of C is denoted by $\text{bar}(C)$:

$$\text{bar}(C) := \{ x^* \mid \text{for some } \beta \in \mathbb{R}, \langle x, x^* \rangle \leq \beta \ \forall x \in C \}.$$

4. *Horizon cone:* The horizon cone [28, Theorem 8.7] of h is denoted by

$$\text{hzn}(h) := \{ y \mid h^\infty(y) \leq 0 \} = [\text{lev}_h(\tau)]^\infty \ \forall \tau > \inf h.$$

3.3. Calculus rules. The conjugate, horizon, and perspective transformations in section 3.1 possess a rich calculus. We use this calculus to obtain explicit expressions for the functions ρ^* , ϕ^* , $(\phi^*)^\infty$ and $(\phi^*)^\pi$ that play a crucial role in the applications of section 6. The calculus for conjugates and horizons is developed in many references, including [28]. We extend these results to the perspective transformation in a straightforward way. Below we review three convex operations and give the associated calculus rules.

Affine composition. Let $p : \mathbb{R}^m \rightarrow \overline{\mathbb{R}}$ be a closed proper and convex function, $A \in \mathbb{R}^{m \times n}$, and $b \in \mathbb{R}^m$, such that $(\text{Ran}(A) - b) \cap \text{ri}(\text{dom } p) \neq \emptyset$. Let

$$h(x) := p(Ax - b).$$

Then

$$\begin{aligned} h^\infty(z) &= p^\infty(Az), \\ h^*(y) &= \inf_{A^T u = y} [\langle b, u \rangle + p^*(u)], \\ h^\pi(x, \lambda) &= p^\pi(Ax - \lambda b, \lambda), \end{aligned}$$

where, for $\lambda = 0$,

$$p^\pi(Ax, 0) = p^\infty(Ax).$$

All three functions are closed proper and convex.

Inverse linear image. (Proof is in section 8.) Let $p : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ be closed proper and convex, and let $A \in \mathbb{R}^{m \times n}$. Define the inverse linear image of p under A to be

$$h(w) = \inf_{Ax=w} p(x).$$

Then $h^*(y) = p^*(A^T y)$, and if $(A^T)^{-1} \text{ri}(\text{dom } p^*) \neq \emptyset$, then

$$h^\infty(z) = \inf_{Ax=z} p^\infty(x) \quad \text{and} \quad h^\pi(w, \lambda) = \inf_{Ax=w} p^\pi(x, \lambda),$$

where all of the functions h , h^* , h^∞ , and h^π are closed proper and convex.

Addition. Let $h_i : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$, for $i = 1, \dots, m$, be closed proper convex functions. If $h := h_1 + \dots + h_m$ is not identically $+\infty$, then

$$h^\infty = h_1^\infty + \dots + h_m^\infty \quad \text{and} \quad h^\pi = h_1^\pi + \dots + h_m^\pi,$$

where both are closed proper and convex. Moreover,

$$\text{if } \bigcap_{i=1}^m \text{ri}(\text{dom } h_i) \neq \emptyset, \quad \text{then } h^* = h_1^* \nabla \dots \nabla h_m^*$$

is closed proper and convex, where ∇ denotes infimal-convolution.

Infimal convolution. Let $h_i : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$, for $i = 1, \dots, m$, be closed proper and convex functions. Set $h = h_1 \nabla \dots \nabla h_m$. Then $h^* = h_1^* + \dots + h_m^*$, and

$$\text{if } \bigcap_{i=1}^m \text{ri}(\text{dom } h_i^*) \neq \emptyset, \quad \text{then } h^\infty = h_1^\infty \nabla \dots \nabla h_m^\infty,$$

and

$$h^\pi(x, \lambda) = \inf_{\sum_{i=1}^m x_i = x} [h_1^\pi(x_1, \lambda) + \dots + h_m^\pi(x_m, \lambda)].$$

All three functions are closed proper and convex.

The results for both the horizon and conjugation transformations can be found in many references [9, 22, 28, 29]. Properties for the perspective transformation are less well documented. Since addition is a special case of affine composition, and infimal convolution is a special case of inverse linear image, we need only establish the perspective calculus formulas for affine composition and the inverse linear image. The affine-composition formula follows from [28, Theorem 9.5] and the definition of the perspective transformation, and the inverse linear image is established in section 8.

4. The dual problem. For our analysis, it is convenient to consider the (equivalent) epigraphical formulation

$$v(b, \tau) = \underset{x}{\text{minimize}} \ f(x, b, \tau) \quad (\mathcal{P})$$

of $P(b, \tau)$, where

$$f(x, b, \tau) := \rho(b - Ax) + \delta((x, \tau) \mid \text{epi } \phi).$$

Because the functions ρ and ϕ are convex, it immediately follows that f is also convex. This fact gives the convexity of the value function v , since it is the inf-projection of the objective function in x [28, Theorem 5.3].

We make use the duality framework described by Rockafellar and Wets [29, Chapter 11, Section H], and associate \mathcal{P} with its dual problem and its corresponding value function

$$\hat{v}(b, \tau) = \underset{u, \mu}{\text{maximize}} \ \langle b, u \rangle + \tau\mu - f^*(0, u, \mu). \quad (\mathcal{D})$$

This dual problem is the key to understanding the variational behavior of the value function. To access these results we must compute the conjugate of f . For this it is useful to have an alternative representation for the support function of the epigraph, which is the conjugate of the indicator function appearing in f .

4.1. Reduced dual problem. In Theorem 4.2 of this section, we derive an equivalent representation of the dual problem \mathcal{D} in terms of u alone. This is the *reduced* dual problem for \mathcal{P} . We first present a result about conjugates for epigraphs and lower level sets.

LEMMA 4.1 (Conjugates for epigraphs and lower level sets). *Let $h : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ be closed proper and convex. Then*

$$\delta^*((y, \mu) \mid \text{epi } h) = (h^*)^\pi(y, -\mu), \quad (4.1)$$

$$\delta^*(y \mid \text{lev}_h(\tau)) = \text{cl} \left(\inf_{\mu \geq 0} [\tau\mu + (h^*)^\pi(y, \mu)] \right). \quad (4.2)$$

Expressions (4.2) and (4.1) are established in [28, Theorem 13.5] and [28, Corollary 13.5.1], respectively, where it is shown that (4.1) is a consequence of (4.2). In section 8 we provide a different proof where it is shown that (4.2) follows from (4.1). The arguments provided in this proof are instructive for later computations.

The conjugate $f^*(y, u, \mu)$ of the perturbation function $f(x, b, \tau)$ defined in \mathcal{P} is now easily computed:

$$\begin{aligned} f^*(y, u, \mu) &= \sup_{x, b, \tau} [\langle y, x \rangle + \langle u, b \rangle + \mu\tau - \rho(b - Ax) - \delta((x, \tau) \mid \text{epi } \phi)] \\ &= \sup_{x, w, \tau} [\langle y, x \rangle + \langle u, w + Ax \rangle + \mu\tau - \rho(w) - \delta((x, \tau) \mid \text{epi } \phi)] \\ &= \sup_{x, \tau} \left[\left\langle y + A^T u, x \right\rangle + \mu\tau - \delta((x, \tau) \mid \text{epi } \phi) \right] + \sup_w [\langle u, w \rangle - \rho(w)] \\ &= (\phi^*)^\pi(y + A^T u, -\mu) + \rho^*(u), \end{aligned} \quad (4.3)$$

where the final equality follows from (4.1). With this representation of the conjugate of f , we obtain the following equivalent representations for the dual problem \mathcal{D} .

THEOREM 4.2 (Dual representations). *For problem \mathcal{P} define the functions*

$$\begin{aligned} g_\tau(u) &:= \rho^*(u) + \delta^*\left(A^T u \mid \text{lev}_\tau(\phi)\right) \\ p_\tau(s, \mu) &:= \tau\mu + (\phi^*)^\tau(s, \mu). \end{aligned}$$

Then the value function for \mathcal{D} has the following equivalent characterizations:

$$\begin{aligned} \hat{v}(b, \tau) &= \sup_u \left[\langle b, u \rangle - \rho^*(u) - \inf_{\mu \geq 0} p_\tau(A^T u, \mu) \right] \\ &= \sup_u \left[\langle b, u \rangle - \rho^*(u) - \delta^*\left(A^T u \mid \text{lev}_\tau(\phi)\right) \right] & (\mathcal{D}_\tau) \\ &= g_\tau^*(b) & (4.4a) \\ &= \text{cl}(v(\cdot, \tau))(b), & (4.4b) \end{aligned}$$

where the closure operation in the (4.4b) refers to the lower semi-continuous hull of the convex function $b \mapsto v(b, \tau)$. In particular, this implies the weak duality inequality $\hat{v}(b, \tau) \leq v(b, \tau)$. Moreover, if the function ρ is differentiable, the solution u to \mathcal{D}_τ is unique.

One implication of these characterizations is that a solution $(\bar{u}, \bar{\mu})$ to the full dual \mathcal{D} can be obtained in two stages: solve \mathcal{D}_τ for \bar{u} , then solve $\inf_{\mu \geq 0} p_\tau(A^T \bar{u}, \mu)$ for $\bar{\mu}$. Indeed, in some cases there is a closed form expression for the solution. This is the case in the examples of section 6. This motivates the following result, which provides the basis for the extended Lagrange multiplier theory.

LEMMA 4.3. *Let ϕ be as in \mathcal{P} with $\tau > \inf \phi$ and $\bar{x} \in \text{lev}_\phi(\tau) \cap \text{dom } \partial\phi$.*

1. *For every s , we have*

$$\delta^*(s \mid \text{lev}_\phi(\tau)) \leq \inf_{\mu \geq 0} p_\tau(s, \mu). \quad (4.5)$$

2. *Let (\bar{x}, \bar{s}) satisfy $\bar{s} \in N(\bar{x} \mid \text{lev}_\phi(\tau))$ and define*

$$S_1 = \arg \min_{\mu \geq 0} p_\tau(\bar{s}, \mu) \quad \text{and} \quad S_2 = \left\{ \bar{\mu} \geq 0 \mid \begin{array}{l} \bar{s} \in \bar{\mu}^+ \partial\phi(\bar{x}) \\ 0 = \bar{\mu}(\phi(\bar{x}) - \tau) \end{array} \right\},$$

where, for $x \in \text{dom } \phi$,

$$\mu^+ \partial\phi(x) := \begin{cases} \{ \mu z \mid z \in \partial\phi(x) \} & \text{if } \mu > 0 \text{ and } x \in \text{dom } \partial\phi, \\ N(x \mid \text{dom } \phi) & \text{if } \mu = 0. \end{cases}$$

If either S_1 or S_2 is non-empty, then $S_1 = S_2$ and equality holds in (4.5).

The second part of the definition for $\mu^+ \partial\phi(x)$ is motivated by two observations. First, by [28, Theorem 8.2], for any non-empty closed convex set C we have $\lim_{\mu \downarrow 0} \mu C =$

C^∞ . Second, it is easily shown that if $\partial\phi(x) \neq \emptyset$, then $\partial\phi(x)^\infty = N(x \mid \text{dom } \phi)$.

LEMMA 4.4 (Coercivity of primal and dual objectives).

