CONSTRAINED BUNDLE METHODS FOR UPPER INEXACT ORACLES WITH APPLICATION TO JOINT CHANCE CONSTRAINED ENERGY PROBLEMS

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Abstract. Joint chance constrained problems give rise to many algorithmic challenges. Even in the convex case, i.e., when an appropriate transformation of the probabilistic constraint is a convex function, its cutting-plane linearization is just an approximation, produced by an oracle providing subgradient and function values that can only be evaluated inexactly. As a result, the cutting-plane model may lie above the true constraint. For dealing with such *upper* inexact oracles, and still solving the problem up to certain precision, a special numerical algorithm must be put in place. We introduce a family of constrained bundle methods, based on the so-called improvement functions, that is shown to be convergent and encompasses many previous approaches as well as new algorithms. Depending on the oracle accuracy, we analyze to which extent the considered methods solve the joint chance constrained program. The approach is assessed on real-life energy problems, arising when dealing with stochastic hydro-reservoir management.

Key words. Joint Chance Constraints, Bundle Methods, Inexact Oracles, Hydro-Reservoir Management

AMS subject classifications. 49M37, 65K05, 90B36, 90C15, 90C25

1. Introduction and Motivation.

Real-life problems are often modeled in an uncertain setting, to better reflect unknown phenomena specific to the application. In particular, such is the case in the energy sector; see [37]. For the numerical experience of this paper we focus on a specific energy problem arising in stochastic unit commitment ([3], [43], [29], [47],[45]). This is the problem of optimally managing reservoirs of a hydro valley in the short term.

A hydro valley is a set of power plants cascaded along the same hydrological bassin. For the considered system, part of Electricité de France mix, uncertainty is mostly related to the amount of melted snow arriving as a streamflow to the most uphill reservoirs. The volume of these reservoirs changes with the inflows and determines the amount of water that can be converted into energy. After turbining water to produce power, the uphill reservoirs release a certain volume that fills the reservoirs downhill, and the process continues until the power plant at the bottom of the bassin. In this interconnected context, it is important to jointly manage the generation of the cascaded plants in a manner that not only is economical but also reliable. More precisely, it is crucial to keep the volume of each reservoir in the valley between prescribed minimum and maximum levels (to prevent floods, to ensure touristic activities, etc). Since it is not realistic to ensure such conditions for every possible streamflow, satisfaction of lower and upper bounds for the volumes can be required in a probabilistic manner.

Introduced by [4], probability constraints are quite an appealing tool for dealing with uncertainty, because they give a physical interpretation to risk. For hydro valley management, chance constraints have been employed in [25, 7, 8, 24, 28, 49, 48, 44]. Most of these works require each component of the uncertain constraint to be satisfied in a probabilistic sense in a separate manner. As explained in [44], a stochastic model with individual chance constraints may sometimes result in unreliable optimal decisions, because there is no guarantee that the whole stochastic inequality will be satisfied with a given probability. In this work we build upon the model in [44], with joint chance constraints, and derive a sound numerical solution procedure, based on bundle methods, [18, 2].

In the classical textbook [32], convexity of chance constraints is ensured for a variety of distributions. Accordingly, for appropriate choices of the stochastic model for the inflows (cf. Sec. 5.4 for details), the hydro valley management problem has the abstract form

$$\min_{x\in X\subseteq\mathbb{R}^n}\{f(x) : c(x)\leq 0\} \tag{1.1}$$

for f and c finite-valued convex continuous functions and X a compact convex polyhedron. Even when c can be

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shown to be differentiable ([44]), it is reported in [27] that chance constraints are occasionally sufficiently stiff for smooth optimization methods to encounter convergence issues. Moreover in many applications c is actually non-smooth ([16]). For this reason it is of interest to make no smoothness assumption on c. For algorithmical purposes, given any $x \in X$, joint chance constraints (corresponding to c in (1.1)) need to be efficiently computed, together with a gradient. Gradient formulæ for multi-variate Gamma, Dirichlet, and Gaussian distributions can be found in [35], [32, 14, 41], and [32, 16, 44], respectively. Similarly to evaluating the function, these formulæ involve computing a probability. In turn, the calculation of a probability amounts to compute, for any given x, an integral in relatively high dimension (for our numerical application in Sec. 6, the corresponding dimension is 48). For multi-variate Gaussian distributions, the code developped by A. Genz can be used to efficiently approximate probabilities; [12, 13]. The numerical method therein outputs values that can be as accurate as required on input, provided enough time can be spent in the calculation. Since the numerical solution of (1.1) requires evaluating the constraint c (and a gradient) at many trial points x, the evaluation is not done exactly, but with some error, whose sign is unknown. As a result, and in spite of convexity, a linearization of the form $c_x + \langle g_{\mathtt{c}x}, \cdot - x\rangle$ with $c_x \approx c(x)$ and $g_{\mathtt{c}x}$ an approximate subgradient, may lie *above* the function $c(\cdot)$.

To circumvent this difficulty, in this paper we present a bundle method specially taylored to solve problems of the form (1.1) when computing f and/or c (as well as respective gradients) is computationally heavy. The algorithm is special because it solves a constrained nonsmooth problem based on the information provided by an inexact oracle, possibly of *upper* type. This means mean that the oracle output provides linearizations for f and c in (1.1) that are inexact and may lie *above* the corresponding function. The simpler case of *lower* oracles, yielding linearizations that always remain below the convex function, is also considered as a corollary. The convergence analysis of bundle methods with lower oracles is simpler, because it fits better the usual exact framework, in which the oracle linearizations define cutting planes for the function of interest (f and c in our case).

For unconstrained problems, bundle methods dealing with inexact oracles can be found in [17, 40, 19, 10, 11, 31]. Most of these works consider only *lower* oracles; we refer to [31] for a discussion on how such a setting considerably simplifies the convergence analysis. For constrained problems like (1.1), inexact bundle methods are more rare; see [20, 22, 30]. These works consider oracles that are either lower ones, or asymptotically exact. In this paper, we give a method suitable for the more general upper setting, and hence, adapted to the hydro application of interest.

Our paper is organized as follows. Sec. 2 is devoted to bundle methods for upper inexact oracles. After giving the initial bundle setting in § 2.1, the attenuation step required for the method to converge in the presence of oracle noise is explained in § 2.2. The new bundle algorithm is given in full details in § 2.3. Based on Sec. 3 asymptotic results, Sec. 4 proves convergence of the method to a solution of (1.1), up to the accuracy provided by the oracle. In particular, we show that for lower oracles that are asymptotically exact, the method finds an exact minimizer whenever (1.1) has a Slater point. Since our setting is more general than previous work (cf. the discussion in § 4.2), our approach significantly generalizes and extends results in the literature. The specific unit-commitment application is considered in the last two sections. Sec. 5 describes the hydro valley considered for the numerical tests, formulates the joint chance constrained problem to be solved, and discusses the upper inexact oracle employed. The different solvers used for comparison and a thorough set of numerical tests showing the interest of the approach are given in Sec. 6. The paper ends with some concluding remarks.

2. Designing Bundle methods for Constrained Optimization. Following the lead of [38], the numerical solution of (1.1) is addressed by means of an *improvement function* $H_\tau : \mathbb{R}^n \to \mathbb{R}$ defined by

$$H_\tau(y) = \max\Big(f(y) - \tau_1, c(y) - \tau_2\Big)\,,\ \text{for suitably chosen scalar targets } \tau_1\,, \tau_2\,. \tag{2.1}$$

However, unlike the exact setting considered in [38], the oracles which provide function and subgradient values for f and c make calculations with some error. For this reason, our method minimizes approximations of H_τ, built using oracle information computed with some inaccuracy. For clarity, at any point $x \in \mathbb{R}^n$ the symbols $f(x)$ and $c(x)$ refer to *exact* function values. Following [19], to denote *inexact* values the argument is put as a subscript, like in f_x, c_x (for functions) and $g_{\mathtt{f}x}$ and $g_{\mathtt{c}x}$ (for subgradients). Accordingly, given an iterate $x^j \in X$, the oracle provides f_{x^j} and $g_{\mathtt{f}x^j}$ shortened for convenience to $f^j = f_{x^j}$ and $g^j_{\mathtt{f}} = g_{\mathtt{f}x^j}$, and similarly for the c-values.

2.1. Initial setting. We assume that at any $x^j \in X$, the oracle provides

$$\begin{array}{lll} f^j \quad \text{and} \quad c^j\,, & \text{estimates for the functional values, as well as} \\ g^j_{\mathtt{f}} \quad \text{and} \quad g^j_{\mathtt{c}}\,, & \text{estimates for the respective subgradients.} \end{array} \tag{2.2}$$

Since in this oracle the signs of the errors, e.g., $f(x^j) - f^j$, are not specified, the true function values can be either overestimated or underestimated, and similarly for the subgradients. In particular, nothing is known on the linearizations, e.g., $f^j + \left\langle g^j_{\mathtt{f}}, \cdot - x^j \right\rangle$, that may lie below or above the corresponding function, e.g., f. Further conditions on the (possibly upper) inexact oracle will be required in what follows, as needed (cf. (2.20) and (4.1) below).

Along iterations, the method keeps an algorithmic center, denoted by $\hat{x}^k$ at iteration k, with function values denoted by $\hat{f}^k$ and $\hat{c}^k$. The center is some past iterate that was singled out because it produced significant progress towards the goal of solving (1.1). Progress is measured with respect to the current approximation of the improvement function, in a sense to be made clear below.

Specifically, the k-th *inexact improvement function* is

$$h^k_y = \max\left(f_y - \tau^k_1, c_y - \tau^k_2\right) \quad \text{where} \quad \left\{ \begin{array}{ll} \tau^k_1 = \hat{f}^k + \rho_k \max(\hat{c}^k, 0) & \text{for } \rho_k \geq 0 \\ \tau^k_2 = \sigma_k \max(\hat{c}^k, 0) & \text{for } \sigma_k \geq 0\,. \end{array} \right. \tag{2.3}$$

In the expression above, the targets τ^k are more general than those considered in [38], which correspond to taking null penalties ρ_k and σ_k. Relations with other improvement functions in the literature that are also covered by the setting (2.3) are discussed in § 4.2.

The oracle output is collected along iterations to form the *Bundle of information*

$$\mathscr{B}^k = \{\hat{x}^k, \hat{f}^k, \hat{c}^k\} \cup \left\{(x^j, f^j, c^j, g^j_{\mathtt{f}}, g^j_{\mathtt{c}}) : j \in J^k\right\} \qquad \text{for } J^k \subset \{1, \ldots, k\}\,.$$

Having this information, the k-th inexact improvement function is modelled by a convex function $\mathscr{M}_k : \mathbb{R}^n \to \mathbb{R}$ which uses approximate cutting-plane functions $\check{f}_k$ and $\check{c}_k$:

$$\mathscr{M}_k(y) = \max\left(\check{f}_k(y) - \tau^k_1, \check{c}_k(y) - \tau^k_2\right) \text{ where } \left\{ \begin{array}{l} \check{f}_k(y) = \max\left\{f^j + \left\langle g^j_{\mathtt{f}}, y - x^j \right\rangle : j \in J^k\right\} \\ \check{c}_k(y) = \max\left\{c^j + \left\langle g^j_{\mathtt{c}}, y - x^j \right\rangle : j \in J^k\right\}\,. \end{array} \right. \tag{2.4}$$

To generate iterates, the algorithm chooses a prox-parameter $\mu_k > 0$ and solves the quadratic program (QP)

$$x^{k+1} = \arg\min_{y \in X} \{\mathscr{M}_k(y) + \frac{1}{2}\mu_k \|y - \hat{x}^k\|^2\}\,. \tag{2.5}$$

As a result, $x^{k+1} \in X$ and

$$x^{k+1} = \hat{x}^k - \frac{1}{\mu^k}(G^k + v^k) \quad \text{where} \quad G^k \in \partial\mathscr{M}_k(x^{k+1}) \text{ and } v^k \in N_X(x^{k+1})\,, \tag{2.6}$$

where $N_X(x^{k+1})$ denotes the normal cone (of convex analysis) of X at the new iterate and $\partial\mathscr{M}_k(x^{k+1})$ the subgradient of $\mathscr{M}_k$ at x^{k+1}. After solving the problem, the *aggregate linearization*

$$\mathtt{M}^k(y) = \mathscr{M}_k(x^{k+1}) + \left\langle G^k, y - x^{k+1} \right\rangle\,, \tag{2.7}$$

which is an affine function, can be defined. Clearly, because $G^k \in \partial\mathscr{M}_k(x^{k+1})$,

$$\mathtt{M}^k(y) \leq \mathscr{M}_k(y) \quad \text{for all } y \in \mathbb{R}^n\,. \tag{2.8}$$

The last ingredient in the bundle method is given by the *aggregate error*, defined by

$$\mathtt{E}^k = h^k_{\hat{x}^k} - \mathtt{M}^k(\hat{x}^k) - \left\langle v^k, \hat{x}^k - x^{k+1} \right\rangle\,. \tag{2.9}$$

In view of (2.7), for any y it holds that $G^k = \nabla \mathtt{M}^k(y)$. Therefore, because $\mathtt{M}^k(\hat{x}^k) = \mathtt{M}^k(y) + \langle G^k, \hat{x}^k - y\rangle$ with $\mathtt{M}^k \le \mathscr{M}_k$ by (2.8), we derive the relation

$$h^k_{\hat{x}^k} - \mathtt{E}^k \le \mathscr{M}_k(y) + \left\langle G^k + v^k, \hat{x}^k - y\right\rangle \quad \text{for all } y \in X\,, \tag{2.10}$$

where we used the fact that the term $\langle v^k, x^{k+1} - y\rangle$ is nonnegative for all $y \in X$, because $v^k \in N_X(x^{k+1})$.

2.2. Handling inexact oracle information. Usually, the noise introduced by the inexact evaluations is deemed “too large” when the function value at the algorithmic center is *below* the minimum model value (a situation that is impossible with an exact oracle, by convexity). For our setting, this amounts to checking if the noise measurement quantity defined below is negative:

$$h^k_{\hat{x}^k} - \left(\mathscr{M}_k(x^{k+1}) + \frac{1}{2}\mu_k\|x^{k+1} - \hat{x}^k\|^2\right) < 0\,.$$

When the relation above holds, the algorithm maintains the model and the center, and reduces the prox-parameter. The new iterate yields a smaller noise measurement quantity, thus *attenuating* the noise induced by the inexact bundle information. For the sake of numerical versatility, we consider here an alternative mechanism that is more general, and checks asymptotic satisfaction of the inequality above, based on a *relative* criterion. More precisely, noise is considered too large if

$$\frac{h^k_{\hat{x}^k} - \left(\mathscr{M}_k(x^{k+1}) + \frac{1}{2}\mu_k\|x^{k+1} - \hat{x}^k\|^2\right)}{\frac{1}{2}\mu_k\|x^{k+1} - \hat{x}^k\|^2} < -\beta_k \tag{2.11}$$

for a parameter β_k satisfying (2.14) below.

