# Stochastic Network Design for Disaster Preparedness

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Abstract: We propose a new stochastic modeling approach for a pre-disaster relief network design problem under uncertain demand and transportation capacities. We determine the size and the location of the response facilities and the inventory levels of relief supplies at each facility with the goal of guaranteeing a certain level of network reliability. The overall objective is to enhance the effectiveness of the post-disaster relief operations. We introduce a probabilistic constraint on the existence of a feasible flow to ensure that the demand for relief supplies across the network is satisfied with a specified high probability. The responsiveness criterion is accounted for by defining multiple regions in the network, and introducing a local probabilistic constraint on satisfying the demand within each region. These local constraints ensure that each region is self-sufficient in terms of providing for their own relief needs with a large probability. The Gale-Hoffman inequalities and a combinatorial method are used to reformulate the probabilistically constrained models as computationally efficient mixed-integer linear programs. The solution method enables the use of a large number of scenarios to model the dependency between uncertain variables affecting disaster relief networks. Computational results for a case study and randomly generated problem instances based on a real disaster network demonstrate the effectiveness of the models and solution methods.

*Keywords*: Humanitarian logistics, Disaster preparedness, Facility location, Stochastic programming, Probabilistic constraint, Feasible flow, Network reliability

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**1. Introduction** Natural disasters lead to increasingly higher death tolls and material losses due to a multitude of factors, such as unplanned urbanization and increase in population and poverty. The recent massive 2010 Haiti earthquake caused hundreds of thousands of people's death and injuries (Walton and Ivers, 2011). The frequent occurrence of severe natural disasters has captured the attention of governments, humanitarian relief organizations and researchers all over the world and highlighted the need for enhancing the effectiveness of humanitarian relief management. Humanitarian logistics plays an important role in this field (Van Wassenhove, 2006) and differs greatly from business logistics (Van Wassenhove, 2006). Humanitarian logistics involves a high level of uncertainty and raises specific issues, such as the quickness and the fairness of the response to the affected areas (Van Wassenhove and Pedraza Martinez, 2010; Huang et al., 2011). Considering the complex structure of humanitarian relief systems and the need of allocating scarce relief resources in a way that improves the effectiveness of the relief operations, humanitarian logistics could greatly benefit from operations research methods (Altay and Green, 2006; Van Wassenhove, 2006; Van Wassenhove and Pedraza Martinez, 2010). It is thus not surprising to see a growing body of literature devoted to the development of optimization models for humanitarian relief logistics (see, e.g., Caunhye et al., 2012).

Our study focuses on the preparedness phase of disaster management for long-term planning purposes and considers the stochastic "pre-disaster relief network design problem" (PRNDP) to respond to sudden-onset natural disasters. This problem determines the location of the response facilities (distribution centers) and the positioning of the relief supplies in order to improve the effectiveness of the post-disaster relief operations. The importance of long-term pre-disaster planning has been emphasized in the recent literature (see e.g., Balcık and Beamon, 2008; Salmerón and Apte, 2010). The number of people affected, and thus, the demand for relief supplies vary with the location, severity and time of the disaster. Moreover, the roads may be damaged, and therefore, the transportation capacities fluctuate according to the location and severity of the disaster. It is crucial to develop models incorporating the inherent uncertainty in order to make sound relief network design decisions. To this end, we use stochastic programming to model the uncertainty in the demand and transportation capacities. The uncertain parameters are represented by a finite set of scenarios as in most applied stochastic programming models. We use the "network reliability" as the performance metric, and define it as the probability that a feasible flow of relief supplies exists in the immediate post-disaster response phase. A flow is said to be feasible if it satisfies the demand for relief supplies across the network. Thus, we obtain a reliable relief network by introducing a global probabilistic constraint to guarantee the existence of a distribution plan under which the network demand is satisfied with a specified high probability.

In a humanitarian relief system, positioning the resources reasonably close to the potential demand locations is essential to decreasing the response times, and consequently, to alleviate the suffering of the people in need. We incorporate the issue of responsiveness by defining multiple regions (e.g., states, counties, district, etc.) in the network, and locating adequate capacity and resources in each region. The goal is to guarantee that the regions are self-sufficient in terms of providing for their relief supply needs with a high probability. This is achieved by introducing local probabilistic constraints on satisfying the demand within each region. By enforcing the same reliability level for every region, we ensure an equitable response and service distribution among regions.

The PRNDP can be modeled as a two-stage stochastic programming problem. The first-stage decisions typically concern the size and the location of the response facilities as well as the inventory level at each facility. The second-stage (recourse) decisions pertain to the distribution of the relief supplies and depend on the predetermined first-stage decisions and the observed realization of the random parameters. Developing such two-stage stochastic models for the PRNDPs with inventory pre-positioning and relief distribution has recently received particular attention (Balcık and Beamon, 2008; Salmerón and Apte, 2010; Rawls and Turnquist, 2010; Mete and Zabinsky, 2010; Noyan, 2012; Döyen et al., 2011). Most of these existing studies propose risk-neutral two-stage stochastic programming models, i.e., they are based on expected values. However, considering expected values only may not be good enough for rarely occurring disaster events. Incorporating the risk concept is crucial in order to model the random variability inherent in disaster relief systems. The importance of managing uncertainty and risk is also discussed in Van Wassenhove (2006) and Noyan (2012). In this spirit, we propose new reliability-based models for the PRNDP.

The standard two-stage stochastic programming formulation enforces the feasibility of the second-stage problem for each joint realization of the random parameters, and may consequently lead to over-conservative solutions. A well-known approach to avoid such conservative solutions is to relax some constraints in the second-stage problem and penalize the amount by which the relaxed constraints are violated. For example, in the disaster management context, Rawls and Turnquist (2010) relax the demand satisfaction constraints and include the expected penalty cost for the unmet demand in the cost function of the second-stage problem. However, it may be challenging to estimate the shortage costs. As an alternative, a qualitative approach can be used in order to control the violation of the relaxed constraints. In line with this approach, we introduce a probabilistic constraint to control the infeasibility of the second-stage problem.

In general, a two-stage stochastic programming problem is formulated as a large-scale mixed integer programming (MIP) model by introducing a potentially huge number of second-stage decision variables and constraints. Noyan (2012) and Rawls and Turnquist (2010) emphasize the computational difficulty of solving such an MIP formulation for the PRNDP, in particular when the number of scenarios is large. As in Noyan (2012) and Rawls and Turnquist (2010), one could use a scenario decomposition method, but the problem remains hard to solve when many scenarios are considered. In this study, we also focus on the two-stage programming framework. We propose a novel modeling approach that ensures the feasibility of the second-stage problem without requiring the introduction of recourse decisions. Indeed, the feasibility of the second stage problem is defined in terms of the first-stage decisions only. This is the key feature of our study and it is accomplished by using the Gale-Hoffman theorem (Gale, 1957; Hoffman, 1960). This theorem represents the conditions on the existence of a feasible network flow with a set of linear inequalities expressed in terms of demand and arc capacities. In particular, we introduce a single-stage stochastic programming model with a joint probabilistic constraint on the feasibility of the second-stage problem. Then, we reformulate the joint probabilistic constraint in terms of the first-stage decision variables only by using the Gale-Hoffman inequalities. This approach allows us to incorporate the effectiveness of the post-disaster relief operations into the strategic level pre-disaster decision making without modeling the operational level relief distribution explicitly.

Besides the presence of integer variables pertaining to the location of facilities, two computational challenges linked to the probabilistic constraints need careful attention. First, the number of inequalities for which we impose a joint probabilistic constraint is extremely large (see Section 4.1). As a remedy, we reduce the number of inequalities involved in a joint probabilistic constraint by adopting the elimination procedures proposed by Prékopa and Boros (1991) and Wallace and Wets (1995), which provide non-redundant Gale-Hoffman inequalities. Second, we use a method based on the concept of combinatorial patterns (Lejeune, 2012a) in order to solve the proposed models. This method reformulates a chance-constrained model as an MIP problem in which the number of binary variables does not depend on the number of scenarios. Being able to solve a stochastic network design model for a large number of scenarios is a significant contribution, since it may be crucial to use many scenarios to represent the uncertainty in humanitarian relief environments (Snyder, 2006; Campbell and Jones, 2011). We also propose a scenario generation method allowing for the representation of dependency structures specific to disaster relief systems.

The contributions of this study are: (i) the introduction of new stochastic programming models for the pre-disaster relief network design problem; (ii) the design of a novel approach to model the reliability of a disaster relief network; and (iii) the development of computationally efficient methods to solve the probabilistically constrained models for moderate-size networks and large number of scenarios.

The paper is organized as follows. In Section 2, we present the related literature. In Section 3, we propose optimization models for the stochastic PRNDP. Section 4 first discusses how to derive the set of non-redundant Gale-Hoffman inequalities. It continues with the presentation of the combinatorial pattern-based method used to reformulate the probabilistically constrained models as MIP problems. Section 5 is devoted to the computational results for a case study and the randomly generated problem instances based on a real disaster network. Section 6 provides managerial implications and concluding remarks.

**2. Literature review** We first review the relevant literature on the design of pre-disaster relief networks in the presence of uncertainty. Then, we briefly review some of the studies related to the methodological aspects of our work.

There is an increasing number of studies that develop mathematical models for designing relief distribution networks. The high level of uncertainty characterizing humanitarian relief environments and the need of allocating scarce relief resources efficiently led to the development of stochastic optimization models for the PRNDP in the recent literature. Within this body of literature, there are few studies that address only the facility location (e.g., Jia et al., 2007; Ukkusuri and Yushimito, 2008) or only the inventory problem (e.g., Beamon and Kotleba, 2006). Among the studies about preparedness, we focus on the ones that propose stochastic optimization models involving both facility location and inventory decisions. In general, such studies propose two-stage stochastic programming models, where the first-stage decisions are for locating the response facilities and pre-positioning the relief supplies, and the second-stage decisions are related to the distribution of the relief supplies. Basically, such models determine the pre-disaster decisions considering the potential post-disaster decisions.

Chang et al. (2007) consider a flood emergency logistics problem and propose two stochastic optimization models considering a prioritized structure of rescue organizations. The first model minimizes the expected shipping distance to classify the disaster rescue areas into several multi-level rescue groups. Then, based on the specified rescue groups, a two-stage stochastic programming model is proposed to determine the location of the local rescue centers only and the amount of allocated rescue equipments. The second-stage decision variables represent the amount of equipments shipped from each rescue center to each demand point and the objective is to minimize the total transportation, supply shortage and demand shortage penalty costs. Balcık and Beamon (2008) develop a maximal covering type model to determine the location of the distribution centers and the inventory levels of multiple relief items at each facility under demand and transportation time uncertainty. The recourse decisions represent the proportion of demand for a particular type of item satisfied by each distribution center. Thus, they assume that there is only one demand location under each scenario. They also assume that the demand is fully satisfied under each scenario, and hence, the model may provide conservative solutions. Rawls and Turnquist (2010) allow multiple locations to be demand points under each disaster scenario and allow for the infeasibility of the demand satisfaction constraints. They consider the PRNDP with uncertainty in the pre-positioned supplies, demand and transportation capacities. Different from a coverage type model as the one proposed in Balcık and Beamon (2008), the second-stage problem in Rawls and Turnquist (2010) involves detailed distribution decisions representing the flow of relief supplies on each arc of the network, and penalizes the demand shortages. Other recent and related studies are by Mete and Zabinsky (2010), Salmerón and Apte (2010) and Döyen et al. (2011). In contrast with the other studies, Döyen et al. (2011) consider facility location decisions in both the first- and second-stage problems.

Almost all the above studies propose risk-neutral models, i.e., are based on expected values. More recently, the literature addresses the importance of incorporating the concept of risk into disaster management decision making (Van Wassenhove, 2006; Noyan, 2012). Beraldi and Bruni (2009) introduces a risk-averse model for a emergency service vehicle location problem under demand uncertainty. They consider a coverage type model and formulate it as a two-stage stochastic program with a probabilistic constraint. The second-stage decisions represent the assignment of supply nodes to demand points under each scenario. The probabilistic constraint enforces a lower bound on the probability that each demand node is covered and the demand at each node is fully served. The idea of developing such a stochastic programming model with a probabilistic constraint on the feasibility of the second-stage problem has been first proposed by Prékopa (1980). Noyan (2012) incorporates a risk measure to develop a risk-averse two-stage stochastic model for the PRNDP. Noyan (2012) extends the model proposed in Rawls and Turnquist (2010) by incorporating the conditional-value-at-risk (CVaR) as the risk measure on the total cost in addition to the expectation criterion. We refer to Noyan (2012) for the definition of the CVaR and further details. Alternatively Rawls and Turnquist (2011) extend their earlier model (Rawls and Turnquist, 2010) by introducing a lower bound on the probability that the network demand is satisfied and the resulting average total shipment distance is less than a specified threshold. Different from the existing studies, our paper proposes reliability-based models with a joint probabilistic constraint on the feasibility of the second-stage problem without introducing any recourse decision. Moreover, the use of a joint probabilistic constraint is an alternative way of modeling "network reliability".

There is a huge literature on modeling network reliability and designing reliable networks. We shall briefly review the relevant studies in the disaster context. Peeta et al. (2010) propose a pre-disaster investment problem that identifies a subset of links to be strengthened. The goals are to maximize the post-disaster connectivity and minimize the expected transportation cost between various origin-destination pairs. They assume that there are only two possible states for each link, i.e., a link either survives or fails after the disaster, and the link failure probabilities are known a priori. Under the assumption of independent link failures, they propose a two-stage stochastic programming where the investment and flow decisions are represented by first- and second-stage decisions, respectively. They characterize the network reliability in terms of its connectivity through non-failed links. Thus, their only focus is to provide access in case of a disaster and they do not take demand for relief supplies and capacity into consideration. Note that the common and simplifying assumption that link failures are independent can be criticized in the disaster context where it is crucial to model the dependency between link failures. Gunnec and Salman (2011) focus on assessing network reliability considering dependency among link failures. According to their dependency model, the links are partitioned into sets. Links in different sets are assumed to fail independently of each other, while links within the same set fail according to a specified dependency model. As Peeta et al. (2010), Gunnec and Salman (2011) view network reliability as the probability of connectedness and define the performance of the network as the expected shortest path distance. Gunnec and Salman (2011) propose methods to calculate these measures in order to evaluate the post-disaster performance of a network. Differing from these two studies, ours takes demand and capacity into consideration. Moreover, each link is allowed to be partially damaged and the dependencies among link failures are accounted for by using a scenario based approach. For a comprehensive review of network reliability measures and reliable network design problems, we refer the reader to Gunnec and Salman (2011) and the references therein. A recent study by Nolz et al. (2011) also considers several risk measures but to design a post-disaster delivery system under the assumption that link failures are independent.

For other types of mathematical models used to develop disaster preparedness policies, we refer to the review papers by Brandeau et al. (2009) and Caunhye et al. (2012), and the references therein. Another major stream of literature in humanitarian logistics is related to the distribution of emergency supplies and vehicle routing in the post-disaster response phase (Haghani and Oh, 1996; Barbarosoğlu and Arda, 2004; Ozdamar et al., 2004). For a recent and comprehensive review of the humanitarian logistics literature, we also refer to the special issues of *International Journal of Production Economics* edited by Boin et al. (2010), *OR Spectrum* edited by Doerner et al. (2011).

As mentioned above, we propose probabilistic constrained optimization problems. Stochastic programming problems with joint probabilistic (chance) constraints in which the uncertainty is represented by a set of scenarios are typically NP-hard, and extremely difficult to solve. The main challenge lies in identifying the optimal set of scenarios for which the desired condition is violated. Developing solution methods for this class of problems has been receiving a sustained attention in the last decade. In general, the proposed solution methods utilize the $p$-efficiency concept (e.g., Dentcheva et al., 2001; Lejeune and Ruszczyński, 2007; Lejeune and Noyan, 2010; Dentcheva and Martinez, 2012), valid inequalities to derive strengthened MIP reformulations (e.g., Ruszczyński, 2002; Luedtke et al., 2010), and conservative convex approximations. In order to be able to consider a large number of scenarios to characterize the uncertainty, we use the method proposed by Lejeune (2012a) to reformulate and solve the chance-constrained optimization problems.

**3. Stochastic Optimization Models** In this study, we consider the problem of long-term pre-disaster planning and develop models that support response facility location and stock pre-positioning decisions. These decisions are to be implemented well before a disaster strikes. Such long-term considerations are markedly different from short-term relief planning, where the goal is to position the supplies near a potentially affected area in anticipation of an imminent disaster. For example, Galindo and Batta (2012) propose a pre-disaster management model, where the pre-positioning decisions are made two days before a foreseeable hurricane, based on the forecasts available about five days prior to the landfall. However, such a short time period might not be sufficient to ensure a desired level of service, and response operations can be difficult to carry out effectively without appropriate long-term pre-disaster planning.

