

On metric regularity for weakly almost piecewise smooth functions and some applications in nonlinear semidefinite programming

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Abstract

The one-to-one relation between the points fulfilling the KKT conditions of an optimization problem and the zeros of the corresponding Kojima function is well-known. In the present paper we study the interplay between metric regularity and strong regularity of this a priori nonsmooth function in the context of semidefinite programming.

Having in mind the topological structure of the positive semidefinite cone we identify a class of locally Lipschitz functions which turn out to have coherently oriented B-subdifferentials if metric regularity is assumed. This class is general enough to contain the Kojima function corresponding to the nonlinear semidefinite programming problem. Using a characterization of strong regularity for semismooth functions in terms of B-subdifferentials we arrive at an equivalence between metric regularity and strong regularity provided that an assumption involving the topological degree is fulfilled. Moreover, we shall show that metric regularity of the Kojima function implies constraint nondegeneracy.

Keywords: metric regularity, strong regularity, Kojima function, piecewise smooth function, coherent orientation, nonlinear semidefinite programming, constraint nondegeneracy.

AMS subject classification: 49J52, 49J53, 90C22, 90C31.

1 Introduction

The Kojima function (see [21]) plays an important role in the theory of optimization. The main reason is the one-to-one correspondence between its zeros and the points fulfilling the Karush-

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Kuhn-Tucker (KKT) conditions of a nonlinear program. In this way one is able to replace the KKT conditions by a system of a priori nonsmooth equations. The nonsmoothness of the Kojima function results from the fact that its definition includes the projection onto the cone describing the constraints, i.e. the positive orthant.

Hence, the local behavior of the Kojima function around a fixed zero is of great importance. The best possible case we can wish for is when the inverse of the Kojima function is locally single-valued and Lipschitz continuous. Robinson (cf. [34]) introduced this property in the context of variational inequalities and he called it strong regularity. Another nice situation occurs when the inverse of the Kojima function is a multifunction fulfilling a certain generalization of Lipschitz continuity (cf. [1]). This property is also known as pseudo-regularity or metric regularity.

In [10] Dontchev and Rockafellar studied the variational inequality describing the first-order optimality conditions for nonlinear programs provided that the objective function and the functions defining the restrictions are twice continuously differentiable. They showed that the polyhedrality of the cone describing the constraints is the main reason for the equivalence between metric regularity and strong regularity of the variational equation. In [22] (see also [19, Chapter 7]) Kummer showed the same equivalence for so-called generalized Kojima functions.

In nonlinear semidefinite programming the situation is different: the cone describing the constraints (the cone of positive semidefinite matrices) is not polyhedral. Hence the foregoing argumentation does not apply and the relation between the two mentioned regularity concepts is unclear.

Our main tool for better understanding of this relation is a well-known equivalent characterization of strong regularity valid for Lipschitz semismooth functions. This characterization consists of two conditions: coherent orientation of the Bouligand subdifferential and a condition involving the topological degree of the function in question (see [16, Corollary 4], cf. [31, Theorem 6]). Therefore we focus our investigation on conditions ensuring that a locally Lipschitz metrically regular function has coherently oriented B-subdifferentials.

In [33] Qi and Tseng introduced a class of locally Lipschitz functions with the additional property that their smooth point sets are locally connected around all nonsmooth points. It turns out that these functions are not piecewise smooth. The authors call them weakly almost smooth. Inspired by this concept we introduce a more general class of locally Lipschitz functions which includes both piecewise smooth and weakly almost smooth functions. More precisely, we say that a locally Lipschitz function is weakly almost piecewise smooth if the sets of points where the function in question is piecewise smooth are locally connected. We will show that this class of functions is small enough to ensure that a locally Lipschitz metrically regular function has coherently oriented B-subdifferentials and at the same time it is general enough to include Kojima functions corresponding to nonlinear semidefinite problems.

In Section 2 we recall the definitions of metric regularity and strong regularity. Some useful properties of metrically regular functions are mentioned. A characterization of metric regularity for piecewise smooth functions turns out to be of great importance. In Section 3 we will introduce the weakly almost piecewise smoothness property ensuring that the principal part of the B-subdifferential of a locally Lipschitz metrically regular function is coherently oriented. Section 4 is devoted to the application of these results to the Kojima function of a nonlinear semidefi-

nite programming problem. At first, we will show that metric regularity of the Kojima function implies constraint nondegeneracy. This is the semidefinite counterpart to the fact that metric regularity of the Kojima function implies LICQ in classical nonlinear programming (see [19, Lemma 7.1]). We will later see that the Kojima function corresponding to a nonlinear semidefinite programming problem is weakly almost piecewise smooth everywhere. This is not surprising because this property was constructed while having in mind the structure of the semidefinite cone. Finally, we will obtain the equivalence between strong regularity and metric regularity of the Kojima function provided a condition including the topological degree is satisfied.

Notation: Let \mathbb{S}^p denote the space of symmetric $p \times p$ matrices with real entries equipped with the inner product $\langle A, B \rangle := \text{tr}(AB)$ and with its induced norm, the Frobenius norm $\|A\| := \langle A, A \rangle^{1/2}$. We shall use lower case letters for vectors and capitals for matrices when possible. For $(x, A) \in \mathbb{R}^n \times \mathbb{S}^p$ we define $\|(x, A)\| := \|x\| + \|A\|$. Let \mathbb{S}_+^p and \mathbb{S}_-^p denote the subsets of \mathbb{S}^p consisting of positive semidefinite and negative semidefinite matrices, respectively. The orthogonal projection of a matrix $A \in \mathbb{S}^p$ onto \mathbb{S}_+^p is denoted by A_+ or $\Pi_+(A)$, similarly the orthogonal projection of A onto \mathbb{S}_-^p is denoted by A_- or $\Pi_-(A)$. By $\lambda(A)$ we denote the vector consisting of eigenvalues of a matrix $A \in \mathbb{S}^p$ sorted in the nonincreasing order, i.e. $\lambda_1(A) \geq \dots \geq \lambda_p(A)$.

The symbols \mathbb{B}_X and S_X denote the closed unit ball and the closed unit sphere in the space X , respectively. We will omit the index X if it is clear from the context which space is meant. For any sets $C, D \subset \mathbb{R}^n$ and $r \in \mathbb{R}$ the notation $C + rD$ is used to describe the set $\{c + rd \mid c \in C, d \in D\}$, the Minkowski sum. For $a \in \mathbb{R}^n, r \in \mathbb{R}$ and $D \subset \mathbb{R}^n$ we write $a + rD$ instead of $\{a\} + rD$. We use the notation $x = o(\delta)$ to describe the fact that $\|x\|/\delta \rightarrow 0$ as $\delta \rightarrow 0$. By $\text{bd } C$, $\text{cl } C$, $\text{int } C$ and $\text{conv } C$ we denote the boundary, closure, interior and convex hull of a given set $C \subset \mathbb{R}^n$, respectively. The graph of a set-valued mapping $f : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is denoted by $\text{gph } f$.

2 Metrically regular functions

2.1 Metric regularity and contingent derivatives

At first we recall some definitions of important notions used later. Let $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ be a set-valued map and let $x^0 \in F(y^0)$. The mapping F is said to have the **Aubin property** at (y^0, x^0) if there exist neighborhoods V of y^0 , U of x^0 and a constant $L > 0$ such that

$$F(y_1) \cap U \subset F(y_2) + L\|y_1 - y_2\|\mathbb{B} \quad \forall y_1, y_2 \in V. \quad (2.1)$$

This notion was introduced in [1] by Aubin under the name **pseudo-Lipschitz property**. The name Aubin property was suggested by Dontchev and Rockafellar in [10]. It is a simple consequence of (2.1) that the Aubin property of F at (y^0, x^0) implies the Aubin property of F at every point $(y, x) \in \text{gph } F$ in a suitable neighborhood of (y^0, x^0) .

Further, consider a multifunction $f : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$. We say that f is **pseudo-regular** at $(x^0, y^0) \in \text{gph } f$ if the set-valued mapping $F := f^{-1}$ has the Aubin property at (y^0, x^0) , cf. [19], [23]. Additionally, let the sets $f^{-1}(y) \cap U$ be singletons for all $y \in V$, where U and V are the

corresponding neighborhoods from (2.1). Then f is said to be **strongly regular** at (x^0, y^0) , cf. [34].

Another notion very closely linked to pseudo-regularity is metric regularity. We call a multi-function $f : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ **metrically regular** at $(x^0, y^0) \in \text{gph} f$ if there are neighborhoods U of x^0 and V of y^0 and a constant $L > 0$ such that

$$d(x, f^{-1}(y)) \leq Ld(y, f(x)) \quad \forall x \in U \quad \forall y \in V.$$

The equivalence between metric regularity of f and pseudo-regularity of f under weak assumptions (closedness of $\text{gph} f$) was shown rather late, perhaps first in [17], [5] and [32]. For further details see [19], [23], [28] or [37]. We prefer to use the name metric regularity instead of pseudo-regularity in the rest of the paper.

