PARTIAL SECOND-ORDER SUBDIFFERENTIALS IN VARIATIONAL ANALYSIS AND OPTIMIZATION

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Abstract. This paper presents a systematic study of partial second-order subdifferentials for extended-real-valued functions, which have already been applied to important issues of variational analysis and constrained optimization in finite-dimensional spaces. The main results concern developing extended calculus rules for these second-order constructions in both finite-dimensional and infinite-dimensional frameworks. We also provide new applications of partial second-order subdifferentials to Lipschitzian stability of stationary point mappings in parametric constrained optimization and discuss some other applications.

Key words. variational analysis, parametric constrained optimization, generalized differentiation, coderivatives, second-order subdifferentials, stability, stationary point mappins

AMS subject classifications. 49J53, 49J52, 90C31

1 Introduction

This paper is devoted to the development and applications of *second-order variational analysis* based on appropriate constructions of *second-order generalized differentiation.* Although the recent years have witnessed a rapidly growing interest in these areas, there are still more questions than answers in the second-order variational theory and its applications.

Several different approaches to second-order variational analysis and generalized differentiation have been recognized in the literature; see, e.g., the books [1, 21, 38] with the commentaries and references therein. In this paper we develop the dual "derivative-of-derivative" approach initiated in [20], which treats a second-order subdifferential $\partial^2\varphi$ of an extended-real-valued function $\varphi\colon X \to \overline{I\!R} := (-\infty,\infty]$ as a dual-space generalized derivative/coderivative $D^*\partial\varphi$ of a (set-valued) first-order subdifferential mapping $\partial\varphi\colon X \rightrightarrows X^*$; see [21] and Section 2 below for more details. There are numerous applications of the basic second-order subdifferential construction from [20], as well as its modifications and partial versions, to important issues of variational analysis and optimization largely related to optimality and stability conditions in problems of constrained and multiobjective optimization, systems control, and mechanics. They mainly concern nonlinear and conic programming, mathematical and equilibrium problems with equilibrium constraints and hierarchical/bilevel structures, stochastic programming with applications to electricity spot market

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modeling, parameterized variational inequalities and complementarity systems, trust-region methods in numerical optimization, optimal control of the sweeping process and generalized gradient systems, contact and shape design problems of continuum mechanics, etc.; see, e.g., [3, 5, 7, 8, 9, 10, 11, 12, 13, 15, 16, 20, 21, 24, 25, 26, 27, 28, 29, 30, 32, 33, 34, 36, 38, 41] for more details and references. We specifically mention a remarkable result of [34], which provides a complete characterization of the so-called *tilt-stable* local minimizers of extended-real-valued functions in finite dimensions as the positive-definiteness of their second-order subdifferential (or generalized Hessian) mappings from [20].

Our major attention in this paper is paid to the study and applications of *partial* second-order subdifferentials of extended-real-valued functions $\varphi\colon X\times Y\to\overline{I\!R}$ of two variables defined by the scheme $\partial^2_x\varphi=D^*\partial_x\varphi$ via coderivatives of first-order partial subdifferential mappings. To the best of our knowledge, such a partial version of the second-order subdifferential from [20] was first used in [15] to characterize *full stability* (a significant generalization of tilt stability) of local minimizers for extended-real-valued functions on finite-dimensional spaces. In contrast to the (total, full) second-order subdifferential of [20] for which various calculus rules have been widely available (see, e.g., [14, 21, 25, 26, 33] and the references therein), it is not the case for its more involved partial counterparts in finite and infinite dimensions. Some results in this direction have been recently obtained in [29, 30] for special classes of functions on finite-dimensional spaces, being mainly motivated by applications to full stability of optimal solutions for various classes of problems in constrained optimization.

The *primary goals* of this paper are to develop new *calculus rules* for the aforementioned *partial second-order subdifferential* and its infinite-dimensional extensions, providing then their *applications* to some important issues in *parametric constrained optimization*. Most of the results obtained are new even in *finite dimensions*, and some of the crucial calculus rules have never been derived before for the *full* second-order subdifferential constructions.

The rest of the paper is organized as follows. In Section 2 we define and discuss the two basic partial second-order subdifferentials of our study (called the *normal* and *mixed* one depending on the respected coderivative), which both reduce to the aforementioned second-order construction $\partial^2_x\varphi$ in finite dimensions. We also present some preliminaries from variational analysis and generalized differentiation widely used below.