In short-term planning situations, forecasts about the location, severity, and time of a disaster are often available. In such cases, replacing an uncertain parameter (such as the random demand for relief commodities) by its expected value (as in Galindo and Batta, 2012) may be justified due to the lower inherent uncertainty. However, as our focus is on long-term planning in the absence of accurate predictions, it is crucial to consider a more detailed way of modeling uncertainty. To this end, we use a finite set of scenarios to represent potential future outcomes. In our models we take into account the uncertainty in the demand for relief supplies, as well as the uncertainty in the arcs capacities in the transportation network in case of a disaster. Accordingly, each scenario consists of a joint realization of the arc capacities and the demand at each node. This approach allows us to incorporate the dependency between uncertain parameters into our models. We discuss the details of the scenario generation process in Section 5.2.3.

We briefly mention here that it is not always possible to implement long-term pre-positioning policies in certain rural regions or underdeveloped countries due to various factors such as budget constraints. Our models are primarily applicable in developed regions with sufficient data availability and financial resources to design a reliable disaster network ahead-of-time.

In the remainder of this section, we present our optimization models for the PRNDP and their MIP reformulations. In Section 3.1, we present a two-stage stochastic programming model. Based on this two-stage model, we develop in Section 3.2 an optimization model enforcing a network-wide (global) probabilistic constraint. This constraint ensures the existence of a feasible flow of relief supplies to satisfy the demand across the network with a specified high probability. Next, the model is extended in Section 3.3 to account for the responsiveness and equity criteria that play a crucial role in disaster management. This is achieved by introducing local probabilistic constraints that ensure that each region provides for its own commodity needs with a certain reliability. In Section 3.2, we utilize the Gale-Hoffman inequalities to define the probabilistic existence of a feasible network flow. Additionally, we implement a preprocessing algorithm to identify non-redundant Gale-Hoffman inequalities and obtain compact chance-constrained models for the PRNDP. In the last step of the modeling approach, we employ a combinatorial method to derive equivalent MIP reformulations for the chance-constrained models proposed in Sections 3.2 and 3.3.

**3.1 Two-Stage Model** We present now a two-stage stochastic programming model for the PRNDP with stochastic demand and arc capacities. This model constitutes a basis for the development of our new chance-constrained optimization models. We assume that each node represents a candidate facility location. This is not a restrictive assumption, since the corresponding formulation can be easily modified to allow the set of candidate facility locations to differ from the set of demand points. We consider a single type of commodity that can be a bundle of critical relief supplies, such as prepackaged food, medical kits, blankets and water. Rawls and Turnquist (2010) define a bundled commodity as a package "that includes specific elements in fixed proportion" and mention that it may be desirable to bundle different types of commodities to ensure that all are provided together.

Here we introduce the notation:

$I$: set of nodes (locations);
$A$: set of arcs;
$L$: set of facility types differentiated by their size;
$\Omega$: set of scenarios;
$\xi_i^d$: random demand for the bundled commodity at location $i$;
$\xi_{ij}^v$: random capacity of arc linking locations $i$ and $j$;
$d_i^s$: realization of demand for the bundled commodity at location $i$ under scenario $s$;
$v_{ij}^s$: realization of the capacity of arc $(i,j)$ expressed in units of the bundled commodity under scenario $s$;
$p$: prespecified probability (reliability) level;
$M_l$: total capacity of a facility of type $l$;
$F_{il}$: fixed cost of opening a type $l$ facility at location $i$;
$q$: unit acquisition cost for the bundled commodity;
$h$: unit shortage cost for the bundled commodity.

Let $|B|$ denote the cardinality of a set $B$. The $|I|$-dimensional random vector associated with the demand at each location and the $|A|$-dimensional random vector associated with the arc capacities are denoted by $\xi^d$ and $\xi^v$, respectively. Using these notations, we represent the inherent uncertainty in the network by the $(|I|+|A|)$-dimensional random vector $\xi = (\xi^d, \xi^v)$ and denote its realization under scenario $s$ by $\omega^s$. Thus, $\omega^s$ is the vector whose components correspond to the elements of $\{d_i^s,\ i \in I,\ v_{ij}^s,\ (i,j) \in A\}$ and denote the joint realizations of the random variables $\xi_i^d,\ i \in I,\ \xi_{ij}^v,\ (i,j) \in A$, under scenario $s$.

The first-stage decision variables define the size and location of the facilities and the amount $r_i$ of commodity pre-positioned at each facility $i \in I$. Let the binary variable $y_{il},\ i \in I,\ l \in L$, be 1 if a facility of type $l$ is located at location $i$, and 0 otherwise. The main second-stage (recourse) decision variables define the distribution of the relief supplies, while the auxiliary second-stage decision variables measure the amount of demand shortages. Let $x_{ij}^s$ designate the amount of the bundled commodity shipped through link $(i,j)$ under scenario $s$, while $w_i^s$ denotes the amount of demand shortage at location $i$ under scenario $s$. The first-stage problem takes the form

$$\min \quad \sum_{i\in I}\sum_{l\in L} F_{il} y_{il} + \sum_{i\in I} q r_i + E\left[Q(\mathbf{r},\mathbf{y},\xi)\right] \tag{1}$$

$$\text{s. t.} \quad r_i \le \sum_{l\in L} M_l y_{il}, \quad i \in I \tag{2}$$

$$\sum_{l\in L} y_{il} \le 1, \quad i \in I \tag{3}$$

$$y_{il} \in \{0,1\}, \quad i \in I, l \in L \tag{4}$$

$$r_i \ge 0, \quad i \in I. \tag{5}$$

Here $Q(\mathbf{r},\mathbf{y},\xi)$ denotes the random objective function value of the second-stage problem and its realization $Q(\mathbf{r},\mathbf{y},\omega^s)$ under scenario $s \in \Omega$ is obtained by solving the following problem:

$$Q\left(\mathbf{r},\mathbf{y},\omega^s\right) = \min \sum_{i\in I} h w_i^s \tag{6}$$

$$\text{s. t.} \quad r_i + \sum_{(j,i)\in A} x_{ji}^s - \sum_{(i,j)\in A} x_{ij}^s \ge d_i^s - w_i^s, \quad i \in I \tag{7}$$

$$w_i^s \geq 0, \quad i \in I \tag{8}$$
$$0 \leq x_{ij}^s \leq v_{ij}^s, \quad (i,j) \in A. \tag{9}$$

The objective function (1) of the first-stage problem minimizes the sum of the cost of opening facilities and purchasing the relief supplies, and the expected total shortage cost. Constraint (2) ensures that each facility is sufficiently large to store the amount of the commodity pre-positioned at that facility. Constraint (3) guarantees that at most one facility is located at each node. Constraints (4) and (5) are the binary and non-negativity restrictions. In the formulation of the second-stage problem, constraints (7) and (8) ensure that the shortfall variable $w_i^s$ is positive if the amount of the commodity available at node $i$ is not sufficient to cover the associated demand $d_i^s$ under scenario $s$: $w_i^s \geq \max\{d_i^s - (r_i + \sum_{(j,i)\in A} x_{ji}^s - \sum_{(i,j)\in A} x_{ij}^s), 0\}$. The structure of the objective function forces $w_i^s$ to be equal to the exact shortage amount for all $i \in I,\ s \in \Omega$, at the optimal solution of the second-stage problem. Constraint (9) prevents the flow on arc $(i,j)$ from exceeding the capacity of the arc under scenario $s$.

The two-stage stochastic programming model (1)-(9) is very similar to the one proposed by Rawls and Turnquist (2010). However, it focuses mainly on satisfying demand for relief supplies, and excludes the surplus and transportation costs. Surplus costs are implicitly accounted for in the model due to the objective of minimizing the acquisition costs. Nevertheless, considering the transportation costs may be helpful in reducing response times. We partially address this issue by proposing a model that incorporates a responsiveness criterion. As the model presented in Rawls and Turnquist (2010), model (1)-(9) penalizes the expected violation amounts associated with the relaxed demand satisfaction constraints, and thus, requires estimating the shortfall cost parameter $h$. While such penalty parameters are often used in the disaster management literature (see, e.g., Barbarosoğlu and Arda, 2004; Chang et al., 2007; Salmerón and Apte, 2010; Rottkemper et al., 2011), it may be challenging to quantify the cost of failing to satisfy the demand (see, e.g., Prékopa, 1995; Rottkemper et al., 2011). As an alternative, we introduce a probabilistic constraint to control the existence of a feasible flow of relief supplies to satisfy the network demand. The proposed probabilistically constrained optimization model is described in the next section.

**3.2 Chance-Constrained Models** We shall now present an alternative modeling approach to the ones that inflict penalty costs for the unsatisfied demand (see the previous subsection and, e.g., Rawls and Turnquist (2010); Noyan (2012)). Our approach focuses on the probability of failing to satisfy the demand in the network and avoid inflicting a penalty per unit of shortfall as in (6). Hence, it does not require the estimation of the shortfall cost parameters. The proposed chance constrained model, which we refer to as **SP1**, probabilistically prevents the demand shortage in case of a disaster.

$$\textbf{SP1} \qquad \min \quad \sum_{i\in I}\sum_{l\in L} F_{il} y_{il} + \sum_{i\in I} q r_i \tag{10}$$

$$\text{s. t.} \quad (2); (3); (4); (5);$$

$$\mathbb{P}\begin{pmatrix} r_i + \sum\limits_{(j,i)\in A} x_{ji}(\omega) - \sum\limits_{(i,j)\in A} x_{ij}(\omega) \geq \xi_i^d, \quad i \in I \\ 0 \leq x_{ij}(\omega) \leq \xi_{ij}^v, \quad (i,j) \in A \end{pmatrix} \geq p \tag{11}$$

Here we elaborate only on the new probabilistic constraint (11). First recall that the corresponding two-stage problem (6)-(9) involves scenario-dependent second-stage variables $x_{ij}^s,\ (i,j) \in A$. Basically, the recourse decisions depend on the realizations of the random vector $\xi$, and are thus random variables. In the above formulation, we drop scenario index $s$ from $x_{ij}^s$ and denote the random recourse decisions by $x_{ij}(\omega),\ (i,j) \in A$. Constraint (11) is a joint chance constraint enforcing the probability of satisfying the second-stage constraints (7) and (9) with $w_i^s = 0$ to be at least $p$. In other words, any location-allocation policy defined by a feasible solution $(\mathbf{y}, \mathbf{r})$ of **SP1** guarantees the satisfaction of the network demand with probability at least $p$. We refer to the risk parameter $p$ as the "network-wide reliability level". Introducing binary variables $\beta^s,\ s \in \Omega$, to identify whether there is such a feasible distribution of relief supplies under each scenario, we reformulate **SP1** as the large-scale MIP problem **SP1a**:

$$\textbf{SP1a} \qquad \min \sum_{i\in I}\sum_{l\in L} F_{il}y_{il} + \sum_{i\in I} qr_i$$

$$\text{s. t.} \quad (2); (3); (4); (5);$$

$$r_i + \sum_{(j,i)\in A} x_{ji}^s - \sum_{(i,j)\in A} x_{ij}^s \geq d_i^s(1-\beta^s), \quad i \in I,\ s\in \Omega \tag{12}$$

$$0 \leq x_{ij}^s \leq v_{ij}^s + Z\beta^s, \quad (i,j)\in A,\ s\in\Omega \tag{13}$$

$$\sum_{s\in\Omega} \pi^s\beta^s \leq 1-p. \tag{14}$$

Here $\pi^s$ denotes the probability of scenario $s$ and $Z$ is a sufficiently large positive number. Constraints (12) and (13) force $\beta^s$ to take value 1 if the network demand cannot be satisfied without exceeding any arc capacity under scenario $s$. Thus, constraint (14) implies that the probability of the flow of relief supplies to be feasible is lower bounded by $p$. Note that MIP formulations stronger than **SP1a** can be obtained. However, any such deterministic equivalent formulation introduces recourse decisions associated with each scenario, and hence, involves a potentially large number of second-stage decision variables. To obtain a computationally efficient reformulation of **SP1**, we propose to avoid adding the scenario-dependent second-stage variables. In particular, we use the Gale-Hoffman theorem (Gale, 1957; Hoffman, 1960) to represent the conditions on the existence of a feasible network flow for the second-stage problem. The feasibility of the network flow is modeled with a set of linear inequalities defined in terms of demand and arc capacities. This, in turn, leads to the reformulation of the probabilistic constraint (11) in terms of first-stage decision variables only. We shall now discuss the concepts of a feasible demand function and a feasible network flow, and present the Gale-Hoffman inequalities which lead to a reformulation equivalent to **SP1**.

Definition 3.1 *Consider a directed network defined by a set of nodes $I$ and a set of arcs $A \subseteq I\times I$. Each arc $(i,j)\in A$ and each node $i\in I$ has an associated nonnegative arc capacity $v_{ij}$ and demand $d(i)$, respectively. A flow on the network is a real valued function $x_{ij}, (i,j)\in A$, satisfying the conditions*

$$0 \leq x_{ij} \leq v_{ij}, \quad \forall\ (i,j)\in A. \tag{15}$$

*A demand function $d(i),\ i\in I$, is a real function on the set of nodes $I$. We say that a demand function defined on the set of nodes is feasible if there exists a flow such that for each node the sum of the incoming flow values is greater than or equal to the demand at that node. Thus, a demand function $d(i),\ i\in I$, is said to be feasible if and only if there exists a flow $\mathbf{x}$ such that*

$$\sum_{j:\ (j,i)\in A} x_{ji} - \sum_{j:(i,j)\in A} x_{ij} \geq d(i),\ \textit{for every } i\in I. \tag{16}$$

*A flow on the network is said to be feasible if it satisfies the conditions (15) and (16).*

For any $H\subseteq I$, we define $d(H) = \sum_{i\in H} d(i)$. Let $\bar{H}$ refer to the complement of a set $H$ and $v(\bar{H},H)$ be the sum of the capacities of the arcs connecting the nodes in $H$ to those in $\bar{H}$. Using these notations, we present the Gale-Hoffman theorem providing the necessary and sufficient conditions on the existence of a feasible demand function for a network.

Theorem 3.1 *(Gale and Hoffman) The demand function $d(i),\ i\in I$, is feasible if and only if the inequality*

$$d(H) \leq v(\bar{H},H) \tag{17}$$

*holds for every $H\subseteq I$.*

The set of constraints (17) will be thereafter referred to as the Gale-Hoffman inequalities. If the stocked inventory is not sufficient to satisfy the demand at a node, the resulting net demand is positive, i.e., the node is in need of relief supplies. Since the total capacity of the incoming arcs to a node is essentially an upper bound on the amount of supplies that can be delivered to that node, the net demand at each node must be smaller than or equal to the total capacity of the incoming arcs in order to guarantee the satisfaction of the network demand. In the stochastic setup, this condition means that the stochastic net demand does not exceed the stochastic total capacity of the incoming arcs at each node and it is guaranteed by the stochastic version of the Gale-Hoffman inequalities.

For our network design problem, the net demand associated with the subset $H$ under scenario $s$ is obtained as $d(H) = \sum_{i \in H}(d_i^s - r_i)$, and a feasible network flow exists if and only if the Gale-Hoffman inequality

$$\sum_{i \in H}(d_i^s - r_i) \leq \sum_{(i,j)\in A \;:\; i \in \bar{H}, j \in H} v_{ij}^s = \sum_{i \in \bar{H}, j \in H} v_{ij}^s \tag{18}$$

holds for every $H \subseteq I$. If this is the case, the demand at all nodes of the network can be satisfied under scenario $s$. We use the convention that $v_{ij}^s = 0,\ s \in \Omega$, for all $(i,j) \notin A$, and therefore, the equality in (18) holds true.

Lemma 3.1, which is a direct consequence of Theorem 3.1, utilizes a stochastic version of the Gale-Hoffman inequalities (17) and provides us with an equivalent formulation of the chance constraint (11).