An important and useful characterization of metric regularity can be formulated in terms of contingent derivatives. The **contingent derivative** $Cf(x, y) : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ of the set-valued mapping $f : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ at the point $(x, y) \in \text{gph} f$ is given by

$$v \in Cf(x, y)(u) \iff \exists \{t_k\}, t_k \searrow 0, \exists (u^k, v^k) \rightarrow (u, v) : y + t_k v^k \in f(x + t_k u^k).$$

In other words,

$$\text{gph} Cf(x, y) = T_{\text{gph} f}(x, y),$$

where $T_C(x)$ denotes the tangent cone to the set C at a point $x \in C$. Therefore, the following property is straightforward:

$$(Cf(x, y))^{-1} = Cf^{-1}(y, x). \quad (2.2)$$

If the mapping f is a directionally differentiable locally Lipschitz function then the contingent derivative $Cf(x, f(x))$ of f coincides with its directional derivative $f'(x; \cdot)$.

The next theorem recalls an equivalent characterization of metric regularity by a surjectivity condition for contingent derivatives.

Theorem 2.1. ([2], [22], [23]) *Let $f : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a set-valued mapping with a closed graph, and let $y^0 \in f(x^0)$. Then the following statements are equivalent:*

- (i) *the mapping f is metrically regular at (x^0, y^0) ,*
- (ii) *there exist a neighborhood W of (x^0, y^0) and a constant $\alpha > 0$ such that*

$$\mathbb{B} \subset \bigcup_{\|u\| \leq \alpha} Cf(x, y)(u) \quad \forall (x, y) \in W \cap \text{gph} f.$$

For locally Lipschitz directionally differentiable functions $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, metric regularity implies the injectivity of the directional derivative:

Theorem 2.2. ([14, Theorem 3.3]) *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a locally Lipschitz function which is metrically regular at x^0 . Further, let f be directionally differentiable. Then there exists a neighborhood U of x^0 such that*

$$f'(x; u) \neq 0 \quad \forall x \in U \quad \forall u \in \mathbb{R}^n \setminus \{0\}.$$

Consequently, the sets $f^{-1}(y) \cap U$ are finite for all $y \in \mathbb{R}^n$.

Thus, preimage sets of the function f consist (locally around x^0) of isolated points only. Moreover, the minimal norm of directional derivatives in directions lying on the unit sphere is uniformly bounded by a positive constant (see [14, Theorem 3.4]). Combining Theorem 2.1 with Theorem 2.2 we conclude that—roughly speaking—under metric regularity, the surjectivity of directional derivatives implies their injectivity (similarly to the case when f is a C^1 function).

By applying Theorem 2.1 to a continuous function $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ one obtains that metric regularity of f at x^0 ensures the regularity of the derivative $Df(x)$ at every point x which is sufficiently close to x^0 and where the derivative $Df(x)$ exists. The following corollary gives a uniform bound for determinants of such matrices.

Corollary 2.3. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a continuous function, metrically regular at x^0 . Then there exist a neighborhood U of x^0 and a constant $c > 0$ such that*

$$|\det Df(x)| \geq c \quad \forall x \in U \cap \mathcal{D}_f,$$

where \mathcal{D}_f denotes the set of all points where the derivative $Df(x)$ exists.

Proof. Theorem 2.1 implies the existence of a constant $\alpha > 0$ such that for all points $x \in \mathcal{D}_f$ near x^0 we obtain that $Df(x)$ is regular and the spectral norm of the matrix $A := (Df(x))^{-1}$ is at most α . Equivalently, the largest eigenvalue of the positive definite matrix $A^T A$ is at most α^2 . This gives

$$(\det A)^2 = \det A^T \cdot \det A = \det (A^T A) \leq \alpha^{2n},$$

hence

$$|\det Df(x)| = |\det(A^{-1})| = \frac{1}{|\det A|} \geq \alpha^{-n}.$$

□

2.2 Semismooth functions and B-subdifferentials

To simplify the notation let us define the following sets corresponding to a locally Lipschitz function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$:

$$\mathcal{S}_f := \{x \in \mathbb{R}^n \mid f \text{ is continuously differentiable on a suitable neighborhood of } x\}, \quad (2.3)$$

$$\mathcal{D}_f := \{x \in \mathbb{R}^n \mid f \text{ is differentiable at } x\}, \quad (2.4)$$

$$\mathcal{N}_f := \{x \in \mathbb{R}^n \mid f \text{ is not differentiable at } x\}. \quad (2.5)$$

Rademacher's theorem (see [37, Section 9.J]) states that a locally Lipschitz function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is differentiable almost everywhere. Hence, the **B-subdifferential** $\partial_B f(x)$ of f at x defined by

$$\partial_B f(x) := \{V \mid \exists \{x^k\} \subset \mathcal{D}_f, x^k \rightarrow x : V = \lim_{k \rightarrow \infty} Df(x^k)\} \quad (2.6)$$

is nonempty for all $x \in \mathbb{R}^n$. Further, consider the **Clarke generalized Jacobian** $\partial f(x)$ of f at x defined by (cf. [8])

$$\partial f(x) := \text{conv } \partial_B f(x).$$

We call a locally Lipschitz function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ **semismooth** at x (cf. [27]) if for any $x^k \rightarrow x$ and $V_k \in \partial f(x^k)$ it holds

$$f(x^k) - f(x) - V_k(x^k - x) = o(\|x^k - x\|).$$

It will turn out that for our purposes another notion is of great importance. For any locally Lipschitz function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and any point $x \in \text{cl } \mathcal{S}_f$ define the **principal part of the B-subdifferential** of f at x by

$$\partial_P f(x) := \{V \mid \exists \{x^k\} \subset \mathcal{S}_f, x^k \rightarrow x : V = \lim_{k \rightarrow \infty} Df(x^k)\}. \quad (2.7)$$

The assumption $\text{cl } \mathcal{S}_f = \mathbb{R}^n$ ensures that $\partial_P f(x)$ is nonempty for all $x \in \mathbb{R}^n$, and $\partial_P f(x) \subset \partial_B f(x)$. This construction has been used by Klatte and Kummer in [19, equation (6.30)] and by Qi and Tseng in [33]. They showed the equality $\partial_P f(x) = \partial_B f(x)$ for real-valued functions which are smooth everywhere except for a set consisting of isolated points only (cf. [33, Theorem 4]).

It is well-known that the mapping $\partial_B f(\cdot)$ of a locally Lipschitz function f has nonempty, compact-valued images and is upper semicontinuous, see e.g. [12, Lemma 7.4.11]. With the same arguments it can be shown that the mapping $\partial_P f(\cdot)$ has the same properties provided \mathcal{S}_f is dense in \mathbb{R}^n .

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be locally Lipschitz and directionally differentiable. The function f is said to be **strongly B-differentiable** at x^0 if for every $\varepsilon > 0$ there exists a neighborhood U of the origin such that the function r_{x^0} given by

$$r_{x^0}(u) := f(x^0 + u) - f(x^0) - f'(x^0; u) \quad (2.8)$$

is locally Lipschitz on U with modulus ε .

2.3 Metric regularity and piecewise smooth functions

A function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be **piecewise smooth** at x^0 if there exist an open set U with $x^0 \in U \subset \mathbb{R}^n$ and a finite collection $\{f_i\}_{i \in I}$ of C^1 functions $f_i : U \rightarrow \mathbb{R}^m$ such that f is continuous on U and

$$f(x) \in \{f_i(x) \mid i \in I\} \quad \forall x \in U.$$

Such a collection of functions is said to form a **local representation** for f at x^0 , and this collection is called **minimal** if no subcollection forms a local representation for f at x^0 . In the case when $U = \mathbb{R}^n$ and the functions f_i , $i \in I$ are linear (i.e. $f_i(x) = A_i x$ for suitable matrices A_i , $i \in I$) we call f **piecewise linear**. It is well-known that for a piecewise linear function, the minimal local representation at origin is unique (cf. [40, Proposition 2.1]). This allows one to call a piecewise linear function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ **coherently oriented** if all the matrices A_i corresponding to the minimal local representation for f at origin have the same nonvanishing determinant sign.