1. The objective function $f(\cdot, b, \tau)$ of \mathcal{P} is coercive if and only if

$$\text{hzn}(\phi) \cap [-A^{-1}\text{hzn}(\rho)] = \{0\}. \quad (4.6)$$

2. The objective function of the reduced dual \mathcal{D}_r is coercive if and only if

$$b \in \text{int}(\text{dom } \rho + A\text{lev}_\phi(\tau)). \quad (4.7)$$

5. Variational properties of the value function. With \mathcal{D} and representation of the conjugate of the objective of \mathcal{P} (cf. (4.3)), we can specialize [29, Theorem 11.39] to obtain a characterization of the subdifferential of the value function, as well as sufficient conditions for strong duality.

THEOREM 5.1 (Strong duality and subgradient of the value function). *Let v and \hat{v} be as in \mathcal{P} and \mathcal{D} , respectively. It is always the case that*

$$v(b, \tau) \geq \hat{v}(b, \tau) \quad (\text{weak duality}).$$

If $(b, \tau) \in \text{int}(\text{dom } v)$, then

$$v(b, \tau) = \hat{v}(b, \tau) \quad (\text{strong duality})$$

and $\partial v(b, \tau) \neq \emptyset$ with

$$\partial v(b, \tau) := \arg \max_{u, \mu \geq 0} [\langle b, u \rangle - \rho^*(u) - p_\tau(A^T u, -\mu)].$$

Furthermore, for fixed $(b, \tau) \in \mathbb{R}^m \times \mathbb{R}$,

$$\text{dom } f(\cdot, b, \tau) \neq \emptyset \iff b \in \text{dom } \rho + A(\text{lev}_\phi(\tau)).$$

In particular, this implies that

$$(b, \tau) \in \text{int}(\text{dom } v) \iff b \in \text{int}(\text{dom } \rho + A(\text{lev}_\phi(\tau))).$$

We now derive a characterization of the subdifferential $\partial v(b, \tau)$ based on the solutions of the reduced dual \mathcal{D}_r .

THEOREM 5.2 (Value function subdifferential). *Suppose that*

$$b \in \text{ri}(\text{dom } \rho) + A \text{ri}(\text{lev}_\phi(\tau)) \quad (5.1a)$$

$$\text{ri}(\text{dom } \rho^*) \cap [A^{-T} \text{ri}(\text{bar}(\text{lev}_\phi(\tau)))] \neq \emptyset. \quad (5.1b)$$

1. If the pair (\bar{x}, \bar{u}) satisfies

$$\bar{x} \in \text{lev}_\phi(\tau), \quad \bar{u} \in \partial\rho(b - A\bar{x}) \text{ and } A^T\bar{u} \in N(\bar{x} \mid \text{lev}_\phi(\tau)), \quad (5.1c)$$

then \bar{x} solves \mathcal{P} and \bar{u} solves \mathcal{D}_r .

2. If \bar{x} solves \mathcal{P} and (5.1a) holds, there exists \bar{u} such that (\bar{x}, \bar{u}) satisfies (5.1c).

3. If \bar{u} solves \mathcal{D}_r and (5.1b) holds, there exists \bar{x} such that (\bar{x}, \bar{u}) satisfies (5.1c).

4. If either (4.6) and (5.1a) holds, or (4.7) and (5.1b) holds, then $\partial v(b, \tau) \neq \emptyset$ and $\arg \min_{\mu \geq 0} p_\tau(A^T\bar{u}, \mu) \neq \emptyset$ for all $(\bar{x}, \bar{u}) \in \mathbb{R}^n \times \mathbb{R}^m$ satisfying (5.1c) with

$$\partial v(b, \tau) = \left\{ \begin{pmatrix} \bar{u} \\ -\bar{\mu} \end{pmatrix} \mid \begin{array}{l} (\bar{x}, \bar{u}) \in \mathbb{R}^n \times \mathbb{R}^m \text{ satisfy (5.1c) and} \\ \bar{\mu} \in \arg \min_{\mu \geq 0} p_\tau(A^T\bar{u}, \mu) \end{array} \right\} \quad (5.1d)$$

$$= \left\{ \begin{pmatrix} \bar{u} \\ -\bar{\mu} \end{pmatrix} \mid \begin{array}{l} \exists \bar{x} \in \text{lev}_\phi(\tau) \text{ s.t. } 0 \in -A^T\bar{u} + \bar{\mu}^+ \partial\phi(\bar{x}) \\ \text{where } \bar{u} \in \partial\rho(b - A\bar{x}), \\ \bar{\mu} \geq 0, \text{ and } \bar{\mu}(\phi(\bar{x}) - \tau) = 0 \end{array} \right\}. \quad (5.1e)$$

The representation (5.1e) expresses the elements of $\partial v(b, \tau)$ in terms of classical Lagrange multipliers when $\bar{\mu} > 0$, and extends the classical theory when $\bar{\mu} = 0$. (See Lemma 4.3 for the definition of $\mu^+ \partial\phi(x)$.) Because v is convex, it is subdifferentially regular, and so for fixed b , we can obtain the subdifferential of v with respect to τ alone [29, Corollary 10.11], i.e.,

$$\partial_\tau v(b, \tau) = \left\{ \omega \mid \begin{pmatrix} u \\ \omega \end{pmatrix} \in \partial v(b, \tau) \right\}.$$

6. Applications. In this section we apply the calculus rules of section 3.3 in conjunction with Theorem 5.2 to evaluate the subdifferential of the value function in three important special cases: where ϕ is a gauge-plus-indicator, a restricted quadratic penalty (RQP) function, and an affine composition with an RQP function. In all cases we allow ρ to be an arbitrary convex function.

6.1. Gauge-plus-indicator. The case where ρ is a linear least-squares objective and ϕ is a gauge function is studied in [8]. We generalize this case by allowing ρ to be a possibly non-smooth and non-finite-valued convex function, and take

$$\phi(x) := \gamma(x \mid U) + \delta(x \mid X), \quad (6.1)$$

where U is a nonempty closed convex set containing the origin. Note that ϕ is a gauge if and only if X is a convex cone.

Observe that the requirement that $x \in X$ is unaffected by varying τ in the constraint $\phi(x) \leq \tau$. Indeed, the problem \mathcal{P} is unchanged if we replace ρ and ϕ by

$$\hat{\rho}(y, x) := \rho(y) + \delta(x \mid X) \quad \text{and} \quad \hat{\phi}(x) := \gamma(x \mid U), \quad (6.2)$$

with A and b replaced by

$$\hat{b} := \begin{pmatrix} b \\ 0 \end{pmatrix} \quad \text{and} \quad \hat{A} := \begin{bmatrix} A \\ -I \end{bmatrix}.$$

Hence, the generalization of [8] discussed here only concerns the application to more general convex functions ρ .

There are two ways one can proceed with this application. One can use ϕ as given in (6.1) or use $\hat{\rho}$ and $\hat{\phi}$ as defined in (6.2). We choose the former in order to highlight the presence of the abstract constraint $x \in X$. But we emphasize—regardless of the formulation chosen—that the end result is the same.

LEMMA 6.1. *Let ϕ be as given in (6.1). The following formulas hold:*

$$\gamma(\cdot | U) = \delta^*(\cdot | U^\circ), \quad (6.3a)$$

$$\text{dom } \gamma(\cdot | U) = \text{cone}(U) = \text{bar}(U^\circ), \quad (6.3b)$$

$$\text{dom } \phi = \text{cone}(U) \cap X, \quad (6.3c)$$

$$\text{lev}_\phi(\tau) = (\tau U) \cap X, \quad (6.3d)$$

$$\text{hzn}(\phi) = U^\infty \cap X^\infty, \text{ and} \quad (6.3e)$$

$$\text{cl}(\text{bar}(\text{lev}_\phi(\tau))) = \text{cl}(\text{bar}(U)) + \text{cl}(\text{bar}(X)). \quad (6.3f)$$

If it is further assumed that

$$\text{ri}(\tau U) \cap \text{ri}(X) \neq \emptyset, \quad (6.4)$$

then we also have

$$\phi^*(z) = \min_s [\delta^*(z - s | X) + \delta(s | U^\circ)], \quad (6.5a)$$

$$(\phi^*)^\pi(z, \mu) = \min_s [\delta^*(z - s | X) + \delta(s | \mu U^\circ)], \quad (6.5b)$$

$$\delta^*(z | \text{lev}_\phi(\tau)) = \min_{\mu \geq 0} [\tau \mu + (\phi^*)^\pi(z, \mu)] \quad (6.5c)$$

$$= \min_s [\delta^*(z - s | X) + \tau \gamma(s | U^\circ)], \text{ and} \quad (6.5d)$$

$$N(x | \text{lev}_\phi(\tau)) = N(x | \tau U) + N(x | X). \quad (6.5e)$$

If \bar{s} minimizes (6.5d), then $\bar{\mu} := \gamma(\bar{s} | U^\circ)$ minimizes (6.5c).

By Theorem 5.1, the subdifferential of $v(b, \tau)$ is obtained by solving the dual problem (8.4) or the reduced dual \mathcal{D}_r . When ϕ is given by (6.1), the results of Lemma 6.1, show that the dual and the reduced dual take the form

$$\sup_{u, \mu} [\langle b, u \rangle + \tau \mu - (\phi^*)^\pi(A^T u, -\mu) - \rho^*(u)] \quad (6.6)$$

$$= \sup_u [\langle b, u \rangle - \rho^*(u) - \delta^*(A^T u | \text{lev}_\phi(\tau))]]$$

$$= \sup_u [\langle b, u \rangle - \rho^*(u) - \min_s [\delta^*(A^T u - s | X) + \tau \gamma(s | U^\circ)]]]$$

$$= \sup_{u, s} [\langle b, u \rangle - \rho^*(u) - \delta^*(A^T u - s | X) - \delta^*(s | \tau U)]. \quad (6.7)$$

Moreover, if (\bar{u}, \bar{s}) solves (6.7), then $(\bar{u}, \bar{\mu})$ solve (6.6) with $\bar{\mu} = -\gamma(\bar{s} | U^\circ)$, and

$$(\bar{u}, -\gamma(\bar{s} | U^\circ)) \in \partial v(b, \tau).$$

We have the following version of Theorem 5.2 when ϕ is given by (6.1).

THEOREM 6.2. *Let ϕ be given by (6.1) under the assumption that (6.4) holds, and consider the following two conditions:*

$$b \in \text{ri}(\text{dom } \rho + A[\tau U \cap X]) = \text{ri}(\text{dom } \rho) + A[\text{ri}(\tau U) \cap \text{ri}(X)] \quad (6.8)$$

and

$$\exists \hat{u} \in \text{ri}(\text{dom } \rho^*) \quad \text{such that} \quad A^T \hat{u} \in \text{ri}(\text{bar}(U)) + \text{ri}(\text{bar}(X)). \quad (6.9)$$

1. *If the triple $(\bar{x}, \bar{u}, \bar{s})$ satisfies*

$$\begin{aligned} \bar{u} &\in \partial \rho(b - A\bar{x}), & \bar{x} &\in X \cap (\tau U), \\ \bar{s} &\in N(\bar{x} | \tau U), & \text{and} & \quad A^T \bar{u} - \bar{s} \in N(\bar{x} | X), \end{aligned}$$

then \bar{x} solves $P(b, \tau)$ and (\bar{u}, \bar{s}) solves (6.7).