Lemma 3.1 *The chance constraint* (11) *can be equivalently rewritten as*

$$\mathbb{P}\left(\sum_{i \in H}(\xi_i^d - r_i) \leq \sum_{i \in \bar{H}, j \in H} \xi_{ij}^v \ , \quad \textit{for every } H \subseteq I\right) \geq p. \tag{19}$$

Proof. The net demand at each location and the capacity of each arc are random. This implies that $d(H)$ and $v(\bar{H}, H)$ are random for a subset $H \subseteq I$. Hence, we utilize a stochastic version of the Gale-Hoffman inequalities to represent the stochastic conditions on the existence of the feasible network flow. Requiring that the Gale-Hoffman inequalities hold jointly with probability at least $p$ gives

$$\mathbb{P}\left(d(H) \leq v(\bar{H}, H), \quad \text{for every } H \subseteq I\right) \geq p. \tag{20}$$

Substituting $d(H) = \sum_{i \in H}(\xi_i^d - r_i)$ and $v(\bar{H}, H) = \sum_{i \in \bar{H}, j \in H} \xi_{ij}^v$ into (20) provides the desired probabilistic constraint (19). Then, applying Theorem 3.1, relation (19) holds true if and only if the demand function $d(i) = \xi_i^d - r_i,\ i \in I$, is feasible with probability at least equal to $p$. Then, the assertion follows by the definition of the feasibility of the demand function $d(i)$ (see Definition 3.1). □

The Gale-Hoffman inequalities in (19) define the sufficient and necessary conditions for the existence of a flow to satisfy the network demand and are written in terms of the first-stage decision variables $r_i, i \in I$, only. This is a marked difference with model **SP1**, where the probabilistic feasibility of the second stage problem (11) is defined in terms of the second-stage distribution variables $x_{ij}(\omega)$. Replacing (11) by (19), we obtain an equivalent formulation of **SP1**, referred to as **SP2**:

$$\textbf{SP2} \qquad \min\{\sum_{i \in I}\sum_{l \in L} F_{il} y_{il} + \sum_{i \in I} q r_i \ : \ (2), (3), (4), (5), (19), \mathbf{r} \in \mathbb{R}^{|I|}, \mathbf{y} \in \mathbb{R}^{|I| \times |L|}\}.$$

Typically, scenario-based approaches use a limited number of scenarios to represent the uncertain future outcomes in order to avoid introducing large number of decision variables and obtain computationally tractable models (see e.g., Snyder, 2006; Campbell and Jones, 2011). A significant contribution of our approach is that it does not require the introduction of the scenario-dependent second-stage decision variables. Thus, it permits the efficient handling of a large number of scenarios, which is crucial to obtain a fine characterization of the uncertainty in humanitarian relief environments (Lempert et al., 2003; Comfort, 2005; Snyder, 2006).

**3.3 Responsiveness and Equity** As for other types of emergency service systems (e.g., Brotcorne et al., 2003), it is also crucial to take the responsiveness and fairness criteria into consideration while designing a humanitarian relief network (Tzeng et al., 2007; Van Wassenhove and Pedraza Martinez, 2010; Huang et al., 2011). Responsiveness refers to the quickness of the response and is related to the positioning of the resources reasonably close to the potential demand locations. We ensure responsiveness by defining multiple regions (e.g., states, counties, districts) in the network, and by requiring the allocation of enough resources to make the regions self-sufficient. This means that each region will be able to provide for its own needs with a large probability, which we call the local (region-wide) reliability. Chang et al. (2007) also focus on dividing the disaster area into several regions and assume that a facility gives a higher priority to the demand points in its own region before extending help to others. As in Chang et al. (2007), a mathematical programming model can be used to cluster the nodes in the network. For each cluster of nodes representing a certain region, we introduce a so-called local probabilistic constraint on satisfying the demand within each region. Moreover, we ensure an equitable service distribution at the region level by enforcing the same reliability level for every region. Noyan (2010) also models equity by setting the risk parameter to be equal for different regions, but considers a different type of risk measure for an EMS design problem. Thus, we propose an alternative way of modeling equity which is based on the reliability level of satisfying the demand within each region.

In this section, we introduce additional stochastic constraints in order to incorporate the responsiveness and equity into the proposed model **SP2**. These constraints are referred to as the local probabilistic constraints. Let $K$ denote the number of regions in the network and $I^k$ be the set of nodes within region $k$. Then, we introduce the following $K$ local (region-wide) chance constraints:

$$\mathbb{P}\left(\sum_{i\in H}(\xi_i^d - r_i) \leq \sum_{i\in \bar{H}^k, j\in H} \xi_{ij}^v \ , \ \text{ for every } H \subseteq I^k\right) \geq p'_k \quad k \in \{1,\ldots,K\}, \tag{21}$$

where $\bar{H}^k = I^k \setminus H$ and $p'_k$ is the specified minimal acceptable probability level of satisfying the demand within region $k$. We assign the same value to each parameter $p'_k$: $p'_k = p',\ k = 1,\ldots,K$. We refer to the risk parameter $p'$ as the "region-wide reliability level". Incorporating (21) into **SP2** we obtain the model with global and local probabilistic constraints, which we refer to as **SP3**:

$$\textbf{SP3} \qquad \min\{\sum_{i\in I}\sum_{l\in L} F_{il}y_{il} + \sum_{i\in I} qr_i \ : \ (2),(3),(4),(5),(19),(21), \mathbf{r}\in\mathbb{R}^{|I|}, \mathbf{y}\in\mathbb{R}^{|I|\times|L|}\}.$$

Constraint (21) ensures that the inventories pre-positioned within a region are sufficient to satisfy the demand in that region with a probability of at least $p'$. This would ensure that the demand for relief commodities can be satisfied from nearby facilities with probability at least $p'$, and thus, the introduction of the local reliability constraints can improve the effectiveness of the immediate post-disaster operations in terms of the response times. This novel model allows the simultaneous control of the reliability-based performances of individual regions and the entire network.

In our computational study we consider local reliability levels $p'$ that are smaller than the network-wide reliability level $p$. This choice is supported by the observation that individual region-wide Gale-Hoffman inequalities $\sum_{i\in H}(\xi_i^d - r_i) \leq \sum_{i\in \bar{H}^k, j\in H} \xi_{ij}^v$ imply their global counterparts $\sum_{i\in H}(\xi_i^d - r_i) \leq \sum_{i\in \bar{H}, j\in H} \xi_{ij}^v$. However, we note that local probabilistic constraints in general do not imply the global one, since the latter is enforced in a joint fashion. Thus, it could also be meaningful to specify a $p'$ value that is higher than $p$.

Remark 3.1 *Regions can be viewed as clusters of locations, and can be constructed based on threshold distances or response times. In particular, clusters can be identified in such a way that the distance between any two pair of locations within each cluster is below a specified value. This guarantees a certain level of service in terms of response times. One may also develop an approach that incorporates the stochastic nature of the response times while constructing the clusters. Thus, the model SP3, which includes local constraints for a set of regions, has the potential to incorporate the uncertainty in response times.*

## 4. Solution Methods

In this section, we first present computationally more efficient formulations of **SP2** and **SP3**. They are obtained by eliminating the redundant Gale-Hoffman inequalities with the procedures proposed by Prékopa and Boros (1991) and Wallace and Wets (1995). Then, we use a recently proposed method based on combinatorial patterns (Lejeune, 2012a) to reformulate the proposed probabilistically constrained models as MIP problems including a limited number of integer variables.

We note that the computational time is not critical in pre-disaster decision making. The main challenge is to be able to solve the models of interest at all. Even moderate sized problems can be intractable due to the memory limit, not necessarily due to the time limits. It is therefore important to develop computationally tractable formulations and solution methods.

**4.1 Elimination of Redundancy** Each Gale-Hoffman inequality corresponds to a subset of $I$. This implies that the number $(2^{|I|}-1)$ of inequalities required to hold jointly by the global probabilistic constraint (19) is very large even for small networks. Thus, solving **SP2** and **SP3** would be extremely challenging, if not hopeless. Fortunately, as mentioned in Prékopa and Boros (1991), the number of Gale-Hoffman inequalities can be drastically reduced by eliminating the redundant ones. For example, Prékopa and Boros (1991) study a 15-node network with a total of $32,767$ Gale-Hoffman inequalities and show that only 13 of those remain after the elimination procedure. Motivated by these discussions, we use the methods proposed by Prékopa and Boros (1991) and Wallace and Wets (1995) to eliminate the redundant Gale-Hoffman inequalities. The elimination procedure uses lower and upper bounds on the random demand and arc capacities, which are assumed to be known. We present the details of the elimination procedure in Appendix A for completeness. However, as it will be shown in Section 5.2.2, the time needed to eliminate the redundant Gale-Hoffman inequalities also increases fast with the size of the network. Therefore, our modeling approach will be most beneficial for networks of small and moderate sizes. In case of a large network, we propose to construct an aggregated version of the original network. This is a commonly used practice for the design of humanitarian logistics networks (Klibi and Martel, 2010; Schulz, 2008; Galindo and Batta, 2012) that pools some of the individual demand nodes to form aggregated nodes. For example, Klibi and Martel (2010) suggest to merge the demand points (based on the characteristics such as geographical and political) into a geographical demand zone with a computable centroid. In this study, we shall pool some of the adjacent individual nodes. The resulting aggregated nodes will form the basis to analyze the relief commodity needs and design the relief network. If a disaster happens, more detailed post-disaster decisions can be made on the basis of the original network.

Suppose that we perform the elimination procedure for the Gale-Hoffman inequalities associated with all the subsets of $I$. Let $V_t \subseteq I, t \in T \subseteq \{1,\dots,2^{|I|}-1\}$, be the subsets of $I$ that remain after performing the elimination procedure. Thus, these are the subsets of $I$ for which the associated Gale-Hoffman inequalities are non-redundant. Then, the model **SP2** is equivalently reformulated as

$$\begin{aligned} \textbf{SP2a} \qquad & \min \sum_{i\in I}\sum_{l\in L} F_{il}y_{il} + \sum_{i\in I} qr_i \\ & \text{s. t. } (2); (3); (4); (5) \\ & \qquad \mathbb{P}\left(\sum_{i\in V_t}(\xi_i^d - r_i) \le \sum_{i\in \bar{V}_t, j\in V_t} \xi_{ij}^v,\ t\in T\right) \ge p\,. \end{aligned} \tag{22}$$

The number of inequalities subjected to the probabilistic condition in (22) is $|T|$ and is typically much smaller than the number $(2^{|I|}-1)$ of the inequalities involved in (19). The elimination procedure provides a much more compact formulation of the probabilistic constraint (19), which, from a computational point of view, is much easier to deal with than (19).

Obviously, we can use the same procedure to obtain a more compact reformulation of **SP3**. Recall that we have defined $K$ regions in the network, where region $k$ comprises the nodes included in the set $I^k$. Let $V_t \subseteq I, t\in T^k \subseteq \{1,\dots,2^{|I|}-1\}$ be the subsets of $I^k$ that remain after performing the elimination procedure for the network defined by the set of nodes $I^k,\ k=1,\dots,K$. Then, an equivalent formulation of **SP3** becomes

$$\begin{aligned} \textbf{SP3a} \qquad & \min \sum_{i\in I}\sum_{l\in L} F_{il}y_{il} + \sum_{i\in I} qr_i \\ & \text{s. t. } (2); (3); (4); (5); (22) \end{aligned}$$

$$\mathbb{P}\left(\sum_{i\in V_t}(\xi_i^d - r_i) \leq \sum_{i\in \bar{V_t}^k, j\in V_t} \xi_{ij}^v \ , \ t\in T^k\right) \geq p'_k, \quad k=1,\ldots,K \ , \tag{23}$$

where $\bar{V_t}^k = I^k \setminus V_t$ for all $k=1,\ldots,K,\ t\in T^k$, and $p'_k = p'$ for all $k=1,\ldots,K$.

The remaining computational challenge is to solve the models **SP2a** and **SP3a** when the number of scenarios characterizing the uncertainty in the network is large, as it is desirable in the disaster context.

**4.2 Combinatorial Pattern-Based Formulations** In this section, we use the combinatorial pattern-based framework proposed by Lejeune (2012a) to reformulate the models **SP2a** and **SP3a**. It has been successfully used to solve probabilistic constrained stochastic programming problems in which the uncertainty is described by a huge (up to 50,000) number of scenarios. For self-containment purpose, we succinctly describe the main components of the method and refer the reader to Lejeune (2012a) for a more thorough description and an illustration of the method.

We shall use the method to reformulate the joint probabilistic constraint (22). Recall that $|T|$ is the number of inequalities in constraint (22). To ease the notation, we introduce the $|T|$-dimensional random vector $\zeta$ whose components, denoted by $\zeta_t,\ t\in T$, are linear combinations of random demand and arc capacities: $\zeta_t = \sum\limits_{i\in V_t}\xi_i^d - \sum\limits_{i\in\bar{V}_t, j\in V_t}\xi_{ij}^v, t\in T$. Moreover, the marginal cumulative probability distribution of $\zeta_t$ is designated by $F_t$. Then, we rewrite the chance constraint (22) in its canonical form with the random variables in the right-hand sides of the inequalities:

$$\mathbb{P}\left(\sum_{i\in V_t} r_i \geq \zeta_t \ , \ t\in T\right) \geq p \ . \tag{24}$$

Let us also introduce the $|T|$-dimensional deterministic vector $\varpi^s, s\in \Omega$, denoting the realization of the random vector $\zeta$ under scenario $s$:

$$\varpi_t^s = \sum_{i\in V_t} d_i^s - \sum_{i\in\bar{V}_t, j\in V_t} v_{ij}^s \ , \ t\in T, s\in\Omega \ .$$

The first step of the combinatorial method involves the partitioning of the set $\Omega$ of scenarios into two disjoints subsets $\Omega^+ = \{s\in\Omega : P(\zeta\leq\varpi^s)\geq p\}$ and $\Omega^- = \{s\in\Omega : P(\zeta\leq\varpi^s) < p\}$. We refer to the scenarios in $\Omega^+$ and $\Omega^-$ as the $p$-sufficient and $p$-insufficient scenarios, respectively.

In order to derive sufficient, possibly minimal, conditions for the chance constraint (22) to be satisfied, we shall consider all the scenarios that can be $p$-sufficient, i.e., satisfy the basic necessary conditions

$$F_t(\varpi_t^s)\geq p,\ t\in T \ . \tag{25}$$

The second step of the pattern-based modeling approach involves the generation of the exhaustive set of the points satisfying (25). This is accomplished by constructing the sets

$$Z_t = \{\varpi_t^s : F_t(\varpi_t^s)\geq p,\ s\in\Omega\},\ \ t\in T,$$

whose direct product $\bar{\Omega} = \prod\limits_{t\in T} Z_t$ provides the set $\bar{\Omega}$ of *relevant points* including all points that can possibly be $p$-sufficient. A relevant point $s\in\bar{\Omega}$ is represented by a vector $\varpi^s$ satisfying the $|T|$ conditions defined by (25). The disjoint sets $\bar{\Omega}^+ = \{s\in\bar{\Omega} : F(\varpi^s)\geq p\}$ and $\bar{\Omega}^- = \{s\in\bar{\Omega} : F(\varpi^s) < p\}$ are called the sets of $p$-sufficient and $p$-insufficient relevant points.

The next step involves the binarization of the probability distribution $F$ of $\zeta$ and is based on the concept of cut point (Boros et al., 1997). The binarization process is based on the set of cut points and provides a partially defined Boolean function (pdBf) that represents the feasibility of the joint chance constraint. The cut points are selected in a way that the pdBf obtained through the binarization process is consistent, i.e., that preserves the disjointedness between the set of binary images of the $p$-sufficient scenarios and the set of binary images of the $p$-insufficient scenarios. In a subsequent reformulation step, the pdBf is characterized by a set of mixed-integer linear inequalities involving a number of binary variables equal to the number of cut points. The notation $\{0,1\}^n$ refers to the $n$-dimensional unit cube.

Definition 4.1 *The binarization process is the mapping* $\mathbb{R}^{|T|} \rightarrow \{0,1\}^n$ *of a real-valued vector* $\varpi^s$ *into a binary one* $\beta^s$ *in such a way that the value of each component* $\beta^s_{jt}$ *is defined with respect to a cut point* $c_{jt}$ *as follows:*

$$\beta^s_{jt} = \begin{cases} 1 & \text{if } \varpi^s_t \geq c_{jt} \\ 0 & \text{otherwise} \end{cases} ,$$

*where* $c_{jt}$ *denotes the* $i^{th}$ *cut point associated with component* $\xi_t$,

$$j' < j \;\Rightarrow\; c_{j't} < c_{jt} \;,\; t \in T,\; j = 1, \ldots, n_t, \tag{26}$$

*and* $n = \sum_{t \in T} n_t$ *is the sum of the number* $n_t$ *of cut points for each component* $\xi_t$.