Analogously to (2.3)–(2.5) we introduce for a locally Lipschitz function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ the notation

$$\mathcal{P}_f := \{x \in \mathbb{R}^n \mid f \text{ is piecewise smooth at } x\}. \quad (2.9)$$

Piecewise smooth functions have nice properties: they are locally Lipschitz (cf. [39], [41]) and semismooth (see [12]). Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be piecewise smooth at x and let $\{f_i\}_{i \in I}$ be any minimal local representation for f at x . It is well-known (cf. [30, Lemma 2], [33], [39, Proposition A.4.1], [41]) that

$$\partial_B f(x) = \partial_P f(x) = \{Df_i(x) \mid i \in I\}. \quad (2.10)$$

Moreover, f is directionally differentiable at x and it holds for all $u \in \mathbb{R}^n$

$$f'(x; u) \in \{Df_i(x)u \mid i \in I\}. \quad (2.11)$$

Consequently, the directional derivative $f'(x; \cdot)$ is piecewise linear (cf. [24], [39], [41]) and

$$\partial_B f'(x^0; \cdot)(0) \subset \partial_B f(x^0). \quad (2.12)$$

Theorem 2.4. (cf. [13, Theorem 3.18]) *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be piecewise smooth at x^0 , and let $\{f_i\}_{i \in I(x^0)}$ denote any minimal local representation for f at x^0 .*

- (i) *If f is metrically regular at x^0 then the directional derivative $f'(x^0; \cdot)$ is coherently oriented.*
- (ii) *If all determinants of the matrices $Df_i(x^0)$, $i \in I(x^0)$ have the same nonvanishing sign then f is metrically regular at x^0 .*

Proof. (i) Metric regularity of f at x^0 is equivalent to the Aubin property of f^{-1} at $(f(x^0), x^0)$. Consequently, the contingent derivative $Cf^{-1}(f(x^0), x^0)$ has the Aubin property everywhere (see e.g. [37, Exercise 9.49]). Because of (2.2) this equivalently means that $Cf(x^0, f(x^0))$ is metrically regular everywhere. From (2.11) we know that the directional derivative $f'(x^0; \cdot)$ exists, is piecewise linear and metrically regular everywhere because $f'(x^0; \cdot) = Cf(x^0, f(x^0))$. Furthermore, the well-known equivalence between metric regularity and linear openness (cf. [37, Theorem 9.43]) implies that $f'(x^0; \cdot)$ is open, too. Using Propositions 2.3.7 and 2.3.5 in [39] (cf. [41]) we obtain that $f'(x^0; \cdot)$ is coherently oriented (see also [40, Theorem 2.1]).

(ii) Because of (2.11) the directional derivative $f'(x^0; \cdot)$ is coherently oriented. Moreover, for continuity reasons there exists a suitable neighborhood U of x^0 such that the determinants of the derivatives $Df_i(x)$ have the same nonvanishing sign for all $i \in I(x^0)$ and all $x \in U$. Further, for every $x \in U$ there exists a minimal local representation $\{f_i\}_{i \in I(x)}$ for f at x such that $I(x) \subset I(x^0)$. Consequently, all directional derivatives $f'(x; \cdot)$ with $x \in U$ are coherently oriented, and hence, surjective (cf. [39, Propositions 2.3.5 and 2.3.6], [41] or [40, Theorem 2.1]).

Make the neighborhood U smaller if necessary in order to get the existence of $Df_i(x)$ for all $x \in \text{cl } U$ and $i \in I(x^0)$. Then, for continuity reasons, the minimum

$$c := \min_{i \in I(x^0)} \min_{(x,u) \in \text{cl } U \times S} \|Df_i(x)u\|$$

exists and is positive. Set $\alpha := c^{-1}$, and take any $v \in \mathbb{B}$. Because of the above shown surjectivity, for every $x \in U$ there exists a direction u such that $v = f'(x; u)$ and

$$1 = \|v\| = \|f'(x; u)\| = \|u\| \left\| f' \left(x; \frac{u}{\|u\|} \right) \right\| \geq \|u\|c = \frac{\|u\|}{\alpha}.$$

In other words, for every $x \in U$ we have $\mathbb{B} \subset \bigcup_{\|u\| \leq \alpha} f'(x; u)$. Finally, Theorem 2.1 gives the assertion. \square

In [15, Theorem 3.5] Gowda and Sznajder show for a piecewise smooth function f the equivalence between metric regularity of f and coherent orientation of its directional derivative provided f is strongly B-differentiable. This statement is a consequence of the fact that strong B-differentiability of f ensures locally the same behavior of f and its directional derivative with respect to metric regularity (cf. [9], [22]).

Using Theorem 2.4 we are able to show the same equivalence as in [15], Theorem 3.5 while assuming

$$\partial_B f(x^0) = \partial_B f'(x^0; \cdot)(0) \quad (2.13)$$

instead of the strong B-differentiability of f at x^0 . It will turn out that condition (2.13) is weaker than the strong B-differentiability of f (see Lemma 2.6 and Example 2.7).

Corollary 2.5. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be piecewise smooth at x^0 . Further assume that (2.13) is fulfilled. Then the following assertions are equivalent:*

- (i) *the function f is metrically regular at x^0 ,*
- (ii) *the directional derivative $f'(x^0; \cdot)$ is coherently oriented,*
- (iii) *$\partial_B f(x^0)$ consists of regular matrices with the same determinant sign.*

Proof. The implications (iii) \Rightarrow (i) \Rightarrow (ii) follow from Theorem 2.4 and (2.10). The implication (ii) \Rightarrow (iii) is a consequence of (2.10) and (2.13). \square

Lemma 2.6. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be piecewise smooth at x^0 and strongly B-differentiable at x^0 . Then (2.13) is fulfilled.*

Proof. Because of (2.12) it is enough to show $\partial_B f(x^0) \subset \partial_B f'(x^0; \cdot)(0)$. Consider any matrix $V \in \partial_B f(x^0)$. Because of (2.6) there exists a sequence $\{x^k\} \subset \mathcal{D}_f$ with $x^k \rightarrow x^0$ and $Df(x^k) \rightarrow V$. Let $\{f_i\}_{i \in I}$ be any minimal local representation for f at x^0 . Clearly f is piecewise smooth at x^k for k large enough with a minimal local representation $\{f_i\}_{i \in I(x^k)}$ such that $I(x^k) \subset I$. Then $Df(x^k) = Df_{i(k)}(x^k)$ for a suitable index $i(k) \in I(x^k)$ (see [38, Lemma 2]). Without loss of generality we may assume that $i(k) \equiv i^*$ for all k as $I(x^k) \subset I$. Lemma 1 in [38] implies that for any k there exists an open set O^k with $x^k \in \text{cl } O^k$ and $f \equiv f_{i^*}$ on O^k .

Let us use the notation $g := f'(x^0; \cdot)$. As g is Lipschitz we know that Rademacher's Theorem ensures that g is differentiable almost everywhere. Now find points $y^k \in O^k$ with $\|x^k - y^k\| \leq 1/k$

such that g is differentiable at $v^k := y^k - x^0$ (it is enough to set $y^k := x^k$ if $x^k - x^0 \in \mathcal{D}_g$). The existence of such points y^k is ensured by the openness of the sets O^k and by the Rademacher Theorem. Thus, r_{x^0} given by (2.8) is differentiable at v^k and we obtain

$$Dr_{x^0}(v^k) = Df(y^k) - Dg(v^k)$$

for k large enough.

The strong B-differentiability of f at x^0 implies that r_{x^0} is strictly differentiable at origin with $Dr_{x^0}(0) = 0$ (cf. [37]). Consequently, the function Dr_{x^0} is continuous at origin relative to the set of points where Dr_{x^0} exists. Hence, as $v^k \rightarrow 0$, we get

$$\|Df(y^k) - Dg(v^k)\| \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Finally, as $y^k \in O^k$ and Df_{i^*} is uniformly continuous on a suitable neighborhood of x^0 , we get

$$\|Df(x^k) - Df(y^k)\| = \|Df_{i^*}(x^k) - Df_{i^*}(y^k)\| \rightarrow 0.$$

Hence $Dg(v^k) \rightarrow V$ for $v^k \rightarrow 0$ and consequently $V \in \partial_B g(0)$. \square

The next example proves the existence of a piecewise smooth function $f : U \rightarrow \mathbb{R}^2$ (where $U \subset \mathbb{R}^2$ is a suitable small neighborhood of the origin) which is not strongly B-differentiable at origin but fulfills the condition $\partial_B f(0) = \partial_B f'(0; \cdot)(0)$.

Example 2.7. Let the function $f : U \rightarrow \mathbb{R}^2$ be given via its components $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ as follows:

$$f_1(x, y) = x,$$

$$f_2(x, y) = \begin{cases} y & \text{if } x \geq 0 \text{ and } y \leq x^4, \\ 2y - x^4 & \text{if } (x \geq 0 \text{ and } x^4 \leq y \leq x^2) \text{ or } (x \leq 0 \text{ and } x \leq y \leq -x), \\ x^2 + y - y^2 & \text{if } x \geq 0 \text{ and } y \geq x^2, \\ y - y^2 + x(y^3 - y - 1) & \text{if } x \leq 0 \text{ and } y \geq -x, \\ y + x(1 - y^3) & \text{if } x \leq 0 \text{ and } y \leq x. \end{cases}$$

Let us simplify the notation by using $g(u, v) := f'((0, 0); (u, v))$. A straightforward calculation shows that $g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$ is given by $g_1(u, v) = u$ and

$$g_2(u, v) = \begin{cases} v & \text{if } u \geq 0, \\ -u + v & \text{if } u \leq 0 \text{ and } v \geq -u, \\ u + v & \text{if } u \leq 0 \text{ and } v \leq u, \\ 2v & \text{if } u \leq 0 \text{ and } u \leq v \leq -u. \end{cases}$$

Hence

$$\partial_B f(0, 0) = \partial_B g(0, 0) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}.$$