To measure progress towards the goal of solving (1.1), certain *predicted decrease* δ^k is employed. Usual definitions for the decrease are $\delta^k = h^k_{\hat{x}^k} - \mathscr{M}_k(x^{k+1})$, or $\delta^k = h^k_{\hat{x}^k} - \mathscr{M}_k(x^{k+1}) - \frac{1}{2}\mu_k\|x^{k+1} - \hat{x}^k\|^2$. We consider a slightly more general variant, and let

$$\delta^k = h^k_{\hat{x}^k} - \mathscr{M}_k(x^{k+1}) - \frac{1}{2}\alpha_k\mu_k\|x^{k+1} - \hat{x}^k\|^2 \quad \text{for a parameter } \alpha_k \text{ satisfying (2.14) below.} \tag{2.12}$$

Since the numerator in (2.11) equals $\delta^k - \frac{1}{2}(1-\alpha_k)\mu_k\|x^{k+1} - \hat{x}^k\|^2$, we see that detecting the need of a noise attenuation step amounts to checking satisfaction of the inequality

$$\delta^k < \frac{1}{2}\Big(1 - (\alpha_k + \beta_k)\Big)\mu_k\|x^{k+1} - \hat{x}^k\|^2\,. \tag{2.13}$$

The choice of parameters $\alpha_k\,,\beta_k$ should ensure that the nominal decrease in (2.15) is nonnegative when noise is not too large. Accordingly, we suppose that

$$\exists b > -1 \text{ and } B > 0 \text{ such that } \beta_k \in [b, 1 - \alpha_k - B] \text{ for } \alpha_k \in [0,2]\,. \tag{2.14}$$

Only when noise is declared acceptable, that is when (2.13) does not hold, the algorithm examines if the new iterate is good enough to become the next center by checking

$$\begin{cases} \text{either if} & f^{k+1} \le \hat{f}^k - m\delta^k \quad \text{and} \quad c^{k+1} \le 0 & \text{when } \hat{c}^k \le 0\,,\\ \text{or if} & c^{k+1} \le \hat{c}^k - m\delta^k & \text{when } \hat{c}^k > 0\,. \end{cases} \tag{2.15}$$

When the relation holds, the iteration is declared *serious*, because it provides a new algorithmic center: $\hat{x}^{k+1} = x^{k+1}$. Otherwise, the center is maintained and the iteration is declared *null*.

The rationale behind (2.15) is to measure progress towards minimization of (1.1) by focusing either in reducing the objective value without losing feasibility if the center is approximately feasible. Otherwise, when $\hat{c}^k > 0$, the emphasis is put in reducing infeasibility by checking the second condition in (2.15).

The parameters $\alpha_k\,,\beta_k$ in the criterion (2.13) make it possible to control the relation between noise attenuation and descent, a flexibility that can help the numerical performance of the algorithm. More specifically, to progress towards a solution, it is preferable for the algorithm to:

- make more serious iterations, because serious iterates converge to a solution, and
- have few noise attenuation steps. Noise attenuation steps are undesirable to occur often, because they prevent the algorithm from having "true" iterates, for which to check the descent condition.
- However, checking (2.13) does not involve any f/c-oracle calculation at x^{k+1}, and can therefore be considered an inexpensive test.

The flexibility introduced by the additional parameters α_k, β_k allows the user to seek for a trade-off between the time spent in oracle calculations and the CPU time required for the algorithm to find a solution to (1.1). By (2.15), more serious iterations are achieved by taking larger α_k's (yielding smaller δ^k's), while a larger β_k reduces the left handside term in (2.13), making less likely noise attenuation. Our numerical experience in Sec. 6 shows how different choices of these parameters impact the numerical performance, both in terms of CPU time and accuracy.

We now list some consequences resulting from the various definitions above, obtained with some simple algebraic manipulations.

First, by (2.7) written with $y = \hat{x}^k$ and (2.12),

$$h^k_{\hat{x}^k} - \mathtt{M}^k(\hat{x}^k) + \left\langle G^k, \hat{x}^k - x^{k+1} \right\rangle = \delta^k + \frac{1}{2}\alpha_k\mu_k\|x^{k+1} - \hat{x}^k\|^2 .$$

Together with (2.9) and (2.6) we see that

$$\mathtt{E}^k = \delta^k - \frac{2-\alpha_k}{2}\mu_k\|x^{k+1} - \hat{x}^k\|^2 \quad \text{and, therefore,} \quad \mathtt{E}^k \leq \delta^k \text{ at all iterations} \tag{2.16}$$

because $\alpha_k \leq 2$, by (2.14).

Second, by adding δ^k to both members of the identity (2.16), we see that

$$\text{inequality (2.13) holds} \iff \delta^k + \mathtt{E}^k < -\frac{(\alpha_k + 2\beta_k)}{2}\mu_k\|x^{k+1} - \hat{x}^k\|^2 . \tag{2.17}$$

2.3. A Nonsmooth Optimization Solver for Inexact Oracles. We now give our bundle algorithm for solving problem (1.1).

Algorithm 2.1. *We assume given an oracle computing approximate f/c values as in* (2.2) *for any $x \in X$, possibly of upper type.*

Step 0 (Input and Initialization) *Select a initial starting point $\hat{x}^0$ a stopping tolerance* TOL≥ 0, *an Armijo-like parameter $m \in (0,1)$. Initialize the iteration counter $k = 0$, the bundle index set $J^0 := \{0\}$, and the first candidate point $x^0 := \hat{x}^0$. Call* (2.2) *to compute f^0, c^0 as well as ${g_\mathtt{f}}^0$ and ${g_\mathtt{c}}^0$. Choose the starting prox-parameter $\mu_0 > 0$, parameters α_0, β_0 satifying* (2.14)*, and penalties $\rho_0, \sigma_0 \geq 0$ satisfying* (2.19) *below.*

Step 1 (Model Generation and QP Subproblem) *Having the current algorithmic center $\hat{x}^k$, the bundle $\mathscr{B}^k$, the prox-parameter μ_k and the penalties ρ_k, σ_k, define the convex piecewise linear model function $\mathscr{M}_k$ and compute $x^{k+1} = \arg\min\{\mathscr{M}_k(y) + \frac{1}{2}\mu_k\|y - \hat{x}^k\|^2 : y \in X\}$. Define the predicted decrease δ^{k+1} as in* (2.12).

Step 2 (Noise attenuation test)
If condition (2.13) *is true, noise is too large: decrease the prox-parameter as in* (2.18b) *below; maintain the center, the bundle, and the penalties:*

$$\left(\hat{x}^{k+1}, \mathscr{B}^{k+1}, \rho_{k+1}, \sigma_{k+1}\right) = \left(\hat{x}^k, \mathscr{B}^k, \rho_k, \sigma_k\right);$$

choose parameters $\alpha_{k+1}, \beta_{k+1}$ satisfying (2.14)*, and loop to Step1.*
Otherwise, if (2.13) *does not hold, proceed to Step 3.*

Step 3 (Stopping Test and New Oracle Information) *If $\delta^{k+1} \leq$*TOL *then stop. Otherwise call the oracle to obtain $f^{k+1}, c^{k+1}, {g_\mathtt{f}}^{k+1}$ and ${g_\mathtt{c}}^{k+1}$.*

Step 4 (Serious step test) *Check the descent condition* (2.15)*. If this condition is true, declare a serious iteration and set $\hat{x}^{k+1} = x^{k+1}$. Otherwise, declare a null step and maintain the center: $\hat{x}^{k+1} = \hat{x}^k$.*

Step 5 (Bundle Management and updates) *Choose a new prox-parameter μ_{k+1} satisfying* (2.18a) *if the iteration was declared serious or satisfying* (2.18c) *whenever the iteration was declared null. In all cases choose* *parameters $\alpha_{k+1}, \beta_{k+1}$ satisfying* (2.14) *and penalties ρ_{k+1}, σ_{k+1} satisfying* (2.19) *below. Define the new bundle $\mathscr{B}^{k+1}$, for example by appending to the index set the last iterate information: $J^{k+1} = J^k \cup \{k+1\}$. Increase k by 1 and loop to Step 1.*

Both in Step 2 and Step 5 there is some freedom in the choice of the new bundle $\mathscr{B}^{k+1}$. When noise is excessive, as in Step 2, the conservative choice of keeping the same cutting-plane models for both f and c seems reasonable. Alternative choices for managing the bundle in Step 5 are discussed after Lem. 3.1, 3.2, and 3.3, for the cases of serious, noisy, and null iterations, respectively.

We now explain how Algorithm 2.1 handles the update of its prox-parameter μ_k and penalties ρ_k, σ_k. For the prox-parameter, the update uses positive constants $\mu_{\max}$ and Δ, as follows:

$$\mu_{k+1} \le \mu_{\max} < +\infty \qquad \text{if iteration } k \text{ was declared serious} \tag{2.18a}$$

$$\mu_{k+1} \le \mu_k - \Delta < \mu_k \qquad \text{if iteration } k \text{ resulted in noise attenuation} \tag{2.18b}$$

$$\mu_{k+1} \in [\mu_k, \mu_{\max}] \qquad \text{if iteration } k \text{ was declared null.} \tag{2.18c}$$

The rule for the penalty parameters uses a some positive constant RS and is given below:

$$0 \le \sigma_{k+1} \le 1 \text{ and } \rho_{k+1} \ge 0 \text{ satisfy } 1 - \sigma_{k+1} + \rho_{k+1} \ge RS > 0\,. \tag{2.19}$$

The main purpose of these conditions is to ensure satisfaction of the relations stated in the proposition below.

Proposition 2.2 (Consequences of penalization updates). *When Algorithm 2.1 uses the rule* (2.19) *for the penalties, the following holds.*

– *At every iteration k,*

$$h^k_{\hat{x}^k} = 0 \quad \textit{if } \hat{c}^k \le 0 \quad \textit{and} \quad h^k_{\hat{x}^k} = \hat{c}^k(1-\sigma_k) \ge 0 \quad \textit{otherwise}\,.$$

– *At every iteration k declared null, the inequality $h^k_{x^{k+1}} > h^k_{\hat{x}^k} - m\delta^k$ is satisfied.*
– *Suppose that, in addition, the oracle* (2.2) *is bounded, in the sense that*

$$\textit{the inexact values } \{f^k, c^k\} \textit{ and } \{\|g^k_{\mathtt{f}}\|, \|g^k_{\mathtt{c}}\|\} \textit{ are bounded for every sequence } \{x^k\} \subset X\,. \tag{2.20}$$

Then, there exist positive constants M and M′ such that for any iteration k that is declared null or needing noise attenuation

$$\mathscr{M}_k(\hat{x}^k) \le \max(M - \hat{f}^k, M) \quad \textit{and} \quad h^k_{\hat{x}^k} \le M'\,. \tag{2.21}$$

Proof. By (2.3), if the center is approximately feasible, $h^k_{\hat{x}^k} = 0$. Otherwise, when $\hat{c}^k > 0$, (2.3) yields that $h^k_{\hat{x}^k} = \hat{c}^k \max(-\rho_k, 1-\sigma_k) = \hat{c}^k(1-\sigma_k) \ge 0$, because $1 - \sigma_k \ge RS > 0 \ge -\rho_k$, by (2.19).

Regarding the second item, we recall that $h^k_{\hat{x}^k} = 0$ by the first item. It therefore suffices to show $h^k_{x^{k+1}} > -m\delta^k$. Indeed assume that (2.15) does not hold and $\hat{c}^k \le 0$, then (2.3) gives that

$$h^k_{x^{k+1}} = \max(f^{k+1} - \hat{f}^k, c^{k+1}) \ge f^{k+1} - \hat{f}^k > -m\delta^k\,.$$

When the center is infeasible, i.e., $\hat{c}^k > 0$, the inequality is derived once more from combining (2.3) with the negation of (2.15), using the previous item, as follows:

$$h^k_{x^{k+1}} = \max(f^{k+1} - \hat{f}^k - \rho_k\hat{c}^k, c^{k+1} - \sigma_k\hat{c}^k) \ge c^{k+1} - \sigma_k\hat{c}^k > \hat{c}^k(1-\sigma_k) - m\delta^k = h^k_{\hat{x}^k} - m\delta^k\,.$$

Finally, to see (2.21), notice that $x^k \in X$ with X bounded, so (2.20) ensures that each linearization $f^k + \langle g^k_{\mathtt{f}}, \cdot - x^k \rangle$ (or its c-counterpart) is bounded over X. In particular, both $\check{f}_k(\hat{x}^k)$ and $\check{c}_k(\hat{x}^k)$ are bounded by some constant M. In view of the model definition (2.4) and (2.3), $\mathscr{M}_k(\hat{x}^k) \le \max(M - \tau^k_1, M - \tau^k_2) = \max(M - \hat{f}^k - \rho_k \max(\hat{c}^k, 0), M - \sigma_k \max(\hat{c}^k, 0))$. The bound for $\mathscr{M}_k(\hat{x}^k)$ follows, because the penalty terms are nonnegative by (2.19). An analogous reasoning gives the bound for $h^k_{\hat{x}^k}$. □

3. Asymptotic Analysis. We now analyse the different cases that can arise when the algorithm in Sec. 2.1 loops forever, i.e., $k \to \infty$. Then only one of the following mutually exclusive cases can occur:

- either there are infinitely many serious iterates,
- or the stability center remains unchanged after a finite number of iterations. In this case,
    - either there is an infinite number of noise attenuation steps,
    - or there is a finite number of noise attenuation steps and eventually only null steps are done.

The first case is considered in Lem. 3.1, the second case in Lem. 3.2 and the last case in Lem. 3.3.

LEMMA 3.1 (Infinitely many serious iterations). *Consider solving* (1.1) *with Algorithm 2.1 using an oracle satisfying* (2.2) *and* (2.20), *with parameters* α_k, β_k *satisfying* (2.14).