The binarization process defines the binary projection $\bar{\Omega}_B \subseteq \{0,1\}^n$ of $\bar{\Omega}$. The set of relevant Boolean points is denoted by $\bar{\Omega}_B = \bar{\Omega}^+_B \bigcup \bar{\Omega}^-_B$, and $\bar{\Omega}^+_B$ (resp., $\bar{\Omega}^-_B$) is the set of $p$-sufficient (resp., insufficient) relevant Boolean points. Evidently, the cut points are parameters whose values must not be defined arbitrarily. In order to identify the conditions that are necessary for (24) to hold, we need to use a consistent set of cut points (Boros et al., 1997) such that the resulting binarization process (26) preserves the disjointedness between the binary projections $\bar{\Omega}^+_B$ and $\bar{\Omega}^-_B$ of $\bar{\Omega}^+$ and $\bar{\Omega}^-$, respectively. This is achieved by carrying out the binarization process by using the sufficient-equivalent set of cut points (Lejeune, 2012a,b)

$$C^e = \bigcup_{t \in T} C_t \;, \quad \text{where} \quad C_t = \{\varpi^s_t : \varpi^s_t \geq F_t^{-1}(p),\; s \in \Omega\}.$$

The notation $F_t^{-1}(p)$ refers to the $p$-quantile of the marginal distribution of $\zeta_t$.

The binarization process of the probability distribution with a consistent set of cut points permits the representation of the combination $(F,p)$ of the probability distribution $F$ of $\zeta$ and of the prescribed probability level $p$ as a partially defined Boolean function (pdBf) defined by the pair of disjoint sets $\Omega^+_B, \Omega^-_B \subseteq \{0,1\}^n$. The pdBf is a mapping $g : (\Omega^+_B \bigcup \Omega^-_B) \rightarrow \{0,1\}$ such that $g(s) = 1$ if $s \in \Omega^+_B$ and $g(s) = 0$ if $s \in \Omega^-_B$. The domain of the mapping $g$ is a cube $\{0,1\}^n$ of dimension $n$ equal to the number of cut points. As demonstrated by Lejeune (2012a), this in turn allows the derivation of a deterministic MIP reformulation **SP2_IP** equivalent to **SP2a** and containing $n$ binary variables.

Theorem 4.1 *Let* $\xi$ *be a* $|T|$*-variate random variable and* $c_{jt},\; t \in T,\; j = 1, \ldots, n_t$, *be the cut points in the sufficient-equivalent set. The mixed-integer programming problem* **SP2_IP**

$$\textbf{SP2\_IP} \qquad \min \quad \sum_{i \in I} \sum_{l \in L} F_{il} y_{il} + \sum_{i \in I} q r_i$$

$$s.\ t. \quad (2); (3); (4); (5)$$

$$\sum_{t \in T} \sum_{j=1}^{n_t} \beta^s_{jt}\; u_{jt} \leq |T| - 1 \;, \quad s \in \bar{\Omega}^-_B \tag{27}$$

$$\sum_{i \in V_t} r_i \geq \sum_{j=1}^{n_t} c_{jt}\; u_{jt} \;, \quad t \in T \tag{28}$$

$$\sum_{j=1}^{n_t} u_{jt} = 1 \;, \quad t \in T \tag{29}$$

$$u_{jt} \in \{0,1\}, \quad t \in T, j = 1, \ldots, n_t \tag{30}$$

*is a deterministic equivalent formulation of the probabilistically constrained model* **SP2a**.

The set of constraints (27) ensures that none of the $p$-insufficient realizations has a binary image identical to any vector $u$ feasible for **SP2_IP**. Hence, any feasible $u$ will coincide with one of the $p$-sufficient realizations. The set of constraints (28) guarantees that the pre-positioned commodities $\sum_{i \in V_t} r_i$ satisfy the requirements

$\sum_{j=1}^{n_t} c_{jt} u_{jt}$ imposed by a $p$-sufficient scenario, which in turn implies that the solution is feasible for the chance constraint (24). Note that the requirements of the $p$-sufficient scenario (say $s$) are defined in terms of its binary image ($u_{jt} = \beta^s_{jt}$) and the cut points $c_{jt}$. The constraints (29) ensure that exactly one variable $u_{jt}$ taking value one is associated with each component $\zeta_t$ of the random vector. Thus, each component of the random vector is accounted for in the inequalities defining the conditions required for (24) to hold. The constraints (30) define the binary character of the variables $u_{jt}$.

We shall now present the MIP formulation of the model **SP3a** including a local probabilistic constraint for each region $k,\ k = 1, \dots, K$. Let $|T^k|$ be the number of non-redundant Gale-Hoffman inequalities associated with region $k$. The components of the $|T^k|$-dimensional random vector $\zeta^k,\ k = 1, \dots, K$, are

$$\zeta^k_t = \sum_{i \in V_t} \xi^d_i - \sum_{i \in \bar{V}^k_t, j \in V_t} \xi^v_{ij}, t \in T^k.$$

Thus, the local chance constraint associated with region $k$ reads

$$\mathbb{P}\left( \sum_{i \in V_t} r_i \geq \zeta^k_t \ ,\ t \in T^k \right) \geq p'. \tag{31}$$

As done for model **SP2a**, we also define the $|T^k|$-dimensional deterministic vector $\varpi^{s,k}, s \in \Omega,\ k = 1, \dots, K$, denoting the realization of the random variable $\zeta^k$ under scenario $s$:

$$\varpi^{s,k}_t = \sum_{i \in V_t} d^s_i - \sum_{i \in \bar{V}^k_t, j \in V_t} v^s_{ij}\ ,\ t \in T^k, s \in \Omega\ . \tag{32}$$

Let $c^k_{jt}, t \in T^k,\ j = 1, \dots, n^k_t$, be the sufficient-equivalent cut points associated with region $k$, the number of which is equal to $n^k = \sum_{t \in T^k} n^k_t$. Additionally, $\beta^{s,k}_{jt}$ is the binary mapping of $\varpi^{s,k}_t$ using the cut points $c^k_{jt},\ t \in T^k,\ j = 1, \dots, n^k_t$, and $u^k_{jt}, t \in T^k,\ j = 1, \dots, n^k_t$, are the binary variables corresponding to the cut points $c^k_{jt}$. The notation $\bar{\Omega}^-_{B^k}$ refers to the set of relevant $p'$-insufficient scenarios for region $k$. The deterministic formulation of **SP3a** with local probabilistic constraints, referred to as **SP3_IP**, is given by

**SP3_IP**

$$\begin{aligned}
\min \quad & \sum_{i \in I} \sum_{l \in L} F_{il} y_{il} + \sum_{i \in I} q r_i \\
\text{s. t.} \quad & (2); (3); (4); (5); (27); (28); (29); (30) \\
& \sum_{t \in T^k} \sum_{j=1}^{n^k_t} \beta^{s,k}_{jt}\ u^k_{jt} \leq |T^k| - 1\ , && s \in \bar{\Omega}^-_{B^k},\ k = 1, \dots, K \\
& \sum_{i \in V_t} r_i \geq \sum_{j=1}^{n^k_t} c^k_{jt}\ u^k_{jt}\ , && t \in T^k,\ k = 1, \dots, K \\
& \sum_{j=1}^{n^k_t} u^k_{jt} = 1\ , && t \in T^k,\ k = 1, \dots, K \\
& u^k_{jt} \in \{0, 1\}, && t \in T^k, j = 1, \dots, n^k_t,\ k = 1, \dots, K.
\end{aligned}$$

The optimal solution of **SP2_IP** (resp., **SP3_IP**) is equivalent to that of the probabilistic programming model **SP2** (resp., **SP3**). A critical feature of the combinatorial pattern method is that the number of binary variables in the MIP reformulations **SP2_IP** and **SP3_IP** is not an increasing function of the number of scenarios. It contains a significantly lower number of binary variables, which is equal to the number of cut points (Definition 4.1) used for the binarization process ($n$ for **SP2_IP** and $n + \sum_{k=1}^{K} n^k$ for **SP3_IP**). This is what makes it applicable to problems in which uncertainty is described with a large number of scenarios.

**5. Computational Study and Modeling Insights** This section is decomposed into two main parts. In the first subsection, we conduct a case study and sensitivity analysis that provide insights about the impact of the main features of the proposed models. The case study focuses on designing a pre-disaster relief network for responding to hurricane disasters in the Southeast part of the US. The description of the structure of the network was first presented in Rawls and Turnquist (2010). The second subsection assesses the computational effectiveness of the proposed modeling approach. We show that it allows to find the optimal location-allocation policies when the uncertainty is characterized with a large number of scenarios.

The pre-processing algorithm used for eliminating the redundant Gale-Hoffman inequalities and the binarization process employed for deriving the proposed MIP formulations are implemented in Matlab. The AMPL modeling language is used to formulate the mathematical programming problems which are solved with the Cplex 12.3 solver. Each problem instance is solved on a 64-bit Dell Precision T5400 Workstation with Quad Core Xeon Processor X5460 3.16GHz CPU, and 4X2GB of RAM.

**5.1 Case Study - Hurricane Threat in the US Southeast Region and Sensitivity Analysis** The case study concerns the PRNDP for the threat of hurricanes in the Southeastern US Region, which is represented by a graph including 30 nodes and 55 links (see Figure 1). The case study focuses on satisfying the demand for water. Each node represents a demand point and facilities of three different sizes can be located at any node in the network. The uncertainty is described using 51 scenarios constructed by Rawls and Turnquist (2010). Each scenario represents a joint realization of random demand and arc capacities. We use the parameter values given by Rawls and Turnquist (2010) and described below. The demand for water is expressed in units of 1000 gallons, and the unit acquisition price is $647.7. Up to 252 (resp., 2823 and 5394) units of water can be stored in a small (resp., medium and large) facility. The setup cost for opening a facility of small (resp., medium and large) size is $19,600 (resp., $188,400 and $300,000). The unit shortage penalty cost and the unit holding cost for unused material are assumed to be ten times the purchasing cost and 25% of the purchase price, respectively. We refer the reader to Rawls and Turnquist (2010) for a more detailed description of the structure of the network.

As discussed in Section 3.3, we introduce one local probabilistic constraint for each region. We have decomposed the US Southeast area into three disjoint regions as illustrated in Figure 1:

- Region 1: Brownsville, Corpus Christi, San Antonio, Dallas, Houston, Little Rock, Memphis, Biloxi, Jackson, Monroe, Lake Charles, Baton Rouge, Hammond, New Orleans, Mobile;
- Region 2: Birmingham, Nashville, Atlanta, Savannah, Columbia, Charlotte, Wilmington, Charleston;
- Region 3: Tallahassee, Lake City, Jacksonville, Orlando, Tampa, Miami, Key West.

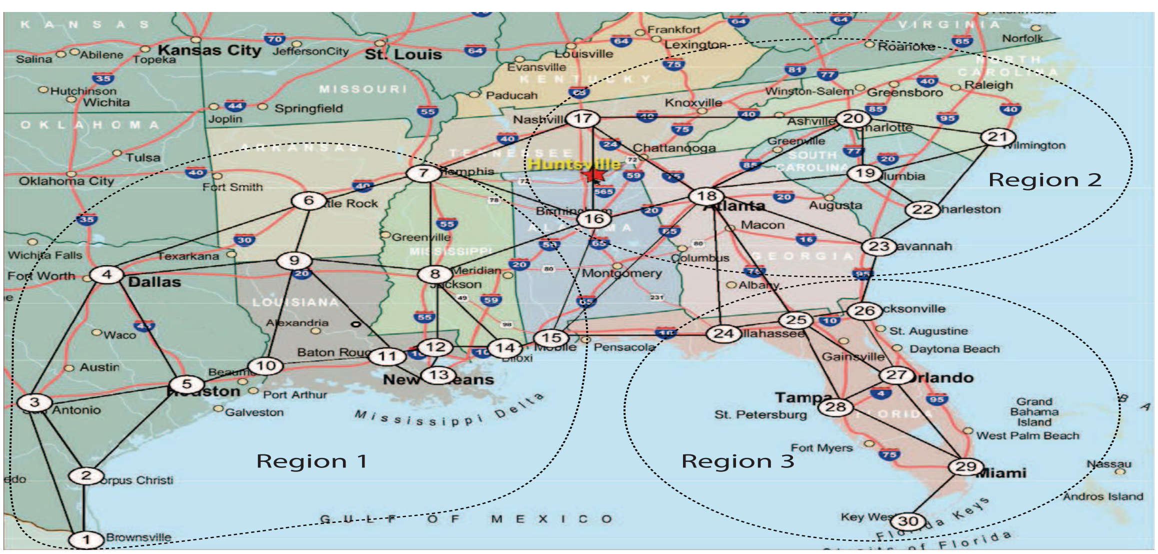


Figure 1: Southeastern US Region Facing Hurricane Risk

We benchmark our results with those obtained by solving the model proposed in Rawls and Turnquist (2010) under the assumption that all the stocked supplies remain usable after a disaster occurs. The benchmark model is thereafter referred to as **RTM**. It is a modified version of the two-stage model (1)-(9) whose objective function (6) is replaced by

$$\min \sum_{i\in I} hw_i^s + \sum_{(i,j)\in A} c_{ij}x_{ij}^s + \sum_{i\in I} \tau \max\{r_i + \sum_{(j,i)\in A} x_{ji}^s - \sum_{(i,j)\in A} x_{ij}^s - d_i^s, 0\} \tag{33}$$

that includes the transportation and holding costs. Here, $c_{ij}$ and $\tau$ denote the unit transportation cost on link $(i,j)$ and the unit holding cost, respectively. In the base problem instance of **RTM**, referred to as the Base Problem, the parameters take the values described above and $c_{ij} = 0.3$ dollar per unit-mile. Note that the objective function of the second-stage problem given in (33) can be linearized by introducing surplus variables. Solving the large-scale MIP formulation of the Base Problem provides an optimal solution at which a total of 8973 units of water is pre-positioned in two small facilities located at Charlotte and Charleston and three medium facilities located at Wilmington, Tallahassee and Miami. The total facility setup and acquisition costs amount to \$6,416,210 dollars. Moreover, the optimal solution of **RTM** leads to the occurrence of demand shortage with a probability of 14.49%. The probability of demand shortage associated with a location-allocation policy is basically computed as

$$1 - \alpha = \sum_{s\in\Omega} \left\{ p_s : \max_{i\in N} \; w_i^{s*} > 0 \right\}, \tag{34}$$

where $w_i^{s*}$ is the amount of demand shortage for node $i$ under scenario $s$ at the specified solution and $p_s$ is the probability of scenario $s$. Clearly, $\max_{i\in N} \; w_i^{s*}$ taking a strictly positive value implies that a demand shortage occurs for at least one of the nodes at the network under scenario $s$. Table 8 (column 6 with $h$ = \$6477) provides an in-depth description of the optimal solution of the Base Problem.

**5.1.1 Model SP2_IP** As mentioned in Section 4.1, the elimination of the redundant Gale-Hoffman inequalities is computationally challenging. Hence, we consider an aggregated version of the network representing the Southeastern US Region. Recall that the original network includes 30 nodes and 55 links. As Klibi and Martel (2010), we use the proximity criterion to construct an aggregated network comprising 16 nodes. The demand at an aggregated node is equal to the sum of the demands at the individual nodes merged into the aggregated one. The aggregated network is only used to obtain non-redundant Gale-Hoffman inequalities, and formulate the network-wide probabilistic constraint. However, the region-wide probabilistic constraints are formulated based on the original network. Thus, we solve the PRNDP for the original 30-node network but we ensure the specified network-wide reliability level for the aggregated version.

We solve thirteen instances of **SP2_IP**, which are constructed by successively setting the parameter $p$ to values between 0.7 and 0.975 in increments of 0.025, in addition to $p = 0.99$. We then conduct a sensitivity analysis to investigate how the location-allocation decisions, the associated costs, and the network and region-wide reliability levels vary with the risk parameter $p$. In what follows, we use excerpts from Table 5 to present a detailed analysis of the optimal solutions of the thirteen **SP2_IP** problems.

First, we focus on how the total cost, defined as the sum of the acquisition (light-grey shaded area in Figure 2) and the facility setup costs (black area in Figure 2), evolves with the parameter $p$. The total facility setup cost accounts for roughly 8% of the total cost for any considered level of reliability. Figure 2 shows that each type of cost is non-decreasing with respect to the enforced reliability level. Note that the costs associated with the optimal location-allocation policies for $p = 0.725$ and $p = 0.75$ are identical. The same comment applies for $p = 0.775$ and $p = 0.80$, and also for $p = 0.875$ and $p = 0.90$. In line with this, the rate at which the total cost increases with the network-wide reliability level is relatively smooth for $p$ ranging between 0.70 and 0.925 (see Figure 2). The optimal network design obtained for $p = 0.925$ costs 1.90 times more than the one for $p = 0.7$. As a sharp contrast to this, the total cost increases by 87% and 25% as we change the value of $p$ from 0.925 to 0.95, and from 0.975 to 0.99, respectively.