Section 3 is devoted to *second-order subdifferential calculus* in general *Banach* spaces. The main result here establishes *exact*/equality type second-order chain rules for both mixed and normal partial second-order subdifferential constructions under a certain *reduction* condition for extended-real-valued functions in the vein of that [1] for set and an appropriate *nondegeneracy* condition for mappings. These second-order chain rule are further elaborated in Section 4 in the case of *Asplund* spaces, i.e., such Banach spaces where every separable subspace has a separable dual; this is automatic when the space in question is reflexive. The chain rules obtained in this direction are new even for the full second-order subdifferential in finite-dimensional spaces extending in this case the corresponding result of [33] given for special compositions. Furthermore, it significantly generalizes the recent chain rule of [29] for partial second-order subdifferentials in finite dimensions obtained under the *full rank* condition on the partial derivative of the inner mapping.

In Section 4 we derive, besides the aforementioned exact second-order chain rules under nondegeneracy, several other sum and chain rules for the second-order partial subdifferentials of extended-real-valued functions in the Asplund space setting. The results obtained in this section hold for significantly broader classes of mappings in comparison with those from Section 3, while they generally ensure only the "right" *inclusions*/upper estimates in contrast to the equalities in the preceding section. Note, in particular, that the partial second-order chain rule derived here in Asplund spaces for *strongly amenable* compositions sharpens the corresponding result of [29] even in finite dimensions.

In Section 5 we specify the calculus rules obtained in Section 3 and Section 4 to deriving efficient representations of the second-order partial subdifferential constructions generated by coderivatives of *perturbed normal cone mappings* of the type

$$F(x,y):=N\big(x;G(y)\big)\;\text{ with }\;G(y):=\big\{x\in X\big|\;g(x,y)\in\Theta\big\}.$$

The results obtained are shown to be useful for complete calculations of these second-order constructions for mappings appearing in *trust-region methods* of numerical optimization.

Section 6 concerns applications of the partial second-order subdifferentials to the study of the so-called *stationary point mapping/multifunction* $S\colon Y\rightrightarrows X$ given by

$$S(y):=\big\{x\in X\big|\;0\in\partial_x\psi_0(x,y)+\partial_x\psi(x,y)\big\},\quad y\in Y,$$

and associated with the stationary condition $0\in\partial_x(\psi_0+\psi)(x,y)$ for the following constrained *parametric optimization problem*:

$$\text{minimize }\;\psi_0(x,y)+\psi(x,y)\;\text{ over }\;x\in X. \tag{1.1}$$

Note that the functions ψ_0 and ψ play essentially different role in (1.1). The former one ψ_0, called the *cost function*, is usually nice (often smooth) while the latter *constraint function* ψ is extended-real-valued and thus allows us to incorporate various constraints in the formally unconstrained format (1.1); see, e.g., [14] for more details and discussions. The above multifunction S is the solution map of a *generalized equation* (in the sense of Robinson [37]), where the second term is always set-valued and depends on the parameter y. Invoking *coderivative characterizations* of Lipschitzian behavior of such multifunctions [21], we derive efficient conditions ensuring *robust Lipschitzian stability* of stationary point mappings via the partial second-order subdifferentials of their initial data.

Section 7 contains some concluding remarks on further developments and applications.

Our notation is basically standard in variational analysis and generalized differentiation; see, e.g., [21, 38]. Unless otherwise stated, all the spaces in question are *Banach space* with the norm $\|\cdot\|$ and the canonical pairing $\langle\cdot,\cdot\rangle$ between, say, X and its topological dual X^*. Given a set-valued mapping/multifunction $F\colon X\rightrightarrows X^*$, the symbol

$$\mathop{\rm Lim\,sup}_{x\to\bar x}F(x):=\Big\{x^*\in X^*\Big|\;\begin{array}{l}\exists\;\text{sequences }\;x_k\to\bar x,\;x_k^*\stackrel{w^*}{\to}x^*\;\text{ with}\\ x^*\in F(x_k)\;\text{ for all }\;k\in I\!\!N:=\{1,2,\ldots\}\end{array}\Big\} \tag{1.2}$$

signifies the *sequential Painlevé-Kuratowski outer/upper limit* of F as $x\to\bar x$, where w^* stands for the weak* topology on X^*. Recall also that the notation $x\stackrel{\Omega}{\to}\bar x$ and $x\stackrel{\varphi}{\to}\bar x$ for $\Omega\subset X$ and $\varphi\colon X\to\overline{I\!\!R}$ means that $x\to\bar x$ with $x\in\Omega$ and with $\varphi(x)\to\varphi(\bar x)$, respectively.