2. *If \bar{x} solves $P(b, \tau)$ and (6.8) holds, then there exists a pair (\bar{u}, \bar{s}) such that $(\bar{x}, \bar{u}, \bar{s})$ satisfies (6.10).*

3. *If (\bar{u}, \bar{s}) solves (6.7) and (6.9) holds, then there exists \bar{x} such that $(\bar{x}, \bar{u}, \bar{s})$ satisfies (6.10).*

4. *If either*

$$U^\infty \cap X^\infty \cap [-A^{-1} \text{hzn}(\rho)] = \{0\} \quad \text{and (6.8) holds,} \quad (6.11)$$

or

$$b \in \text{int}(\text{dom } \rho + A[\tau U \cap X]) \quad \text{and (6.9) holds,} \quad (6.12)$$

then $\partial v(b, \tau) \neq \emptyset$ and is given by

$$\begin{aligned} \partial v(b, \tau) &= \left\{ \left(\begin{array}{c} \bar{u} \\ -\gamma(\bar{s} | U^\circ) \end{array} \right) \left| \begin{array}{l} (\bar{x}, \bar{u}, \bar{s}) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \text{ satisfy (6.10)} \end{array} \right. \right\} \quad (6.13) \\ &= \left\{ \left(\begin{array}{c} \bar{u} \\ -\bar{\mu} \end{array} \right) \left| \begin{array}{l} \exists \bar{x} \in X \cap (\tau U) \text{ s.t.} \\ 0 \in -A^T \bar{u} + N(\bar{x} | X) + \bar{\mu}^+ \partial \gamma(\bar{x} | U) \text{ where} \\ \bar{u} \in \partial \rho(b - A\bar{x}), \quad 0 \leq \bar{\mu} \text{ and } \bar{\mu}(\gamma(\bar{x} | U) - \tau) = 0 \end{array} \right. \right\}. \end{aligned}$$

6.1.1. Gauge penalties. In [8], the authors study the case where ρ is a linear least-squares objective, ϕ is a gauge functional, and $X = \mathbb{R}^n$. In this case, [8, Lemma 2.1] and [8, Corollary 2.2(b)] can be deduced from (6.7) and (6.13), respectively. Another application is to the case where ρ is finite-valued and smooth, ϕ is a norm, and X is a generalized box. In this case, all of the conditions of Theorems 5.1 and 6.2 are satisfied, solutions to both $P(b, \tau)$ and (6.7) exist, and v is differentiable. In particular, consider the non-negative 1-norm-constrained inversion, where

$$\phi(x) = \|x\|_1 + \delta(x | \mathbb{R}_+^n),$$

and ρ is any differentiable convex function. The subdifferential characterization given in Theorem 5.1 can be explicitly computed via Theorem 6.2. In the notation of (6.1),

$$U = \{ x \mid \|x\|_\infty \leq 1 \} ,$$

the dual ball \mathbb{B}_∞ , and X in (6.1) is \mathbb{R}_+^n . Since the function ρ is differentiable, the solution \bar{u} to the dual (6.7) is unique [28, Theorem 26.3]. Therefore, Theorem 6.2 gives the existence of a *unique* gradient

$$\nabla_b v(b, \tau) = -A^T \nabla \rho(b - A\bar{x}),$$

where \bar{x} is any solution that achieves the optimal value. The derivative with respect to τ is immediately given by Theorem 6.2 as

$$\nabla_\tau v(b, \tau) = -\gamma \left(A^T \nabla \rho(b - A\bar{x}) \mid U^\circ \right) = -\|A^T \nabla \rho(b - A\bar{x})\|_\infty . \quad (6.14)$$

6.2. Restricted quadratic penalty functions. We now consider the case

$$\phi(x) := \sup_{w \in U} [\langle x, w \rangle - \frac{1}{2} \langle w, Bw \rangle] , \quad (6.15)$$

where $U \subset \mathbb{R}^n$ is nonempty closed and convex with $0 \in U$, and $B \in \mathbb{R}^{n \times n}$ is positive semi-definite. We call this class of functions restricted quadratic penalty (RQP) functions since their conjugates are given by

$$\phi^*(w) = \frac{1}{2} \langle w, Bw \rangle + \delta(w \mid U) . \quad (6.16)$$

If the set U is polyhedral convex, then the function ϕ is called a piecewise linear-quadratic penalty function [29, Example 11.18]. Since B is positive semi-definite there is a matrix $L \in \mathbb{R}^{n \times k}$ such that $B = LL^T$ where k is the rank of B . Using L , the calculus rules in section 3.3 give the following alternative representation for ϕ :

$$\begin{aligned} \phi(x) &= \sup_{w \in U} [\langle w, x \rangle - \frac{1}{2} \|L^T w\|_2^2 - \delta(w \mid U)] \\ &= \inf_{x_1 + x_2 = x} \left[\delta^*(x_1 \mid U) + \inf_{Ls = x_2} \frac{1}{2} \|s\|_2^2 \right] \\ &= \inf_s \left[\frac{1}{2} \|s\|_2^2 + \delta^*(x - Ls \mid U) \right] \\ &= \inf_s \left[\frac{1}{2} \|s\|_2^2 + \gamma(x - Ls \mid U^\circ) \right] , \end{aligned} \quad (6.17)$$

where the final equality follows from [28, Theorem 14.5] since $0 \in U$. Note that the function class (6.15) includes all gauge functionals for sets containing the origin. By (6.16), it easily follows that

$$(\phi^*)^\pi(w, \mu) = \begin{cases} \frac{1}{2\mu} \|w\|_B^2 + \delta(w \mid \mu U) & \text{if } \mu > 0, \\ \delta(w \mid U^\infty \cap \text{Nul}(B)) & \text{if } \mu = 0, \\ +\infty & \text{if } \mu < 0, \end{cases}$$

where $\|\cdot\|_B$ denotes the seminorm induced by B , i.e.,

$$\|w\|_B := \sqrt{w^T B w} .$$

The next result catalogues important properties of the function ϕ given in (6.15).

LEMMA 6.3. *Let ϕ be given by (6.15) with $\tau > 0$. Then*

$$\begin{aligned} \text{dom } \phi &= \text{cone}(U^\circ) + \text{Ran}(B) \quad \text{and} \\ \text{hzn}(\phi) &= \text{cone}(U)^\circ, \end{aligned}$$

in particular, ϕ is coercive if and only if $0 \in \text{int}(U)$. Moreover,

$$\delta^*(w \mid \text{lev}_\phi(\tau)) = \min_{\lambda \geq 0} [\tau\lambda + (\phi^*)^\pi(w, \lambda)] \quad (6.18)$$

$$= \begin{cases} \tau\gamma(w \mid U) + \frac{\|w\|_B^2}{2\gamma(w \mid U)} & \text{if } \gamma(w \mid U) > \|w\|_B/\sqrt{2\tau}, \\ \sqrt{2\tau}\|w\|_B & \text{if } \gamma(w \mid U) \leq \|w\|_B/\sqrt{2\tau}, \end{cases} \quad (6.19)$$

where the minimizing λ in (6.18) is given by

$$\lambda = \max \left\{ \gamma(w \mid U), \frac{\|w\|_B}{\sqrt{2\tau}} \right\}. \quad (6.20)$$

In particular, the formula (6.19) implies that

$$\text{bar}(\text{lev}_\phi(\tau)) = \text{dom}(\delta^*(\cdot \mid \text{lev}_\phi(\tau))) = \text{dom}(\gamma(\cdot \mid U)) = \text{cone}(U).$$

We now apply Theorem 5.2 to the case where ϕ is given by (6.15).

THEOREM 6.4. *Let ϕ be given by (6.15), and consider the following two conditions:*

$$\exists \hat{x} \in \text{ri}(\text{dom } \phi) \quad \text{such that} \quad \phi(\hat{x}) < \tau \quad \text{and} \quad b - A\hat{x} \in \text{ri}(\text{dom } \rho) \quad (6.21)$$

and

$$\exists \hat{u} \in \text{ri}(\text{dom } \rho^*) \quad \text{such that} \quad A^T \hat{u} \in \text{ri}(\text{cone}(U)). \quad (6.22)$$

1. *If the pair (\bar{x}, \bar{u}) satisfy*

$$\bar{x} \in \text{lev}_\phi(\tau), \quad \bar{u} \in \partial\rho(b - A\bar{x}) \quad \text{and} \quad A^T \bar{u} \in N(\bar{x} \mid \text{lev}_\phi(\tau)), \quad (6.23)$$

then \bar{x} solves $P(b, \tau)$ and \bar{u} solves \mathcal{D}_r .

2. *If \bar{x} solves $P(b, \tau)$ and (6.21) holds, then there exists \bar{u} such that (6.23) holds.*

3. *If \bar{u} solves \mathcal{D}_r and (6.22) holds, then there exists \bar{x} such that (6.23) holds.*

4. *If either*

$$\text{cone}(U)^\circ \cap [-A^{-1}\text{hzn}(\rho)] = \{0\} \quad \text{and} \quad (6.21) \quad \text{holds,} \quad (6.24)$$

or

$$b \in \text{int}(\text{dom } \rho + A\text{lev}_\phi(\tau)) \quad \text{and} \quad (6.22) \quad \text{holds,} \quad (6.25)$$

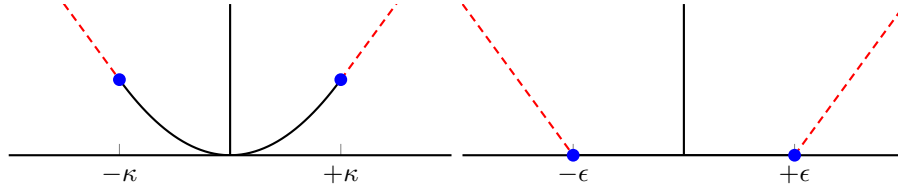


FIG. 6.1. Huber (left) and Vapnik (right) penalties

then $\partial v(b, \tau) \neq \emptyset$ and is given by

$$\begin{aligned} \partial v(b, \tau) &= \left\{ \begin{pmatrix} \bar{u} \\ -\bar{\mu} \end{pmatrix} \mid \begin{array}{l} \exists \bar{x} \text{ s.t. } (\bar{x}, \bar{u}) \text{ satisfy (6.23) and} \\ \bar{\mu} = \max \left\{ \gamma \left(A^T \bar{u} \mid U \right), \|A^T \bar{u}\|_B / \sqrt{2\tau} \right\} \end{array} \right\} \\ &= \left\{ \begin{pmatrix} \bar{u} \\ -\bar{\mu} \end{pmatrix} \mid \begin{array}{l} \exists \bar{x} \in \text{lev}_\phi(\tau) \text{ s.t. } 0 \in -A^T \bar{u} + \bar{\mu}^+ \partial \phi(\bar{x}) \text{ where} \\ \bar{u} \in \partial \rho(b - A\bar{x}), 0 \leq \bar{\mu} \text{ and } \bar{\mu}(\phi(\bar{x}) - \tau) = 0 \end{array} \right\}. \end{aligned} \quad (6.26)$$

In the following corollary we exploit the structure of ϕ to refine the multiplier description of the $\partial v(b, \tau)$ given in (6.26).