The function f is metrically regular at origin as all corresponding determinants are positive. On the other hand, f is not strongly B-differentiable at origin. In order to see this take sequences $\{u^k\}, \{v^k\}, \{w^k\}$, with $u^k > 0, u^k \rightarrow 0$ and $v^k, w^k \in [(u^k)^4, (u^k)^2]$ for all $k \in \mathbb{N}$. This choice ensures that (cf. (2.8))

$$r_{(0,0)}(u^k, v^k) = f(u^k, v^k) - g(u^k, v^k) = \begin{pmatrix} u^k \\ 2v^k - (u^k)^4 \end{pmatrix} - \begin{pmatrix} u^k \\ v^k \end{pmatrix} = \begin{pmatrix} 0 \\ v^k - (u^k)^4 \end{pmatrix},$$

and analogously

$$r_{(0,0)}(u^k, w^k) = \begin{pmatrix} 0 \\ w^k - (u^k)^4 \end{pmatrix}.$$

The function $r_{(0,0)}$ has a Lipschitz modulus of at least 1 on any small neighborhood of the origin because of

$$\frac{\|r_{(0,0)}(u^k, v^k) - r_{(0,0)}(u^k, w^k)\|}{|v^k - w^k|} = 1.$$

Hence, f is not strongly B-differentiable at origin, cf. (2.8). □

2.4 Some properties of the topological degree for metrically regular functions

Let $f : \text{cl}\Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a continuous function where $\Omega \subset \mathbb{R}^n$ is an open, bounded set and let $y \in \mathbb{R}^n \setminus f(\text{bd}\Omega)$. By $\text{deg}(f, \Omega, y)$ we denote the topological degree of f on Ω at y . For the definition and properties we refer to [25] and [29]. Let us mention the following properties of the degree assuming that $y \notin f(\text{bd}\Omega)$ (see [16, Section 2.5]):

- (i) If $\text{deg}(f, \Omega, y) \neq 0$, then there exists a solution of the equation $f(x) = y$ in Ω .
- (ii) Nearness property: If $\|z - y\| < d(y, f(\text{bd}\Omega))$, then $\text{deg}(f, \Omega, z)$ is defined and $\text{deg}(f, \Omega, z) = \text{deg}(f, \Omega, y)$.
- (iii) Let x^0 be an isolated solution of the equation $f(x) = y$ in Ω , and let U be any open subset of Ω containing x^0 such that x^0 is the only solution of $f(x) = y$ in $\text{cl}U$. Then $\text{deg}(f, U, y)$ is independent of U . We call this common degree the **index** of f at x^0 and denote it by $\text{ind}(f, x^0)$.
- (iv) If f is differentiable at x and $Df(x)$ is nonsingular, then

$$\text{ind}(f, x) = \text{sgn} \det Df(x). \tag{2.14}$$

- (v) If the equation $f(x) = y$ has finitely many solutions x^1, \dots, x^k in Ω , then

$$\text{deg}(f, \Omega, y) = \sum_{i=1}^k \text{ind}(f, x^i). \tag{2.15}$$

Recall that Theorem 2.2 ensures that preimage sets of a locally Lipschitz and metrically regular function are locally finite provided the function in question is directionally differentiable. Combining this fact with the above mentioned properties gives the following assertion:

Lemma 2.8. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be locally Lipschitz, metrically regular at x^0 . Further, let f be directionally differentiable. Then there exist a neighborhood U of x^0 and a neighborhood V of $f(x^0)$ such that $\text{ind}(f, x)$ exists for all $x \in U$ and*

$$\text{ind}(f, x^0) = \sum_{x \in f^{-1}(y) \cap U} \text{ind}(f, x) \quad \forall y \in V. \quad (2.16)$$

Proof. Let U be the neighborhood of x^0 from Theorem 2.2. Make this neighborhood U smaller if necessary to ensure that x^0 is the only solution of the equation $f(x) = f(x^0)$ for $x \in \text{cl } U$ and set $\Omega := U$. While Ω remains fixed decrease U further if necessary to ensure that $f(x) \notin f(\text{bd } \Omega)$ for all $x \in U$. Then Property (iii) implies the existence of $\text{ind}(f, x)$ for all $x \in U$.

Further we have $\text{ind}(f, x^0) = \text{deg}(f, U, f(x^0))$. Define a small neighborhood V of $f(x^0)$ such that for $y \in V$ we obtain $\|y - f(x^0)\| < d(f(x^0), f(\text{bd } U))$ (which is possible as $f(x^0) \notin f(\text{bd } U)$). Then, the Nearness property (ii) together with Property (v) give for $y \in V$

$$\text{ind}(f, x^0) = \text{deg}(f, U, f(x^0)) = \text{deg}(f, U, y) = \sum_{x \in f^{-1}(y) \cap U} \text{ind}(f, x). \quad (2.17)$$

The last equality is a consequence of Theorem 2.2. □

3 Metric regularity and coherent orientation of the Bouligand subdifferential

Our first goal is the generalization of Corollary 2.5 for a more general class of locally Lipschitz functions. Specifically, this class should be broad enough to include the projection on the semidefinite cone.

It is well-known that the semidefinite projection Π_+ is not piecewise smooth at matrices $Y \in \mathbb{S}^p$ with $\text{rk } Y \leq p - 2$ as the corresponding B-subdifferential is infinite (cf. [26]). However, in Section 4 we will show that this mapping is piecewise smooth at any $Y \in \mathbb{S}^p$ with $\text{rk } Y \geq p - 1$ (cf. Corollary 4.5). Additionally, it possesses certain topological properties which can help to identify the class of functions we are looking for.

Definition 3.1. *A function $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is said to be **weakly almost piecewise smooth** at x^0 if there exists $\varepsilon^* > 0$ such that*

$$\text{the set } (x^0 + \varepsilon \mathbb{B}) \cap \mathcal{P}_f \text{ is nonempty and connected for all } \varepsilon \in (0, \varepsilon^*). \quad (3.1)$$

The name of the above property is inspired by [33] where Qi and Tseng introduced the class of weakly almost smooth functions in this manner. Definition 3.1 allows us to formulate the following result:

Theorem 3.2. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be locally Lipschitz, metrically regular at x^0 , and let $x^0 \in \text{cl } \mathcal{S}_f$. Assume further that f is weakly almost piecewise smooth at x^0 . Additionally, let the inclusion*

$$\partial_B f(x) \subset \partial_B f'(x; \cdot)(0) \quad (3.2)$$

be fulfilled for all $x \in \mathcal{P}_f$. Then, all matrices $V \in \partial_P f(x^0)$ are regular and have the same determinant sign.

Proof. Notice that $\partial_P f(x^0)$ is nonempty because of $x^0 \in \text{cl } \mathcal{S}_f$. At first, we will show that all matrices in $\partial_P f(x^0)$ are regular. For this purpose take any matrix $V \in \partial_P f(x^0)$. Due to (2.7) there exists a sequence $\{x^k\} \subset \mathcal{S}_f$ such that $x^k \rightarrow x^0$ and $Df(x^k) \rightarrow V$. Because of Corollary 2.3 all matrices $Df(x^k)$ are regular and fulfill $|\det Df(x^k)| \geq c$ for k sufficiently large and a suitable positive constant c . This also means that $|\det V| \geq c$.

Further, assume the contrary, i.e. suppose that there exist two matrices $V_1, V_2 \in \partial_P f(x^0)$ such that $\det V_1 \cdot \det V_2 < 0$. Definition (2.7) ensures the existence of two sequences $\{x^k\}, \{y^k\} \subset \mathcal{S}_f$ such that $x^k, y^k \rightarrow x^0$ and $Df(x^k) \rightarrow V_1, Df(y^k) \rightarrow V_2$. As f is metrically regular at x^0 it is possible to choose $\varepsilon \in (0, \varepsilon^*)$ (with ε^* from the condition (3.1)) small enough such that f remains metrically regular at every $x \in x^0 + \varepsilon\mathbb{B}$. Consequently, it is possible to find two points $\tilde{x}, \tilde{y} \in (x^0 + \varepsilon\mathbb{B}) \cap \mathcal{S}_f$ such that

$$\det Df(\tilde{x}) \cdot \det Df(\tilde{y}) < 0 \quad (3.3)$$

(take $\tilde{x} := x^k$ and $\tilde{y} := y^k$ for k large enough). Define the multifunction $F : (x^0 + \varepsilon\mathbb{B}) \cap \mathcal{P}_f \rightrightarrows \{-1, 1\}$ by

$$F(x) := \{\text{sgn } \det V \mid V \in \partial_B f(x)\}.$$

Consider any minimal local representation $\{f_i\}_{i \in I}$ for f at $x \in (x^0 + \varepsilon\mathbb{B}) \cap \mathcal{P}_f$. Because of (2.10), (2.12) and (3.2) we have

$$\partial_B f(x) = \partial_B f'(x; \cdot)(0) = \{Df_i(x) \mid i \in I\}.$$

Corollary 2.5 implies that all matrices $Df_i(x)$, $i \in I$ are regular and have the same determinant sign. Hence F is well-defined and the sets $F(x)$ are singletons for $x \in (x^0 + \varepsilon\mathbb{B}) \cap \mathcal{P}_f$. Consequently the mapping F is a function.