Figure 3 displays the inventory level at each region for different network-wide reliability levels. It is clear that the total inventory level and cost associated with a more reliable network design (more reliable policy) are in general higher. The network-wide inventory level increases at a quasi linear rhythm for $p \leq 0.925$, and increases sharply for larger reliability levels.

We shall now compare the optimal location-allocation policies provided by **SP2_IP** for different values of

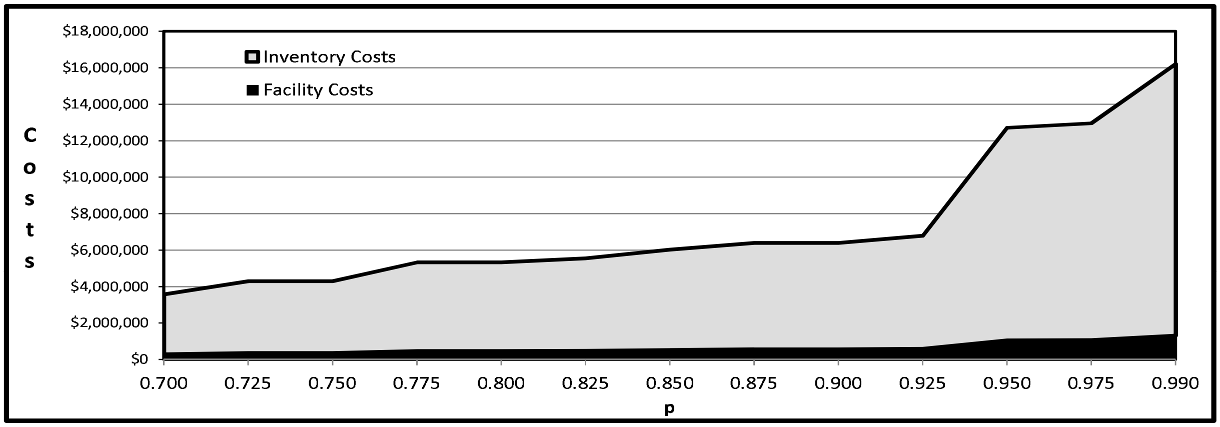


Figure 2: Total Cost, Inventory and Facility Setup Costs Versus Network-Wide Reliability Level $p$

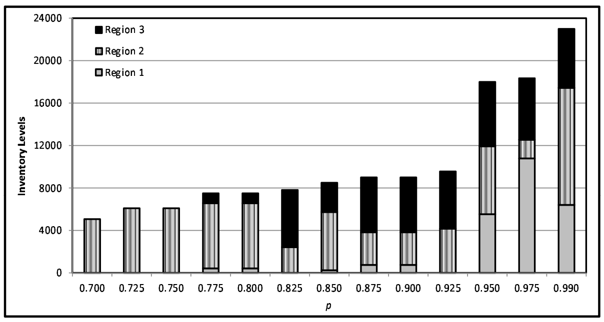


Figure 3: Inventory Levels Versus Network-Wide Reliability Level $p$

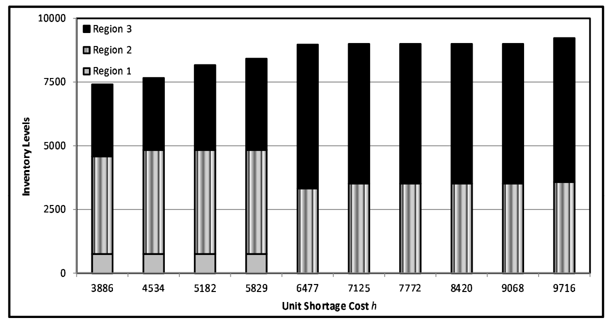


Figure 4: Model **RTM**: Inventory Level in Each Region Versus Unit Shortage Cost Parameter $h$

$p$ with the optimal policy obtained by solving the large-scale MIP formulation of the base problem instance of **RTM** (Base Problem). While the proportion of the total cost incurred by opening the facilities never exceeds 8.93% with **SP2_IP**, the proportion of the facility setup cost in the total first-stage cost reaches 9.80% at the optimal solution of the Base Problem. Another difference is that the optimal solution of the Base Problem does not involve opening any large facility, while the optimal solution of **SP2_IP** requires the opening of at least one large facility for all the considered values of $p$. At the optimal solution of the Base Problem, the facility setup and acquisition costs amount to \$6,416,210, and the probability of satisfying the demand across the network, i.e., the network-wide reliability, is 0.8611. The optimal solution of **SP2_IP** (with $p = 0.8611$) shows that it is possible to design a network with a reliability exceeding 0.8611 at a total cost lower than \$6,321,510. This network design policy suggests to open one medium size facility in Corpus Christi, three small facilities in Nashville, Atlanta, and Wilmington, and one large facility in Tampa. The decrease in the total cost is achieved by savings in both the facility setup and acquisition costs. The decrease in the facility setup cost by 9.46% is more noticeable.

In our model, we do not consider the operational costs of the second-stage problem while determining the location-allocation policies. In practice, when a disaster happens, we can make the post-disaster decisions on distributing the relief supplies by solving the second-stage problem under the realized uncertain parameters and the location-allocation decisions determined a priori by our models. The objective function of the second-stage problem could then include the operational costs. For consistency with practice, we shall evaluate our optimal policies in terms of the operational costs. This requires the solution of the second-stage problem of **RTM** under each scenario, where the first-stage decisions are set equal to the optimal solution $(\mathbf{r}^*, \mathbf{y}^*)$ of **SP2_IP**. The solutions of the second-stage problems and the resulting expected operational costs can be obtained all together by solving a large-scale model which we call **RTM2**. Basically, **RTM2** is the large-scale linear programming formulation of **RTM** with fixed first-stage decisions. The expected value of the resulting second-stage transportation, shortage, and holding costs are displayed in Table 5. The holding costs increase as the network-wide reliability level increases. Such an increase in the holding costs is expected, since the proposed models focus only on satisfying the demand. However, the transportation and, most interestingly, the shortage costs do not change monotonically with respect to $p$. For example, the expected shortage costs decrease as $p$ increases in the range between 0.7 and 0.95. However, the expected shortage costs amount to $\$4,590,140$ for $p = 0.975$, and are more than twice larger than the ones ($\$2,112,030$) for $p = 0.925$. The expected second stage costs tend to decrease as $p$ increases, except for very high reliability levels ($p = 0.975$ and $p = 0.99$) when the shortage costs increase sharply.

**5.1.2 Model SP3_IP** We shall now analyze the effect of adding a local reliability constraint for each region. Recall that $|T^k|$ is the number of non-redundant Gale-Hoffman inequalities for region $k$. The achieved region-wide reliability for region $k$ associated with an optimal location-allocation policy represented by $(\mathbf{r}^*, \mathbf{y}^*)$ is computed as:

$$\alpha'_k = 1 - \sum_{s \in \Omega} \left\{ p_s : \max_{t \in T^k} \ \varpi_t^{s,k} - \sum_{i \in V_t} r_i^* \ > 0 \right\}. \tag{35}$$

Observe that by (31)-(32), $\varpi_t^{s,k} - \sum\limits_{i \in V_t} r_i^*$ indicates by how much the Gale-Hoffman inequality corresponding to the set of nodes $V_t, \ t \in T^k$, is violated at the optimal solution $(r^*, y^*)$ under scenario $s$. While calculating $\alpha'_k$ we consider the scenarios under which at least one of the Gale-Hoffman inequalities associated with region $k$ is violated, i.e., a demand shortage occurs within the region. Table 5 reports the region-wide reliability level for each region achieved by the optimal solution of **SP2_IP** for various values of the network-wide reliability level $p$. It appears clearly that satisfying the network-wide reliability constraint does not guarantee that the local probabilistic constraints are also satisfied. Thus, it is not guaranteed that the demand at a node is satisfied by the relief supplies pre-positioned within the same region with a specified high probability. For example, at the optimal solution of **SP2_IP**, the region-wide reliability level for region 2 ($\alpha'_2$) is only 0.32 (resp., 0.47) when the network-wide reliability parameter $p$ is 0.75 (resp., 0.99). Whereas, when we solve **SP3_IP** enforcing the local probabilistic constraints for $p' = 0.60$, the optimal policy leads to a region-wide reliability level of 0.909 (resp., 0.915) for region 2 when $p$ is 0.75 (resp., 0.99%). These observations emphasize the contribution of incorporating local probabilistic constraints into **SP2_IP**. Table 6 (resp. 7) provides the detailed solutions of **SP3_IP** for the region-wide reliability is constrained to be at least equal to 0.60 (resp., 0.7) at each region when $p$ ranges from 0.7 to 0.99.

Let $z_p^*$ be the optimal objective function value of **SP2_IP** enforcing a network-wide reliability level of $p$. Further, let $z_{p,p'}^*$ denote the optimal objective value of **SP3_IP** enforcing a network-wide reliability of $p$ and region-wide reliability level of $p'$ for each region. The values of the ratio $\frac{z_{p,p'}^*}{z_p^*}$ showing the relative cost increases due to the incorporation of the local probabilistic constraints are reported in Table 1.

| $p$ | 0.7 | 0.725 | 0.75 | 0.775 | 0.8 | 0.825 | 0.85 | 0.875 | 0.9 | 0.925 | 0.95 | 0.975 | 0.99 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\frac{z_{p,0.6}^*}{z_p^*}$ | 1.011 | 1 | 1 | 1 | 1 | 1.001 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\frac{z_{p,0.7}^*}{z_p^*}$ | 1.207 | 1.005 | 1.005 | 1 | 1 | 1.001 | 1.032 | 1 | 1 | 1 | 1 | 1 | 1 |

Table 1: Impact of Incorporating Local Probabilistic Constraints on Total Cost

As shown by Table 1, **SP3_IP** provides network design policies under which the regions are self-sufficient (i.e., in terms of fulfilling their own commodity needs with probability at least 0.6 or 0.7) without incurring almost any additional costs when the network-wide reliability is reasonably large ($p \geq 0.725$). For instance, the total cost at the optimal solution of **SP3_IP** with $p = 0.975$ and $p' = 0.70$ is the same as the one at the optimal solution of **SP2_IP** with $p = 0.975$ for which the achieved region-wide reliability is 0.47 for region 2. The impact of the local constraints on the total cost diminishes as $p$ increases, i.e., the global probabilistic constraint becomes more restrictive. Figures 5 and 6 display the optimal inventory level at each region determined by solving **SP3_IP** when the region-wide reliability level is 0.6 and 0.7, respectively.

Here we briefly discuss some comparative results for **SP3_IP** and **RTM**. At the optimal solution of the Base Problem, the achieved network-wide reliability is 0.86 and the achieved region-wide reliability levels for regions 1, 2, and 3 are 0.57, 0.81, and 0.78, respectively. However, the solution of **SP3_IP** with required reliability levels $p = 0.86$, $p'_1 = 0.57$, $p'_2 = 0.81$, and $p'_3 = 0.78$ indicates that a network with reliability levels

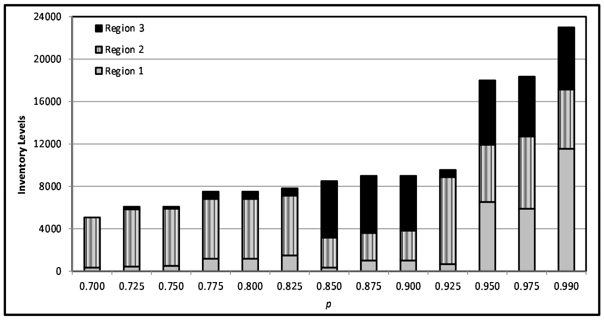


Figure 5: Inventory Level in Each Region Versus Network-Wide Reliability Level $p$ When $p' = 0.6$

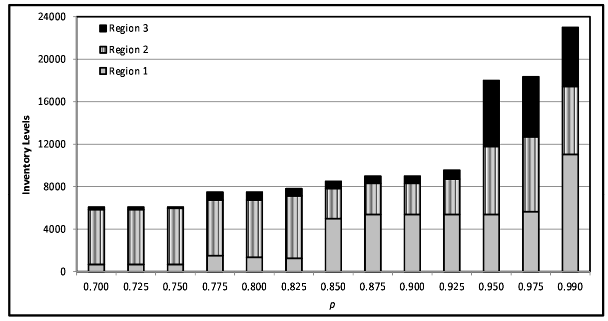


Figure 6: Inventory Level in Each Region versus the Network-Wide Reliability Level $p$ When $p' = 0.7$

exceeding the required ones can be constructed at a total cost of \$6,321,510. Note that the facility setup and acquisition costs associated with the optimal solution of the Base Problem reach \$6,416,210.

As above for **SP2_IP**, we fix the first-stage decisions of **RTM** equal to the optimal solution $(r^*, y^*)$ of **SP3_IP** and solve the model **RTM2**. The expected values of the resulting second-stage transportation, shortage, and holding costs are displayed in Tables 6 and 7 for $p' = 0.6$ and $p' = 0.7$, respectively. Similar to **SP2_IP**, it can be seen that the resulting holding costs in general increase and the expected second-stage transportation and shortage costs do not change monotonically with respect to the risk parameter $p$. When the required network-wide reliability level $p$ is larger than or equal to 0.875, the solution of **SP3_IP** is the same regardless of whether the required local reliability levels are equal to 0.6 or 0.7. The same comment applies when $p$ is equal to 0.775, 0.8, and 0.825. A larger local reliability level $p' = 0.7$ (instead of 0.6) triggers a cost increase of 0.3% (resp., 0.4%, 0.4%, and 19%) when the network-wide reliability level is equal to 0.85 (resp., 0.75, 0.725, and 0.70). Clearly, the effect of enforcing a higher local reliability level becomes more visible as $p$ decreases.

Considering **RTM** as the benchmark, not involving the transportation costs in the second-stage problem of the proposed models (see Tables 5-8) leads to an increase of almost 10% in the total expected second stage costs. However, the total cost is only about 1-2% more than the one associated with **RTM**. Moreover, **SP3_IP** allows us to incorporate the equity and responsiveness criteria in addition to the network reliability.

**5.1.3 Alternate approach based on penalizing the amount of demand shortage** One can argue that penalizing the demand shortage amounts and enforcing an upper bound on the probability of a demand shortage are alternate ways of controlling the existence of a feasible flow to satisfy the demand. Thus, **RTM** and our proposed models involve alternate stockout measures. We analyze the policies obtained with **RTM** and our proposed models by varying the shortage penalty cost in **RTM** and the reliability levels in our models. We solve **RTM** for various values of the unit shortage cost, which is defined as $h = \lambda \cdot 6477$ with $\lambda$ being 1 in the Base Problem. We construct eleven problem instances by setting the coefficient $\lambda$ to values between 0.5 and 1.5 in increments of 0.1. Table 8 presents the detailed results for the optimal solutions of **RTM** and Figure 4 displays the inventory level in each region for different values of the unit shortage cost. It is significant to observe that **RTM** provides policies in which no relief supplies are pre-positioned within region 1 for $h \geq \$6477$. This is due to the fact that, in the constructed set of scenarios, the demand in region 1 is in general smaller than this in the other regions. However, not pre-positioning relief supplies within region 1 can be an issue to respond quickly, in particular if a disaster affects the western part of region 1 (see Figure 1). In that respect, our policies, which require the pre-positioning of commodities in region 1, may perform better in terms of responsiveness.

Recall that **RTM** does not impose any explicit constraint on the occurrence of a demand shortage. At the optimal solution of **RTM**, the achieved network-wide reliability ranges from 0.7685 (for $h =\$3886$) to 0.9148 (for $h$ is between \$7125 and \$9716). Next, given the optimal first-stage decisions of **RTM** which define the number, location and size of the facilities and the allocation of commodities, we compute the largest network- and region-wide reliability levels that can be achieved. This is done by solving four MIP problems in which the network-wide reliability (resp., the local reliability for each region) is maximized. The decision variables are the transportation, shortage, and surplus variables, while the inventory pre-positioning and facility location variables are fixed. The largest (resp., smallest) network-wide reliability level is 0.9148 (resp., 0.7741) and obtained when $h$ takes the values \$7125, \$7772, \$8420, \$9068, and \$9716 (resp., \$3886). The local reliability for region 1 varies between 0.57 and 0.77, and is constant and equal to 0.81 and 0.78 for regions 2 and 3, respectively. These results show that a very high network-wide reliability level is not achieved with **RTM** even if the unit shortage cost is increased by 50% (from \$6477 to \$9718). Thus, it can be challenging to implement a (very) high-reliability policy with the **RTM** model. When reliability levels are less than 0.95 (e.g., 0.9148 as mentioned above) are targeted, **RTM** and our models **SP2_IP** and **SP3_IP** produce policies which perform closely in terms of the network-wide reliability. Our modeling approach might be preferable, as it avoids the difficult problem of estimating the cost of unsatisfied demand (required by **RTM**). Moreover, **SP3_IP** allows the enforcement of high levels of region-wide reliability.