COROLLARY 6.5. Consider the problem $P(b, \tau)$ with ϕ given by (6.15). A pair (\bar{x}, \bar{u}) satisfies (6.23) if and only if $\bar{x} \in \text{lev}_\phi(\tau)$, $\bar{u} \in \partial \rho(b - A\bar{x})$, and either

$$A^T \bar{u} \in N(\bar{x} \mid \text{dom } \phi), \quad \text{or} \quad (6.27a)$$

$$\exists \bar{\mu} > 0, \bar{w} \in U \quad \text{such that} \quad \bar{x} \in B\bar{w} + N(\bar{w} \mid U) \text{ and } A^T \bar{u} = \bar{\mu} \bar{w}. \quad (6.27b)$$

6.2.1. Huber penalty. A popular function in the PLQ class is the Huber penalty [23]:

$$\phi(x) = \sup_{w \in [-\kappa, \kappa]^n} \left[\langle x, w \rangle - \frac{1}{2} \|w\|_2^2 \right] = \sum_{i=1}^n \phi_i(x_i); \quad \phi_i(x_i) := \begin{cases} x_i^2, & \text{if } x_i \leq \kappa, \\ \kappa |x_i| - \kappa^2/2, & \text{otherwise.} \end{cases}$$

The Huber function is of form (6.15), with $B = I$ and $U = [-\kappa, \kappa]^n$. In this case, $U^\infty \cap \text{Nul}(B) = \{0\}$ so that the conditions of Corollary 6.5 hold.

A graph of the scalar component function ϕ_i is shown in Figure 6.1. The Huber penalty is robust to outliers, since it increases linearly rather than quadratically. For any misfit function ρ , Theorem 6.4 can be used to easily compute the subgradient $\partial v(b, \tau)$ of the value function. If the regularity condition (6.21) is satisfied (e.g., if ρ is finite valued), then Theorem 6.4 implies that

$$\partial v(b, \tau) = \left\{ \begin{pmatrix} \bar{u} \\ -\bar{\mu} \end{pmatrix} \mid \begin{array}{l} (\bar{x}, \bar{u}) \text{ satisfy (6.23) and} \\ \bar{\mu} = \max \left\{ \kappa \|A^T \bar{u}\|_\infty, \|A^T \bar{u}\|_2 / \sqrt{2\tau} \right\} \end{array} \right\}.$$

In particular, if ρ is differentiable finite-valued, $\bar{u} = \nabla \rho(b - A\bar{x})$ is unique and

$$\nabla v(b, \tau) = \begin{pmatrix} \bar{u} \\ -\bar{\mu} \end{pmatrix}.$$

6.3. Affine composition with RQP functions. Next consider the case where ϕ takes the form

$$\phi(x) := \psi(Hx + c), \quad \text{where} \quad \psi(y) := \sup_{w \in U} [\langle y, w \rangle - \frac{1}{2} \langle w, Bw \rangle], \quad (6.28)$$

$H \in \mathbb{R}^{\nu \times n}$ is injective, $c \in \mathbb{R}^\nu$, $U \subset \mathbb{R}^\nu$ is nonempty closed and convex with $0 \in U$ and $B \in \mathbb{R}^{\nu \times \nu}$ is symmetric and positive semi-definite. We assume that

$$\exists \hat{x} \text{ such that } H\hat{x} + c \in \text{ri}(\text{dom } \psi),$$

where $\text{dom } \psi = \text{cone}(U^\circ) + \text{Ran}(B)$ (Lemma 6.3). We show that the function ϕ in (6.28) is an instance of the piecewise linear quadratic function considered in section 6.2. To see this we make the following definitions:

$$\begin{aligned} \tilde{x} &= \begin{pmatrix} x \\ s \end{pmatrix}, \quad \tilde{y} = \begin{pmatrix} y \\ z \end{pmatrix}, \quad \tilde{w} = \begin{pmatrix} v \\ w \end{pmatrix}, \quad \tilde{U} = \{0\} \times U, \\ \tilde{b} &= \begin{pmatrix} b \\ c \end{pmatrix}, \quad \tilde{A} = \begin{bmatrix} A & 0 \\ -H & I \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix}, \quad \tilde{\rho} \begin{pmatrix} y \\ z \end{pmatrix} = \rho(y) + \delta(z \mid \{0\}), \quad \text{and} \\ \tilde{\phi} \begin{pmatrix} x \\ s \end{pmatrix} &= \sup_{\substack{(v) \\ (w)} \in \tilde{U}} \left[\left\langle \begin{pmatrix} v \\ w \end{pmatrix}, \begin{pmatrix} x \\ s \end{pmatrix} \right\rangle - \frac{1}{2} \left\langle \begin{pmatrix} v \\ w \end{pmatrix}, \tilde{B} \begin{pmatrix} v \\ w \end{pmatrix} \right\rangle \right] = \delta^*(x \mid \{0\}) + \psi(s). \end{aligned}$$

With these definitions, the two problems $P(b, \tau)$ and

$$\text{minimize} \quad \hat{\rho}(\tilde{b} - \tilde{A}\tilde{x}) \quad \text{subject to} \quad \tilde{\phi}(\tilde{x}) \leq \tau$$

are equivalent. In addition, we have the relationships

$$\begin{aligned} \tilde{\rho}^* \begin{pmatrix} u \\ r \end{pmatrix} &= \rho^*(u) + \delta^*(r \mid \{0\}), \quad \tilde{\phi}^* \begin{pmatrix} v \\ w \end{pmatrix} = \delta(v \mid \{0\}) + \psi^*(w), \\ \gamma \left(\begin{pmatrix} v \\ w \end{pmatrix} \mid \tilde{U} \right) &= \delta(v \mid \{0\}) + \gamma(w \mid U), \quad \text{and} \quad \left\| \begin{pmatrix} v \\ w \end{pmatrix} \right\|_{\tilde{B}} = \delta^*(v \mid \{0\}) + \|w\|_B. \end{aligned}$$

Moreover, the reduced dual \mathcal{D}_r becomes

$$\sup_{H^T r = A^T u} [\langle b, u \rangle + \langle c, r \rangle - \rho^*(u) - \delta^*(r \mid \text{lev}_\psi(\tau))]. \quad (6.29)$$

Using standard methods of convex analysis, we obtain the following result as a direct consequence of Theorem 6.4 and [29, Corollary 10.11].

THEOREM 6.6. *Let ϕ be given by (6.28), and consider the following two conditions:*

$$\exists \hat{x} \text{ such that } H\hat{x} + c \in \text{ri}(\text{dom } \psi), \psi(H\hat{x} + c) < \tau, \text{ and } b - A\hat{x} \in \text{ri}(\text{dom } \rho) \quad (6.30)$$

and

$$\exists \hat{u} \in \text{ri}(\text{dom } \rho^*) \text{ and } \hat{r} \in \text{ri}(\text{cone}(U)) \text{ such that } \begin{pmatrix} \hat{u} \\ \hat{r} \end{pmatrix} \in \text{Nul} \left(\begin{bmatrix} A \\ -H \end{bmatrix}^T \right). \quad (6.31)$$

1. *If the triple $(\bar{x}, \bar{u}, \bar{r})$ satisfies*

$$\bar{x} \in \text{lev}_\phi(\tau), \bar{u} \in \partial\rho(b - A\bar{x}), \bar{r} \in N(H\bar{x} + c \mid \text{lev}_\psi(\tau)) \text{ and } A^T\bar{u} = H^T\bar{r}, \quad (6.32)$$

then \bar{x} solves $P(b, \tau)$ and (\bar{u}, \bar{r}) solves (6.29).

2. *If \bar{x} solves $P(b, \tau)$ and (6.30) holds, there exists (\bar{u}, \bar{r}) such that (6.32) holds.*

3. *If (\bar{u}, \bar{r}) solves (6.29) and (6.31) holds, there exists \bar{x} such that (6.32) holds.*

4. *If either*

$$H^{-1}[\text{cone}(U)^\circ] \cap [-A^{-1}\text{hzn}(\rho)] = \{0\} \text{ and (6.30) holds,}$$

or

$$\begin{pmatrix} b \\ c \end{pmatrix} \in \text{int} \left(\text{dom } \rho \times \text{lev}_\psi(\tau) + \text{Ran} \left(\begin{bmatrix} A \\ -H \end{bmatrix} \right) \right) \text{ and (6.31) holds,}$$

then $\partial v(b, \tau) \neq \emptyset$ and is given by

$$\begin{aligned} \partial v(b, c, \tau) &= \left\{ \begin{pmatrix} \bar{u} \\ \bar{r} \\ -\bar{\mu} \end{pmatrix} \mid \begin{array}{l} \exists \bar{x} \in \mathbb{R}^n \text{ s.t. } (\bar{x}, \bar{u}, \bar{r}) \text{ satisfy (6.32) and} \\ \bar{\mu} = \max \{ \gamma(\bar{r} \mid U), \|\bar{r}\|_B / \sqrt{2\tau} \} \end{array} \right\} \\ &= \left\{ \begin{pmatrix} \bar{u} \\ \bar{r} \\ -\bar{\mu} \end{pmatrix} \mid \begin{array}{l} \exists \bar{x} \in \mathbb{R}^n \text{ s.t. } c + H\bar{x} \in \text{lev}_\psi(\tau), \\ \bar{u} \in \partial\rho(b - A\bar{x}), \bar{r} \in \bar{\mu}^+ \partial\psi(c + H\bar{x}), \bar{\mu} \geq 0, \\ \bar{\mu}(\psi(c + H\bar{x}) - \tau) = 0, \text{ and } A^T\bar{u} = H^T\bar{r} \end{array} \right\}. \end{aligned}$$

COROLLARY 6.7. *Consider the problem $P(b, \tau)$ with ϕ given by (6.28). Then $(\bar{x}, \bar{u}, \bar{r})$ satisfies (6.32) if and only if*

$$\begin{aligned} H\bar{x} + c \in \text{lev}_\psi(\tau), \quad \bar{u} \in \partial\rho(b - A\bar{x}), \quad A^T\bar{u} = H^T\bar{r}, \\ \text{and either } \bar{r} \in N(H\bar{x} + c \mid \text{dom } \psi), \quad \text{or} \\ \exists \bar{\mu} \geq 0, \bar{w} \in U \text{ such that } H\bar{x} + c \in B\bar{w} + N(\bar{w} \mid U) \text{ and } \bar{r} = \bar{\mu}\bar{w}. \end{aligned}$$

6.3.1. Vapnik penalty. The Vapnik penalty

$$\rho(r) = \sup_{u \in [0,1]^{2n}} \left\{ \left\langle \begin{bmatrix} r - \epsilon \\ -r - \epsilon \end{bmatrix}, u \right\rangle \right\} = (r - \epsilon)_+ + (-r - \epsilon)_+$$

is an important example in the extended PLQ class which cannot represent using the PLQ class of the previous section. The scalar version is shown in the right panel of Figure 6.1. In this case,

$$H = \begin{bmatrix} I \\ -I \end{bmatrix}, \quad c = - \begin{bmatrix} \epsilon \mathbf{1} \\ \epsilon \mathbf{1} \end{bmatrix}, \quad B = \mathbf{0} \in \mathbb{R}^{2n \times 2n}, \quad \text{and } U = [0, 1]^{2n}.$$