Let K_s denote the set gathering indices of serious iterations. If there are infinitely many of such indices, and the prox-parameter sequence satisfies (2.18a), *then*

$$h^k_{\hat{x}^k} \le \mathscr{M}_k(y) + o(1/k) \quad \textit{for all } y \in X \textit{ and } k \in K_s \textit{ sufficiently large.}$$

Proof. Consider $k \in K_s$. We first show that $\delta^k \to 0$. If for all serious steps we have $c^{k+1} > 0$ then (2.15) implies that

$$\hat{c}^{k+1} = c^{k+1} \le c^k - m\delta^k = \hat{c}^k - m\delta^k\,, \tag{3.1}$$

noting that $\delta^k \ge 0$, because serious steps can only take place when no noise attenuation occurs. Since the non-increasing sequence $\{\hat{c}^k\}_{K_s}$ is bounded below by 0, it converges. From (3.1) we deduce that $0 \le \delta^k \le \frac{c^k - c^{k+1}}{m}$. Since $\{\hat{c}^k\}_{K_s}$ converges this gives gives that $\delta^k \to 0$ as $K_s \ni k \to \infty$.

Otherwise, there is some $\bar{k} \in K_s$ such that $\hat{c}_{\bar{k}} \le 0$. In view of (2.15), all subsequent serious iterates are feasible: for each serious step $\bar{k} \le k \in K_s$ we have $\hat{c}^{k+1} = c^{k+1} \le 0$ and

$$\hat{f}^{k+1} \le \hat{f}^k - m\delta^k \quad \text{with} \quad \delta^k \ge 0\,. \tag{3.2}$$

Thus, the nonincreasing sequence $\{\hat{f}^k\}_{\bar{k} \le k \in K_s}$ is either unbounded below or converges. The first case is ruled out by (2.20). As for the second case, it implies that $\delta^k \to 0$ since $0 \le \delta^k \le \frac{\hat{f}^k - \hat{f}^{k+1}}{m}$.

We now show that both $G^k + \nu^k \to 0$ and $\mathtt{E}^k \to 0$. Since $(1-(\alpha_k+\beta_k)) \ge B > 0$ by (2.14), the negation of (2.13) and (2.6) imply that

$$\begin{aligned} 0 \leftarrow \delta^k &\ge \frac{1}{2}\Big(1-(\alpha_k+\beta_k)\Big)\mu_k\|x^{k+1}-\hat{x}^k\|^2 = \frac{1}{2\mu_k}\Big(1-(\alpha_k+\beta_k)\Big)\|G^k+\nu^k\|^2 \\ &\ge \frac{B}{2\mu_k}\|G^k+\nu^k\|^2 \ge \frac{B}{2\mu_{max}}\|G^k+\nu^k\|^2 \ge 0 \end{aligned}$$

where we used (2.18a) for the last inequality. As a result, both $\|G^k+\nu^k\|^2/\mu_k \to 0$ and $G^k+\nu^k \to 0$ as $K_s \ni k \to \infty$.

Since $\alpha_k \ge 0$ and $\alpha_k+\beta_k \le 1-B$ by (2.14), we deduce that $-(\alpha_k+2\beta_k)/2 = -(\alpha_k+\beta_k)+\alpha_k/2 \ge B-1$. To show that the aggregate error goes to 0, pass to the limit in the negation of (2.17) and use the above estimate to see that

$$\lim_{k\in K_s} \mathtt{E}^k \ge (B-1)\lim_{k\in K_s}\mu_k\|x^{k+1}-\hat{x}^k\|^2\,.$$

Since by (2.6), $\mu_k\|x^{k+1}-\hat{x}^k\|^2 = \|G^k+\nu^k\|^2/\mu_k$ and we have just shown this term vanishes asymptotically, the error limit is nonnegative. Then $\mathtt{E}^k \to 0$, because $\mathtt{E}^k \le \delta^k$ by (2.16) and $\delta^k \to 0$.

The stated inequality holds follows from (2.10) by boundedness of X and the fact that both $G^k+\nu^k$ and $\mathtt{E}^k \to 0$. □

The analysis above shows that Step 5 of Algorithm 2.1 can freely manage the bundle at serious iterations. Of course, a richer bundle yields better cutting-plane models for f and c, so having larger index-sets J^k should improve the speed of the method (keeping in mind that large sets J^k make harder the QP subproblem solution).

Notice also that in the inequality in Lem. 3.1 the remainder $o(1/k)$ corresponds in fact to both $G^k + v^k$ and $\mathtt{E}^k$ going to zero. Similar relations will hold when there is a finite number of serious iterations, as shown below.

In the next two cases, eventually no more serious steps occur and after a finite number of iterations the algorithmic center remains unchanged. As a result, there exists $\hat{k}$ and $\hat{x}$ such that $\hat{x}^k = \hat{x}$ for all $k > \hat{k}$. Unlike Lem. 3.1, the results below rely on conditions (2.19), required for the penalty parameters defining the inexact improvement function (2.3).

LEMMA 3.2 (Infinitely many noisy iterations). *Consider solving* (1.1) *with Algorithm 2.1 using an oracle satisfying* (2.2) *and* (2.20), *with parameters α_k, β_k satisfying* (2.14) *and penalties as in* (2.19).

Suppose at iteration $\hat{k}$ there is a last serious iterate $\hat{x}$ and let K_a denote the set gathering indices of iterations larger than $\hat{k}$ for which (2.13) *holds and noise is deemed too large. If there are infinitely many of such indices and* (2.18b) *holds then*

$$h^k_{\hat{x}^k} = h^k_{\hat{x}} \leq \mathscr{M}_k(y) + o(1/k) \quad \textit{for all } y \in X \textit{ and } k \in K_a \textit{ sufficiently large.}$$

Proof. Consider $k \in K_a$. By (2.16) and (2.13) we obtain the following estimate

$$2\mathtt{E}^k = 2\delta^k - (2-\alpha_k)\mu_k \|x^{k+1} - \hat{x}^k\|^2 < -(1+\beta_k)\mu_k\|x^{k+1} - \hat{x}^k\|^2 \leq 0, \tag{3.3}$$

since $\beta_k \geq b > -1$ by (2.14). Since $\mathtt{E}^k \leq 0$, $h^k_{\hat{x}^k} - \mathtt{E}^k \geq h^k_{\hat{x}^k}$, and with (2.10) we see that

$$h^k_{\hat{x}^k} \leq \mathscr{M}_k(y) + \left\langle G^k + v^k, \hat{x} - y \right\rangle \quad \text{for all } y \in X.$$

Since the set X is bounded, the stated inequality would hold if $G^k + v^k \to 0$. To show this result, first note that (2.9) and (2.7) imply that

$$-\mathtt{E}^k = \mathscr{M}_k(x^{k+1}) + \left\langle G^k + v^k, \hat{x} - x^{k+1} \right\rangle - h^k_{\hat{x}^k} \leq \mathscr{M}_k(\hat{x}) - h^k_{\hat{x}^k},$$

where the inequality comes from (2.6). Let $\hat{f}$ denote the f-value computed by the oracle at $\hat{x}$. By the first item in Prop. 2.2, $h^k_{\hat{x}^k} \geq 0$ because $\sigma_k \leq 1$ by (2.19), so $-\mathtt{E}^k \leq \mathscr{M}_k$. Since $\mathscr{M}_k(\hat{x}) \leq \max(\hat{f} - M, \hat{f})$ by the third item in the proposition, for some constant $\hat{M}$

$$-\mathtt{E}^k \leq \hat{M} \text{ for all } k \in K_a.$$

From (3.3) and the condition $\beta_k \geq b$ from (2.14) we obtain the inequality $\mathtt{E}^k < -\frac{1}{2}(1+b)\mu_k\|x^{k+1} - \hat{x}^k\|^2$. Moreover, using (2.6) and the fact that $1 + b > 0$ by (2.14), this means that

$$\frac{2}{1+b}\mu_k \mathtt{E}^k < -\|G^k + v^k\|^2$$

or, equivalently, that

$$0 \leq \|G^k + v^k\| \leq \sqrt{-\frac{2}{1+b}\mu_k \mathtt{E}^k} \leq \sqrt{\frac{2}{1+b}\mu_k \hat{M}}\,.$$

Since the update in (2.18b) ensures that the sequence $\{\mu_k\}_{k \in K_a}$ is strictly decreasing, when $K_a \ni k \to \infty$ we obtain that $\mu_k \to 0$ and $G^k + v^k \to 0$, as desired. □

For the result above to hold, Step 2 in Algorithm 2.1 only needs the sequence $\{\mu_k\}_{K_a}$ to be strictly decreasing, and the left bound in (2.21) to hold. So, as for the case of infinitely many serious iterations, there is freedom in how the bundle is managed when noise is deemed too large. Since in this case there is no new oracle information, it seems reasonable to maintain the current bundle.

In a manner similar to Lem. 3.1, the remainder term in Lem. 3.2 corresponds in fact to $G^k + v^k \to 0$ with $\mathtt{E}^k \leq 0$.

The final result considers there are finitely many serious and noise attenuation steps, which implies the algorithm makes infinitely many consecutive null steps. For this case, the model is required to satisfy the following conditions whenever the iteration k is declared null:

$$\mathscr{M}_{k+1}(y) \geq \mathtt{M}^k(y) \text{ and} \tag{3.4a}$$

$$\mathscr{M}_{k+1}(y) \geq \max\left(f^{k+1} + \left\langle g_{\mathtt{f}}^{k+1}, y - x^{k+1} \right\rangle - \tau_1^k, c^{k+1} + \left\langle g_{\mathtt{c}}^{k+1}, y - x^{k+1} \right\rangle - \tau_2^k \right) \tag{3.4b}$$

The result below uses standard arguments in bundle methods [6], taking advantage of the boundedness relations in (2.20) and (2.21) to adapt those arguments to the inexact improvement function setting.

LEMMA 3.3 (Infinitely many consecutive null iteration). *Consider solving* (1.1) *with Algorithm 2.1 using an oracle satisfying* (2.2) *and* (2.20)*, with parameters α_k, β_k satisfying* (2.14) *and penalties as in* (2.19)*.*

Suppose at iteration $\hat{k}$ there is a last serious step, denoted by $\hat{x}$ and after an iteration $\overline{k} > \hat{k}$ there are no more noise attenuation steps: eventually only null steps occur. Let K_n denote the set gathering indices of iterations larger than $\underline{k}$. If there are infinitely many of such indices and both (2.18c) *and* (3.4) *hold, then*

$$h^k_{\hat{x}^k} = h^k_{\hat{x}} \leq \mathscr{M}_k(y) + o(1/k) \textit{ for all } y \in X \textit{ and } k \in K_n \textit{ sufficiently large.}$$

Furthermore, $x^k \to \hat{x}$ as $K_n \ni k \to \infty$.

Proof. Consider $k \in K_n$. Once again, the stated inequality will follow from (2.10) by boundedness of X, if we show that both $G^k + v^k \to 0$ and $\mathtt{E}^k \to 0$. In turn, these results follow from showing that $\delta^k \to 0$.

To see that $\delta^k \to 0$, we start by expanding squares and using (2.6) to write the identity

$$2\langle G^k + v^k, y - x^{k+1}\rangle = 2\mu_k\langle \hat{x} - x^{k+1}, y - x^{k+1}\rangle = \mu_k\|x^{k+1} - \hat{x}\|^2 + \mu_k\|y - x^{k+1}\|^2 - \mu_k\|y - \hat{x}\|^2 \tag{3.5}$$

for all $y \in \mathbb{R}^n$. Since the function $\mathtt{M}^k$ is affine with gradient G^k, for all $y \in \mathbb{R}^n$

$$\begin{aligned}
\mathtt{M}^k(y) &= \mathtt{M}^k(x^{k+1}) + \langle G^k, y - x^{k+1}\rangle \\
&= \mathtt{M}^k(x^{k+1}) + \langle G^k, y - x^{k+1}\rangle + \langle v^k, y - x^{k+1}\rangle - \langle v^k, y - x^{k+1}\rangle \\
&= \mathtt{M}^k(x^{k+1}) + \frac{1}{2}\mu_k\|x^{k+1} - \hat{x}\|^2 + \frac{1}{2}\mu_k\|y - x^{k+1}\|^2 - \frac{1}{2}\mu_k\|y - \hat{x}\|^2 - \langle v^k, y - x^{k+1}\rangle \\
&= \mathtt{OV}^k + \frac{1}{2}\mu_k\|y - x^{k+1}\|^2 - \frac{1}{2}\mu_k\|y - \hat{x}\|^2 - \langle v^k, y - x^{k+1}\rangle,
\end{aligned} \tag{3.6}$$

where in the last equality we define $\mathtt{OV}^k := \mathscr{M}_k(x^{k+1}) + \frac{1}{2}\mu_k\|x^{k+1} - \hat{x}\|^2$ and use the relation $\mathtt{M}^k(x^{k+1}) = \mathscr{M}_k(x^{k+1})$ from (2.7) (the value $\mathtt{OV}^k$ is the optimal value of the QP subproblem (2.5) defining x^{k+1}). Since $v^k \in N_X(x^{k+1})$ and $y \in X$, the term $-\langle v^k, y - x^{k+1}\rangle \geq 0$ and, hence, we derive from (3.6) that

$$\forall y \in X \quad \mathtt{M}^k(y) + \frac{1}{2}\mu_k\|y - \hat{x}\|^2 \geq \mathtt{OV}^k + \frac{1}{2}\mu_k\|y - x^{k+1}\|^2\,. \tag{3.7}$$

By evaluating at $y = \hat{x} \in X$, using (2.8) and the third item in Prop. 2.2, there exists some constant $\hat{M}$ such that

$$\mathtt{OV}^k + \frac{1}{2}\mu_k\|\hat{x} - x^{k+1}\|^2 \leq \mathtt{M}^k(\hat{x}) \leq \mathscr{M}_k(\hat{x}) \leq \hat{M}\,.$$

In particular, the sequence $\{\mathtt{OV}^k\}$ is bounded from above.

Assumption (3.4a) in (3.7) yields that

$$\mathscr{M}_{k+1}(y) + \frac{1}{2}\mu_k\|y - \hat{x}\|^2 \geq \mathtt{OV}^k + \frac{1}{2}\mu_k\|y - x^{k+1}\|^2$$

for all $y \in X$. Since $\mu_{k+1} \geq \mu_k$ by (2.18c), by evaluating the inequality above at $y = x^{k+2}$ we see that

$$\mathtt{OV}^{k+1} \geq \mathtt{OV}^k + \frac{1}{2}\mu_k\|x^{k+2} - x^{k+1}\|^2\,,$$

and, being bounded above, the non-decreasing sequence $\{\mathtt{OV}^k\}$ converges. Since the righthand side is larger than $\mathtt{OV}^k$, we conclude that

$$\mathtt{OV}^{k+1} - \left(\mathtt{OV}^k + \frac{1}{2}\mu_k\|x^{k+2} - x^{k+1}\|^2\right) \to 0 \quad \text{and, furthermore,} \quad \|x^{k+2} - x^{k+1}\|^2 \to 0 \tag{3.8}$$

because $\mu_k \geq \mu^-_{k+1}$ by (2.18c).