Recall that $\partial_B f(\cdot)$ is upper semicontinuous. Together with $\det(\cdot)$ being continuous and $\text{sgn}(\cdot)$ being continuous on $\mathbb{R} \setminus \{0\}$ we get that F is an upper semicontinuous mapping which is a function at the same time. Hence, F is continuous.

Summing up, we have a continuous function on a connected set which has images in $\{-1, 1\}$. Hence, F is constant on its domain, in particular it is constant on $(x^0 + \varepsilon\mathbb{B}) \cap \mathcal{S}_f$. This is a contradiction to (3.3). \square

Remark: The assumptions of Theorem 3.2 can be weakened in the following way:

- (i) instead of f being weakly almost piecewise smooth at x^0 it is enough to assume that there exist $\varepsilon^* > 0$ and a set X with $\mathcal{S}_f \subset X \subset \mathcal{P}_f$ such that

$$\text{the set } (x^0 + \varepsilon\mathbb{B}) \cap X \text{ is nonempty and connected for all } \varepsilon \in (0, \varepsilon^*) \quad (3.4)$$

and at the same time

(ii) assume that the inclusion (3.2) holds for $x \in X$ only.

Making use of the above remark let us set $X := \mathcal{P}_f^2$, where

$$\mathcal{P}_f^2 = \left\{ x \in \mathcal{P}_f \mid \begin{array}{l} \text{there exists a minimal local representation} \\ \text{for } f \text{ at } x \text{ consisting of at most 2 functions} \end{array} \right\}. \quad (3.5)$$

Corollary 3.3. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be locally Lipschitz, metrically regular at x^0 , and let $\mathcal{S}_f = \mathcal{D}_f$. Assume further that there exists $\varepsilon^* > 0$ such that*

$$\text{the set } (x^0 + \varepsilon\mathbb{B}) \cap \mathcal{P}_f^2 \text{ is nonempty and connected for all } \varepsilon \in (0, \varepsilon^*). \quad (3.6)$$

Then, all matrices $V \in \partial_B f(x^0)$ are regular and have the same determinant sign.

Proof. Rademacher's Theorem and $\mathcal{S}_f = \mathcal{D}_f$ ensure that $x^0 \in \text{cl } \mathcal{S}_f$. It is enough to show that the inclusion (3.2) is fulfilled for all $x \in \mathcal{P}_f^2$. The case $x \in \mathcal{S}_f$ is trivial, so assume that $x \in \mathcal{P}_f^2 \setminus \mathcal{S}_f$. Because of $\mathcal{S}_f = \mathcal{D}_f$ we have $x \in \mathcal{N}_f$. Let $\{f_1, f_2\}$ be a minimal local representation for f at x with $Df_1(x) \neq Df_2(x)$. Hence $\partial_B f'(x; \cdot)(0)$ consists of at least two elements. Together with (2.12) and (2.10) we get $\partial_B f'(x; \cdot)(0) = \partial_B f(x) = \{Df_1(x), Df_2(x)\}$. The assertion follows from Theorem 3.2. \square

Combining Lemma 2.8 and Corollary 3.3 we get our main result which puts strong regularity, metric regularity and coherent orientation of B-subdifferentials into relation. A similar result (without metric regularity) was shown in [16, Theorem 3 and Corollary 4], see also [31, Theorem 6].

Theorem 3.4. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a locally Lipschitz function which fulfills the assumption (3.6) at x^0 for a suitable $\varepsilon^* > 0$. Further, let f be directionally differentiable, and let $\mathcal{S}_f = \mathcal{D}_f$. Consider the following statements:*

- (i) f is strongly regular at x^0 ,
- (ii) f is metrically regular at x^0 and $|\text{ind}(f, x^0)| = 1$,
- (iii) $\partial_B f(x^0)$ consists of matrices with positive (negative) determinants and $\text{ind}(f, x^0) = 1$ (-1 , respectively).

Then it holds (i) \Leftrightarrow (ii) \Rightarrow (iii). If, additionally, f is semismooth then all three statements are equivalent.

Proof. (i) \Rightarrow (ii): Metric regularity is a trivial consequence of strong regularity. As f is locally one-to-one, $\text{ind}(f, x^0) = \pm 1$ ([25]).

(ii) \Rightarrow (iii), (ii) \Rightarrow (i): Using $\mathcal{S}_f = \mathcal{D}_f$ gives $\partial_P f(x^0) = \partial_B f(x^0)$. All the assumptions of Corollary 3.3 are fulfilled, hence all matrices in $\partial_P f(x^0) = \partial_B f(x^0)$ have the same nonvanishing determinant sign. Let us assume that the common determinant sign is positive.

As the mapping $\partial_B f(\cdot)$ is upper semicontinuous (cf. [8], [12]), we can choose a neighborhood U of x^0 such that for $x \in U$ all matrices in $\partial_B f(x)$ have positive determinants. Further, because of Lemma 2.8 it is possible to decrease this neighborhood U and to choose the corresponding neighborhood V of $f(x^0)$ such that (2.16) is fulfilled. As f is locally Lipschitz, both $\mathcal{N}_f \cap U$ and $f(\mathcal{N}_f \cap U)$ have a measure zero. Take any $y \in V \setminus f(\mathcal{N}_f \cap U) \neq \emptyset$. Metric regularity of f at x^0 ensures that $\emptyset \neq f^{-1}(y) \cap U \subset \mathcal{D}_f$. Because of (2.14), the equation (2.16) reads as

$$\text{ind}(f, x^0) = \sum_{x \in f^{-1}(y) \cap U} \text{sgn det} Df(x) \quad \forall y \in V \setminus f(\mathcal{N}_f \cap U). \quad (3.7)$$

As $Df(x) \in \partial_B f(x)$ we get $\text{det } Df(x) > 0$ for all $x \in f^{-1}(y) \cap U$. Equation (3.7) implies $\text{ind}(f, x^0) > 0$, hence $\text{ind}(f, x^0) = 1$. Consequently, the set $f^{-1}(y) \cap U$ is a singleton for $y \in V \setminus f(\mathcal{N}_f \cap U)$.

Assume further that $f^{-1}(y) \cap U$ contains at least two elements for a suitable $y \in V \cap f(\mathcal{N}_f \cap U)$. Recalling that $f(\mathcal{N}_f \cap U)$ has measure zero we are able to find $\tilde{y} \in V \setminus f(\mathcal{N}_f \cap U)$ which is as close to y as we wish. Metric regularity of f (i.e Aubin property of f^{-1} , cf. (2.1)) implies that $f^{-1}(\tilde{y}) \cap U$ contains at least 2 elements if \tilde{y} is close enough to y . This is a contradiction to the previous paragraph. Thus, any set $f^{-1}(y) \cap U$ is a singleton for $y \in V$ and f is strongly regular at x^0 .

(iii) \Rightarrow (i): Additionally, assume that f is semismooth. The statement was shown in [16, Corollary 4]. \square

Remark: The assumptions of Theorem 3.4 can be alternated as follows: instead of condition (3.6) we can assume that f is weakly almost piecewise smooth and fulfills the condition (3.2) for all $x \in \mathcal{P}_f$.

4 Some applications to nonlinear semidefinite programming

Consider the optimization problem

$$\min_{x \in X} f(x) \quad \text{s.t.} \quad G(x) \in K, \quad (4.1)$$

where $f : X \rightarrow \mathbb{R}$ and $G : X \rightarrow Z$ are twice continuously differentiable functions, X and Z are two finitely dimensional real Hilbert spaces each equipped with the scalar product $\langle \cdot, \cdot \rangle$ and its induced norm $\| \cdot \|$, and K is a closed convex set in Z . Define the Lagrangian function $L : X \times Z \rightarrow \mathbb{R}$ by

$$L(x, \gamma) := f(x) + \langle \gamma, G(x) \rangle, \quad (x, \gamma) \in X \times Z.$$

We are looking for points (x, γ) fulfilling the first-order optimality condition

$$D_x L(x, \gamma) = 0, \quad \gamma \in N_K(G(x)), \quad (4.2)$$

where $N_K(z)$ is the normal cone of K at z . The point (x, γ) is said to be a **KKT point** of the problem (4.1) if it satisfies (4.2).

When K is a polyhedral set (this is the case in classical nonlinear programming), the theory concerning sensitivity and stability of solutions of the problem (4.1) subject to parameter perturbations is quite complete. We are interested in semidefinite programming, i.e. in the case when the set K is the cone of negative semidefinite symmetric $p \times p$ matrices denoted by \mathbb{S}_-^p .