In order to satisfy (6.32), we need to find a triple $(\bar{x}, \bar{u}, \bar{w})$ with $\bar{w} = [\bar{w}_1 \ \bar{w}_2]^T \in [0, 1]^{2n}$ so that $\bar{u} \in \partial \rho(b - A\bar{x})$ and $A^T \bar{u} = H^T \bar{w} = \bar{w}_1 - \bar{w}_2$. We claim that either $\bar{w}_1(i) = 0$ or $\bar{w}_2(i) = 0$ for all i . To see this, observe that $\bar{w} \in N(H\bar{x} + c \mid \text{lev}_\psi(\tau))$, so

$$\left\langle \bar{w}, y - \begin{bmatrix} \bar{x} - \epsilon \\ -\bar{x} - \epsilon \end{bmatrix} \right\rangle \leq 0$$

whenever $\psi(y) \leq \tau$. Taking y first with $-\epsilon$ as the only non-zero in the i th coordinate, and then with $-\epsilon$ in the only nonzero in the $(n+i)$ th coordinate, we get

$$\bar{w}_1(i)(-\bar{x}(i)) \leq 0 \quad \text{and} \quad \bar{w}_2(i)(\bar{x}(i)) \leq 0.$$

If $x(i) < 0$, from the first equation we get $\bar{w}_1(i) = 0$, while if $x(i) > 0$, we get $\bar{w}_2(i) = 0$ from the second equation. If $x(i) = 0$, then taking $y = 0$ gives

$$\bar{w}_1(i)\epsilon \leq 0 \quad \text{and} \quad \bar{w}_2(i)\epsilon \leq 0,$$

so $\bar{w}_1(i) = \bar{w}_2(i) = 0$. Since $A^T \bar{u} = \bar{w}_1 - \bar{w}_2$, and $\bar{w}_1(i)$ or $\bar{w}_2(i)$ is 0 for each i , we get $\mu = \gamma(\bar{w} \mid [0, 1]^{2n}) = \|A^T \bar{u}\|_\infty$. Hence, the subdifferential $\partial \nu$ is computed in precisely the same way for the Vapnik regularization as for the 1-norm.

7. Numerical example: robust nonnegative BPDN. In this example, we recover a non-negative undersampled sparse signal from a set of very noisy measurements using several formulations of \mathcal{P} . We compare the performance of three different penalty functions ρ : least-squares, Huber (see section 6.2.1), and a nonconvex penalty arising from the Student's t distribution (see e.g., [3]). The regularizing function ϕ in all of the examples is the sum of the 1-norm and the indicator of the positive orthant (see section 6.1.1).

The formulations using Huber or Student's t misfits are robust alternatives to the nonnegative basis pursuit problem [13]. The Huber misfit agrees with the quadratic penalty for small residuals, but is relatively insensitive to larger residuals. The Student's t misfit is the negative likelihood of the Student's t distribution,

$$\rho_s(x) = \log(1 + x^2/\nu), \tag{7.1}$$

where ν is the degrees of freedom parameter.

For each penalty ρ , our aim is to solve the problem

$$\underset{x \geq 0}{\text{minimize}} \quad \|x\|_1 \quad \text{subject to} \quad \rho(b - Ax) \leq \sigma,$$

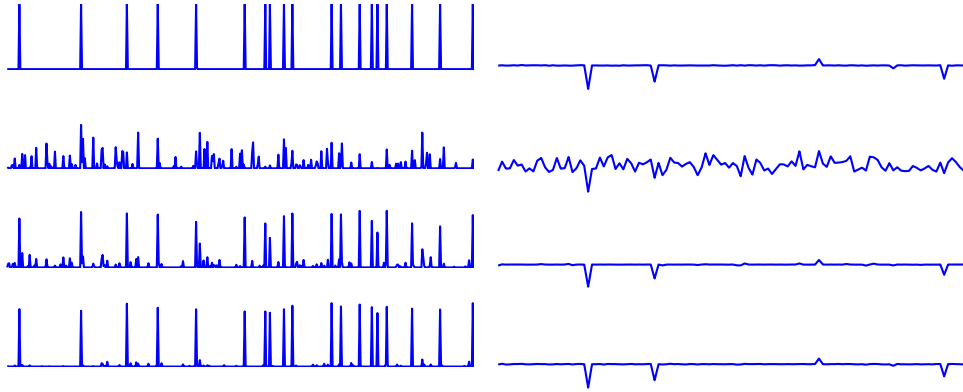


FIG. 7.1. *Left, top to bottom: True signal, and reconstructions via least-squares, Huber, and Student's t. Right, top to bottom: true errors, and least-squares, Huber, and Student's t residuals.*

via a series of approximate solution of \mathcal{P} . The 1-norm regularizer on x encourages a sparse solution. In particular, we solve the nonlinear equation (1.5), where v is the value function of \mathcal{P} . This is the approach used by the SPGL1 software package [8]; the underlying theory, however, does not cover the Huber function. Also, ϕ is not everywhere finite valued, which violates [8, Assumption 3.1]. Finally, the Student's t misfit (7.1) is nonconvex; however, the inverse function relationship (cf. Theorem 2.1) still holds, so we can achieve our goal, provided we can solve the root-finding problem.

Formula (6.14) computes the derivative of the value function associated with $P(b, \tau)$ for any convex differentiable ρ . The derivative requires $\nabla\rho$, evaluated at the optimal residual associated with $P(b, \tau)$. For the Huber case, this is given by

$$(\nabla\rho(b - A\bar{x}))_i = \text{sign}(b_i - A_i\bar{x}) \cdot \min(|b_i - A_i\bar{x}|, \kappa).$$

The Student's t misfit is also smooth, but nonconvex. Therefore, the formula (6.14) may still be applied—with the caveat that there is no guarantee of success. However, in all of the numerical experiments, we are able to find the root of (1.5).

We consider a common compressive sensing example: we want to recover a 20-sparse vector in \mathbb{R}_+^{512} from 120 measurements. We use a Gaussian measurement matrix $A \in \mathbb{R}^{100 \times 1024}$, by which we mean that each entry is sampled from the distribution $\mathbf{N}(0, \frac{1}{10})$. While measurements to test the BPDN formulation are typically generated by contaminating Ax with small Gaussian noise, we generate them according to

$$b = Ax + w + \zeta,$$

where $w \sim N(0, \sigma^2)$ with $\sigma = 0.005$ matches common practice, and ζ describes a small randomly placed set of 5 outliers, sampled from $N(0, 4)$. For each penalty ρ , the σ parameter is the true measure of the error in that penalty, i.e., $\sigma_\rho = \rho(\zeta)$. This allows a fair comparison between the penalties.

We expect the Huber function to out-perform the least squares penalty by budgeting the error level σ to allow a few large outliers, which will never happen with the quadratic. We expect the Student's t penalty to work even better, because it is non convex, and the grows sublinearly as outliers increase. The results in Figure 7.1 demonstrate that this is indeed the case. In many instances the Huber function is able to do just as well as the Student's t ; however, often the Student's

t does better (and never worse). Both robust penalties always do better than the least squares fit. The code is implemented in SPGL1, and can be downloaded from <https://github.com/saravkin/spgl1>. The particular experiment presented here can be found in `tests/spgl1TestNN.m`.

8. Appendix: Proofs of results.

Proof of Theorem 2.1. Let $\sigma \in \mathcal{S}_{1,2}$ and set $\tau_\sigma = v_1(\sigma)$. By assumption, $\arg \min \mathcal{P}_{1,2}(\sigma) \neq \emptyset$. Let $x_\sigma \in \arg \min \mathcal{P}_{1,2}(\sigma)$, so that $\psi_1(x_\sigma) = \tau_\sigma$ and $\psi_2(x_\sigma) = \sigma$. In particular, x_σ is feasible for $\mathcal{P}_{2,1}(\tau_\sigma)$. Let \hat{x} be any other feasible point for $\mathcal{P}_{2,1}(\tau_\sigma)$ so that $\psi_1(\hat{x}) \leq \tau_\sigma = v_1(\sigma) = \psi_1(x_\sigma)$. If $\psi_1(\hat{x}) < \tau_\sigma = v_1(\sigma)$, then $\psi_2(\hat{x}) > \sigma$ since otherwise we contradict the definition of $v_1(\sigma)$. If $\psi_1(\hat{x}) = \tau_\sigma$, then we claim that $\psi_2(\hat{x}) \geq \sigma$. Indeed, if $\psi_2(\hat{x}) < \sigma$, then $\hat{x} \in \arg \min \mathcal{P}_{1,2}(\sigma)$ but $\psi_2(\hat{x}) < \sigma$, which contradicts the fact that $\sigma \in \mathcal{S}_{1,2}$. Hence, every feasible point for $\mathcal{P}_{2,1}(\tau_\sigma)$ has $\psi_2(\hat{x}) \geq \sigma$ with equality only if $\psi_1(\hat{x}) = \tau_\sigma$. But x_σ is feasible for $\mathcal{P}_{2,1}(\tau_\sigma)$ with $\psi_2(x_\sigma) = \sigma$. Therefore, $x_\sigma \in \arg \min \mathcal{P}_{2,1}(\tau_\sigma) \subset \{x \in X \mid \psi_1(x) = \tau_\sigma\}$. Consequently, $v_2(v_1(\sigma)) = \sigma$ and

$$\emptyset \neq \arg \min \mathcal{P}_{1,2}(\sigma) \subset \arg \min \mathcal{P}_{2,1}(\tau_\sigma) \subset \{x \in X \mid \psi_1(x) = \tau_\sigma\}. \quad (8.1)$$

We now show that $\arg \min \mathcal{P}_{2,1}(\tau_\sigma) \subset \arg \min \mathcal{P}_{1,2}(\sigma)$. Let $\hat{x} \in \arg \min \mathcal{P}_{2,1}(\tau_\sigma)$. In particular, \hat{x} is feasible for $\mathcal{P}_{2,1}(\tau_\sigma)$, so, by what we have already shown, $\psi_2(\hat{x}) \geq \sigma$ with equality only if $\psi_1(\hat{x}) = \tau_\sigma$. But, by our choice of \hat{x} , $\psi_2(\hat{x}) = v_2(v_1(\sigma)) = \sigma$, so $\psi_1(\hat{x}) = \tau_\sigma$, i.e., $\hat{x} \in \arg \min \mathcal{P}_{1,2}(\sigma)$.

It remains to establish the final statement of the theorem. By (8.1), we already have that $\{v_1(\sigma) \mid \sigma \in \mathcal{S}_{1,2}\} \subset \mathcal{S}_{2,1}$, so we need only establish the reverse inclusion. For this, let $\tau \in \mathcal{S}_{2,1}$ and set $\sigma_\tau = v_2(\tau)$. By interchanging the indices and applying the first part of the theorem, we have from (8.1) that

$$\emptyset \neq \arg \min \mathcal{P}_{2,1}(\tau) \subset \arg \min \mathcal{P}_{1,2}(\sigma_\tau) \subset \{x \in X \mid \psi_2(x) = \sigma_\tau\}.$$

That is, $\sigma_\tau \in \mathcal{S}_{1,2}$ and, by (a), $\tau = v_1(v_2(\tau)) = v_1(\sigma_\tau)$.