Recall that condition (2.14) implies that $\delta_k \geq 0$ when (2.13) fails, i.e., no noise is detected. The descent test also fails, so by the second item in Prop. 2.2,

$$h^k_{x^{k+1}} > h^k_{\hat{x}^k} - m\delta^k = h^k_{\hat{x}} - m\delta^k .$$

Adding δ^k to both terms and using the definition in (2.12) we obtain that

$$\begin{aligned}
0 \leq (1-m)\delta^k &< \delta^k + h^k_{x^{k+1}} - h^k_{\hat{x}^k} \\
&= h^k_{x^{k+1}} - \mathscr{M}_k(x^{k+1}) - \frac{1}{2}\alpha_k\mu_k\|x^{k+1} - \hat{x}\|^2 \\
&\leq h^k_{x^{k+1}} - \mathscr{M}_k(x^{k+1}) \\
&= h^k_{x^{k+1}} - \mathscr{M}_{k+1}(x^{k+2}) + \mathscr{M}_{k+1}(x^{k+2}) - \mathscr{M}_k(x^{k+1}) .
\end{aligned}$$

Writing (3.4b) for $y = x^{k+2}$ and using the Cauchy-Schwarz inequality yields that

$$\mathscr{M}_{k+1}(x^{k+2}) \geq \max(f^{k+1} - \tau^k_1 - \|g^{k+1}_{\mathtt{f}}\|\|x^{k+2} - x^{k+1}\|, c^{k+1} - \tau^k_2 - \|g^{k+1}_{\mathtt{c}}\|\|x^{k+2} - x^{k+1}\|) .$$

By (2.20), there exists a constant Γ such that $\|g^{k+1}_{\mathtt{f}}\|, \|g^{k+1}_{\mathtt{c}}\| \leq \Gamma$ and, hence,

$$\mathscr{M}_{k+1}(x^{k+2}) \geq \max(f^{k+1} - \tau^k_1, c^{k+1} - \tau^k_2) - \Gamma\|x^{k+2} - x^{k+1}\| .$$

The first righthand side term above equals $h^k_{x^{k+1}}$, by (2.3). Continuing and using the definition of $\mathtt{OV}^k$ and (2.18c),

$$\begin{aligned}
0 \leq (1-m)\delta^k &\leq \Gamma\|x^{k+2} - x^{k+1}\| + \mathscr{M}_{k+1}(x^{k+2}) - \mathscr{M}_k(x^{k+1}) \\
&= \Gamma\|x^{k+2} - x^{k+1}\| + \mathtt{OV}^{k+1} - \frac{1}{2}\mu_{k+1}\|x^{k+2} - \hat{x}\|^2 - \mathtt{OV}^k + \frac{1}{2}\mu_k\|x^{k+1} - \hat{x}\|^2 \\
&= \Gamma\|x^{k+2} - x^{k+1}\| + \mathtt{OV}^{k+1} - \mathtt{OV}^k - \frac{1}{2}\mu_k\|x^{k+2} - \hat{x}\|^2 + \frac{1}{2}\mu_k\|x^{k+1} - \hat{x}\|^2 \\
&= \Gamma\|x^{k+2} - x^{k+1}\| + \mathtt{OV}^{k+1} - (\mathtt{OV}^k + \frac{1}{2}\mu_k\|x^{k+2} - x^{k+1}\|^2) \\
&\quad + \frac{1}{2}\mu_k\left(\|x^{k+2} - x^{k+1}\|^2 - \|x^{k+2} - \hat{x}\|^2 + \|x^{k+1} - \hat{x}\|^2\right) .
\end{aligned}$$

Following (3.5), we can observe that the last three terms equal $\mu_k\left\langle x^{k+2} - x^{k+1}, \hat{x} - x^{k+1}\right\rangle$. By (2.6), $\mu_k(\hat{x} - x^{k+1}) = G^k + \nu^k$ and, since $\nu^k \in N_X(x^{k+1})$ and $x^{k+2} \in X$,

$$\mu_k\left\langle x^{k+2} - x^{k+1}, \hat{x} - x^{k+1}\right\rangle = \left\langle x^{k+2} - x^{k+1}, G^k + \nu^k\right\rangle \leq \left\langle x^{k+2} - x^{k+1}, G^k\right\rangle \leq \|G^k\|\|x^{k+2} - x^{k+1}\| .$$

Since $G^k \in \partial\mathscr{M}_k(x^{k+1}) \subset conv\{g^j_{\mathtt{f}}, g^j_{\mathtt{c}} : j \in J_k\}$, assumption (2.20) implies that $\|G^k\| \leq \Gamma$ and, hence,

$$0 \leq (1-m)\delta^k \leq 2\Gamma\|x^{k+2} - x^{k+1}\| + \mathtt{OV}^{k+1} - (\mathtt{OV}^k + \frac{1}{2}\mu_k\|x^{k+2} - x^{k+1}\|^2) .$$

In view of (3.8), the right handside above tends to zero. Since $m \in (0,1)$, $\delta^k \to 0$, as desired.

The negation of (2.13), condition $1 - (\alpha_k + \beta_k) \geq B > 0$ from (2.14), and the fact that $\mu_k \geq \mu^-_{k+1}$ from (2.18c) imply that

$$0 \leftarrow \delta^k \geq \frac{1}{2}\mu^-_{k+1}B\|x^{k+1} - \hat{x}\|^2 \geq 0 ,$$

so $x^{k+1} \to \hat{x}$, as stated. Since by (2.6) $G^k + \nu^k = \mu_k(\hat{x} - x^{k+1})$ and by (2.18c) $\mu_k \leq \mu_{\max}$, we obtain that

$$0 \leq \frac{1}{\mu_{\max}} \|G^k + \nu^k\| \leq \frac{1}{\mu_k} \|G^k + \nu^k\| = \|x^{k+1} - \hat{x}\| \to 0 \text{ and, hence, } G^k + \nu^k \to 0\,.$$

Finally, using in (2.16) that $\alpha_k \geq 0$ by (2.14) gives

$$0 \leftarrow \delta^k \geq \mathtt{E}^k = \delta^k - \frac{2-\alpha_k}{2}\mu_k \|x^{k+1} - \hat{x}\|^2 \geq \delta^k - \mu_k \|x^{k+1} - \hat{x}\|^2 \geq \delta^k - \mu_{\max} \|x^{k+1} - \hat{x}\|^2.$$

Since both righthand side terms go to zero, so does $\mathtt{E}^k$ and the proof is finished. □

For the result above to hold, at null steps the bundle needs to be managed in a manner ensuring the relations (3.4). Condition (3.4b) holds if the last generated information enters the bundle, that is,

$$k+1 \in J^{k+1} \quad \text{ which is equivalent to having } \left(x^{k+1}, f^{k+1}, c^{k+1}, g_{\mathtt{f}}{}^{k+1}, g_{\mathtt{c}}{}^{k+1}\right) \in \mathscr{B}^{k+1}\,.$$

As for (3.4a), the inequality is typically ensured by introducing aggregate information in the bundle. In our improvement function setting, this can be done by splitting the aggregate information into their f and c parts. Specifically, in view of the model definition (2.4), there is a multiplier $\lambda^k \in [0,1]$ such that

$$\mathscr{M}_k(x^{k+1}) = \lambda^k(\check{f}_k(x^{k+1}) - \tau_1^k) + (1-\lambda^k)(\check{c}_k(x^{k+1}) - \tau_2^k)\,,$$

and

$$G^k = \lambda^k G_f^k + (1-\lambda^k) G_c^k \quad \text{ with } G_f^k \in \partial \check{f}_k(x^{k+1})\,, G_c^k \in \partial \check{c}_k(x^{k+1})\,.$$

For (3.4a) to hold it is then enough to take

$$\left(x^{k+1}, \check{f}_k(x^{k+1}), \check{c}_k(x^{k+1}), G_f^k, G_c^k\right) \in \mathscr{B}^{k+1}\,.$$

Similar calculations can be derived for *economic* bundles which, in the information $(x^j, f^j, c^j, g_{\mathtt{f}}^j, g_{\mathtt{c}}^j)$, replace the knowledge of the vector x^j by two scalars, referred to as linearization errors for f and c at $\hat{x}^k$. We refer to [38] for more details.

Finally, like in the case of infinitely many serious steps, the remainder term corresponds to $G^k + \nu^k \to 0$ with $\mathtt{E}^k \to 0$. For this reason, instead of the stopping criteria in Step 3 of Algorithm 2.1, one could stop when $G^k + \nu^k$ is sufficiently small, as long as $\mathtt{E}^k$ is also small (serious and null cases) or nonpositive (noisy case).

4. Link with the Original Problem.

In Sec. 3 we established the limiting behaviour of Algorithm 2.1. We still need to analyze in which sense the method converges to an approximate solution to problem (1.1). It is at this stage that the oracle errors play a major role. Our results below show what can be expected in terms of solving (1.1) for oracles that are of upper or lower type. Partly assymptotically exact and exact oracles are also considered.

4.1. Convergence Results. We start with a general result, suitable for oracles yielding inexact linearizations that may lie *above* the function. Specifically, we suppose that for any $x^k \in X$ the oracle output (2.2) is of *upper* type, in the sense that errors ε^k at iteration k and asymptotic error $\varepsilon \geq 0$ satisfy:

$$\forall y \in X \quad \left\{ \begin{array}{l} f^k + \langle g_{\mathtt{f}}^k, y - x^k \rangle \leq f(y) + \varepsilon^k \\ c^k + \langle g_{\mathtt{c}}^k, y - x^k \rangle \leq c(y) + \varepsilon^k \end{array} \right. \text{ with } \varepsilon := \limsup \varepsilon^k\,. \tag{4.1}$$

Theorem 4.1 (Asymptotic Bounds for Upper Oracles). *Consider solving* (1.1) *with Algorithm 2.1 using an oracle* (2.2) *such that* (2.20) *and* (4.1) *hold, with parameters* α_k, β_k *satisfying* (2.14) *and penalties as in* (2.19). *Suppose the prox-parameter is updated according to* (2.18)*, and the model satisfies* (3.4).

When the algorithm loops forever, for any accumulation point $\bar{x}$ of the (bounded) sequence $\{\hat{x}^k\}$ and its limiting inexact f/c-values and parameters ($\bar{f}$, $\bar{c}$, $\bar{\rho}$, $\bar{\sigma}$) it holds that

$$\forall y \in X \quad \max(\bar{c},0)(1-\bar{\sigma}) \le \max\Big(f(y)-\bar{f}-\bar{\rho}\max(\bar{c},0), c(y)-\bar{\sigma}\max(\bar{c},0)\Big)+\varepsilon. \tag{4.2}$$

Moreover, :

(i) If $\bar{c} > 0$ then

$$\exists R : \bar{\rho} \ge R \Longrightarrow \bar{c} \le c(y)+\varepsilon \textit{ for all } y \in X.$$

(ii) If $\bar{c} \le 0$ and the set $X_\varepsilon = \{y \in X : c(y) \le -2\varepsilon\}$ is not empty, then

$$\bar{f} \le f(y)+\varepsilon \textit{ for all } y \in X_\varepsilon.$$

Proof. When the algorithm loops forever, one of the index sets K_s, K_a, K_n, defined respectively in Lem. 3.1, 3.2, 3.3, is infinite. Since $\hat{x}^k \in X$ and X is compact, for any of such sets there exists a subset K' such that $\{\hat{x}^k\}_{k\in K'} \to \bar{x}$, recalling that when $K' = K_a$ and $K' = K_n$ eventually $\hat{x}^k = \hat{x} = \bar{x}$ remains fixed. By the same three lemmas, and the first item in Prop. 2.2,

$$\max(\hat{c}^k,0)(1-\sigma_k) = h^k_{\hat{x}^k} \le \mathscr{M}_k(y)+o(1/k) \text{ for all } y \in X \text{ and } k \in K' \text{ sufficiently large.}$$

By (4.1), $\mathscr{M}_k(y) \le H_{\tau^k}(y)+\varepsilon^k$ for the exact improvement function (2.1) written with target $\tau = \tau^k$, so

$$\forall y \in X \quad \max(\hat{c}^k,0)(1-\sigma_k) \le \max(f(y)-\tau^k_1, c(y)-\tau^k_2)+o(1/k)+\varepsilon^k. \tag{4.3}$$

By (2.3), the target is $\tau^k = (\hat{f}^k+\rho_k\max(\hat{c}^k,0), \sigma_k\max(\hat{c}^k,0))$ and by (2.20) the sequences $\{\hat{f}^k\}$ and $\{\hat{c}^k\}$ are bounded. Extracting from K' a further subsequence K if needed, we let

$$\bar{f} = \lim_{k\in K}\hat{f}^k, \quad \bar{c} = \lim_{k\in K}\hat{c}^k, \quad \bar{\rho} = \lim_{k\in K}\rho_k, \quad \bar{\sigma} = \lim_{k\in K}\sigma_k,$$

noting that $\bar{\rho}$ is not necessarily finite. Passing to the limit as $K \ni k \to \infty$ in (4.3) yields (4.2).

Consider first the case $\bar{c} > 0$. Since X is bounded and both f and c are real-valued, the constant $R = \frac{1}{\bar{c}}\Big(\max_{y\in X}(f(y)-c(y))-\bar{f}\Big)+\bar{\sigma}$ is well defined and any $\bar{\rho} > R$ satisfies $(\bar{\rho}-\bar{\sigma})\bar{c} > f(y)-c(y)-\bar{f}$ for all $y \in X$. Since the inequality is equivalent to having $f(y)-\bar{f}-\bar{\rho}\max(\bar{c},0) < c(y)-\bar{\sigma}\max(\bar{c},0)$, the stated results follows from (4.2).

When $\bar{c} \le 0$, (4.2) becomes $0 \le \max\Big(f(y)-\bar{f}, c(y)\Big)+\varepsilon$ and evaluating at any $y \in X_\varepsilon$ gives the final result. □

Thm. 4.1 states that, as long as ρ_k is managed as a penalty parameter (for instance $\rho_{k+1} = 2\rho_k$ if $c_{\hat{x}^{k+1}} > 0$, and $\rho_{k+1} = \rho_k$ otherwise), Algorithm 2.1 will eventually detect problem (1.1) as infeasible up to the accuracy ε, or it will find an approximate minimizer in the sense stated by the second item in the theorem. For the latter to happen, the accuracy should be small enough to ensure nonemptiness of the set X_ε (noting that in this set the factor -2 could be replaced by any value strictly smaller than -1).