Consider the problem

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad h(x) = 0, \quad G(x) \preceq 0, \quad (4.3)$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}, G : \mathbb{R}^n \rightarrow \mathbb{S}^p$ and $h : \mathbb{R}^n \rightarrow \mathbb{R}^m$ are twice continuously differentiable. The KKT conditions can be written as

$$D_x L(x, y, \Gamma) = 0 \quad (4.4)$$

$$h(x) = 0 \quad (4.5)$$

$$\Gamma \in N_{\mathbb{S}_-^p}(G(x)), \quad (4.6)$$

where the Lagrangian function $L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{S}^p \rightarrow \mathbb{R} \times \mathbb{R}^m \times \mathbb{S}^p$ is defined by $L(x, y, \Gamma) := f(x) + \langle y, h(x) \rangle + \langle \Gamma, G(x) \rangle$. It is well-known that

$$\begin{aligned} \Gamma \in N_{\mathbb{S}_-^p}(G(x)) &\iff G(x) = (G(x) + \Gamma)_- \iff \Gamma = (G(x) + \Gamma)_+ \iff \\ &\iff G(x) \preceq 0, \Gamma \succeq 0, \langle \Gamma, G(x) \rangle = 0. \end{aligned}$$

Hence, the KKT conditions (4.4)–(4.6) can be re-written equivalently as an equation

$$H(x, y, \Gamma) := \begin{pmatrix} D_x L(x, y, \Gamma) \\ h(x) \\ G(x) - (G(x) + \Gamma)_- \end{pmatrix} = 0 \quad (4.7)$$

or as a generalized equation:

$$\begin{pmatrix} D_x L(x, y, \Gamma) \\ h(x) \\ G(x) \end{pmatrix} \in \begin{pmatrix} N_{\mathbb{R}^n}(x) \\ N_{\mathbb{R}^m}(y) \\ N_{\mathbb{S}_+^p}(\Gamma) \end{pmatrix}. \quad (4.8)$$

According to Kojima [21], the following function $F : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{S}^p \rightarrow \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{S}^p$ can be assigned to the problem (4.3) by defining

$$F(x, y, Z) := \begin{pmatrix} D_x L(x, y, Z_+) \\ h(x) \\ G(x) - Z_- \end{pmatrix} = \begin{pmatrix} Df(x) + [Dh(x)]^T y + [DG(x)]^* Z_+ \\ h(x) \\ G(x) - Z_- \end{pmatrix}. \quad (4.9)$$

The function F is called the **Kojima function** to the program (4.3). If $F(x, y, Z) = 0$, we call the point (x, y, Z) a **critical point** of F . One immediately sees that

$$\begin{aligned} (x, y, \Gamma) \text{ is a KKT point} &\implies (x, y, \Gamma + G(x)) \text{ is a critical point,} \\ (x, y, Z) \text{ is a critical point} &\implies (x, y, Z_+) \text{ is a KKT point.} \end{aligned} \quad (4.10)$$

Both transformations are locally Lipschitz which is important for regularity considerations. Consider now the perturbed program

$$\min_{x \in \mathbb{R}^n} f(x) - a^T x \quad \text{s.t.} \quad h(x) = b, \quad G(x) \preceq C, \quad (4.11)$$

where $a \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and $C \in \mathbb{S}^p$. It is easy to see that the critical points (x, y, Z) of the Kojima function corresponding to (4.11) are the solutions of the equation

$$F(x, y, Z) = \begin{pmatrix} a \\ b \\ C \end{pmatrix}. \quad (4.12)$$

Similarly, the KKT points (x, y, Γ) of (4.11) satisfy

$$\begin{pmatrix} D_x L(x, y, \Gamma) \\ h(x) \\ G(x) \end{pmatrix} - \begin{pmatrix} a \\ b \\ C \end{pmatrix} \in \begin{pmatrix} N_{\mathbb{R}^n}(x) \\ N_{\mathbb{R}^m}(y) \\ N_{\mathbb{S}_+^p}(\Gamma) \end{pmatrix}. \quad (4.13)$$

It is well-known that in the context of classical nonlinear programming, metric regularity of the Kojima function implies its strong regularity provided the problem data are twice continuously differentiable (cf. [19, Corollary 7.22]). Moreover, LICQ is a necessary consequence of metric regularity in this case (cf. [19, Lemma 7.1]).

One might ask whether this statement is also true for nonlinear semidefinite programs for a suitable generalization of LICQ. For this purpose we use the notion of constraint nondegeneracy, introduced in [36]. We say that a feasible point x^0 to the problem (4.3) is **constraint nondegenerate** if

$$\begin{pmatrix} Dh(x^0) \\ DG(x^0) \end{pmatrix} \mathbb{R}^n + \begin{pmatrix} \{0\} \\ \text{lin}(T_{\mathbb{S}_+^p}(G(x^0))) \end{pmatrix} = \begin{pmatrix} \mathbb{R}^m \\ \mathbb{S}^p \end{pmatrix}, \quad (4.14)$$

(cf. [43, equation (49)]), where $\text{lin}(C)$ denotes the largest linear subspace of the closed convex cone C , also called lineality space of C . This formulation is consistent with the one in [35]. For classical nonlinear programming, it coincides with LICQ.

The following theorem states that—in the semidefinite case, too—constraint nondegeneracy is necessary for metric regularity of the Kojima function under quite weak assumptions on f , G and h .

Theorem 4.1. *Let F be the Kojima function (4.9) corresponding to the problem (4.3) with $f, G, h \in C^1$ and let $F(x^0, y^0, Z^0) = 0$. If F is metrically regular at (x^0, y^0, Z^0) , then x^0 is a constraint nondegenerate feasible point of the problem (4.3).*

Proof. Let Z^0 have the spectral decomposition $Z^0 = P\Lambda P^T$, where Λ is the diagonal matrix of eigenvalues of Z^0 , $\Lambda = \text{Diag}[\lambda_1, \dots, \lambda_p]$ and P is a corresponding orthogonal matrix of orthonormal eigenvectors. Define two index sets corresponding to non-zero and zero eigenvalues of Z^0 by

$$I := \{i \in \{1, \dots, p\} \mid \lambda_i \neq 0\}, \quad J := \{i \in \{1, \dots, p\} \mid \lambda_i = 0\}.$$

Further, define a matrix Z near Z^0 by

$$Z := Z^0 + PZ(\varepsilon)P^T \quad (4.15)$$

where $Z(\varepsilon) := \text{Diag} [z_1(\varepsilon), \dots, z_p(\varepsilon)]$ and

$$z_i(\varepsilon) := \begin{cases} 0 & \text{if } i \in I, \\ \varepsilon & \text{if } i \in J, \end{cases} \quad (4.16)$$

for $\varepsilon > 0$. It follows $(\Lambda + Z(\varepsilon))_+ = \Lambda_+ + Z(\varepsilon)$ and $(\Lambda + Z(\varepsilon))_- = \Lambda_-$, hence

$$Z_+ = P(\Lambda + Z(\varepsilon))_+ P^T = P(\Lambda_+ + Z(\varepsilon)) P^T = Z_+^0 + PZ(\varepsilon)P^T$$

and

$$Z_- = P(\Lambda + Z(\varepsilon))_- P^T = P\Lambda_- P^T = Z_-^0.$$

Further, define the vector y by prescribing its components via $y_i := y_i^0 + \varepsilon$, $i = 1, \dots, m$. For $a_1(\varepsilon) := [DG(x^0)]^* PZ(\varepsilon)P^T$ and $a_2(\varepsilon) := [Dh(x^0)]^T (y - y^0)$ we get

$$\begin{aligned} F(x^0, y, Z) &= \begin{pmatrix} Df(x^0) + [Dh(x^0)]^T y^0 + a_2(\varepsilon) + [DG(x^0)]^* (Z_+^0 + PZ(\varepsilon)P^T) \\ h(x^0) \\ G(x^0) - Z_-^0 \end{pmatrix} \\ &= \begin{pmatrix} a_2(\varepsilon) + a_1(\varepsilon) \\ 0 \\ 0 \end{pmatrix} \end{aligned} \quad (4.17)$$

because of $F(x^0, y^0, Z^0) = 0$. We assumed that F is metrically regular at (x^0, y^0, Z^0) , i.e. F^{-1} has the Aubin property at $(0, (x^0, y^0, Z^0))$. Let U and V be the corresponding neighborhoods from (2.1), i.e. U is a neighborhood of (x^0, y^0, Z^0) and V is a neighborhood of $0 \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{S}^p$.

For any $b \in \mathbb{R}^m$ and any matrix $C \in \mathbb{S}^p$, consider points of the form $\begin{pmatrix} a_1(\varepsilon) + a_2(\varepsilon) \\ \delta b \\ \delta C \end{pmatrix}$, where

$\delta > 0$. Because of

$$\|a_1(\varepsilon)\| \leq \|[DG(x^0)]^*\| \|PZ(\varepsilon)P^T\| = \|DG(x^0)\| \|Z(\varepsilon)\| \leq \|DG(x^0)\| \varepsilon \sqrt{p}$$

and

$$\|a_2(\varepsilon)\| \leq \|Dh(x^0)\| \|y - y^0\| = \|Dh(x^0)\| \varepsilon \sqrt{m}$$

we can choose ε and δ sufficiently small such that

$$\begin{pmatrix} a_1(\varepsilon) + a_2(\varepsilon) \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} a_1(\varepsilon) + a_2(\varepsilon) \\ \delta b \\ \delta C \end{pmatrix} \in V \text{ and } (x^0, y, Z) \in U.$$

Applying (2.1) and (4.17) gives the existence of suitable $(x^\delta, y^\delta, Z^\delta) \in F^{-1} \begin{pmatrix} a_1(\varepsilon) + a_2(\varepsilon) \\ \delta b \\ \delta C \end{pmatrix}$

and $L > 0$ such that

$$\|x^\delta - x^0\| + \|y^\delta - y\| + \|Z^\delta - Z\| \leq L\delta(\|b\| + \|C\|). \quad (4.18)$$