Proof of the inverse linear image (section 3.3). For $\lambda > 0$, observe that

$$\begin{aligned} h^\pi(w, \lambda) &= \lambda \inf_{Ax = \lambda^{-1}w} p(x) \\ &= \lambda \inf_{As = w} p(\lambda^{-1}s) \quad (s := \lambda x) \\ &= \inf_{As = w} p^\pi(s, \lambda) \end{aligned} \quad (8.2)$$

$$= \inf \left\{ p^\pi(s, \zeta) \mid \hat{A} \begin{pmatrix} s \\ \zeta \end{pmatrix} = \begin{pmatrix} w \\ \lambda \end{pmatrix} \right\}, \quad (8.3)$$

where

$$\hat{A} = \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}.$$

Again by [28, Theorem 9.2] in conjunction with [28, Corollary 16.2.1], the function in (8.3) is closed if $(\hat{A}^T)^{-1} \text{dom}(p^\pi)^* \neq \emptyset$. Since, by [28, Corollary 13.5.1], $\text{dom}(p^\pi)^* = \{(u, \eta) \mid p^*(u) \leq -\eta\}$, we have

$$(\hat{A}^T)^{-1} \text{dom}(p^\pi)^* \neq \emptyset \quad \text{if and only if} \quad (A^T)^{-1} \text{dom} p^* \neq \emptyset.$$

Hence, by assumption, the function in (8.3) is closed proper and convex and equals $h^\pi(w, \lambda)$ on the relative interior of its domain. Since $h^\pi(w, \lambda)$ is closed, (8.2) implies that these functions must coincide.

Proof of Lemma 4.1. We first prove (4.1). The conjugate of $\delta((x, \tau) \mid \text{epi } h)$ is obtained as follows:

$$\begin{aligned}
\delta^*((y, \mu) \mid \text{epi } h) &= \sup_{x, \tau} [\langle y, x \rangle + \mu\tau - \delta((x, \tau) \mid \text{epi } h)] \\
&= \sup_{x \in \text{dom } h, \tau} [\langle y, x \rangle + \mu\tau - \delta(h(x) - \tau \mid \mathbb{R}_-)] \\
&= \sup_{x, \omega} [\langle y, x \rangle + \mu(h(x) - \omega) - \delta(\omega \mid \mathbb{R}_-)] \quad (\omega := h(x) - \tau) \\
&= \sup_x [\langle y, x \rangle + \mu h(x) + \sup_{\omega} [-\mu\omega - \delta(\omega \mid \mathbb{R}_-)]] \\
&= \sup_x [\langle y, x \rangle + \mu h(x) + \delta(\mu \mid \mathbb{R}_-)].
\end{aligned}$$

For $\mu < 0$, we obtain

$$\delta^*((y, \mu) \mid \text{epi } h) = -\mu \sup_x [\langle -\mu^{-1}y, x \rangle - h(x)] = -\mu h^*(-\mu^{-1}y).$$

Since h^* is necessarily a closed proper convex function, we obtain the result.

To see (4.2), first note that the function

$$q(y) := \inf_{\mu > 0} [\tau\mu + \mu h^*(y/\mu)] = \inf_{\mu \geq 0} [\tau\mu + (h^*)^\pi(y, \mu)]$$

is the positively homogeneous function generated by the function $y \mapsto \tau + h(y)$ [28, page 35], and so is convex in y . Next observe that the conjugate of q is given by

$$\begin{aligned}
q^*(x) &= \sup_y \left[\langle x, y \rangle - \inf_{\mu \geq 0} [\tau\mu + (h^*)^\pi(y, \mu)] \right] \\
&= \sup_{y, \mu \geq 0} [\langle x, y \rangle + \tau(-\mu) - (h^*)^\pi(y, \mu)] \\
&= \sup_{(y, \mu)} [\langle x, y \rangle + \tau\mu - (h^*)^\pi(y, -\mu)] \quad (\text{exchange } -\mu \text{ for } \mu) \\
&= \sup_{(y, \mu)} [\langle x, y \rangle + \tau\mu - \delta^*((y, \mu) \mid \text{epi } h)] \quad (\text{by (4.1)}) \\
&= \delta((x, \tau) \mid \text{epi } h) = \delta(x \mid \text{lev}_\phi(\tau)).
\end{aligned}$$

The result now follows from the Bi-Conjugate Theorem [28, Theorem 12.2].

Proof of Theorem 4.2. Combining \mathcal{D} with (4.2) and (4.3) gives

$$\begin{aligned}
\hat{v}(b, \tau) &:= \sup_{u, \mu} [\langle b, u \rangle + \tau\mu - (\phi^*)^\pi(A^T u, -\mu) - \rho^*(u)] \tag{8.4} \\
&= \sup_u \left[\langle b, u \rangle - \rho^*(u) - \inf_{\mu \leq 0} [\tau(-\mu) + (\phi^*)^\pi(A^T u, -\mu)] \right] \\
&= \sup_u \left[\langle b, u \rangle - \rho^*(u) - \inf_{\mu \geq 0} [\tau(\mu) + (\phi^*)^\pi(A^T u, \mu)] \right] \\
&= \sup_u \left[\langle b, u \rangle - \rho^*(u) - \delta^*(A^T u \mid \text{lev}_\phi(\tau)) \right],
\end{aligned}$$

where the final equality follows from (4.2). The equivalence (4.4a) follows from the definition of the conjugate and the equivalence (4.4b) follows from [28, Theorems 16.3

and 16.4] which tell us that

$$\begin{aligned}
 g_\tau^*(b) &= \text{cl} \left(\rho \nabla \left[\delta^* \left(A^T \cdot \mid \text{lev}_\tau(\phi) \right) \right]^* \right) (b) \\
 &= \text{cl} \left(\inf_{w_1+w_2=\cdot} \left[\rho(w_1) + \inf_{Ax=w_2} \delta(x \mid \text{lev}_\tau(\phi)) \right] \right) (b) \\
 &= \text{cl} \left(\inf_{\phi(x) \leq \tau} \rho(\cdot - Ax) \right) (b) \\
 &= \text{cl}(v(\cdot, \tau)) (b).
 \end{aligned}$$

The uniqueness of u when ρ is differentiable follows from the essential strict convexity of ρ^* [28, Theorem 26.3].

Proof of Lemma 4.3.

Part 1. The inequality follows immediately from (4.2). But it is also easily derived from the observation that if $\mu > 0$ and $x \in \text{lev}_\phi(\tau)$, then

$$\begin{aligned}
 \tau\mu + \mu\phi^*(s/\mu) &\geq \tau\mu + \mu[\langle x, s/\mu \rangle - \phi(x)] \quad [\text{Fenchel-Young inequality}] \\
 &\geq \phi(x)\mu + \langle x, s \rangle - \mu\phi(x) \\
 &= \langle x, s \rangle.
 \end{aligned}$$

Taking the sup over $x \in \text{lev}_\phi(\tau)$ gives the result.

Part 2. A key fact [28, Theorems 23.5 and 23.7] used repeatedly throughout our proof is that for any non-empty closed convex set U and $\bar{u} \in U$, we have

$$\bar{v} \in N(\bar{u} \mid U) \iff \bar{u} \in \partial\delta^*(\bar{v} \mid U) = \arg \max_{u \in U} \langle \bar{v}, u \rangle. \quad (8.5)$$

Also, from the Fenchel-Young inequality,

$$\tau + \phi^*(\bar{s}) \geq \phi(\bar{x}) + \phi^*(\bar{s}) \geq \langle \bar{s}, \bar{x} \rangle. \quad (8.6)$$

We divide this part of the proof into two parts: (i) if $S_1 \neq \emptyset$, show $S_1 \subset S_2$, and (ii) if $S_2 \neq \emptyset$, show $S_2 \subset S_1$ and equality holds in (4.5). Combined, these implications establish Part 2 of the lemma.

(i) Let $\bar{\mu} \in S_1$. We show that $\bar{\mu} \in S_2$. Let us first consider the case where $\phi(\bar{x}) < \tau$. We begin by showing that $T(\bar{x} \mid \text{lev}_\phi(\tau)) = T(\bar{x} \mid \text{dom } \phi)$. Clearly, $T(\bar{x} \mid \text{lev}_\phi(\tau)) \subseteq T(\bar{x} \mid \text{dom } \phi)$. To obtain the reverse inclusion, [29, Theorem 6.9] tells us that we need only show

$$\bigcup_{\lambda > 0} \lambda(\text{dom } \phi - \bar{x}) \subseteq \bigcup_{\lambda > 0} \lambda(\text{lev}_\phi(\tau) - \bar{x}).$$

Let $\lambda > 0$ and $x \in \text{dom } \phi$. We need to show that there is a $\sigma > 0$ and $z \in \text{lev}_\phi(\tau)$ such that $\sigma(z - \bar{x}) = \lambda(x - \bar{x})$. Observe that $\phi(\bar{x} + \eta(x - \bar{x})) \leq (1 - \eta)\phi(\bar{x}) + \eta\phi(x) < (1 - \eta)\tau + \eta\phi(x)$ for all $0 \leq \eta \leq 1$. Hence, there is an $\bar{\eta} > 0$ such that $z = \bar{x} + \bar{\eta}(x - \bar{x}) \in \text{lev}_\phi(\tau)$ ($\bar{\eta} = 1$ if $x \in \text{lev}_\phi(\tau)$; otherwise $\bar{\eta} = (\tau - \phi(\bar{x})) / (\phi(x) - \phi(\bar{x})) \in (0, 1)$). Setting $\sigma = \lambda / \bar{\eta}$ gives $\sigma(z - \bar{x}) = \sigma\bar{\eta}(x - \bar{x}) = \lambda(x - \bar{x})$. Therefore, $T(\bar{x} \mid \text{lev}_\phi(\tau)) = T(\bar{x} \mid \text{dom } \phi)$, and consequently $N(\bar{x} \mid \text{lev}_\phi(\tau)) = N(\bar{x} \mid \text{dom } \phi)$. Hence, by (8.5), $\bar{s} \in N(\bar{x} \mid \text{dom } \phi)$. Therefore, if $\bar{\mu} = 0$, we have $\bar{\mu} \in S_2$. On the other hand, if $\bar{\mu} > 0$, by (8.5) and the

fact that $N(\bar{x} \mid \text{lev}_\phi(\tau)) = N(\bar{x} \mid \text{dom } \phi)$, we have

$$\begin{aligned}
\langle \bar{s}, \bar{x} \rangle &= \delta^*(\bar{s} \mid \text{dom } \phi) \\
&= (\phi^*)^\infty(\bar{s}) && [28, \text{Theorem 13.3}] \\
&= \tau 0 + (\phi^*)^\pi(\bar{s}, 0) \\
&\geq \tau \bar{\mu} + (\phi^*)^\pi(\bar{s}, \bar{\mu}) \\
&> \bar{\mu} \phi(\bar{x}) + \bar{\mu} \phi^*(\bar{s}/\bar{\mu}) \\
&\geq \bar{\mu} \phi(\bar{x}) + \bar{\mu} [\langle \bar{s}/\bar{\mu}, \bar{x} \rangle - \phi(\bar{x})] && [\text{Fenchel-Young inequality}] \\
&= \langle \bar{s}, \bar{x} \rangle.
\end{aligned}$$

Since this cannot occur, it must be the case that $\bar{\mu} = 0$ and $\bar{\mu} \in S_2$.