We now state a refinement of this result, for the case when inaccurate linearizations eventually stay *below* the functions f and c. We refer to this situation as having "lower" oracles.

COROLLARY 4.1 (Convergence for lower oracles). *In the setting of Thm. 4.1, suppose $\varepsilon = 0$ in* (4.1). *If problem* (1.1) *has a Slater point and $\bar{\rho}$ is sufficiently large, then $\bar{c} \le 0$ and $\bar{f} \le f(y)$ for all y feasible in* (1.1).

Proof. The case $\bar{c} \le 0$ is Thm. 4.1(ii) with $\varepsilon = 0$. As for the case $\bar{c} > 0$, it is excluded by observing that for $\bar{\rho}$ sufficiently large (4.2) becomes

$$\bar{c}(1-\bar{\sigma}) \le \max\Big(f(y)-\bar{f}-\bar{\rho}\bar{c}, c(y)-\bar{\sigma}\bar{c}\Big) = c(y)-\bar{\sigma}\bar{c},$$

so $0 < \bar{c} \le c(y)$ for all $y \in X$, an inequality that cannot hold when y is the Slater point. □

A further refinement is possible for lower oracles that are *partly asymptotically exact*, see [26], [21], [30]. Specifically, these are oracles that become progressively more and more accurate at serious steps. We now show that, unlike the previous cases, in this situation the penalty ρ_k does not need to increase to infinity when the center is deemed infeasible (that is, when $c_{\hat{x}^k} > 0$).

COROLLARY 4.2 (Convergence for partly asymptotically exact lower oracles). *In the setting of Thm. 4.1, suppose* $\varepsilon = 0$ *in* (4.1) *and calculations are eventually exact for serious steps:* $f(\bar{x}) = \bar{f}$ *and* $c(\bar{x}) = \bar{c}$. *The following holds:*

(i) Either (1.1) *is feasible and* $c(\bar{x}) \le 0$ *with* $\bar{x}$ *solving* (1.1).
(ii) Or $c(\bar{x}) > 0$ *and* (1.1) *is unfeasible with* $\bar{x}$ *minimizing infeasibility.*

As a result, when (1.1) *has a Slater point, only item (i) is possible.*

Proof. When $\bar{c} \le 0$, the first result is Thm. 4.1(ii), recalling that $\bar{f} = f(\bar{x})$ and $\varepsilon = 0$.

Similarly for $\bar{c} > 0$ and $\bar{\rho} = +\infty$, applying the first item in Thm. 4.1 with $\bar{c} = c(\bar{x})$ and $\varepsilon = 0$. When $\bar{c} > 0$ and $\bar{\rho}$ is finite (not necessarily larger than R in the theorem), adding $\bar{\sigma}\bar{c}$ to both sides of (4.2) and using that $\varepsilon = 0$ together with the assumption that $c(\bar{x}) = \bar{c}$ and $f(\bar{x}) = \bar{f}$ gives that

$$c(\bar{x}) \le H(y) := \max\Big(f(y) - f(\bar{x}) - (\bar{\rho} - \bar{\sigma})c(\bar{x}), c(y)\Big) \text{ for all } y \in X. \tag{4.4}$$

In the right handside above we use the notation $H(\cdot)$ for the (convex) exact improvement function (2.1), written with target $\tau = (f(\bar{x}) + (\bar{\rho} - \bar{\sigma})c(\bar{x}), 0)$. In particular, when $y = \bar{x}$ the inequality (4.4) becomes

$$0 < c(\bar{x}) \le H(\bar{x}) = \max\Big(-(\bar{\rho} - \bar{\sigma})c(\bar{x}), c(\bar{x})\Big).$$

Since by (2.19) the penalties satisfy $\rho_k \ge RS + \sigma_k - 1 > \sigma_k - 1$, this means that $-(\bar{\rho} - \bar{\sigma}) < 1$ and the maximum above is attained at $c(\bar{x})$ and, hence, $H(\bar{x}) = c(\bar{x})$. As a result, (4.4) states that $0 \in \partial(H + i_X)(\bar{x})$, where i_X denotes the indicator function of the set X. Being a max-function, the subgradients of H at $\bar{x}$ are of the form

$$\begin{gathered}\lambda g_{\mathtt{f}} + (1-\lambda)g_{\mathtt{c}} \text{ for } \lambda \in [0,1] \text{ with } g_{\mathtt{f}} \in \partial f(\bar{x}) \text{ and } g_{\mathtt{c}} \in \partial c(\bar{x}) \\ \text{with } \lambda \in (0,1] \iff -(\bar{\rho} - \bar{\sigma})c(\bar{x}) \ge c(\bar{x}).\end{gathered}$$

Since $c(\bar{x}) > 0$ and $-(\bar{\rho} - \bar{\sigma}) \le 1 - RS < 1$ by (2.19), the only possibility is $\lambda = 0$, i.e., $0 \in \partial c(\bar{x})$ and, therefore, $0 < c(\bar{x}) \le c(y)$ for all $y \in X$, as stated. □

For completeness, and to relate our method with previous algorithms in the literature, we finish with a result for oracles that make exact calculations. In this case, there is no noise attenuation step and the nominal decrease (2.12) is defined with $\alpha_k \in [0,1]$.

COROLLARY 4.3 (Convergence for exact oracles). *Consider an exact oracle: for any* $x^j \in X$ (2.2) *returns* $f^j = f(x^j)$, $c^j = c(x^j)$ *and subgradients* $g^j_{\mathtt{f}} \in \partial f(x^j)$ *and* $g^j_{\mathtt{c}} \in \partial c(x^j)$. *Then Algorithm 2.1 with* $\beta_k \equiv 0$ *never executes Step 2 (noise attenuation).*

Furthermore, suppose $\alpha_k \in [0,1]$, *the penalties satisfy* (2.19), *the prox-parameter is updated according to* (2.18a) *and* (2.18c), *and the model satisfies* (3.4). *Then the statements in Cor. 4.2 apply.*

Proof. By convexity, an exact oracle is of the lower type, so (4.1) holds with $\varepsilon = 0$. In particular, writing (2.3) and (2.4) with $y = \hat{x}^k$ we see that $h^k_{\hat{x}^k} \ge \mathscr{M}_k(\hat{x}^k)$. Since, by (2.5), $\mathscr{M}_k(\hat{x}^k) \ge \mathscr{M}_k(x^{k+1}) + \frac{1}{2}\mu_k\|x^{k+1} - \hat{x}^k\|^2$, the left handside in (2.11) is always nonnegative and, as long as $\beta_k \ge 0$, the algorithm will never consider noise too large (naturally so, since for exact oracles there is no noise to be detected).

The set X is bounded and the subdifferential mapping of a convex function is locally bounded, so (2.20) holds for exact oracles. Since in addition, (2.14) holds by taking for example $B = \frac{1}{2} = -b$, Cor. 4.2 completes the proof. □

4.2. Relation with previous work. Cor. 4.3 covers methods based on improvement functions already considered in the literature for solving constrained nonsmooth problems using exact or lower oracles. More precisely, conditions (2.19) are satisfied by at least the following three choices:

$$\rho_k = \sigma_k \equiv 0, \quad \text{or } \rho_k \equiv \bar{\rho} < +\infty \text{ and } \sigma_k \equiv 1, \quad \text{or } \bar{\rho} = +\infty \text{ and } \sigma_k \equiv 0,$$

corresponding to the improvement functions in [38], [1], and [20], respectively. The first two methods were developed for exact oracles ([1] deals with nonconvex functions). The method of centers in [20] considers a setting corresponding to a lower oracle, and is addressed by Cor. 4.2.

To decide if the iterate gives a serious step, the three methods ([38, 1, 20]) use in Step 4 a criterion based on descent of the improvement function. Namely,

$$\text{if } h^k_{x^{k+1}} = \max(f^{k+1} - \tau_1^k, c^{k+1} - \tau_2^k) \le h^k_{\hat{x}^k} - m\delta^k \quad \text{the iteration is declared serious} \tag{4.5}$$

(noting that the test (2.15) is also mentioned in [20] as a possibility). In view of the second item in Prop. 2.2, the criterion (2.15) is stronger for null steps. Depending on the problem, one criterion or the other might be preferable. By checking the proofs of Lem. 3.1 and 3.3, it is not difficult to see that the asymptotic results in Sec. 3 still hold with the alternative test.

It should also be mentioned that [20] uses an alternative stopping test, suitable for unbounded feasible sets (which is not the case in (1.1)). More precisely, instead of checking if $\delta^k \le \text{TOL}$, the method of centers uses the conditions

$$\max\left\{\|G^k + v^k\|, \mathtt{E}^k + \left\langle G^k + v^k, \hat{x}^k\right\rangle\right\} \le \text{TOL}_{[20]} \text{ and } c_{\hat{x}^k} \le 0. \tag{4.6}$$

For bounded feasible sets in (1.1), (4.6) is equivalent to the stopping test $\delta^k \le$TOL. Indeed, following the definition of $\mathtt{E}^k$ in (2.9) and δ^k in (2.12) with $\alpha_k = 0$, we see that $\mathtt{E}^k + \left\langle G^k + v^k, \hat{x}^k\right\rangle = \delta^k + \left\langle G^k + v^k, \hat{x}^{k+1}\right\rangle$. Using the fact that X is compact and in particular bounded together with $\|G^k + v^k\| \le \text{TOL}_{[20]}$, the stopping criteria (4.6) implies that $\delta^k \le \text{TOL}$ for certain modified tolerance.

Another stopping criterion is the one used in [10]:

$$\max\left\{\mu_k\|x^{k+1} - \hat{x}^k\|, \mathtt{E}^k\right\} \le \text{TOL}_{[10]}. \tag{4.7}$$

It was already mentioned that Lem. 3.1, 3.2, 3.3, ensure that $\|G^k + v^k\| = \mu_k\|x^{k+1} - \hat{x}^k\| \to 0$. Together with (2.16) the stopping test $\delta^k \le \text{TOL}$ implies (4.7) for an appropriate tolerance $\text{TOL}_{[10]}$. On the other hand, keeping the prox-parameters uniformly bounded from above in (2.18), together with (2.16) and (2.6) gives

$$\delta^k = \mathtt{E}^k + \frac{2-\alpha_k}{2}\mu_k\|x^{k+1} - \hat{x}^k\|^2 \le 2\max\left\{\mathtt{E}^k, \mu_k\|x^{k+1} - \hat{x}^k\|^2\right\}, \tag{4.8}$$

showing that (4.7) also implies $\delta^k \le \text{TOL}$ for an appropriately chosen tolerance TOL.

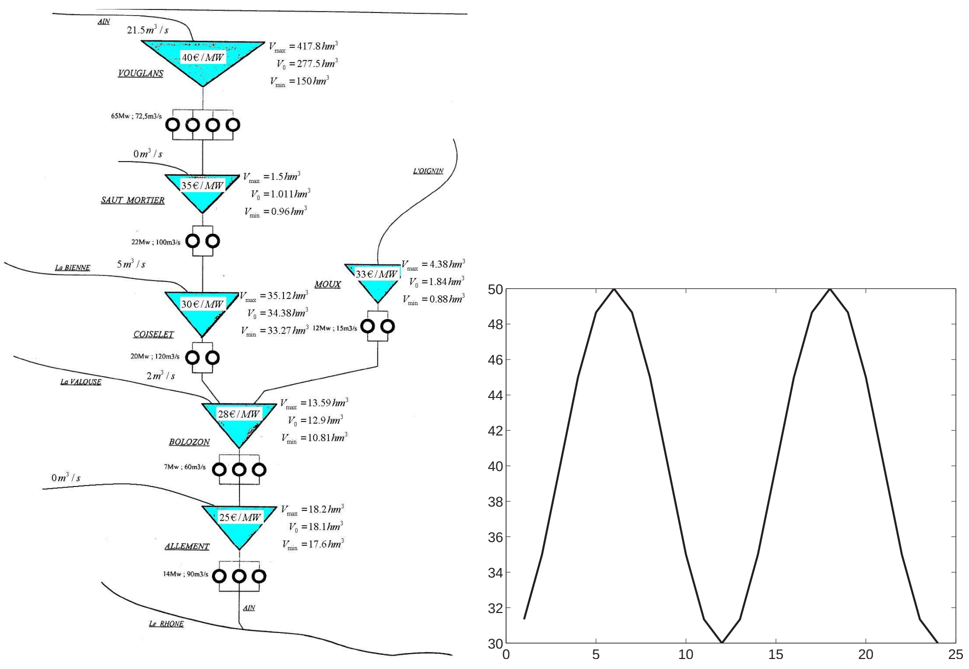
Therefore, for convex problems as (1.1), our approach includes previous work, significantly extending the applicability of algorithms in the literature:

– not only exact and lower oracles can be used, but also upper ones, satisfying (4.1);
– the criterion for serious steps can be (2.15) or based on the improvement function;
– the descent and noise parameters α_k, β_k can be chosen in any manner satisfying (2.14). This versatility has a positive impact on the numerical results.
– any choice for the penalty parameters ρ_k and σ_k in (2.3) satisfying (2.19) is possible.

Regarding the last item, the role/utility of the penalty in second target (in τ_2^k) is not clear. At least in the convex setting, taking $\sigma_k = 0$ and forgetting about this additional parameter seems sufficient. It is argued in [1], however, that a positive σ_k can be beneficial in the numerical performance of the algorithm, so the situation may be different for nonconvex problems.

5. Energy Application: Hydro Reservoir Management. In this section we give the main elements of the application considered in the numerical experience. For convenience, and without loss of generality, we explain here a specific instance, and refer to [45] for more details.

Generally, an individual hydro valley is part of a larger hydro-thermal system which acts on a price signal. The latter can either come from some price decomposition scheme ([5, 23, 37]) or from the market.



(a) The Hydro Valley (b) The Price Signal

FIG. 5.1. *Numerical Instance Data*

5.1. Initial description of the system. Consider a hydro valley as in Fig. (5.1(a)).

The system topology, that is the relation between the six power reservoirs in the valley, is expressed by the reservoir connection matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Likewise, the vector $\sigma_{\mathscr{T}} = (1,1,1,1,2,2,3,3,4,4,5,5,5,6,6,6)$ associates each one of the 16 turbines in the valley to a specific reservoir. Finally, the sets

$$\mathscr{A}(n) = \{m \in \{1,\dots,6\} \ : A_{m,n} = 1\} \text{ and } \mathscr{F}(n) = \{m \in \{1,\dots,6\} \ : A_{n,m} = 1\}$$

for $n \in \{1,\dots,6\}$ identify uphill and downhill connections. In particular, the fact that $\mathscr{A}(1), \mathscr{A}(4)$ are empty is due to the reservoirs $n = 1,4$ being in the top, while emptiness of $\mathscr{F}(6)$ means that $n = 6$ is a bottom reservoir.