Thus

$$h(x^\delta) = \delta b \quad \text{and} \quad G(x^\delta) - Z_-^\delta = \delta C. \quad (4.19)$$

Using the fact that Z is regular (cf. (4.15) and (4.16)) we know that $D\Pi_-(Z)$ exists (cf. [42, Theorem 4.6]), hence from (4.19) we get

$$h(x^0) + Dh(x^0)(x^\delta - x^0) + o_1(\|x^\delta - x^0\|) = \delta b$$

and

$$G(x^0) + DG(x^0)(x^\delta - x^0) + o_2(\|x^\delta - x^0\|) - [Z_- + D\Pi_-(Z)(Z^\delta - Z) + o_3(\|Z^\delta - Z\|)] = \delta C$$

for suitable functions $o_i(\cdot)$, $i = 1, 2, 3$. Because of $h(x^0) = 0$, $G(x^0) = Z_-$ and (4.18) there exist functions $o_4(\cdot)$, $o_5(\cdot)$, such that

$$Dh(x^0)(x^\delta - x^0) + o_4(\delta) = \delta b \quad (4.20)$$

$$DG(x^0)(x^\delta - x^0) - D\Pi_-(Z)(Z^\delta - Z) + o_5(\delta) = \delta C \quad (4.21)$$

Now, let $\{\delta_k\}$ be a sequence, converging to zero. Define

$$u := \lim_{\delta_k \searrow 0} \frac{1}{\delta_k} (x^{\delta_k} - x^0), \quad W := \lim_{\delta_k \searrow 0} \frac{1}{\delta_k} (Z^{\delta_k} - Z),$$

(take a subsequence of $\{\delta_k\}$ if necessary); the existence of both limits follows from (4.18). From (4.20) and (4.21) we obtain

$$Dh(x^0)u = b \quad (4.22)$$

$$DG(x^0)u - D\Pi_-(Z)W = C. \quad (4.23)$$

Recall that the vector $b \in \mathbb{R}^m$ and the matrix $C \in \mathbb{S}^p$ can be chosen freely. Thus, because of (4.22) we have $Dh(x^0)\mathbb{R}^n = \mathbb{R}^m$, and because of (4.23) it remains only to show that $D\Pi_-(Z)W \in \text{lin} \left(T_{\mathbb{S}_-^p}(G(x^0)) \right)$.

As Π_- is differentiable at Z we get both $D\Pi_-(Z)W \in T_{\mathbb{S}_-^p}(Z_-)$ and $-D\Pi_-(Z)W \in T_{\mathbb{S}_-^p}(Z_-)$, thus

$$D\Pi_-(Z)W \in \text{lin} \left(T_{\mathbb{S}_-^p}(Z_-) \right) = \text{lin} \left(T_{\mathbb{S}_-^p}(G(x^0)) \right),$$

where the last equality follows from $G(x^0) = Z_-$, cf. (4.17). \square

Remark: In a recent paper [20] Klatte and Kummer extended this result to the case when the set describing the constraints is a closed convex set $K \subset \mathbb{R}^m$.

Before we can apply Corollary 3.3 to the Kojima function of a semidefinite programming problem we need to show some properties of the semidefinite projection. Let us start with recalling the following result of Weyl for eigenvalues of symmetric matrices (cf. [3, p. 63] and [18, p. 367]):

Lemma 4.2. *Let $Y, Z \in \mathbb{S}^p$. Then*

$$|\lambda_i(Y) - \lambda_i(Z)| \leq \|Y - Z\|, \quad i = 1, \dots, p.$$

\square

Let \mathcal{O} denote the set of all orthogonal $p \times p$ matrices and let \mathcal{O}_Y be the set

$$\mathcal{O}_Y := \{P \in \mathcal{O} \mid Y = P \text{Diag} [\lambda(Y)] P^T\}.$$

For a continuous function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ one can define Löwner's function $\varphi : \mathbb{S}^p \rightarrow \mathbb{S}^p$ by

$$\varphi(Y) := P \text{Diag} [\varphi(\lambda_1(Y)), \dots, \varphi(\lambda_p(Y))] P^T, \quad P \in \mathcal{O}_Y. \quad (4.24)$$

Here we follow the convention of using φ to denote both the scalar-valued and matrix-valued versions of the function. At this point we need the following result which was proven in [7, Lemma 4] and [6, Proposition 4.4]:

Lemma 4.3. *The matrix function $\varphi : \mathbb{S}^p \rightarrow \mathbb{S}^p$ defined by (4.24) is continuously differentiable if and only if the function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is continuously differentiable.*

The next lemma states conditions under which the matrix-valued function $\varphi : \mathbb{S}^p \rightarrow \mathbb{S}^p$ inherits the piecewise smoothness from the scalar function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$.

Lemma 4.4. *Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be continuously differentiable everywhere except for one point λ^* where φ is piecewise smooth. Further, let $Y \in \mathbb{S}^p$ be any matrix such that λ^* is an eigenvalue of Y with multiplicity one. Then the matrix-valued function $\varphi : \mathbb{S}^p \rightarrow \mathbb{S}^p$ is piecewise smooth at Y , and there exists a minimal local representation for φ at Y consisting of at most two functions.*

Proof. Let $\lambda_\ell(Y) = \lambda^*$ with $\ell \in \{1, \dots, p\}$ be the particular eigenvalue of Y where φ is piecewise smooth but not continuously differentiable. If $p \geq 2$ define the number

$$\delta := \min_{i \in \{1, \dots, p\} \setminus \{\ell\}} |\lambda_i(Y) - \lambda^*|$$

which is positive. (For the case $p = 1$ set $\delta := 1$.) Further, find two continuously differentiable functions $g, h : \mathbb{R} \rightarrow \mathbb{R}$ with the following properties:

- the functions g and φ coincide on $(-\infty, \lambda^* - \frac{\delta}{2}] \cup [\lambda^*, \infty)$,
- the functions h and φ coincide on $(-\infty, \lambda^*] \cup [\lambda^* + \frac{\delta}{2}, \infty)$.

The existence of such functions g, h is ensured by Theorem 2 in [38]. For every matrix Z lying in $U := Y + \frac{\delta}{2}\mathbb{B}$ we get

$$|\lambda_i(Z) - \lambda_i(Y)| \leq \|Z - Y\| \leq \frac{\delta}{2} \quad i = 1, \dots, p,$$

which follows from Lemma 4.2. Hence, for all $i \in \{1, \dots, p\} \setminus \{\ell\}$ and all $Z \in U$ we obtain

$$|\lambda_i(Z) - \lambda^*| \geq |\lambda_i(Y) - \lambda^*| - \frac{\delta}{2} \geq \delta - \frac{\delta}{2} = \frac{\delta}{2}. \quad (4.25)$$

Further, define the sets

$$U_+ := U \cap \{Z \mid \lambda_\ell(Z) \geq \lambda^*\} \quad \text{and} \quad U_- := U \cap \{Z \mid \lambda_\ell(Z) \leq \lambda^*\}.$$

The definition of g and h ensures that $\varphi(\lambda_\ell(Z)) = g(\lambda_\ell(Z))$ for $Z \in U_+$ and $\varphi(\lambda_\ell(Z)) = h(\lambda_\ell(Z))$ for $Z \in U_-$. Because of (4.25) we have $\varphi(\lambda_i(Z)) = g(\lambda_i(Z)) = h(\lambda_i(Z))$ for $i \in \{1, \dots, p\} \setminus \{\ell\}$. Hence,

$$\varphi(Z) = \begin{cases} g(Z) & \text{if } Z \in U_+ \\ h(Z) & \text{if } Z \in U_- \end{cases}$$

is piecewise smooth at Y with a local representation consisting of continuously differentiable matrix functions g and h , cf. Lemma 4.3. \square

Corollary 4.5. *Both Π_+ and Π_- are piecewise smooth at any $Y \in \mathbb{S}^p$ with $\text{rk } Y \geq p - 1$, and for both of them there exists a minimal local representation consisting of at most two functions.*

Proof. It is enough to show the statement for Π_+ as $\Pi_- = id - \Pi_+$.

For the case when Y is regular we get that Π_+ is continuously differentiable on a suitable neighborhood of Y (use Lemma 4.3 with a similar argument as in the proof of Lemma 4.4). Further, assume that $\text{rk } Y = p - 1$. In this case the statement follows from Lemma 4.4 by setting $\varphi(\lambda) := \max\{0, \lambda\}$ and $\lambda^* := 0$. \square

In this particular case, the functions g and h can be defined as follows:

$$g(\lambda) := \begin{cases} \max\{0, \lambda\} & \text{if } \lambda \in (-\infty, -\delta/2] \cup [0, \infty), \\ 4\lambda^3/\delta^2 + 4\lambda^2/\delta + \lambda & \text{if } \lambda \in [-\delta/2, 0], \end{cases}$$

and

$$h(\lambda) := \begin{cases} \max\{0, \lambda\} & \text{if } \lambda \in (-\infty, 0] \cup [\delta/2, \infty), \\ -4\lambda^3/\delta^2 + 4\lambda^2/\delta & \text{if } \lambda \in [0, \delta/2]. \end{cases}$$

Lemma 4.6. *The following statements are true:*

(i) $\mathcal{S}_{\Pi_+} = \mathcal{D}_{\Pi_+}$,

(ii) for all $Z \in \mathbb{S}^p$

$$\text{the set } (Z + \varepsilon\mathbb{B}) \cap \mathcal{P}_{\Pi_+}^2 \text{ is nonempty and connected for all } \varepsilon > 0, \quad (4.26)$$

(iii) Π_+ is weakly almost piecewise smooth everywhere.