**5.2 Scalability of Modeling Approach** In this section, we study the scalability of the modeling approach. First, we assess the reduction in the number of Gale-Hoffman inequalities achieved by using the elimination process. Second, we analyze the computational efficiency of the proposed combinatorial reformulation method used to solve the PRNDP models. We also propose a scenario generation method that takes into account the dependency structures inherent in disaster relief networks. The scalability study is based on the computational time needed to solve the MIP formulations **SP2_IP** and **SP3_IP** to optimality and on the striking contrast between the (large) number of considered scenarios and the (small) number of binary variables in the MIP formulations.

**5.2.1 Test Laboratory** The computational study is based on the hurricane network discussed in Section 5.1. Several network configurations detailed in the next subsections are generated for the PRNDP problem and will be used to conduct the computational analysis.

**5.2.2 Elimination of Gale-Hoffman Inequalities** We consider four configuration variants of the hurricane network in the Southeastern US region. The configurations include 16 aggregated nodes and 29 links and differ in terms of the upper ($v_{ij}^u$) and lower ($v_{ij}^l$) bounds of the capacity of each arc $(i, j)$. For each network configuration, we derive non-redundant Gale-Hoffman inequalities using the elimination process described in Section 4.1. The first two columns of Table 2 define the network configuration. Column 4 displays the total number $2^{|I|} - 1$ of Gale-Hoffman inequalities, while columns 5 to 8 report the number $S_i$ of Gale-Hoffman inequalities eliminated at each stage $i$ of the elimination process. The notation $|T|$ (column 9) denotes the number of non-redundant Gale-Hoffman inequalities, i.e., those that have not been eliminated through the elimination process. The last column indicates the CPU time in seconds to carry out the elimination process. For each network configuration, we report the results for the entire network and for each region. Note that the number of Gale-Hoffman inequalities is invariant to the set of scenarios.

Table 2 shows that the first stage in the elimination process removes more than 85% of the 65,535 Gale-Hoffman inequalities describing the conditions for the feasible flow of the entire network. At the end of the elimination process, more than 99.97% of the Gale-Hoffman inequalities have been discarded. Note that the computational time increases fast with the number of nodes in the network. While the elimination process is completed in less than two minutes for region 1 that includes 15 nodes (see Figure 1), the process takes more than 105 minutes for the aggregated network that comprises 16 nodes.

**5.2.3 Scenario Generation** Before analyzing the computational results on the scalability of the proposed approach based on combinatorial patterns, we recall the importance of using a fine characterization of the uncertainty in the context of disaster management. This is accomplished in our study by representing uncertainty with a large number of scenarios which accounts for the dependency structure specific to disaster relief systems.

The identification of scenarios is a challenging task (as mentioned by Snyder (2006)) and has been extensively studied in the stochastic programming literature. There has been significant recent interest in the generation of relevant scenarios in the context of disaster management. Disaster scenarios can be generated based on the available historical data, information provided by specialized centers, and opinions of experts. For example,

| Network Configurations | | Reliability for | Gale-Hoffman Inequalities: Elimination Process | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $v_{ij}^u$ | $v_{ij}^l$ | | $2^{\lvert I \rvert}-1$ | $S_1$ | $S_2$ | $S_3$ | $S_4$ | $\lvert T \rvert$ | Time (sec) |
| 12000 | 6000 | Entire Network | 65,535 | 55,772 | 7605 | 2114 | 34 | 10 | 6394 |
| | | Region 1 | 32,767 | 28,100 | 2309 | 2325 | 18 | 15 | 86 |
| | | Region 2 | 255 | 102 | 148 | 4 | 0 | 1 | 1 |
| | | Region 3 | 127 | 67 | 34 | 24 | 1 | 1 | 1 |
| 10000 | 5000 | Entire Network | 65,535 | 55,772 | 9445 | 292 | 21 | 5 | 6393 |
| | | Region 1 | 32,767 | 28,100 | 4141 | 489 | 26 | 11 | 86 |
| | | Region 2 | 255 | 102 | 152 | 0 | 0 | 1 | 1 |
| | | Region 3 | 127 | 67 | 51 | 7 | 0 | 2 | 1 |
| 8000 | 4000 | Entire Network | 65,535 | 55,772 | 9002 | 695 | 58 | 8 | 6396 |
| | | Region 1 | 32,767 | 28,100 | 3549 | 985 | 107 | 26 | 93 |
| | | Region 2 | 255 | 102 | 151 | 1 | 0 | 1 | 1 |
| | | Region 3 | 127 | 67 | 42 | 10 | 4 | 4 | 1 |
| 7000 | 3500 | Entire Network | 65,535 | 55,772 | 8491 | 1111 | 143 | 18 | 6392 |
| | | Region 1 | 32,767 | 28,100 | 3028 | 1372 | 220 | 47 | 112 |
| | | Region 2 | 255 | 102 | 150 | 2 | 0 | 1 | 1 |
| | | Region 3 | 127 | 67 | 38 | 10 | 8 | 4 | 1 |

Table 2: Gale-Hoffman Inequalities for the Hurricane Network in the US Southeastern Region

Balcık and Beamon (2008) analyze historical natural hazards data from the National Geophysical Data Center to obtain the network-related parameters, and estimate the demand scenarios based on mortality statistics from previous disasters and area population. Geographical information system (GIS)-based software, such as HAZUS (FEMA, ND) and Consequences Assessment Tool Set (CATS) (DTRA, ND), can also be employed to obtain relevant disaster scenarios. These software packages are developed to estimate the potential losses from natural disasters and provide valuable information about the effect of impending natural hazards. Future implementations of our models could certainly benefit from ongoing developments in scenario generation.

We now present a scenario generation method that is specifically suited for our modeling framework, and enables us to accurately represent the dependency structure within a relief network. The method has minimal data requirements, relying for the most part on publicly available geographical and population data. In our numerical study we utilize this method to randomly generate problem instances involving up to 20000 scenarios.

As Rawls and Turnquist (2010), we consider five severity levels for hurricanes. We refer to levels 1 and 5 as the least and the most severe levels, respectively. We postulate that the magnitude of the commodity demand at a node and the damage inflicted to the capacity of an arc depend on the proximity of the node or the arc to the epicenter of the disaster. Further, we assume that the epicenter is at a coastal location.

To generate a scenario, the first step is to randomly select a coastal node as the epicenter and determine the severity level of the hurricane. In the second step, we define the nodes and arcs that are affected by the disaster and its impact on them. To that end, we determine three areas $A_1$, $A_2$, and $A_3$ that are defined as concentric areas with radius $R_1$, $R_2$, and $R_3$ centered at the epicenter $A$ of the disaster (see Figure 7). Next, an intensity level is assigned to each concentric area. The innermost area is most severely impacted by the disaster. Nodes within $R_1$ (resp., within $R_2$ but not within $R_1$; within $R_3$ but not within $R_2$) are impacted with the intensity assigned to $A_1$ (resp., $A_2$; $A_3$). Nodes that are not within distance $R_3$ of the epicenter are not affected. Considering the dependency structure based on the proximity to the epicenter, the intensity levels of three areas under five severity levels of hurricanes are specified. The demand at a node is a function of the population size associated with the node, the severity of the disaster and a random coefficient based on the intensity level of the area that the node belongs. This random coefficient is generated from a uniform distribution with lower and upper limits defined according to the corresponding intensity level. To determine how the disaster affects the capacity of an arc $(i, j)$, we verify whether $(i, j)$ crosses the areas $A_1$, $A_2$, and $A_3$. If the link $(i, j)$ does not cross $A_1$, $A_2$, and $A_3$, than its capacity is not affected. Otherwise, we calculate the segment $c_{ij}^1$ (resp. $c_{ij}^2$, $c_{ij}^3$) of link $(i, j)$ that spans over $A_1$ (resp., $A_2$, $A_3$), and define $\kappa^{ij} = \arg\max_l c_{ij}^l$. Thus, $A_{\kappa^{ij}}$ is the disaster-stricken area that is "most crossed" by link $(i, j)$. We use the intensity level of $A_{\kappa^{ij}}$ to define the proportion by which the nominal capacity of link $(i, j)$ is reduced. For example, for the capacity of the link $BC$ (i.e., connecting $B$ and $C$) in Figure 7, the segment $BD$ is longer than $DC$. Thus, the largest proportion of link $BC$ is in area $A_2$ and $BC$ is affected by the intensity level of $A_2$. We calculate the length of segments using the cosine formula for triangles.

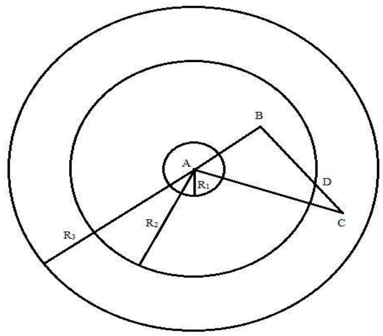


Figure 7: Illustration of the Region for the Scenario Generation Method

**5.2.4 Scalability and Computational Efficiency of Combinatorial Reformulation Method** To evaluate the scalability of our modeling approach, we consider 80 problem instances. For each network configuration of the hurricane network, we consider 20 problem instances that differ in terms of *(i)* the network-wide reliability level $p$ (0.85, 0.9, 0.95, 0.975, 0.99), *(ii)* the number $|\Omega|$ of scenarios (5000, 20000), and *(iii)* the inclusion of local reliability constraints. Table 3 reports the solution times and the number $n$ and $n' = n + \sum_{k=1}^{3} n^k$ of binary variables included in **SP2_IP** and **SP3_IP**, respectively.

It can be seen that the number of integer variables increases as the enforced reliability level decreases. Even for low values taken by $p$ (i.e., 0.85) and $p'$ (i.e., 0.70), the number of integer variables remains low. We also observe that the number of binary variables is significantly smaller than the number of scenarios used to represent uncertainty. Indeed, when 5000 (resp., 20000) scenarios are considered, the maximum number of binary variables is 42 (resp., 46) in **SP2_IP** and 150 (resp., 164) in **SP3_IP**. This also indicates that the number of binary variables is about the same regardless of the number (5000, 20000) of scenarios. This comment applies to both **SP2_IP** and **SP3_IP**, and points out the scalability of the method, as well as its applicability to problems in which uncertainty is finely represented. This is confirmed by the computational times needed to solve the problem instances to optimality. Each of the 80 problem instances is solved in less than 2 seconds. The reason for this is that the Boolean modeling approach represents the satisfiability of the probabilistic constraint as a pdBf $g : (\Omega_B^+ \bigcup \Omega_B^-) \to \{0, 1\}$ such that $g(s) = 1$ if $s \in \Omega_B^+$ and $g(s) = 0$ if $s \in \Omega_B^-$. Since the domain of the mapping of $g$ is in $\{0, 1\}^n$, we are able to derive a deterministic equivalent reformulation comprising a number $n$ of binary variables that does not increase with the number of scenarios.

As a benchmark, we attempt to solve the MIP reformulation **SP1a**, equivalent to **SP2_IP**, for the problem instance with $v_{ij}^u$=12000, $v_{ij}^l$=6000, $p$=0.9, and $|\Omega| = 20000$. While **SP1a** includes 135 binary variables (90 variables $y_{il}$ indicating whether a facility of size $l$ is located at node $i$ and 45 variables $u_{jt}$ used for the binarization process), problem **SP1a** contains 20090 binary variables. Moreover, it also includes a very large number of continuous decision variables and constraints that are not needed in **SP2_IP**. It is thus not surprising that **SP1a** for the above-described instance could not be solved within 2 hours of computing time. This is a striking contrast with the formulation **SP2_IP** that can be solved to optimality in less than 0.1 seconds. The overall reformulation process takes for that instance 6394.05 seconds: 6394 seconds are needed for the elimination of the redundant inequalities (Table 2).

The computational study shows that our solution method enables the use of a large number of scenarios, and can be efficiently used to design highly reliable, fair and reasonably responsive pre-disaster relief networks.

**6. Conclusions, Implications, and Future Research** The principal goal of this study is to support relief planners in making long-term facility location and stock pre-positioning decisions that enable an efficient

| Problem Instances | | SP2_IP | | SP3_IP | | Problem Instances | | SP2_IP | | SP3_IP | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $v_{ij}^u = 12000$, $p$ | $v_{ij}^l = 6000$ $\lvert\Omega\rvert$ | $n$ | Time | $n'$ | Time | $v_{ij}^u = 10000$, $p$ | $v_{ij}^l = 5000$ $\lvert\Omega\rvert$ | $n$ | Time | $n'$ | Time |
| 0.85 | 5000 | 42 | 0.05 | 98 | 0.06 | 0.85 | 5000 | 25 | 0.02 | 59 | 0.03 |
| | 20000 | 46 | 0.08 | 107 | 0.11 | | 20000 | 26 | 0.02 | 60 | 0.04 |
| 0.9 | 5000 | 41 | 0.02 | 97 | 0.03 | 0.9 | 5000 | 23 | 0.03 | 57 | 0.04 |
| | 20000 | 45 | 0.03 | 106 | 0.05 | | 20000 | 26 | 0.02 | 60 | 0.04 |
| 0.95 | 5000 | 41 | 0.03 | 97 | 0.03 | 0.95 | 5000 | 20 | 0.06 | 54 | 0.06 |
| | 20000 | 45 | 0.05 | 106 | 0.06 | | 20000 | 26 | 0.05 | 60 | 0.06 |
| 0.975 | 5000 | 40 | 1.73 | 96 | 1.73 | 0.975 | 5000 | 20 | 0.03 | 54 | 0.04 |
| | 20000 | 44 | 0.36 | 105 | 0.38 | | 20000 | 25 | 0.02 | 59 | 0.03 |
| 0.99 | 5000 | 33 | 0.06 | 89 | 0.85 | 0.99 | 5000 | 13 | 0.05 | 47 | 0.05 |
| | 20000 | 39 | 0.03 | 100 | 0.03 | | 20000 | 20 | 0.03 | 54 | 0.04 |
| $v_{ij}^u = 8000$, $p$ | $v_{ij}^l = 4000$ $\lvert\Omega\rvert$ | $n$ | Time | $n'$ | Time | $v_{ij}^u = 7000$, $p$ | $v_{ij}^l = 3500$ $\lvert\Omega\rvert$ | $n$ | Time | $n'$ | Time |
| 0.85 | 5000 | 37 | 0.01 | 99 | 0.02 | 0.85 | 5000 | 41 | 0.05 | 150 | 0.07 |
| | 20000 | 37 | 0.02 | 104 | 0.04 | | 20000 | 46 | 0.02 | 164 | 0.06 |
| 0.9 | 5000 | 35 | 0.28 | 97 | 0.31 | 0.9 | 5000 | 40 | 0.02 | 149 | 0.08 |
| | 20000 | 35 | 0.02 | 103 | 0.05 | | 20000 | 45 | 0.05 | 163 | 0.08 |
| 0.95 | 5000 | 33 | 0.03 | 95 | 0.04 | 0.95 | 5000 | 40 | 0.05 | 149 | 0.03 |
| | 20000 | 35 | 0.08 | 102 | 0.11 | | 20000 | 45 | 0.03 | 163 | 0.09 |
| 0.975 | 5000 | 29 | 0.03 | 91 | 0.10 | 0.975 | 5000 | 40 | 0.05 | 149 | 0.06 |
| | 20000 | 29 | 0.02 | 96 | 0.05 | | 20000 | 45 | 0.03 | 163 | 0.06 |
| 0.99 | 5000 | 18 | 0.11 | 80 | 0.15 | 0.99 | 5000 | 35 | 0.06 | 144 | 0.14 |
| | 20000 | 22 | 0.06 | 89 | 0.07 | | 20000 | 39 | 0.08 | 157 | 0.12 |

Table 3: Scalability of Approach - Hurricane Network without Local Reliability Constraints ($p' = 0.7$)

immediate response to sudden-onset natural disasters. To this end, we propose a new scenario-based modeling approach for the design of a reliable relief network. The stochastic nature of our models allows us to take into account the uncertainty in the severity of the disasters, their impact on the transportation network, and on the demand for relief commodities. Our models feature probabilistic constraints that ensure that the demand for relief supplies across the network is satisfied with a high probability. In addition, we consider local probabilistic constraints that can be incorporated in order to guarantee a certain level of fairness and enhance the responsiveness of the system. The local constraints ensure that each region can provide for its needs with a certain probability.