For simplicity, we do not consider flow delays (also known as "water travel" times), which can easily be incorporated in the model. The time horizon $\mathfrak{T}$ is discretized into 24 time steps, each one with length $\Delta t = 2$ hours.

For $i = 1,\dots,16$ the control of the i-th turbine is done by variables $x^i(t)$ for each $t \in \mathfrak{T}$. These variables, expressed in units m^3/h also satisfy:

$$0 \le x^i(t) \le \bar{x}^i, \ \forall t \in \mathfrak{T}, i = 1,...,16. \tag{5.1}$$

5.2. Random Inflows: modelling and data. Reservoir inflows (in m^3/h) are modelled by a stochastic process. Accordingly, $A^n(t)$ stands for the stochastic process for reservoir $n \in \{1,\dots,6\}$.

Some of the reservoirs have deterministic inflows (typically zero). Indeed, while uphill reservoirs have random inflows due to the melting of snow in the high mountains, rain can be neglected for downhill reservoirs, which can be modelled in a deterministic framework. The set $\mathcal{N}^r \subseteq \{1,\dots,6\}$ gathers all reservoirs with random inflows.

For $n \in \mathcal{N}^r$, the stochastic inflow process is modelled as a sum of a deterministic trend s_t^n and a causal process ([39]) generated by Gaussian innovations. To this end let $\zeta^n(t)$ be a Gaussian white noise process: $(\zeta^{k_1}(t),\dots,\zeta^{k_l})$ is a Gaussian random vector of zero average and variance-covariance matrix $\Sigma(t)$ ($\{k_1,\dots,k_l\} = \mathcal{N}^r$). The ζ vector is assumed to have independent t-components.

Since $A^n(t)$ is a causal process, we obtain the characterization

$$A^n(t) = s_t^n + \sum_{j=0}^{t-1} \psi_j^n \zeta^n(t-j), \text{ for all } n \in \mathcal{N}^r, t \in \mathfrak{T}$$

for some coefficient vector ψ^n.

In this instance, both reservoirs 1 and 2 have random inflows, with a standard deviation equal to 20% of the nominal values. This implies that the standard deviations are 4.24 m^3/s for reservoir 1 and 0.3 m^3/s for reservoir 2. Inflows to reservoir 2 are modelled as an autoregressive process of order three, AR(3), with coefficients $(0.9, 0.7, -0.7)$. For reservoir 1, inflows follow an AR(1) process with coefficient 0.9.

To express the hydro valley relations in the presence of uncertainty, we need to write down the variance-covariance matrix of the stochastic process. This needs the introduction of some notation. Specifically, we let ψ be a 2×24 matrix such that $\psi_{1,j} = 0.9^j$ and $\psi_{2,j} = 0.9\psi_{2,j-1} + 0.7\psi_{2,j-2} - 0.7\psi_{2,j-3}$, $j = 1,\dots,24$, where in the last formula any negative indexed terms are assumed 0. We also define the elementary covariance matrix Σ_a:

$$\Sigma_a = \begin{pmatrix} 4.24 & 0 \\ 0 & 0.3 \end{pmatrix} \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix} \begin{pmatrix} 4.24 & 0 \\ 0 & 0.3 \end{pmatrix}$$

and let Σ_ζ be the block diagonal matrix containing 24 copies of Σ_a. The expression for matrix Σ can be obtain by letting C^ψ be the 48×48 block diagonal matrix, where the first block is obtained from the first row of ψ by accumulating elements, i.e., element (i,j) of the block is $\sum_{k=0}^{i-j} \psi_k$. As a result, if R is the 48×48 matrix with $R_{i,mod(i-1,24)2+1+\lfloor \frac{i-1}{24} \rfloor} = 1$, it follows that

$$\Sigma = (7200)^2 (R^{-1} C^\psi R) \Sigma_\zeta (R^{-1} C^\psi R)^\mathsf{T}, \tag{5.2}$$

is the variance-covariance matrix of the global random inflow vector, that we shall shorten to η. The vector η is a Gaussian multi-variate random variable in dimension 48.

5.3. Flow constraints and Volume bounds. In the hydro valley, reservoirs are in a cascade, and there are flow constraints induced by turbining water in each reservoir. The following water balance equation states such relations:

$$V^n(t) = V^n(t-1) + \sum_{m \in \mathcal{A}(n)} \sum_{i \in \sigma_{\mathcal{T}}^{-1}[m]} x^i(t)\Delta t - \sum_{i \in \sigma_{\mathcal{T}}^{-1}[n]} x^i(t)\Delta t s_t^n \Delta t + \eta_{24(n-1)+t}, n = 1,\dots,6 \tag{5.3}$$

The above equation is entirely deterministic, except for the reservoirs $n \in \mathcal{N}^r$. To deal with the underlying randomness while respecting each reservoir bounds, we introduce the constraints

$$\mathbb{P}\left[V_{min}^n(t) \le V^n(t) \le V_{max}^n(t) \text{ for all } t \in \mathfrak{T}, n \in \mathcal{N}^r\right] \ge p \tag{5.4}$$

$$V_{min}^n(t) \le V^n(t) \le V_{max}^n(t) \text{ for all } t \in \mathfrak{T}, n \in \mathcal{N} \setminus \mathcal{N}^r. \tag{5.5}$$

Relation (5.4) is a joint chance constraint. It states that, for any level of uncertain inflows, reservoirs 1 and 2 must keep their volumes between the prescribed bounds, with probability higher than p. We have taken $p = 0.8$ in our application.

5.4. Objective function and problem structure. In this stylized numerical example, water is priced by a constant factor W^n; a more realistic description, with volume dependent water values, can be found in [45]. Since such modelling is represented by a set of affine constraints, our setting also covers the more general case. Generation is valued by using (given) time dependent price signal $\lambda(t), t \in \mathfrak{T}$ (in €/MWh) like the one shown in Fig. (5.1(b)).

As a result, the objective function in (1.1) is given by

$$\sum_{n=1}^{6}(W^n(V^n(0)-\mathbb{E}(V^n(T)))-\sum_{t\in\mathfrak{T}}\lambda(t)\Delta t\sum_{i=1}^{16}\rho_i(t)x^i(t),$$

where

$$\rho = \frac{1}{3600}(2.4627, 0.5563, 0.3309, 0.2363, 0.3749, 0.0651, 0.2857,\\ 0.0476, 1.2350, 0.3650, 0.2592, 0.0570, 0.0338, 0.3553, 0.0700, 0.0414)$$

denotes the efficiency in MWh/m^3 of each turbine $i = 1,\dots,16$.

Therefore, our problem fits the following abstract structure:

$$\begin{aligned}&\min_{x\geq 0}\langle f,x\rangle\\ &s.t.\ \tilde{A}x \leq \tilde{b}\\ &\qquad p \leq \mathbb{P}[a^r + A^r x \leq \eta \leq b^r + A^r x],\end{aligned} \tag{5.6}$$

where $\eta \in \mathbb{R}^m$ is a Gaussian random vector with variance-covariance matrix Σ and zero mean (we have explicitly extracted the non-zero average in a^r, b^r).

By a classical result (Thm. 4.2.4 [32]), the feasible set induced by the joint chance constraint (5.4) is convex. Moreover from the same result it follows that the function

$$\log(p) - \log(\mathbb{P}[a^r + A^r x \leq \eta \leq b^r + A^r x]) \tag{5.7}$$

is a convex function.

So (5.6) corresponds to (1.1) with

$$f(x) := \langle f,x\rangle\ ,\ X := \{x\in\mathbb{R}^n\ : x\geq 0, \tilde{A}x\leq\tilde{b}\}\,,\ \text{and}\ \ c(x) := \log(p) - \log(\mathbb{P}[a^r + A^r x \leq \eta \leq b^r + A^r x])\,.$$

It is moreover clear that X is bounded according to (5.1). In this setting, the f-oracle is exact. As for the c-oracle, it falls into the framework (4.1) with $\varepsilon > 0$, as explained below.

5.5. Devising an inexact upper oracle for the constraint. It is shown in [44, Thm. 1], (see also [45, Cor. 1]) that the mapping

$$\varphi : \mathbb{R}^n \to [0,1] \quad \text{defined by} \quad \varphi(x) = \mathbb{P}[a^r + A^r x \leq \eta \leq b^r + A^r x]$$

is differentiable with gradient $\nabla\varphi(x) = \nabla_{\hat{a}}F_\eta(\hat{a},\hat{b})^\mathsf{T}A^r + \nabla_{\hat{b}}F_\eta(\hat{a},\hat{b})^\mathsf{T}A^r$, with $\hat{a} = A^r x + a^r$, $\hat{b} = A^r x + b^r$ and $F_\eta : \mathbb{R}^m\times\mathbb{R}^m \to [0,1]$ defined by $F_\eta(\hat{a},\hat{b}) = \mathbb{P}[\hat{a}\leq\eta\leq\hat{b}]$. The results of [44] can be applied since η is a non-degenerate (i.e., with positive definite covariance matrix) Gaussian random variable. In the potentially degenerate case we can refer to [16]. The analysis of what follows will be completely similar as again the computation of the gradient is reduced to evaluation mappings with a structure close to the one of φ above.

Accordingly, the i-th component of $\nabla_{\hat{a}}F_\eta(\hat{a},\hat{b})$ ($-\nabla_{\hat{b}}F_\eta(\hat{a},\hat{b})$ respectively) is equal to $-\psi(z)\mathbb{P}[\tilde{a}^r + \tilde{A}^r x \leq \tilde{\eta} \leq \tilde{b}^r + \tilde{A}^r x]$. In this expression, ψ is the density of a standard Gaussian random variable in 1 dimension; z a specific point depending on x, and $\tilde{a}^r, \tilde{b}^r, \tilde{A}^r, \tilde{\eta}$ of a^r, b^r, A^r are appropriate modifications of the respective objects. These modifications depend on whether the derivative is taken with respect to $\hat{a}$ or $\hat{b}$.

We see that, in order to compute one component of the gradient, one needs to evaluate a mapping of the form $F_\zeta(\hat{a},\hat{b})$ at specific points $\hat{a}\leq\hat{b}\in\mathbb{R}^m$, with $\zeta\in\mathbb{R}^m$ a non-degenerate multi-variate Gaussian random variable.

Since the evaluation of the c-function also requires to compute a similar probability, the core of the c-oracle is to make the involved multidimensional calculations in a fast and efficient manner. This is achieved by using the code [12], developped by A. Genz for multivariate normal probabilities. Having as input a requested accuracy $\varepsilon^g > 0$, the code either returns a value $\tilde{F}$ such that $\left|\tilde{F} - F_\zeta(\hat{a},\hat{b})\right| < \varepsilon^g$, or issues an error message, stating the impossibility of making the calculation with the requested precision. In the latter case, it is possible to increase the number of quasi Monte Carlo particles used in the numerical integration and make another attempt to obtain the desired accuracy.

Since the integral approximation estimate can be larger or smaller than the exact value, the c-linearizations may lie below or above the exact function c, noting that asymptotically exact (up to say floating point precision) calculations would be possible (although they might considerably increase the time spent in the oracle).

5.6. Convergence results for the application. According to the model set up in this section, the objective function f of problem (1.1) is linear and therefore has an exact oracle. Inexactness arises from evaluating the probabilistic constraint c. The corresponding oracle may be of the upper type, because no information is available on the sign of the incurred error, neither for the function values nor for the subgradients. However, as explained in [15], the user can control such error, keeping it sufficiently small if desired (provided enough CPU time is spent in the calculations).

We now derive an explicit expression for $\varepsilon > 0$ in (4.1). Consider a constant $\Phi > 0$, for which $\varphi(x^k) > \Phi > 0$ for all iterations k. For example, one can take as initial point $\hat{x}^0$ an appropriate convex combination with the Slater point $x^s \in \mathbb{R}^n$ ensuring that $\varphi(\hat{x}^0) > \Phi$ for any given $p > \Phi > 0$. Then, as far as serious iterates are concerned, (2.15) ensures that $\varphi(\hat{x}^k) > \Phi$. Since ε in (4.1) depends on the limiting behaviour, clearly one can choose μ_k in (2.5) according to rule (2.18c) in such a way that $\varphi(x^k) > \Phi$ also in null-steps.

The logarithm being a uniformly continuous mapping on $[\Phi,1]$, X being a compact set and $\varphi : X \to [0,1]$ also being uniformly continuous, it is clear that for any $\varepsilon' > 0$ a precision ε^g can be chosen such that $\left|c(x^k) - c_{x^k}\right| \leq \varepsilon'$ for any iteration k. As discussed at the end end of Sec.5.5, $\|\nabla\varphi(x) - \nabla\varphi_x\|_\infty \leq \frac{2}{\sqrt{2\pi}}\|A^r\|_\infty\varepsilon^g$, since the standard Gaussian density ψ satisfies $|\psi(z)| \leq \frac{1}{\sqrt{2\pi}}\ \forall z \in \mathbb{R}$. By the Cauchy-Schwarz inequality and compactness of X we then obtain that

$$\left|\left\langle g_c(x^k), y - x^k\right\rangle - \left\langle g_{c_{x^k}}, y - x^k\right\rangle\right| \leq M\varepsilon^g$$

for an appropriate constant $M > 0$ and any $y \in X$. We conclude that (4.1) holds with $\varepsilon \geq \varepsilon' + M\varepsilon^g$. Moreover, whenever ε^g is taken small enough, ε' can also be made arbitrarily small.

We are in the conditions of Thm. 4.1. Since in the hydro valley application it is reasonable to assume the existence of a Slater point x^s, we pick ε^g, the user-defined precision of Genz' code in such a way that $c(x^s) \leq -2\varepsilon$. This ensures that the set X_ε in Thm. 4.1 is not empty. In the setting of the theorem, let $\bar{x}$ be the limiting point generated by Algorithm 2.1.

When $\bar{x}$ is infeasible according to the oracle information, $\bar{c} > 0$ and item (i) of Thm. 4.1 leads to the contradiction $\bar{c} < c(x^s) + \varepsilon \leq -\varepsilon < 0$. Therefore, $\bar{x}$ is approximately feasible ($\bar{c} < 0$) and satisfies the relation $\bar{f} \leq f(y) + \varepsilon$ for all $y \in X_\varepsilon$ by item (ii) of the same theorem. Consequently, $\bar{x}$ solves approximately (1.1).