Now suppose that $\phi(\bar{x}) = \tau$, and let us first consider the case where $\bar{s} = 0$. Then, for $\mu > 0$, $p_\tau(\bar{s}, \mu) = (\tau + \phi^*(0))\mu \geq 0$ by (8.6), and, for $\mu = 0$, $p_\tau(\bar{s}, \mu) = (\phi^*)^\infty(0) = 0$. Therefore, $0 = \inf_{0 \leq \mu} p_\tau(\bar{s}, \mu)$ with $\mu = 0 \in S_1$. But, in this case, it is also clear $0 \in S_2 \neq \emptyset$, since $\bar{s} = 0 \in N(\bar{x} \mid \text{dom } \phi)$. Thus, if $\bar{\mu} = 0$ we have $\bar{\mu} \in S_2$. If $\bar{\mu} > 0$, then $0 = \tau + \phi^*(0)$ since $0 = p_\tau(0, \bar{\mu}) = (\tau + \phi^*(0))\bar{\mu}$. But then, by (8.6), $\phi(\bar{x}) + \phi^*(0) = \langle \bar{s}, \bar{x} \rangle = 0$ so that $\bar{s} = 0 \in \partial\phi(\bar{x})$. However, $\phi(\bar{x}) = \tau > \inf \phi$, so $0 \notin \partial\phi(\bar{x})$ [28, Theorem 23.5(b)]. This contradiction implies that if $\bar{s} = 0$, then we must also have $\bar{\mu} = 0$, and, in particular, we have $S_1 \subset S_2$.

Let us now consider the case where $\phi(\bar{x}) = \tau$ and $\bar{s} \neq 0$. By [28, Theorem 23.7],

$$N(\bar{x} \mid \text{lev}_\phi(\tau)) = \text{cl}(\text{cone}(\partial\phi(\bar{x}))), \quad (8.7)$$

and, by [29, Proposition 8.12],

$$\partial\phi(x)^\infty = N(x \mid \text{dom } \phi). \quad (8.8)$$

By (8.5) and (8.7), $\bar{s} \in \text{cl}(\text{cone}(\partial\phi(\bar{x})))$. That is, there exists $\{v^k\} \subset \partial\phi(\bar{x})$ and $\{\mu_k\} \subset \mathbb{R}_+$ such that $\mu_k v^k \rightarrow \bar{s}$. Because $\partial\phi(\bar{x})$ is closed, we can assume with no loss in generality that either (a) there exist $v \in \partial\phi(\bar{x})$ such that $v^k \rightarrow v$, or (b) $\|v^k\| \uparrow \infty$ and there exists $v \neq 0$ such that $v^k/\|v^k\| \rightarrow v$. Since $\bar{s} \neq 0$, (a) implies that there is a $\bar{\mu} \neq 0$ such that $\mu_k \rightarrow \bar{\mu}$ and $\bar{s} = \bar{\mu}v \in \text{cone}(\partial\phi(\bar{x}))$, while (b) and (8.8) implies that $\bar{s} \in N(\bar{x} \mid \text{dom } \phi)$. In summary, we have

$$\text{either (a) } \bar{s} \in \text{cone}(\partial\phi(\bar{x})) \quad \text{or} \quad \text{(b) } \bar{s} \in N(\bar{x} \mid \text{dom } \phi). \quad (8.9)$$

Using (8.9), let us first suppose that $\bar{s} \notin N(\bar{x} \mid \text{dom } \phi)$ so, in particular, $\bar{s} \in \text{cone}(\partial\phi(\bar{x}))$. As an immediate consequence, we have that $S_2 \neq \emptyset$ and the only values of μ for which $\bar{s} \in \mu^+ \partial\phi(\bar{x})$ have $\mu > 0$ since $\bar{s} \notin N(\bar{x} \mid \text{dom } \phi)$. Let $0 < \hat{\mu} \in S_2$. If $\bar{\mu} = 0$, then

$$\begin{aligned}
\delta^*(\bar{s} \mid \text{dom } \phi) &= (\phi^*)^\infty(\bar{s}) \\
&= \inf_{0 \leq \mu} p_\tau(\bar{s}, \mu) \\
&\leq \tau \hat{\mu} + \hat{\mu} \phi^*(\bar{s}/\hat{\mu}) \\
&= \hat{\mu} \phi(\bar{x}) + \hat{\mu} [\langle \bar{s}/\hat{\mu}, \bar{x} \rangle - \phi(\bar{x})] && [\text{Fenchel-Young inequality}] \\
&= \langle \bar{s}, \bar{x} \rangle \\
&\leq \delta^*(\bar{s} \mid \text{dom } \phi)
\end{aligned}$$

so that $\langle \bar{s}, \bar{x} \rangle = \delta^*(\bar{s} | \text{dom } \phi)$, or equivalently, $\bar{s} \in N(\bar{x} | \text{dom } \phi)$ contradicting the choice of \bar{s} . Hence, it must be the case that $\bar{\mu} > 0$. Again let $0 < \hat{\mu} \in S_2$. Then, by Part 1,

$$\begin{aligned} \delta^*(\bar{s} | \text{lev}_\phi(\tau)) &\leq p_\tau(\bar{s}, \bar{\mu}) \\ &= \inf_{0 \leq \mu} p_\tau(\bar{s}, \mu) \\ &\leq \tau \hat{\mu} + \hat{\mu} \phi^*(\bar{s}/\hat{\mu}) \\ &= \hat{\mu} \phi(\bar{x}) + \hat{\mu} [\langle \bar{s}/\hat{\mu}, \bar{x} \rangle - \phi(\bar{x})] \quad [\text{Fenchel-Young inequality}] \\ &= \langle \bar{s}, \bar{x} \rangle \\ &\leq \delta^*(\bar{s} | \text{lev}_\phi(\tau)) \end{aligned}$$

so that $\langle \bar{s}, \bar{x} \rangle = \bar{\mu}[\phi(\bar{x}) + \phi^*(\bar{s}/\bar{\mu})]$, or equivalently, $\bar{s} \in \bar{\mu} \partial \phi(\bar{x})$. Hence, $\bar{\mu} \in S_2$.

Finally, consider the case where $0 \neq \bar{s} \in N(\bar{x} | \text{dom } \phi)$. Then

$$\begin{aligned} \inf_{\mu \geq 0} p_\tau(\bar{s}, \mu) &\leq p_\tau(\bar{s}, 0) \\ &= (\phi^*)^\infty(\bar{s}) \\ &= \delta^*(\bar{s} | \text{dom } \phi) \quad [28, \text{Theorem 13.3}] \\ &= \langle \bar{s}, \bar{x} \rangle \quad [\text{by (8.5)}] \\ &= \delta^*(\bar{s} | \text{lev}_\phi(\tau)) \quad [\text{again by (8.5)}] \\ &\leq \inf_{\mu \geq 0} p_\tau(\bar{s}, \mu), \quad [\text{Part 1}] \end{aligned}$$

so $0 \in S_1$ and $0 \in S_2$. If $\bar{\mu} > 0$, then this string of equivalences also implies that $\langle \bar{s}, \bar{x} \rangle = p_\tau(\bar{s}, \bar{\mu}) = \bar{\mu}[\phi(\bar{x}) + \phi^*(\bar{s}/\bar{\mu})]$, or equivalently, $\bar{s} \in \bar{\mu} \partial \phi(\bar{x})$ so that $\bar{\mu} \in S_2$. Putting this all together, we get that $S_1 \subset S_2$.

(ii) Let $\bar{\mu} \in S_2$. If $\bar{\mu} = 0$, then

$$\begin{aligned} p_\tau(\bar{s}, 0) &= (\phi^*)^\infty(\bar{s}) \\ &= \delta^*(\bar{s} | \text{dom } \phi) \quad [28, \text{Theorem 13.3}] \\ &= \langle \bar{s}, \bar{x} \rangle \\ &\leq \delta^*(\bar{s} | \text{lev}_\phi(\tau)) \\ &\leq \inf_{\mu \geq 0} p_\tau(\bar{s}, \mu). \quad [\text{Part 1}] \end{aligned}$$

Therefore, $\bar{\mu} = 0 \in S_1$ and equality holds in (4.5).

On the other hand, if $\bar{\mu} > 0$, then $\bar{s}/\bar{\mu} \in \partial \phi(\bar{x})$, and so

$$\begin{aligned} \tau \bar{\mu} + (\phi^*)^\pi(\bar{s}, \bar{\mu}) &= \bar{\mu}[\phi(\bar{x}) + \phi^*(\bar{s}/\bar{\mu})] \\ &= \bar{\mu} \langle \bar{x}, \bar{s}/\bar{\mu} \rangle \quad [\text{Fenchel-Young inequality}] \\ &= \langle \bar{x}, \bar{s} \rangle \\ &\leq \delta^*(\bar{s} | \text{lev}_\phi(\tau)) \\ &\leq \inf_{\mu \geq 0} [\tau \mu + (\phi^*)^\pi(\bar{s}, \mu)]. \quad [\text{Part 1}] \end{aligned}$$

Hence, $\bar{\mu} \in S_1$ and equality holds in (4.5).

Proof of Lemma 4.4.

Part 1. The primal coercivity equivalence follows from [28, Theorems 8.4 and 8.7] since $\text{hzn}(f(\cdot, b, \tau)) = \text{hzn}(\phi) \cap [-A^{-1}\text{hzn}(\rho)]$.

Part 2. For the dual coercivity equivalence, let $\hat{g}(u) = g_\tau(u) - \langle b, u \rangle$, which is the objective of the reduced dual \mathcal{D}_r . By (4.4b), $\hat{g}^*(0) = g_\tau^*(b) = \text{cl}(v(\cdot, \tau)) \leq v(b, \tau)$. Therefore, the result follows from [28, Corollary 14.2.2] since by (8.10), $\text{dom } v(\cdot, \tau) = \text{dom } \rho + A \text{dom } \phi$.

Proof of Theorem 5.1. The expression for f^* is derived in (4.3). The weak and strong duality relationships as well as the expression for ∂v follow immediately from [29, Theorem 11.39].