Our modeling and solution methods rest on three pillars:

- We use a stochastic version of the Gale-Hoffman inequalities to characterize the probabilistic existence of a feasible flow of relief commodities in the network.
- We implement a preprocessing algorithm (Prékopa and Boros, 1991; Wallace and Wets, 1995) to identify non-redundant Gale-Hoffman inequalities.
- We use a method based on combinatorial patterns (Lejeune, 2012a) to reformulate the stochastic programming models.

The reformulated models take the form of MIP problems that can be solved extremely quickly; more than 97% of the MIPs are solved within one second. The reason for this is that the number of binary variables is typically several orders of magnitude smaller than the number of scenarios. For example, we have reformulated one of the stochastic models for an instance featuring 20,000 scenarios as an MIP with 45 binary variables. The ability to account for a very large number of scenarios is a key contribution of our approach, allowing for a fine characterization of the underlying uncertainties, including the dependencies between various random parameters. In the current implementation, the computational bottleneck is the elimination of the redundant Gale-Hoffman inequalities; reducing the necessary computational burden for larger networks is a subject of ongoing research.

A sensitivity analysis of our models can provide valuable managerial insights on the role of various input parameters. Our numerical results show that the facility setup and acquisition costs are mainly driven by the global reliability level. In particular, even small additional increases to high reliability levels can lead to substantial increases in cost. This highlights the need for decision makers to perform a careful sensitivity analysis before committing to a particular global reliability level. On the other hand, while the local reliability level significantly affects the pre-positioning policies, it appears to have little impact on the total cost. This shows the possibility of significantly enhancing the responsiveness of the relief system without a corresponding budget increase, which is a key insight for decision makers. Our results underline the importance of pre-disaster planning, which is frequently underestimated compared to that post-disaster decisions.

A key feature of our models is that they do not require estimating the cost of unsatisfied demand. Obtaining these cost estimates may be challenging in the context of disaster management when a shortfall in relief commodities could lead to the death of human beings. We have performed a computational experiment to highlight the differences between our reliability-based models and models in which infeasibility is not constrained but is instead penalized. This experiment concerns a case study that focuses on the risk of hurricanes in the Southeastern US region. The results reveal that the penalty-based benchmark model does not guarantee a highly reliable network, even with significantly increased unit shortage costs. Moreover, we have observed that our models can provide policies with higher reliability levels and lower costs than benchmark policies.

Acquiring accurate input data is crucial for ensuring the validity and effectiveness of our proposed models. The relevant data falls into four broad categories:

- Network data (geographical coordinates of demand nodes and candidate facility locations, transportation infrastructure) is publicly available for developed countries.
- Facility setup and acquisition costs are typically available to the decision makers.
- The cost of unsatisfied demand, which can be hard to quantify, does not factor into our models, due to our reliability-based approach.
- The data necessary for disaster scenario generation, including epicenters and intensities of potential disasters as well as the size of the affected population, can be obtained from historical data sources such as the Emergency Events Database EM-DAT (ND).

In addition to sensitivity analysis and data acquisition, two other issues deserve special attention. First, an appropriate scenario generation method is essential if we are to obtain meaningful results. We have developed a detailed scenario generation method that takes into consideration the dependencies between the random parameters of the network, and can be parameterized to be consistent with historical disaster data. Alternatively, GIS-based software tools such as HAZUS can be utilized to generate scenarios. Second, the grouping of locations into clusters, or regions, can be used to express various preferences of the decision makers. For instance, systematically defining regions to guarantee limits on stochastic response times is an interesting future research topic.

Our models can support relief organizations in making both strategic (facility location) and tactical (inventory level) decisions. While facility location decisions typically imply long-term commitments, inventory level decisions are easier to adjust if more accurate information becomes available. Short-term tactical concerns are particularly relevant for certain types of natural disasters, such as hurricanes. For example, the National Hurricane Center provides predictions about the path of a hurricane about five days in advance (Galindo and Batta, 2012). Such predictions allow us to identify facilities that could be destroyed by the disaster, and to update our estimates of the potential damage sustained by the transportation network. Based on this updated information, it is possible to make recourse decisions about modifying the existing relief network by repositioning supplies, either to existing facilities or to temporary locations. Studying the joint use and interaction of short-term and long-term planning approaches is an exciting direction for future research.

**Acknowledgment.** X. Hong and M. Lejeune are grateful for the support of the Army Research Office (Grant # W911NF-09-1-0497). The third author has been supported by The Scientific and Technological Research Council of Turkey (TUBITAK) Career Award # 111M543.

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**Appendix A. Elimination Process** We briefly discuss the preprocessing method to eliminate the redundant Gale-Hoffman inequalities. The elimination procedure uses lower and upper bounds on the demand and capacity functions, which are assumed to be known. We denote the lower and upper bounds of the random capacity of arc $(i,j)$ by $v^l_{ij}$ and $v^u_{ij}$, respectively: $P(v^l_{ij} \leq \xi^v_{ij} \leq v^u_{ij}) = 1, \quad \forall (i,j) \in A$. Recall that the random net demand at location $i$ is obtained as $\xi^d_i - r_i$ and $M_l$ is the capacity of a facility of type $l$. Denoting the largest and the smallest possible values of the demand at node $i$ by $u^d_i$ and $u^l_i$, respectively, the random net demand is bounded from below by $l_i = -\max_{l \in L} M_l$ and bounded from above by $u_i = u^d_i$. The lower and upper bounds of the random total net demand associated with a subset $B \subseteq I$ of nodes are:

$$\begin{aligned} l(B) &= \sum_{i \in B} l_i, & u(B) &= \sum_{i \in B} u_i, \\ v^l(\bar{B}, B) &= \textstyle\sum_{i \in \bar{B}, j \in B} v^l_{ij} & v^u(\bar{B}, B) &= \textstyle\sum_{i \in \bar{B}, j \in B} v^u_{ij}. \end{aligned}$$

Example A.1 *Consider the four-node network presented in Figure 8. The lower and upper bounds of the random demand at each node and the random arc capacities are given in Table 4.*

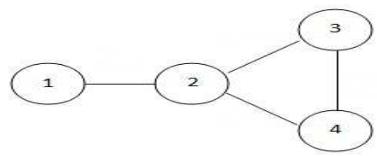

Figure 8: Four-Node Network

| | Lower Bound ($l^d_i$) | Upper Bound ($u^d_i$) | | Lower Bound ($v^l_{ij}$) | Upper Bound ($v^u_{ij}$) |
|---|---|---|---|---|---|
| $\xi^d_1$ | -10 | 20 | $\xi^v_{12}$ | 10 | 20 |
| $\xi^d_2$ | -25 | 15 | $\xi^v_{23}$ | 5 | 10 |
| $\xi^d_3$ | -15 | 18 | $\xi^v_{24}$ | 5 | 10 |
| $\xi^d_4$ | -25 | 15 | $\xi^v_{34}$ | 5 | 10 |

Table 4: Upper and Lower Bounds of the Random Demand and Arc Capacities for the Four-Node Network

*Assuming $\xi^v_{ij} = \xi^v_{ji}$ for all $i < j$ and denoting the pre-positioned inventory levels by $r_i$ , $i = 1, \ldots, 4$, the corresponding stochastic Gale-Hoffman inequalities are given by*

$$\begin{aligned}
\xi^d_1 - r_1 + \xi^d_2 - r_2 + \xi^d_3 - r_3 + \xi^d_4 - r_4 &\leq 0 && (36)\\
\xi^d_1 - r_1 &\leq \xi^v_{12} && (37)\\
\xi^d_2 - r_2 &\leq \xi^v_{12} + \xi^v_{23} + \xi^v_{24} && (38)\\
\xi^d_3 - r_3 &\leq \xi^v_{23} + \xi^v_{34} && (39)\\
\xi^d_4 - r_4 &\leq \xi^v_{24} + \xi^v_{34} && (40)\\
\xi^d_1 - r_1 + \xi^d_2 - r_2 &\leq \xi^v_{23} + \xi^v_{24} && (41)\\
\xi^d_1 - r_1 + \xi^d_3 - r_3 &\leq \xi^v_{12} + \xi^v_{23} + \xi^v_{34} && (42)\\
\xi^d_1 - r_1 + \xi^d_4 - r_4 &\leq \xi^v_{12} + \xi^v_{24} + \xi^v_{34} && (43)\\
\xi^d_2 - r_2 + \xi^d_3 - r_3 &\leq \xi^v_{12} + \xi^v_{24} + \xi^v_{34} && (44)\\
\xi^d_2 - r_2 + \xi^d_4 - r_4 &\leq \xi^v_{12} + \xi^v_{23} + \xi^v_{34} && (45)\\
\xi^d_3 - r_3 + \xi^d_4 - r_4 &\leq \xi^v_{23} + \xi^v_{24} && (46)\\
\xi^d_1 - r_1 + \xi^d_2 - r_2 + \xi^d_3 - r_3 &\leq \xi^v_{24} + \xi^v_{34} && (47)\\
\xi^d_1 - r_1 + \xi^d_2 - r_2 + \xi^d_4 - r_4 &\leq \xi^v_{23} + \xi^v_{34} && (48)\\
\xi^d_1 - r_1 + \xi^d_3 - r_3 + \xi^d_4 - r_4 &\leq \xi^v_{12} + \xi^v_{23} + \xi^v_{24} && (49)\\
\xi^d_2 - r_2 + \xi^d_3 - r_3 + \xi^d_4 - r_4 &\leq \xi^v_{12} && (50)
\end{aligned}$$

For this example, we can show that six of fifteen Gale-Hoffman inequalities are redundant.

The elimination procedure presented here consists of four stages; generating facets, elimination by upper bounds, elimination by lower bounds and elimination by linear programming.

**Stage 1: Generation of all the facets.** We generate the facet inequalities using the recursive algorithm proposed by Wallace and Wets (1995). The method involves the introduction of a slack node with uncapacitated arcs connecting it to all other nodes. This permits to formulate the so-called facet generating problem as a balanced supply and demand problem used to generate the facets of the feasible set of a network flow problem. In Stages 2, 3, and 4, we then withdraw the facets that become superfluous when we take into consideration the additional information about the lower and upper bounds of the demand and arc capacities. For the four-node network example, 12 inequalities ((36) to (41), (44) to (48) and (50)) are generated.

**Stage 2: Elimination by upper bounds** (Prékopa and Boros, 1991). Stage 2 uses the arc capacities to eliminate the redundant inequalities. In contrast to Prékopa and Boros (1991), we model the arc capacities as random variables. We refer to the Gale-Hoffman inequality corresponding to the subset of nodes $F \subseteq I$ as "inequality $(F)$". We eliminate an inequality $(B)$ if $u(B) \leq v^l(\bar{B}, B)$. Indeed, since

$$v^l(\bar{B}, B) = \sum_{i \in \bar{B}, j \in B} v^l_{ij} \leq \sum_{i \in \bar{B}, j \in B} \xi^v_{ij} = v(\bar{B}, B) \qquad \text{and} \qquad u(B) \leq v^l(\bar{B}, B),$$

we have

$$u(B) \leq v(\bar{B}, B).$$

The three inequalities (38), (40) and (45) are eliminated after executing Stage 2.

**Stage 3: Elimination by lower bounds.** If $l(F)$ is finite, the inequality $(F)$ can be rewritten as:

$$d(F) - l(F) \leq v(\bar{F}, F) - l(F). \tag{51}$$

If $G \subset F$, then $l(G)$ is also finite. Furthermore, if we have the inequality

$$v^u(\bar{F}, F) - l(F) \leq v^l(\bar{G}, G) - l(G), \tag{52}$$

then (51) implies that

$$d(G) - l(G) \leq v(\bar{G}, G) - l(G)$$

holds true. Indeed, the sum in $d(F) - l(F)$ has nonnegative terms, and thus,

$$d(G) - l(G) \leq \; d(F) - l(F) \leq \; v(\bar{F}, F) - l(F) \leq \; v^u(\bar{F}, F) - l(F) \leq \; v^l(\bar{G}, G) - l(G) \leq \; v(\bar{G}, G) - l(G).$$

The above result is the basis to eliminate the inequalities using the lower bounds of the arc capacities. Let $H$ be the collection of the subsets of $I$, which have not been eliminated so far. For each $F \in H$, we identify all subsets $G \subseteq F$ for which (52) holds. Such subsets $G \in H$ are eliminated and the set $H$ representing the remaining inequalities are updated accordingly. Stage 3 does not eliminate any inequality for our example.

**Stage 4: Elimination by linear programming.** Suppose that the set $H$ is composed of the subsets of $I$ which have not been eliminated so far. To check whether a subset $F_0 \in H$ can be eliminated, we solve a linear programming problem. Let us introduce the auxiliary decision variables $z_i, i \in I$. Considering the random arc capacities, it is easy to see that the inequality $(F_0)$ is a consequence of the other remaining inequalities (all $F \in H,\ F \neq F_0$) only when the optimum value of the following linear problem

$$\begin{aligned} \max \ & \sum_{i \in F_0} z_i \\ \text{subject to } & \sum_{i \in F} z_i \leq v^l(\bar{F}, F), \quad \forall F \in H,\ F \neq F_0 \\ & l_i \leq z_i \leq u_i, \quad i \in I \end{aligned}$$

is smaller than or equal to $v^l(\bar{F}_0, F_0)$. In our example, the linear programming stage does not remove any inequality. Nine of the fifteen inequalities remain after the elimination procedure.

Table 5: Location-Allocation Decisions with Model **SP2_IP** [a]

| Enforced Network-Wide Reliability Level $p$ | | 0.7 | 0.725 | 0.75 | 0.775 | 0.8 | 0.825 | 0.85 | 0.875 | 0.9 | 0.925 | 0.95 | 0.975 | 0.99 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Total Facility Setup Cost | | 300,000 | 358,800 | 358,800 | 476,400 | 476,400 | 488,400 | 527,600 | 566,800 | 566,800 | 600,000 | 1,056,800 | 1,076,400 | 1,317,600 |
| Total Acquisition Cost | | 3,282,540 | 3,944,040 | 3,944,040 | 4,857,170 | 4,857,170 | 5,063,070 | 5,504,220 | 5,828,650 | 5,828,650 | 6,191,880 | 11,658,600 | 11,885,000 | 14,896,500 |
| Total Cost | | 3,582,540 | 4,302,840 | 4,302,840 | 5,333,570 | 5,333,570 | 5,551,470 | 6,031,820 | 6,395,450 | 6,395,450 | 6,791,880 | 12,715,400 | 12,961,400 | 16,214,100 |
| Total Inventory Level | | 5,068 | 6,089 | 6,089 | 7,499 | 7,499 | 7,817 | 8,498 | 8,999 | 8,999 | 9,560 | 18,000 | 18,350 | 22,999 |
| | Number of | | | | | | | | | | | | | |
| Region 1 | Small Facilities | 0 | 0 | 0 | 2 | 2 | 0 | 1 | 3 | 3 | 0 | 1 | 0 | 4 |
| | Medium Facilities | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | Large Facilities | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 2 | 1 |
| Region 2 | Small Facilities | 0 | 3 | 3 | 3 | 3 | 0 | 1 | 1 | 1 | 0 | 4 | 7 | 1 |
| | Medium Facilities | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 |
| | Large Facilities | 0 | 1 | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 2 |
| Region 3 | Small Facilities | 0 | 0 | 0 | 4 | 4 | 0 | 0 | 0 | 0 | 0 | 3 | 2 | 1 |
| | Medium Facilities | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| | Large Facilities | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| Achieved Region-Wide Reliability Level | $\alpha'_1$ | 57.22% | 57.22% | 57.22% | 60.66% | 74.586% | 57.21% | 57.21% | 89.64% | 89.64% | 57.22% | 89.64% | 89.64% | 89.64% |
| | $\alpha'_2$ | 80.97% | 32.29% | 32.29% | 38.84% | 38.84% | 66.17% | 80.98% | 46.60% | 46.60% | 25.27% | 88.28% | 47.21% | 47.21% |
| (see (35)) | $\alpha'_3$ | 69.82% | 85.29% | 85.29% | 85.29% | 85.29% | 78.48% | 78.48% | 78.48% | 78.48% | 69.82% | 85.29% | 93.11% | 93.11% |
| Achieved Network-Wide Reliability Level | | 70.03% | 75.60% | 75.60% | 81.66% | 81.66% | 83.16% | 85.51% | 90.73% | 90.73% | 92.55% | 97.27% | 97.79% | 99.39% |
| Solve **RTM** where the first-stage decisions are fixed to be the ones presented above (Solve **RTM2**) | | | | | | | | | | | | | | |
| Total First-Stage Cost (=Total Cost of **SP2_IP**) | | 3,582,540 | 4,302,840 | 4,302,840 | 5,333,570 | 5,333,570 | 5,551,470 | 6,031,820 | 6,395,450 | 6,395,450 | 6,791,880 | 12,715,400 | 12,961,400 | 16,214,100 |
| Expected Second-Stage Costs | Transportation | 540,794 | 569,931 | 563,112 | 506,961 | 506,961 | 519,050 | 564,073 | 506,215 | 506,215 | 569,772 | 496,427 | 505,266 | 344,756 |
| | Holding | 320,937 | 439,436 | 439,436 | 614,628 | 614,628 | 651,304 | 743,109 | 813,291 | 813,291 | 899,855 | 2,206,820 | 2,325,370 | 3,051,770 |
| | Shortage | 10,437,200 | 8,580,640 | 8,562,180 | 6,438,610 | 6,438,610 | 5,846,610 | 5,107,330 | 4,670,280 | 4,670,280 | 4,500,510 | 2,112030 | 4,590,140 | 3,531,290 |
| Expected Total Second-Stage Cost | | 11,298,931 | 9,590,007 | 9,564,728 | 7,560,199 | 7,560,199 | 7,016,964 | 6,414,512 | 5,989,786 | 5,989,786 | 5,970,137 | 4,815,277 | 7,420,776 | 6,927,816 |
| Total Cost of Model **RTM2** | | 14,881,471 | 13,892,847 | 13,867,568 | 12,893,769 | 12,893,769 | 12,568,434 | 12,446,332 | 12,385,236 | 12,3852,36 | 12,762,017 | 17,530,677 | 20,382,176 | 23,141,916 |
| Achieved Network-Wide Reliability Level $\alpha$ (see (34)) | | 69.48% | 72.18% | 72.18% | 81.85% | 81.85% | 81.85% | 85.51% | 90.92% | 90.92% | 91.99% | 92.55% | 86.67% | 93.11% |