Proof. Assume $p > 1$ as the statements are trivial for $p = 1$.

(i) Let $Z \in \mathbb{S}^p$. If $\text{rk } Z = p$ then both Π_+ and Π_- are smooth at Z (cf. Lemma 4.3). Moreover, both Π_+ and Π_- are not differentiable at Z with $\text{rk } Z < p$ (see [42, Theorem 4.8] or [6, Proposition 4.3]). Hence

$$\{Z \in \mathbb{S}^p \mid \text{rk } Z = p\} = \mathcal{S}_{\Pi_+} = \mathcal{D}_{\Pi_+}. \quad (4.27)$$

(ii) For the case $\text{rk } Z = p - 1$ we know that Π_+ (and Π_-) are piecewise smooth at Z with a minimal local representation consisting of at most 2 functions. Further, it is well-known that for Z with $\text{rk } Z \leq p - 2$ the projections Π_+ and Π_- are not piecewise smooth at Z as the corresponding B-subdifferential is infinite (see, e.g. [26]). Consequently

$$\{Z \in \mathbb{S}^p \mid \text{rk } Z \geq p - 1\} = \mathcal{P}_{\Pi_+}^2 = \mathcal{P}_{\Pi_+}. \quad (4.28)$$

Let $Z^0 \in \mathbb{S}^p$. We claim that no open ball $Z^0 + \varepsilon\mathbb{B} \subset \mathbb{S}^p$, $\varepsilon > 0$ can be separated by the set $\{Z \in \mathbb{S}^p \mid \text{rk } Z \leq p-2\}$, i. e. the set $(Z^0 + \varepsilon\mathbb{B}) \cap \{Z \mid \text{rk } Z \geq p-1\}$ is nonempty and connected for any $\varepsilon > 0$. To show this we will need the fact (cf. [4, Proposition 5.69]) that the set $W_r := \{Z \in \mathbb{S}^p \mid \text{rk } Z = r\}$ is a smooth manifold with

$$\dim W_r = \frac{p(p+1)}{2} - \frac{(p-r)(p-r+1)}{2}. \quad (4.29)$$

According to [11, p. 312, Exercise e)] it is enough to show that the closed set $\bigcup_{r=0}^{p-2} W_r$ has a dimension of at most $\frac{p(p+1)}{2} - 2$ (in [11] the authors use the small inductive dimension ind , which is equal to \dim for the case of manifolds). Because of [11, p. 304, Exercise d)] we only need to show that $\dim W_r \leq \frac{p(p+1)}{2} - 2$ for $r \leq p-2$. But this is a consequence of (4.29), hence the claim is true.

Thus, the set $(Z^0 + \varepsilon\mathbb{B}) \cap \{Z \mid \text{rk } Z \geq p-1\}$ is nonempty and connected for any $\varepsilon > 0$. Recalling (4.28) we get that the set $(Z^0 + \varepsilon\mathbb{B}) \cap \mathcal{P}_{\Pi_+}^2$ is nonempty and connected for any $\varepsilon > 0$, hence (4.26) is fulfilled at Z^0 .

(iii) The assertion follows from $\mathcal{P}_{\Pi_+}^2 = \mathcal{P}_{\Pi_+}$, cf. (4.28). \square

All of these statements can be shown for the Kojima function F , too:

Lemma 4.7. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$, $G : \mathbb{R}^n \rightarrow \mathbb{S}^p$, $h : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be twice continuously differentiable. Then for the corresponding Kojima function F it holds*

$$(i) \quad \mathcal{S}_F = \mathcal{D}_F,$$

$$(ii) \quad \text{for all } (x, y, Z) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{S}^p$$

$$\text{the set } [(x, y, Z) + \varepsilon\mathbb{B}] \cap \mathcal{P}_F^2 \text{ is nonempty and connected for all } \varepsilon > 0, \quad (4.30)$$

(iii) F is weakly almost piecewise smooth everywhere.

Proof. Again, the case $p = 1$ is trivial, so assume $p > 1$.

(i) Because of (4.27) we have $\{(x, y, Z) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{S}^p \mid \text{rk } Z = p\} \subset \mathcal{S}_F \subset \mathcal{D}_F$. From $(x, y, Z) \in \mathcal{D}_F$ we have that, in particular, the third component of F is differentiable at the point (x, y, Z) . Hence, Π_- is differentiable at Z , and the rank of Z has to be full. Thus

$$\{(x, y, Z) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{S}^p \mid \text{rk } Z = p\} = \mathcal{S}_F = \mathcal{D}_F. \quad (4.31)$$

(ii) Similarly, from (4.28) we have $\{(x, y, Z) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{S}^p \mid \text{rk } Z \geq p-1\} \subset \mathcal{P}_F^2 \subset \mathcal{P}_F$. Again, $(x, y, Z) \in \mathcal{P}_F$ implies $\text{rk } Z \geq p-1$ (because of the third component of F). Hence

$$\{(x, y, Z) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{S}^p \mid \text{rk } Z \geq p-1\} = \mathcal{P}_F^2 = \mathcal{P}_F. \quad (4.32)$$

Let us consider any point $(x^0, y^0, Z^0) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{S}^p$. Our goal is to show that the Kojima function fulfills the condition (4.30) at this point. From the dimension considerations in the

proof of Lemma 4.6(ii) it follows that no open ball $(x^0, y^0, Z^0) + \varepsilon\mathbb{B}$ with $\varepsilon > 0$ can be separated by the set $\{(x, y, Z) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{S}^p \mid \text{rk } Z \leq p - 2\}$. Hence, the set $[(x^0, y^0, Z^0) + \varepsilon\mathbb{B}] \cap \{(x, y, Z) \mid \text{rk } Z \geq p - 1\}$ is nonempty and connected for any $\varepsilon > 0$. Recalling (4.32) we get that the set $[(x^0, y^0, Z^0) + \varepsilon\mathbb{B}] \cap \mathcal{P}_F^2$ is nonempty and connected for any $\varepsilon > 0$, hence (4.30) is fulfilled at (x^0, y^0, Z^0) .

(iii) The assertion follows from $\mathcal{P}_F^2 = \mathcal{P}_F$, cf. (4.32). \square

Theorem 4.8. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$, $G : \mathbb{R}^n \rightarrow \mathbb{S}^p$, $h : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be twice continuously differentiable, and let the corresponding Kojima function F be metrically regular at (x^0, y^0, Z^0) . Then, all matrices $V \in \partial_B F(x^0, y^0, Z^0)$ are regular and have the same determinant sign.*

Proof. Lemma 4.7 implies that the Kojima function F fulfills the condition (4.30) at (x^0, y^0, Z^0) and $\mathcal{D}_F = \mathcal{S}_F$. Corollary 3.3 gives then the statement. \square

In [31, Lemma 11] the authors show that

$$\partial_B \Pi_+(Z) = \partial_B \Pi'_+(Z; \cdot)(0) \quad \forall Z \in \mathbb{S}^p$$

and in [31, Proposition 7] they prove that this equation remains valid if we produce a composite function from Π_+ and a continuously differentiable function. Together with the fact that the Kojima function is weakly almost piecewise smooth we get an alternative way for proving Theorem 4.8.

Theorem 4.9. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$, $G : \mathbb{R}^n \rightarrow \mathbb{S}^p$, $h : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be twice continuously differentiable, and let F denote the corresponding Kojima function. Then the following statements are equivalent:*

- (i) F is strongly regular at $w^0 := (x^0, y^0, Z^0)$,
- (ii) F is metrically regular at w^0 and $|\text{ind}(F, w^0)| = 1$,
- (iii) $\partial_B F(w^0)$ consists of matrices with positive (negative) determinants and $\text{ind}(F, w^0) = 1$ (-1 , respectively).

Proof. Both $\mathcal{D}_F = \mathcal{S}_F$ and the condition (4.30) are fulfilled because of Lemma 4.7. Recall that the semidefinite projections Π_+ and Π_- are directionally differentiable and semismooth (cf. [42]), hence the Kojima function F is directionally differentiable (cf. [19, Section 6.4]) and semismooth (cf. [27]), too. The assertion follows from Theorem 3.4. \square

In [43, Theorem 4.1] Sun formulated equivalent descriptions for the strong regularity of the generalized equation (4.8) provided that the point under consideration is locally optimal and constraint nondegenerate. One of the equivalent formulations states that the function H from (4.7) fulfills the condition (iii) in Theorem 4.9. Our result produces a link to the metric regularity of F , cf. (4.10).

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