6. Numerical experience. For our numerical experience, the starting point uses x^d, a solution to the linear program

$$\begin{aligned} \min_{x\geq 0, x\in\mathbb{R}^n}\ & \langle f, x\rangle \\ s.t.\quad & \tilde{A}x \leq \tilde{b} \\ & a^r + A^r x \leq 0, b^r + A^r x \geq 0, \end{aligned} \tag{6.1}$$

the "deterministic" counterpart of (5.6) wherein the random vector η is replaced by its expectation. In general x^d will not be feasible for (5.6) (unless it solves the problem).

Since improvement functions are scale-dependent, [38], it is useful to scale the constraint. Accordingly, we consider the constraint $Kp \leq K\mathbb{P}[a^r + A^r x \leq \eta \leq b^r + A^r x]$ for some value of $K > 0$. A natural choice for K would be $\left|\langle f,x\rangle^d\right|$. Finally rules (2.18a), (2.18c) and (2.18b) are dealt with in the following way. In serious steps we take $\mu_{k+1} = \mu_k$ without making any changes. When noise is detected, we take $\mu_{k+1} = \kappa\mu_k$ for a parameter $\kappa \in (0,1)$. Finally when null steps are made we choose $\mu_{k+1} = \min\{\mu_s\mu_k, \mu_{max}\}$ for a parameter $\mu_s > 1$.

6.1. Algorithms in the Benchmark. We first compare the performance of several methods, including variants of Algorithm 2.1 using different choices for the parameters. More precisely, we consider:

– A configuration of Alg.2.1 with a strong noise test:

$$Alg.2.1\texttt{SEV}: \quad \sigma_k \equiv 0\,, \quad \rho_k \equiv 0\,, \quad \alpha_k \equiv 1\,, \quad \beta_k \equiv -1+\varepsilon^m\,, \quad \text{StopTest (4.7)}\,,\ \text{DescTest (4.5)}\,,$$

where ε^m is the machine precision.

– A configuration of Alg.2.1 with null parameters:

$$Alg.2.1\texttt{NUL}: \quad \sigma_k \equiv 0\,, \quad \rho_k \equiv 0\,, \quad \alpha_k \equiv 0\,, \quad \beta_k \equiv 0\,, \quad \text{StopTest (4.6)}\,,\ \text{DescTest (4.5)}\,,$$

– The method of centers from [20], bearing some similarities with Alg. 2.1 provided parameters are properly set:

$$Alg.[20]: \quad \sigma_k \equiv 0\,, \rho_k \to \infty \text{ if } \{\hat{x}^{k+1}\} \text{ infeasible}\,, \alpha_k \equiv 0\,, \beta_k = \text{ad-hoc}\,, \text{StopTest (4.6)}\,,\ \text{DescTest (4.5)}\,.$$

– The conic bundle method from [22]:

$$Alg.[22]: \quad \text{No } \sigma_k, \rho_k\,, \quad \alpha_k \equiv 0\,, \quad \text{ad-hoc QP in Step 1, } \beta_k\text{, Stop and Descent Test}\,.$$

– The supporting hyperplanes method from [33] (see also [46, 34, 42]):

$$Alg.[33]: \quad \text{No } \sigma_k, \rho_k, \alpha_k, \beta_k\,, \qquad \text{Linear Program in Step 1, ad-hoc Stop and Descent Test}\,.$$

In both configurations of Alg.2.1 above taking $RS = 1$ ensures satisfaction of (2.19) for any choice of ρ_{k+1}; as for (2.14), it suffices to take any positive B and $b = \beta_k$ (constant in k). Since the conditions in Section 4 are satisfied, eventual convergence to an approximate solution is assured with these methods.

We now describe the specific ad-hoc updates of the last three methods. To help the comparison, the algorithms steps are numbered as in Alg. 2.1.

Alg.[20] Specifics. The penalty ρ_{k+1} is halved at serious steps and doubled if $c_{\hat{x}^{k+1}} > 0$. Once a feasible stability center is found, the numerical value of this penalty becomes irrelevant. For an additional Armijo-like parameter $m' \in (0,1]$, the method of centers modifies the noise management Step 2 in Alg. 2.1 as follows.

Step 2'a (Infeasible serious point) If $\hat{c}^k > 0$ and $\delta^k < m'\hat{c}^k$, noise is too large. Decrease the prox-parameter as in (2.18b), take $\rho_{k+1} = 2\rho_k$, maintain the bundle, and loop to Step 1 (solve a new QP subproblem).

Step 2'b (Noise attenuation). In the other cases, that is, if $\hat{c}^k \le 0$ or $\delta^k \ge m'\hat{c}^k$, check if condition (2.13) holds. If this condition holds, noise is too large. Decrease the prox-parameter as in (2.18b), take $\rho_{k+1} > \rho_k$ if $\hat{c}^k > 0$, maintain the bundle, and loop to Step 1 (solve a new QP subproblem).

The difference with our noise management step is in the first item: when the current serious point is infeasible, instead of (2.13) the condition $\delta^k < m'\hat{c}^k$ is checked. Such condition amounts to verifying if (2.13) holds with $\beta_k = 1 - \frac{2m'\hat{c}^k}{\mu_k\|x^{k+1}-\hat{x}^k\|^2}$. Suppose such value of β_k also satisfies (2.14) (for example suppose m' is sufficiently small). Then this alternative can also be considered a special case of Algorithm 2.1, with a specific updating of β_k for infeasible serious points.

Alg.[22] Specifics. This method is applicable when, like in our application, the objective function in (1.1) is linear or quadratic: the QP subproblem approximates (1.1) with a cutting-plane model for the constraint. Furthermore, the method needs the knowledge of a Slater point x^s for (1.1), which is also its starting point: $\hat{x}^0 = x^s$. The algorithm performs the following steps.

Step 1' (Alternative subproblem, δ^k,$\mathtt{E}^k$) Let (x^{k+1}, η^{k+1}) be a primal-dual solution to the QP

$$\min_{y\in X\subseteq\mathbb{R}^n} \langle f,y\rangle + \frac{1}{2}\mu_k\|y-\hat{x}^k\|^2$$
$$s.t. \quad \check{c}_k(y) \le 0\,.$$

This amounts to taking in (2.6), instead of subgradient $G^k \in \partial\mathscr{M}^k(x^{k+1})$, a vector $G^k \in f + \eta^{k+1}\partial\check{c}_k(x^{k+1})$ given by the QP optimality conditions. Taking $\delta^k := \left\langle f, \hat{x}^k - x^{k+1}\right\rangle$ and $\mathtt{E}^k = \delta^k - \mu_k\|x^{k+1} - \hat{x}^k\|^2$ ensures satisfaction of the left handside identity in (2.16) with $\alpha_k = 0$.

Step 3'a (Stopping test) Stop if $\|G^k + v^k\| \le tol$ and $\mathtt{E}^k + \left\langle G^k + v^k, \hat{x}^k\right\rangle \le 0$,

Step 3'b (Interpolation Step) If $c_{x^{k+1}} \le 0$, set $\gamma^k := 1$, otherwise $\gamma^k := \frac{-c_{x^s}}{c_{x^{k+1}} - c_{x^s}}$. Define $\check{x}^k := \gamma^k x^{k+1} + (1-\gamma^k)x^s$.

Step 4' (Serious step Test) If $\left\langle f, \check{x}^k\right\rangle \left\langle f, \hat{x}^k\right\rangle - m\delta^k$ then declare a serious step, taking as next center $\hat{x}^{k+1} = \check{x}^k$. Otherwise, declare a null step.

When the c-oracle is exact or lower[*], each stability center will be feasible, by convexity. When the c-oracle is of the upper type, like in our case, by the discussion in Sec. 5.5 it is possible to take a slightly larger γ^k so that $\varphi(\check{x}^k) \ge p + \varepsilon^g$. This is a potentially costly fix that may require several evaluations (our implementation uses the original definition for γ^k.) The analysis in [22] does not cover upper oracles, so convergence of this method is unclear, although it numerical behaviour was reasonable for our tests.

Alg.[33] Specifics. Like Alg.[22], the cutting-planes model for the constraint is built using for the c-evaluation points, points obtained from interpolating with the Slater point. The difference is that now the constraint is required to be active at the interpolation point ($x_c^0 = (1-\lambda^0)x^0 + \lambda^0 x^s$ satisfies $c_{x_c^0} = 0$ for some $\gamma^0 \in [0,1]$.) For a linear objective function the algorithm performs the following steps.

Step 1' (Linear Programming subproblem) Let x^{k+1} be a solution to

$$\begin{aligned} \min_{y \in X \subseteq \mathbb{R}^n} \; & \langle f, y\rangle \\ s.t. \quad & \check{c}_k(y) \le 0\,. \end{aligned}$$

Step 3'a (Interpolation and Oracle) Determine $\gamma^k \in [0,1]$ for which $x_c^{k+1} = (1-\gamma^k)x^{k+1} + \gamma^k x^s$ satisfies $c_{x_c^{k+1}} = 0$. Call the c-oracle and add the linearization to the cutting-plane model.

Step 3'b (Stopping Test and Loop) If $\left\langle f, x_c^{k+1} - x^{k+1}\right\rangle < \left\langle f, x^{k+1}\right\rangle \mathtt{TOL}$ then stop. Otherwise, set $k = k+1$ and loop to Step 1'.

For exact c-oracles, the method is a specialization of the cutting-plane algorithm in nonsmooth optimization, hence it converges. For inexact c-oracles, convergence results are not known.

6.2. Computational results. We now comment some of the numerical results (more details can be found in the Appendix). The reported CPU times are to be taken as a measure for comparing different algorithms, rather than as a measure of performance. Our implementation does not contain several improvements and tweaks that could optimize the code (oracle parallelization or multi-threading, for instance). Another important issue is that computing a Slater point is very consuming in terms of CPU time: this amounts to solving a problem as difficult at the original one (5.6) (i.e., solving (5.6) with objective function replaced by maximizing the probability level).

Results reported in Tables 6.1,6.5,6.2,6.3 were obtained on a HP xw6200 workstation with 8 Gb memory, whereas results of Table 6.4 were obtained on a HP z600 workstation.

6.3. Feasible Start using a Slater Point and Infeasible Start. In order to put all algorithms on equal foot, a first comparison supposes a Slater point is available to all of the methods in the benchmark. In this case, for a suitable convex combination of x^d and x^s, the starting point $\hat{x}^0$ is always feasible (Alg.[22] starts at x^s). Computing this convex multiplier requires executing step 3'a of Alg.[33]. This involves calling the c-oracle several times and might be costly. Table 6.5 in the Appendix contains the output of all the runs, with various parameter settings for each algorithm (to decide on the best choice of parameters for the benchmark). We report here a shorter Table 6.1, with some illustrative results.

The best objective function value in this test was -104162, found by Alg.2.1SEV for different settings. When $K = 1e^4$ Alg.2.1SEV found in 239 minutes a similar[†] solution than Alg.[33], which took is approximately the same CPU time. Taking $K = 5e^4$ the same point is reached in only 193 minutes and with $K = 1e^4$ that point is reached

[*] In this case one requires the c-oracle to be exactly evaluated at the Slater point

[†] i.e., feasible with same objective function value

TABLE 6.1
Comparison of Algorithms (nne^x stands for nn10^x), assuming a Slater Point available. Precision of Genz' code $\varepsilon^g = 1e^{-4}$.

method	Obj. Value	Nb. Iter.	CPU time (mins)	parameters
Alg.[33]	-103197	17	247.7	$tol = 1e^{-2}$
Alg.[33]	-104070	45	940.3	$tol = 1e^{-3}$
Alg.[33]	-104154	94	2079.17	$tol = 1e^{-4}$
Alg.2.1`SEV`	-104162	294	1028.43	$K = 1e^5, \mu_0 = 1e^{-6}, \mu_s = 2, \kappa = 0.7$
Alg.2.1`SEV`	-104160	215	723.55	$K = 5e^4, \mu_0 = 1e^{-6}, \mu_s = 2, \kappa = 0.25$
Alg.2.1`SEV`	-104162	239	827.26	$K = 5e^4, \mu_0 = 1e^{-6}, \mu_s = 4, \kappa = 0.5$
Alg.2.1`SEV`	-104162	265	932.19	$K = 1e^4, \mu_0 = 1e^{-6}, \mu_s = 2, \kappa = 0.7$
Alg.2.1`NUL`	-104089	277	1106.43	$K = 5e^4, \mu_0 = 1e^{-6}, \mu_s = 2.0, \kappa = 0.25$
Alg.2.1`NUL`	-104076	234	887.38	$K = 1e^4, \mu_0 = 1e^{-6}, \mu_s = 2.0, \kappa = 0.25$
Alg.[22]	-104026	273	1030.21	$K = 1, \mu_0 = 1e^{-9}, \mu_s = 1.05, \kappa = 0.05$
Alg.[22]	-104024	241	761.37	$K = 1, \mu_0 = 1e^{-9}, \mu_s = 1.001, \kappa = 0.01$

in 178 minutes. Table 6.1 shows the importance of the scaling parameter in the bundle method. It also shows that the supporting hyperplane method reaches good points early, but takes a very long time to converge. Basically each iteration takes approximately 15 to 20 minutes. Each bundle iteration takes less time (approximately 3 to 4 minutes per iteration). This can be explained by the fact that the supporting hyperplane method computes the exact interpolation. Now typically the cutting-plane method would yield a new iterate x^k far from the probability boundary $\mathbb{P}[a^r + A^r x \leq \eta \leq b^r + A^r x] = p$ and it would seem that Genz' code takes more time on such points than on strictly feasible points. It is also interesting to note that each stability center from the bundle methods is feasible. Thereby empirically providing support for the initial discussion in Sec. 5.6. Since in this setting Alg.[20] and Alg.2.1`NUL` would have similar results, we did not include them in this comparison.

The results in Table 6.2 were obtained with an infeasible starting point, $\hat{x}^0 = x^d$ from (6.1), using for each algorithm the best parameter settings in Table 6.5.