Table 6: Location-Allocation Decisions with Model **SP3_IP** - $p' = 0.60$

| Enforced Network-Wide Reliability Level $p$ | | 0.7 | 0.725 | 0.75 | 0.775 | 0.8 | 0.825 | 0.85 | 0.875 | 0.9 | 0.925 | 0.95 | 0.975 | 0.99 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Total Facility Setup Cost | | 339,200 | 358,800 | 358,800 | 476,400 | 476,400 | 496,000 | 527,600 | 566,800 | 566,800 | 606,000 | 1,056,800 | 1,076,400 | 1,317,600 |
| Total Acquisition Cost | | 3,282,540 | 3,944,040 | 3,944,040 | 4,857,170 | 4,857,170 | 5,063,070 | 5,504,220 | 5,828,650 | 5,828,650 | 6,191,880 | 11,658,600 | 11,885,000 | 14,896,500 |
| Total Cost | | 3,621,740 | 4,302,840 | 4,302,840 | 5,333,570 | 5,333,570 | 5,559,070 | 6,031,820 | 6,395,450 | 6,395,450 | 6,797,880 | 12,715,400 | 12,961,400 | 16,214,100 |
| Total Inventory Level | | 5,068 | 6,089 | 6,089 | 7,499 | 7,499 | 7,817 | 8,498 | 8,999 | 8,999 | 9,560 | 18,000 | 18,350 | 22,999 |
| | Number of | | | | | | | | | | | | | |
| Region 1 | Small Facilities | 2 | 2 | 2 | 5 | 5 | 6 | 2 | 3 | 3 | 3 | 5 | 2 | 3 |
| | Medium Facilities | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | Large Facilities | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 2 |
| Region 2 | Small Facilities | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 6 | 1 |
| | Medium Facilities | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 |
| | Large Facilities | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| Region 3 | Small Facilities | 0 | 1 | 1 | 3 | 3 | 3 | 0 | 0 | 0 | 3 | 3 | 1 | 2 |
| | Medium Facilities | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | Large Facilities | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 1 |
| Achieved Region-Wide Reliability Level | $\alpha'_1$ | 60.66% | 60.66% | 60.66% | 81.26% | 81.26% | 82.34% | 60.66% | 81.26% | 81.26% | 66.91% | 89.64% | 93.67% | 93.67% |
| | $\alpha'_2$ | 83.32% | 90.89% | 90.89% | 84.33% | 84.33% | 84.33% | 80.41% | 66.17% | 80.41% | 92.96% | 83.59% | 91.46% | 91.46% |
| (see (35)) | $\alpha'_3$ | 69.82% | 74.59% | 74.59% | 74.59% | 74.59% | 74.59% | 78.48% | 78.48% | 78.48% | 74.59% | 85.29% | 85.29% | 85.29% |
| Achieved Network-Wide Reliability Level | | 70.03% | 75.60% | 75.60% | 82.41% | 82.41% | 83.16% | 85.51% | 90.73% | 90.73% | 92.55% | 97.27% | 97.79% | 99.39% |
| Solve **RTM** where the first-stage decisions are fixed to be the ones presented above (Solve **RTM2**) | | | | | | | | | | | | | | |
| Total First-Stage Cost (=Total Cost of **SP3_IP**) | | 3,621,740 | 4,302,840 | 4,302,840 | 5,333,570 | 5,333,570 | 5,559,070 | 6,031,820 | 6,395,450 | 6,395,450 | 6,797,880 | 12,715,400 | 12,961,400 | 16,214,100 |
| Expected Second-Stage Costs | Transportation | 507,846 | 538,621 | 572,713 | 597,545 | 601,373 | 593,853 | 532,370 | 587,094 | 577,105 | 607,162 | 400,438 | 371,881 | 344,491 |
| | Holding | 319,873 | 435,218 | 435,218 | 620,833 | 620,833 | 666,279 | 743,659 | 814,147 | 813,403 | 898,315 | 2,245,180 | 2,299,270 | 3,009,820 |
| | Shortage | 10,394,600 | 8,393,470 | 8,393,470 | 6,686,790 | 6,686,790 | 6,445,620 | 5,129,330 | 4,704,490 | 4,674,750 | 4,438,920 | 3,646,170 | 3,546,310 | 1,853,400 |
| Expected Total Second-Stage Cost | | 11,222,319 | 9,367,309 | 9,401,401 | 7,905,168 | 7,908,996 | 7,705,752 | 6,405,359 | 6,105,731 | 6,065,258 | 5,944,397 | 6,291,788 | 6,217,461 | 5,207,711 |
| Total Cost of Model **RTM2** | | 14,844,059 | 13,670,149 | 13,704,241 | 13,238,738 | 13,242,566 | 13,264,822 | 12,437,179 | 12,501,181 | 12,460,708 | 12,742,277 | 19,007,188 | 19,178,861 | 21,421,811 |
| Achieved Network-Wide Reliability Level $\alpha$ (see (34)) | | 69.47% | 75.04% | 75.04% | 81.85% | 81.85% | 81.85% | 84.95% | 84.80% | 90.17% | 91.99% | 93.11% | 93.11% | 93.11% |

[a] In Tables 5, 6, 7 and 8, the costs reported are expressed in US dollars and the unit holding cost is \$647.7/4. In Tables 5, 6, and 7, the unit shortage cost is \$6477.

Table 7: Location-Allocation Decisions with Model **SP3_IP** - $p' = 0.70$

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Enforced Network-Wide Reliability Level $p$ | | 0.7 | 0.725 | 0.75 | 0.775 | 0.8 | 0.825 | 0.85 | 0.875 | 0.9 | 0.925 | 0.95 | 0.975 | 0.99 |
| Total Facility Setup Cost | | 378,400 | 378,400 | 378,400 | 476,400 | 476,400 | 496,000 | 547,200 | 566,800 | 566,800 | 606,000 | 1,056,800 | 1,076,400 | 1,317,600 |
| Total Acquisition Cost | | 3,944,040 | 3,944,040 | 3,944,040 | 4,857,170 | 4,857,170 | 5,063,070 | 5,504,220 | 5,828,650 | 5,828,650 | 6,191,880 | 11,658,600 | 11,885,000 | 14,896,500 |
| Total Cost | | 4,322,440 | 4,322,440 | 4,322,440 | 5,333,570 | 5,333,570 | 5,559,070 | 6,051,420 | 6,395,450 | 6,395,450 | 6,797,880 | 12,715,400 | 12,961,400 | 16,214,100 |
| Total Inventory Level | | 6,089 | 6,089 | 6,089 | 7,499 | 7,499 | 7,817 | 8,498 | 8,999 | 8,999 | 9,560 | 18,000 | 18,350 | 22,999 |
| | Number of | | | | | | | | | | | | | |
| Region 1 | Small Facilities | 3 | 3 | 3 | 6 | 6 | 5 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| | Medium Facilities | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | Large Facilities | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 2 |
| | Number of | | | | | | | | | | | | | |
| Region 2 | Small Facilities | 0 | 0 | 0 | 0 | 0 | 2 | 0 | 1 | 1 | 2 | 4 | 7 | 4 |
| | Medium Facilities | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 |
| | Large Facilities | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 |
| Region 3 | Small Facilities | 1 | 1 | 1 | 3 | 3 | 3 | 3 | 3 | 3 | 4 | 4 | 1 | 1 |
| | Medium Facilities | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | Large Facilities | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 |
| Achieved Region-Wide Reliability Level (see (35)) | $\alpha'_1$ | 71.22% | 71.22% | 71.22% | 80.70% | 80.70% | 81.26% | 89.64% | 89.64% | 89.64% | 89.64% | 89.64% | 93.67% | 93.67% |
| | $\alpha'_2$ | 90.89% | 90.89% | 90.89% | 90.89% | 90.89% | 84.33% | 80.41% | 80.41% | 80.41% | 80.98% | 91.46% | 91.46% | 88.28% |
| | $\alpha'_3$ | 74.59% | 74.59% | 74.59% | 74.59% | 74.59% | 74.59% | 74.59% | 74.59% | 74.59% | 74.59% | 85.29% | 85.29% | 85.29% |
| Achieved Network-Wide Reliability Level | | 75.60% | 75.60% | 75.60% | 82.41% | 82.41% | 83.16% | 85.51% | 90.73% | 90.73% | 92.55% | 97.73% | 97.79% | 99.94% |
| Solve **RTM** where the first-stage decisions are fixed to be the ones presented above (Solve **RTM2**) | | | | | | | | | | | | | | |
| Total First-Stage Cost (=Total Cost of **SP3_IP**) | | 4,322,440 | 4,322,440 | 4,322,440 | 5,333,570 | 5,333,570 | 5,559,070 | 6,051,420 | 6,395,450 | 6,395,450 | 6,797,880 | 12,715,400 | 12,961,400 | 16,214,100 |
| Expected Second-Stage Costs | Transportation | 535,715 | 535,715 | 546,214 | 739,471 | 746,064 | 669,506 | 671,388 | 662,664 | 662,664 | 913,548 | 501,835 | 458,648 | 472,113 |
| | Holding | 435,218 | 435,218 | 435,218 | 620,258 | 620,258 | 660,229 | 745,309 | 840,875 | 840,875 | 897,830 | 2,266,550 | 2,320,390 | 3,076,550 |
| | Shortage | 8,393,466 | 8,393,466 | 8,393,466 | 6,663,818 | 6,663,818 | 6,203,603 | 5,195,322 | 5,773,628 | 5,773,628 | 4,419,518 | 4,501,230 | 4,390,910 | 4,522,800 |
| Expected Total Second-Stage Costs | | 9,364,399 | 9,364,399 | 9,374,898 | 8,023,548 | 8,030,141 | 7,533,338 | 6,612,019 | 7,277,167 | 7,277,167 | 6,230,896 | 7,269,615 | 7,169,948 | 8,071,463 |
| Total Cost of Model **RTM2** | | 13,686,838 | 13,686,838 | 13,697,337 | 13,357,118 | 13,363,711 | 13,092,408 | 12,663,439 | 13,672,617 | 13,672,617 | 13,028,776 | 19,985,015 | 20,131,348 | 24,285,563 |
| Achieved Network-Wide Reliability Level $\alpha$ (see (34)) | | 75.60% | 75.60% | 75.60% | 82.41% | 82.41% | 83.16% | 85.51% | 90.73% | 90.73% | 92.55% | 97.73% | 97.79% | 99.94% |

Table 8: Location-Allocation Policies Obtained by Model **RTM** for Various Values of the Unit Shortage Cost

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Unit Shortage Cost $h$ | | 3,886 | 4,534 | 5,182 | 5,829 | 6,477 | 7,125 | 7,772 | 8,420 | 9,068 | 9,716 |
| Total Cost | | 10,088,691 | 10,704,270 | 11,271,522 | 11,793,649 | 12,278,757 | 12,741,014 | 13,202,764 | 13,664,514 | 14,126,274 | 14,578,082 |
| Total Inventory Level | | 7,410 | 7,662 | 8,166 | 8,418 | 8,973 | 8,999 | 8,999 | 8,999 | 8,999 | 9,225 |
| | Number of | | | | | | | | | | |
| Region 1 | Small Facilities | 3 | 3 | 3 | 3 | 0 | 0 | 0 | 0 | 0 | 0 |
| | Medium Facilities | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | Large Facilities | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Region 2 | Small Facilities | 4 | 5 | 5 | 5 | 2 | 3 | 3 | 3 | 3 | 3 |
| | Medium Facilities | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | Large Facilities | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Region 3 | Small Facilities | 0 | 0 | 2 | 3 | 0 | 0 | 0 | 0 | 0 | 0 |
| | Medium Facilities | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 2 | 2 |
| | Large Facilities | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| First-Stage Costs | Facility Setup | 514,000 | 533,600 | 572,800 | 592,400 | 604,400 | 624,000 | 624,000 | 624,000 | 624,000 | 624,000 |
| | Acquisition | 4,799,460 | 4,962,680 | 5,289,120 | 5,452,340 | 5,811,810 | 5,828,650 | 5,828,650 | 5,828,650 | 5,828,650 | 5,975,030 |
| Total First-Stage Cost | | 5,313,460 | 5,496,280 | 5,861,920 | 6,044,740 | 6,416,210 | 6,452,650 | 6,452,650 | 6,452,650 | 6,452,650 | 6,599,030 |
| Expected Second-Stage Costs | Transportation | 380,860 | 375,650 | 367,191 | 366,228 | 412,275 | 397,122 | 397,122 | 397,122 | 397,122 | 394,464 |
| | Holding | 597,441 | 630,120 | 697,431 | 731,651 | 808,372 | 811,972 | 811,972 | 811,972 | 811,972 | 845,448 |
| | Shortage | 3,796,930 | 4,202,220 | 4,344,980 | 4,651,030 | 4,641,900 | 5,079,270 | 5,541,020 | 6,002,770 | 6,464,530 | 6,739,140 |
| Expected Total Second-Stage Cost | | 4,775,231 | 5,207,990 | 5,409,602 | 5,748,909 | 5,862,547 | 6,288,364 | 6,750,114 | 7,211,864 | 7,673,624 | 7,979,052 |
| Achieved Network-Wide Reliability Level $\alpha$ (see (34)) | | 76.85% | 81.85% | 83.10% | 84.91% | 85.51% | 91.48% | 91.48% | 91.48% | 91.48% | 91.48% |
| Highest Possible Region-Wide Reliability Level (see (35)) | $\alpha'_1$ | 77.18% | 77.18% | 77.18% | 77.18% | 57.22% | 57.22% | 57.22% | 57.22% | 57.22% | 57.22% |
| | $\alpha'_2$ | 80.98% | 80.98% | 80.98% | 80.98% | 80.98% | 80.98% | 80.98% | 80.98% | 80.98% | 80.98% |
| | $\alpha'_3$ | 78.48% | 78.48% | 78.48% | 78.48% | 78.48% | 78.48% | 78.48% | 78.48% | 78.48% | 78.48% |
| Highest Possible Network-Wide Reliability Level | | 77.41% | 82.41% | 83.66% | 84.91% | 86.11% | 91.48% | 91.48% | 91.48% | 91.48% | 91.48% |