Next, note that

$$\text{dom } f(\cdot, b, \tau) \neq \emptyset \iff \left[\begin{array}{c} \exists x \in \text{lev}_\phi(\tau) \\ b - Ax \in \text{dom } \rho \end{array} \right] \iff b \in \text{dom } \rho + A \text{lev}_\phi(\tau). \quad (8.10)$$

Now assume that $b \in \text{int}(\text{dom } \rho + A(\text{lev}_\phi(\tau)))$. Recall from [28, Theorem 6.6 and Corollary 6.6.2] that

$$\text{int}(\text{dom } \rho + A(\text{lev}_\phi(\tau))) = \text{ri}(\text{dom } \rho) + A(\text{ri}(\text{lev}_\phi(\tau))). \quad (8.11)$$

Moreover, by [28, Lemma 7.3], for any convex function p ,

$$\text{ri}(\text{epi } p) = \{ (x, \mu) \mid x \in \text{ri}(\text{dom } p) \text{ and } p(x) < \mu \}.$$

Since $\text{lev}_p(\tau) = P(\text{epi } p \cap \{ (x, \mu) \mid \mu \leq \tau \})$, where P is the projection $(x, \mu) \mapsto x$, [28, Theorems 6.5 and 6.6] tells us that

$$\text{ri}(\text{lev}_p(\tau)) = P(\text{ri}(\text{epi } p) \cap \text{ri}(\{ (x, \mu) \mid \mu \leq \tau \})) = \{ x \in \text{ri}(\text{dom } p) \mid p(x) < \tau \}. \quad (8.12)$$

Since $b \in \text{int}(\text{dom } \rho + A(\text{lev}_\phi(\tau)))$, (8.11)–(8.12) imply the existence of $\bar{w} \in \text{ri}(\text{dom } \rho)$ and $\bar{x} \in \text{ri}(\text{dom } \phi)$ with $\phi(\bar{x}) < \tau$ such that $b = \bar{w} + A\bar{x}$. Since ϕ is relatively continuous on the relative interior of its domain [28, Theorem 10.1], there exists $\delta > 0$ such that

$$\begin{aligned} (\bar{w} + \delta \mathcal{B}) \cap \text{dom } \rho &\subset \text{ri}(\text{dom } \rho), \\ (\bar{x} + \delta \mathcal{B}) \cap \text{dom } \phi &\subset \text{ri}(\text{dom } \phi), \\ \phi(x) &< \frac{1}{2}(\phi(\bar{x}) + \tau) \quad \forall x \in (\bar{x} + \delta \mathcal{B}) \cap \text{dom } \phi. \end{aligned}$$

Set $S_\rho = (\bar{w} + \delta \mathcal{B}) \cap \text{dom } \rho$ and $S_\phi = (\bar{x} + \delta \mathcal{B}) \cap \text{dom } \phi$. Since

$$\begin{aligned} \text{cone}(S_\rho + AS_\phi - b) &= \text{cone}(S_\rho - \bar{w}) + A \text{cone}(S_\phi - \bar{x}) \\ &= \text{span}(\text{dom } \rho - \bar{w}) + A \text{span}(\text{dom } \phi - \bar{x}) \\ &= \text{span}(\text{dom } \rho + A \text{dom } \phi - b) \\ &\supset \text{cone}(\text{dom } \rho + A \text{dom } \phi - b) \\ &= \mathbb{R}^m \quad (b \in \text{int}(\text{dom } \rho + A(\text{lev}_\phi(\tau)))) , \end{aligned}$$

we have $0 \in \text{int}(S_\rho + AS_\phi - b)$. Therefore, there exists an $\epsilon > 0$ such that $b + \epsilon \mathcal{B} \subset S_\rho + AS_\phi$. Consequently, if $\hat{b} \in b + \epsilon \mathcal{B}$ and $|\hat{\tau} - \tau| < \frac{1}{2}(\phi(\bar{x}) + \tau)$, then $\text{dom } f(\cdot, \hat{b}, \hat{\tau}) \neq \emptyset$ and so $(\hat{b}, \hat{\tau}) \in \text{dom } v$.

On the other hand, if $(b, \tau) \in \text{int}(\text{dom } v)$, then $\text{dom } f(\cdot, \hat{b}, \hat{\tau}) \neq \emptyset$ for all $(\hat{b}, \hat{\tau})$ near (b, τ) so that $\text{dom } f(\cdot, \hat{b}, \tau) \neq \emptyset$ for all \hat{b} near b . Hence $b \in \text{int}(\text{dom } \rho + A(\text{lev}_\phi(\tau)))$.

Proof of Theorem 5.2.

Part 1. First note that (5.1c) is equivalent to the optimality condition

$$0 \in -A^T \partial \rho(b - A\bar{x}) + \partial \delta(\bar{x} \mid \text{lev}_\phi(\tau)) \quad (8.13)$$

for the problem \mathcal{P} , and hence by [28, Theorem 23.8], \bar{x} solves \mathcal{P} . Moreover, by [28, Theorem 23.5], (5.1c) is equivalent to

$$b - A\bar{x} \in \partial \rho^*(\bar{u}), \quad \bar{x} \in \partial \delta^*\left(A^T \bar{u} \mid \text{lev}_\phi(\tau)\right),$$

or, equivalently,

$$b \in \partial \rho^*(\bar{u}) + A \partial \delta^*\left(A^T \bar{u} \mid \text{lev}_\phi(\tau)\right), \quad (8.14)$$

which by [28, Theorem 23.8] implies that \bar{u} solves the reduced dual \mathcal{D}_r .

Part 2. If \bar{x} solves \mathcal{P} , then

$$0 \in \partial[\rho(b - A(\cdot)) + \delta(\cdot \mid \text{lev}_\phi(\tau))](\bar{x}),$$

which by [28, Theorems 23.8, 23.9] and (5.1a) is equivalent to (8.13), which in turn is equivalent to (5.1c).

Part 3. If \bar{u} solves \mathcal{D}_r , then

$$b \in \partial\left[\rho^*(\cdot) + \delta^*\left(A^T(\cdot) \mid \text{lev}_\phi(\tau)\right)\right](\bar{u}),$$

which by [28, Theorems 23.8, 23.9] and (5.1b) is equivalent to (8.14), which in turn is equivalent to (5.1c).

Part 4. The equivalence (5.1e) follows from (5.1d), Part 2 of Lemma 4.3, and the fact that $A^T \bar{u} \in N(\bar{x} \mid \text{lev}_\phi(\tau))$ if and only if $\bar{x} \in \partial \delta^*(A^T \bar{u} \mid \text{lev}_\phi(\tau))$.

To see (5.1d), note that (4.6), (5.1a), and Part 1 of Lemma 4.4 imply that the primal objective is coercive, so a solution \bar{x} exists. Hence, by Part 2, there exists \bar{u} so that (\bar{x}, \bar{u}) satisfies (5.1c).

Analogously, (4.7), (5.1b), and Part 2 of Lemma 4.4 imply that the solution \bar{u} to the dual exists, and so by Part 3, there exists \bar{x} such that the pair (\bar{x}, \bar{u}) satisfies (5.1c). In either case, the subdifferential is nonempty and is given by (5.1d).

Proof of Lemma 6.1. Formula (6.3a) is just [28, Theorem 14.5]. The first equation in (6.3b) is obvious and the second follows from (6.3a) and the definition of the barrier cone. The formula (6.3c) is now obvious. Formulas (6.3d) and (6.3e) follow immediately from the definitions and [28, Corollary 8.3.3]. Formula (6.3f) follows from (6.3e), [28, Corollary 14.2.1], and [28, Corollary 16.4.2].

First note that (6.4) implies that $\text{ri}(\text{cone}(U)) \cap \text{ri}(X) \neq \emptyset$. Hence, the formula (6.5a) follows from [28, Theorem 16.4] and (6.3c). To see (6.5b), observe that the expression on the RHS is again an infimal convolution for which $\inf = \min$ for the same reason as for (6.5a). The equivalence with $(\phi^*)^\pi(z, \mu)$ follows from the calculus rules in section 3.3. For formula (6.5d), first note that

$$\begin{aligned} \inf_{\mu \geq 0} [\tau \mu + (\phi^*)^\pi(z, \mu)] &= \inf_{\mu \geq 0} [\tau \mu + \inf_s [\delta^*(z - s \mid X) + \delta(s \mid \mu U^\circ)]] \\ &= \inf_s [\delta^*(z - s \mid X) + \inf_{\mu \geq 0} [\tau \mu + \delta(s \mid \mu U^\circ)]] \\ &= \inf_s [\delta^*(z - s \mid X) + \tau \gamma(s \mid U^\circ)]. \end{aligned}$$

Again, the final infimum in this derivation is an infimal convolution for which $\inf = \min$ for the same reasons as in (6.5a) since, by (6.3c) and [28, Theorem 14.5],

$$\text{dom}((\tau\gamma(\cdot | U^\circ))^*) = \text{dom}((\delta^*(\cdot | \tau U))^*) = \text{dom} \delta(\cdot | \tau U) = \tau U.$$

Therefore, an optimal \bar{s} in this infimal convolution exists giving $\bar{\mu} = \gamma(\bar{s} | U^\circ)$ as the optimal solution to the first min in (6.5d).

Formula (6.5e) is an immediate consequence of (6.3d), (6.4), and [28, Corollary 23.8.1].

Proof of Theorem 6.2. By (6.3d) and the calculus rules for the relative interior [28, Section 6], (5.1a) and (6.8) are equivalent. Similarly, by (6.3f) and [28, Theorem 6.3], (5.1b) and (6.9) are equivalent.

Part 1. Since (6.4) holds, the formula (6.5e) holds and so (6.10) and (5.1c) are equivalent. Hence, the result follows from Part 1 of Theorem 5.2.

Part 2. Since (5.1a) and (6.8) are equivalent, the result follows from Part 2 of Theorem 5.2.

Part 3. Since (5.1b) and (6.9) are equivalent, the result follows from Part 3 of Theorem 5.2.

Part 4. By (6.3e), (6.11) is equivalent to (4.6) and (5.1a), and, by (6.3c), (6.12) is equivalent to (4.7) and (5.1b). Therefore, by Theorem 5.2, (6.13) is equivalent to (5.1d) since $\tau\gamma(s | U^\circ) = \inf_{\mu \geq 0} [\tau\mu + \delta(s | \mu U^\circ)]$. The final equivalence is identical to that of Theorem 5.2.

Proof of Theorem 6.3. The formula for $\text{dom} \phi$ follows from (6.17). Indeed, by (6.17), $x \in \text{dom} \phi$ if and only if there exists $s \in \mathbb{R}^k$ such that $x - Ls \in \text{dom} \gamma(\cdot | U^\circ) = \text{cone}(U^\circ)$, or equivalently, $x \in \text{cone}(U^\circ) + \text{Ran}(L) = \text{cone}(U^\circ) + \text{Ran}(B)$. The formula for $\text{hzn}(\phi)$ follows immediately from [28, Theorem 14.2] and (6.16). In particular, ϕ is coercive if and only if $\{0\} = \text{hzn}(\phi)$, or equivalently, $\text{cone}(U) = \mathbb{R}^n$, i.e., $0 \in \text{int}(U)$.

Next we show that the λ given in (6.20) solves (6.18). First observe that the optimal λ must be greater than $\gamma(w | U)$, and from elementary calculus, the minimizer of the hyperbola $\frac{1}{2\lambda} \|w\|_B^2 + \tau\lambda$ for $\lambda \geq 0$ is given by $\|w\|_B / \sqrt{2\tau}$. Therefore, the minimizing λ is given by (6.20). Substituting this value into (6.18) gives (6.19).

It is now easily shown that the function in (6.19) is lower semi-continuous. Therefore, the equivalence in (6.18) follows from (4.2).

Proof of Theorem 6.4. By [28, Theorem 7.6],

$$\text{ri}(\text{lev}_\phi(\tau)) = \{x \mid x \in \text{ri}(\text{dom} \phi), \phi(x) < \tau\}.$$

Hence, by Lemma 6.3, the equivalence between (5.1) and (6.21), (6.22), (6.24), (6.25), respectively, is easily seen. Therefore, Parts 1–4 follow immediately from Theorem 5.2.

Proof of Corollary 6.5. Condition (6.27a) occurs when $\bar{\mu} = 0$ since $0^+ \partial\phi(\bar{x}) = N(\bar{x} | \text{dom} \phi)$. When $\bar{\mu} > 0$, by [28, Theorem 23.5], $\partial\phi(x) = \arg \max_{w \in U} [\langle x, w \rangle - \frac{1}{2} \langle w, Bw \rangle]$, so that $w \in \partial\phi(x)$ if and only if $x \in Bw + N(w | U)$.

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