TABLE 6.2
Comparison of Algorithms (nne^x stands for nn10^x), infeasible starting point. Precision of Genz' code $\varepsilon^g = 1e^{-4}$.

method	Obj. Value	Nb. Iter.	CPU time (mins)	parameters
Alg.2.1`SEV`	-104162	133	379.45	$K = 5e^4, \mu_0 = 1e^{-6}, \mu_s = 2.0, \kappa = 0.25$
Alg.2.1`NUL`	-104154	73	167.25	$K = 1e^4, \mu_0 = 1e^{-5}, \mu_s = 2.0, \kappa = 0.1$
Alg.[22]	-104153	55	114.52	$K = 1e^4, \mu_0 = 1e^{-5}, \mu_s = 2.0, \kappa = 0.1$

For these runs, both Alg.2.1`NUL` and Alg.[22] modify the serious step test in Step 4, as follows. When the current stability center is infeasible, in addition to the usual test for acceptance, any improvement in feasibility by at least the c-oracle precision ("Genz precision" ε^g) declares the current iterate a new stability center. The modification was done to prevent the methods from stalling in the early stages of the algorithmic process and can only be active a finite number of iterations. Alg.2.1`SEV` does not stall and needs no modification. Once more, this variant is the best one in the benchmark, since it finds a feasible point with objective function value -104153 in only 56 iterations (comparable to the other two algorithms in the comparison). The best objective value of -104162 was obtained at the stake of many iterations. This table shows that important speed ups can be gained when starting with infeasible points and that. In particular, Alg.[33] requiring a Slater point and reaching a feasible solution with objective function value -104154 in 2079.17 minutes (see Table 6.1) is definitely out-performed by the Bundle method settings of Table 6.2.

The comparison above shows a strong dependence of the CPU time on the initial point, for all methods.

6.4. Different variants of Alg.2.1`SEV`. In order to determine the impact on the convergence, we varied the precision of Genz code. This test also allows us to check the potential of varying this precision along the iterations. We took the best method, Alg.2.1`SEV`, with the same parameter setting, for different oracle precision. Table 6.3 reports the corresponding results.

TABLE 6.3
*Effect of "Genz Precision" (nne^x stands for nn10^x), Alg.2.1*SEV *with* $K = 5e^4, \mu_0 = 1e^{-6}, \mu_s = 2.0\kappa = 0.5$.

Starting Point	Genz Precision	objective Value	Nb. Iterations	CPU time (mins)
Feasible	$1e^{-2}$	-104177	216	83.55
Feasible	$1e^{-3}$	-104163	216	87.13
Feasible	$1e^{-4}$	-104162	222	722.59
Feasible	$1e^{-5}$	-104161	261	54390.5
Infeasible	$1e^{-2}$	-104177	71	27.48
Infeasible	$1e^{-3}$	-104163	108	41.36
Infeasible	$1e^{-4}$	-104162	128	338.37

The solution obtained when using Genz' code with precision $1e^{-2}$ is slightly infeasible. This explains the "over"-optimal objective function value. From this table, we see that if the oracle accuracy is too high, CPU times can reach inacceptably large values.

Continuing with our analysis of the best variant, Alg.2.1SEV with fixed settings, in Table 6.4 we consider different values of β_k, ranging between the severe noise test (close to -1) to the permissive one (close to 1), and likewise for α_k, which varies from a severe serious step test (close to 0) to a permissive on (close to 2).

TABLE 6.4
*Effect of noise test (nne^x stands for for nn10^x), Alg.2.1*SEV *with* $K = 5e^4, \mu_0 = 1e^{-6}, \mu_s = 2.0\kappa = 0.5$, TOL $= 0.5$, *and Genz precision* $5e^{-4}$

Starting Point	α	β	objective Value	Nb. Iterations	CPU time (mins)
Infeasible	0	$-1+\varepsilon^m$	-104160	88	21.33
Infeasible	0	0	-104157	87	21.0
Infeasible	0	$1-\varepsilon^m$	-104159	111	30.30
Infeasible	1	$-1+\varepsilon^m$	-104158	60	12.8
Infeasible	1	$-\varepsilon^m$	-104158	70	15.57
Infeasible	$2-2\varepsilon^m$	$-1+\varepsilon^m$	-104077	24	5.35

Higher α values lead to more serious steps, until the extreme of declaring all iterates serious ($\alpha = 2 - 2\varepsilon^m$). In our tests, a severe noise test has resulted in a more stable management of the prox-parameter. Indeed, we have observed that if noise gets detected early on, the value of μ_k remains unchanged a significant number of iterations. By contrast, the permissive choice for β_k leads to some chaotic changes throughout the iterative process.

We also tested the two alternative stopping criteria and the alternative serious step condition and observed no significant differences.

Finally, we also examined the impact of penalty parameter ρ_k, comparing taking $\rho_k = 0$ (as in the previous runs in this subsection) with an update as for Alg.[20]. For the sake of brevity, we omit the table with results. Instead, we observe that with a nonnull ρ_k a feasible stability center is found in half the number of iterations. Unfortunately this quest for feasibility strongly deteriorates the objective function value, and eventually more iterations are required to converge to very similar solutions in the end.

We conclude that, at least for our runs, the best variant is Alg.2.1SEV with the settings in Table 6.4.

6.5. Solution Quality. To assess the obtained solutions, we simulate the reservoirs evolution and check if the stipulated probability level is satisfied numerically. Figure 6.1 shows the reservoir levels for 100 simulated inflow scenarios, using the expected-value strategy obtained from solving problem (6.1) and Alg.2.1SEV. The figure clearly shows the importance of integrating uncertainty in order to obtain robust turbining strategies.

Except for the deterministic solution, which violates constraints for almost all scenarios, the various methods in our benchmark provide in fact very similar solutions. When slight differences arise, they account for increased robustness (with respect to the deterministic solution) and for increased optimality (with respect to various solution methods).

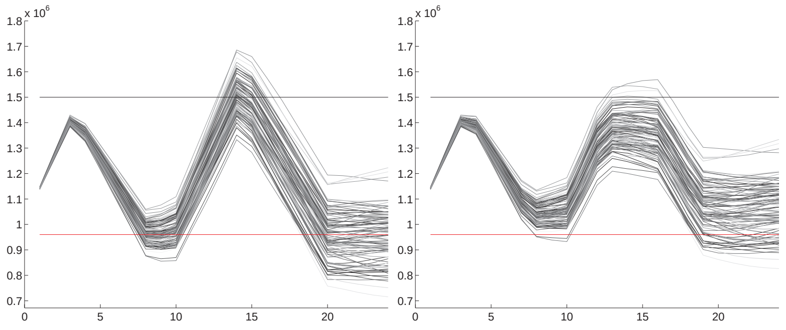


FIG. 6.1. *Evolution of reservoir "Saut Mortier", for the expected-value estrategy (left) and for Alg.2.1*`SEV` *strategy (right).*

Conclusions. For convex nonsmooth constrained problems we have presented a new bundle algorithm capable of dealing with inexact upper oracles. The latter oracles provide linearizations that may not lie below the true convex function. Such a setting naturally arises when dealing with certain classes of Joint Chance Constrained Programming. Indeed, in many cases of interest, an appropriate transformation of the Joint Chance Constraint is a convex function for which obtaining a subgradient or function value involves (quasi-)Monte Carlo sampling and multi-variate numerical integration. A similar situation arises in stochastic programs with second order stochastic dominance, Conditional Value-at-Risk, or robust constraints. As long as the resulting optimization problems remain convex with a finite number of constraints, the bundle method presented in this paper can be applied, taking advantage of its ability of handling inexact oracles to accelerate the calculations.

The good qualities of the proposed method was illustrated on a real-life numerical example coming from unit-commitment: the hydro-reservoir problem. This problem, when integrating uncertainty on inflows can be formulated as a Joint Chance Constrained Program. The bottleneck for an efficient numerical resolution lies in the cost of the oracle calculations for the Joint Chance Constraint. For inflows modelled as a causal time series with Gaussian innovations, and using an efficient code like Genz', the oracle evaluation typically takes up to 98% of the total CPU running time. The methods considered in this work do not need the (potentially time consuming) calculation of a Slater point. With respect to the well-known Supporting Hyperplane method, the bundle method presented in this paper was shown to accelerate the CPU time by a factor of 20 on the typical instance considered in this paper. This shows a great potential for these bundle methods. With a varying precision for the oracle ("asymptotically exact" variant, not tested in this work), an additional factor of 3 seems to be reachable.

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Appendix with full table.

Table 6.5
Comparison of Algorithms (nne^x stands for nn10^x), assuming a Slater Point available.

method	Obj. Value	Nb. Iter.	CPU time (mins)	parameters
Alg.[33]	-103197	17	247.7	$tol = 1e^{-2}$
Alg.[33]	-104070	45	940.3	$tol = 1e^{-3}$
Alg.[33]	-104154	94	2079.17	$tol = 1e^{-4}$
Alg.2.1SEV	-104162	294	1028.43	$K = 1e^5, \mu_0 = 1e^{-6}, \mu_s = 2, \kappa = 0.7$
Alg.2.1SEV	-104160	300	1060.35	$K = 2e^5, \mu_0 = 1e^{-6}, \mu_s = 2, \kappa = 0.7$
Alg.2.1SEV	-104162	286	991.33	$K = 5e^4, \mu_0 = 1e^{-6}, \mu_s = 1.05, \kappa = 0.95$
Alg.2.1SEV	-104162	243	846.20	$K = 5e^4, \mu_0 = 1e^{-6}, \mu_s = 2, \kappa = 0.7$
Alg.2.1SEV	-104160	215	723.55	$K = 5e^4, \mu_0 = 1e^{-6}, \mu_s = 2, \kappa = 0.25$
Alg.2.1SEV	-104162	255	887.07	$K = 5e^4, \mu_0 = 1e^{-5}, \mu_s = 2, \kappa = 0.1$
Alg.2.1SEV	-104162	257	907.20	$K = 5e^4, \mu_0 = 1e^{-6}, \mu_s = 2, \kappa = 0.1$
Alg.2.1SEV	-104162	239	827.26	$K = 5e^4, \mu_0 = 1e^{-6}, \mu_s = 4, \kappa = 0.5$
Alg.2.1SEV	-104162	265	932.19	$K = 1e^4, \mu_0 = 1e^{-6}, \mu_s = 2, \kappa = 0.7$
Alg.2.1NUL	-104109	417	1686.12	$K = 1e^5, \mu_0 = 1e^{-6}, \mu_s = 1.05, \kappa = 0.95$
Alg.2.1NUL	-104096	402	1539.16	$K = 5e^4, \mu_0 = 1e^{-6}, \mu_s = 1.05, \kappa = 0.95$
Alg.2.1NUL	-104102	395	1639.40	$K = 5e^4, \mu_0 = 1e^{-6}, \mu_s = 2, \kappa = 0.7$
Alg.2.1NUL	-104089	277	1106.43	$K = 5e^4, \mu_0 = 1e^{-6}, \mu_s = 2.0, \kappa = 0.25$
Alg.2.1NUL	-104091	323	1257.09	$K = 5e^4, \mu_0 = 1e^{-6}, \mu_s = 4.0, \kappa = 0.5$
Alg.2.1NUL	-104078	366	1480.21	$K = 1e^4, \mu_0 = 1e^{-6}, \mu_s = 1.05, \kappa = 0.95$
Alg.2.1NUL	-104077	395	1591.13	$K = 1e^4, \mu_0 = 1e^{-6}, \mu_s = 2.0, \kappa = 0.7$
Alg.2.1NUL	-104078	278	1063.49	$K = 1e^4, \mu_0 = 1e^{-6}, \mu_s = 2.0, \kappa = 0.5$
Alg.2.1NUL	-104076	234	887.38	$K = 1e^4, \mu_0 = 1e^{-6}, \mu_s = 2.0, \kappa = 0.25$
Alg.2.1NUL	-104078	253	981.13	$K = 1e^4, \mu_0 = 1e^{-6}, \mu_s = 2.0, \kappa = 0.35$
Alg.2.1NUL	-104079	144	549.19	$K = 1e^4, \mu_0 = 1e^{-5}, \mu_s = 2.0, \kappa = 0.1$
Alg.2.1NUL	-104072	208	792.17	$K = 1e^4, \mu_0 = 1e^{-6}, \mu_s = 2.0, \kappa = 0.1$
Alg.2.1NUL	-104078	218	828.58	$K = 1e^4, \mu_0 = 1e^{-7}, \mu_s = 2.0, \kappa = 0.1$
Alg.2.1NUL	-104078	278	1126.51	$K = 1e^4, \mu_0 = 1e^{-6}, \mu_s = 4.0, \kappa = 0.5$
Alg.[22]	-104026	525	1933.29	$K = 1, \mu_0 = 1e^{-5}, \mu_s = 1.05, \kappa = 0.7$
Alg.[22]	-104026	481	1691.07	$K = 1, \mu_0 = 1e^{-6}, \mu_s = 1.05, \kappa = 0.7$
Alg.[22]	-104027	447	1749.57	$K = 1, \mu_0 = 1e^{-7}, \mu_s = 1.05, \kappa = 0.7$
Alg.[22]	-104027	454	1852.24	$K = 1, \mu_0 = 1e^{-8}, \mu_s = 1.05, \kappa = 0.7$
Alg.[22]	-104025	339	1311.24	$K = 1, \mu_0 = 1e^{-8}, \mu_s = 1.05, \kappa = 0.1$
Alg.[22]	-104024	297	1170.13	$K = 1, \mu_0 = 5e^{-9}, \mu_s = 1.05, \kappa = 0.1$
Alg.[22]	-104027	418	1602.25	$K = 1, \mu_0 = 1e^{-9}, \mu_s = 1.05, \kappa = 0.7$
Alg.[22]	-104022	293	1089.25	$K = 1, \mu_0 = 1e^{-9}, \mu_s = 1.05, \kappa = 0.1$
Alg.[22]	-104026	273	1030.21	$K = 1, \mu_0 = 1e^{-9}, \mu_s = 1.05, \kappa = 0.05$
Alg.[22]	-104024	254	893.40	$K = 1, \mu_0 = 1e^{-9}, \mu_s = 1.01, \kappa = 0.05$
Alg.[22]	-104024	254	798.32	$K = 1, \mu_0 = 1e^{-9}, \mu_s = 1.01, \kappa = 0.01$
Alg.[22]	-104024	241	761.37	$K = 1, \mu_0 = 1e^{-9}, \mu_s = 1.001, \kappa = 0.01$
Alg.[22]	-104026	303	1150.23	$K = 1, \mu_0 = 1e^{-9}, \mu_s = 4.0, \kappa = 0.1$
Alg.[22]	-104025	391	1456.13	$K = 1, \mu_0 = 1e^{-10}, \mu_s = 1.05, \kappa = 0.7$
Alg.[22]	-104026	329	1193.51	$K = 1, \mu_0 = 1e^{-10}, \mu_s = 1.05, \kappa = 0.1$
Alg.[22]	-104028	595	2340.03	$K = 1, \mu_0 = 1e^{-10}, \mu_s = 2.0, \kappa = 0.7$
Alg.[22]	-104020	360	1348.14	$K = 1, \mu_0 = 1e^{-10}, \mu_s = 2.0, \kappa = 0.1$
Alg.[22]	-104026	325	1250.17	$K = 1, \mu_0 = 1e^{-10}, \mu_s = 4.0, \kappa = 0.1$