Throughout the paper we deal with mappings on product spaces and use the sum norm $|(x,y)\| := \|x\| + \|y\|$ on the product $X \times Y$. By $\mathcal{L}(X \times Y, Z)$ we denote the Banach space of linear bounded operators from $X \times Y$ to Z and observe that $A \in \mathcal{L}(X \times Y, Z)$ can be uniquely represented as $A(u,v) = A_1(u) + A_2(v)$ with $A_1 \in \mathcal{L}(X,Z)$ and $A_2 \in \mathcal{L}(Y,Z)$. This allows us to treat $\mathcal{L}(X \times Y, Z)$ as $\mathcal{L}(X,Z) \times \mathcal{L}(Y,Z)$ and write $A^*z^* = (A_1^*z^*, A_2^*z^*)$ for $z^* \in Z^*$. Applying this to the Fréchet derivative of $f\colon X \times Y \to Z$, we get that $\nabla f(\bar{x},\bar{y})(u,v) = \nabla_x f(\bar{x},\bar{y})(u) + \nabla_y f(\bar{x},\bar{y})(v)$.

2 Basic Constructions and Preliminary Results

We mainly follow here the two-volume monograph [21] referring the reader also to the books [1, 2, 38, 39] for related material, more discussions, alternative notation and terminology.

In the vein of the *dual-space geometric approach* to variational analysis and generalized differentiation, define first generalized normals to nonempty sets. Given $\Omega \subset X$ and $\varepsilon \ge 0$, the set of *ε-normals* to Ω at $\bar{x} \in \Omega$ is

$$\widehat{N}_\varepsilon(\bar{x};\Omega) := \Big\{ x^* \in X^* \Big| \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle x^*, x - \bar{x}\rangle}{\|x - \bar{x}\|} \le \varepsilon \Big\} \tag{2.1}$$

with $\widehat{N}_\varepsilon(\bar{x};\Omega) := \emptyset$ for $\bar{x} \notin \Omega$. If $\varepsilon = 0$, the set (2.1) is a convex cone denoted by $\widehat{N}(\bar{x};\Omega)$ and known as the *prenormal cone* or the regular/Fréchet normal cone to Ω at $\bar{x}$. Reducing to normal cone of convex analysis for convex sets Ω, the cone $\widehat{N}(\bar{x};\Omega)$ may be empty at boundary points of simple nonconvex sets in $I\!R^2$ and does not possess a reasonable calculus. The situation dramatically changes when we apply the sequential limiting procedure (1.2) to $\widehat{N}_\varepsilon(\cdot;\Omega)$ and define the (basic, limiting, Mordukhovich) *normal cone*

$$N(\bar{x};\Omega) := \mathop{\mathrm{Lim\,sup}}_{\substack{x \to \bar{x} \\ \varepsilon \downarrow 0}} \widehat{N}_\varepsilon(x;\Omega) \tag{2.2}$$

to Ω at $\bar{x} \in \Omega$ first introduced in [18] in finite dimensions. Note that ε can be omitted in (2.2) if X is Asplund and if the set Ω is locally closed around $\bar{x}$. Observe furthermore that the normal cone (2.2) can be *nonconvex* for simple nonconvex sets Ω in $I\!R^2$, while it is not the case if Ω is *normally regular* at $\bar{x}$, i.e., $\widehat{N}(\bar{x};\Omega) = N(\bar{x};\Omega)$.

Given next a set-valued mapping $F\colon X \rightrightarrows Y$ with the *graph*

$$\mathrm{gph}\, F := \big\{ (x,y) \in X \times Y \big| \, y \in F(x) \big\},$$

we define the *normal coderivative* of F at $(\bar{x},\bar{y}) \in \mathrm{gph}\, F$ by

$$D_N^*F(\bar{x},\bar{y})(y^*) := \big\{ x^* \in X^* \big| \, (x^*, -y^*) \in N\big((\bar{x},\bar{y}); \mathrm{gph}\, F\big) \big\}, \quad y^* \in Y^*, \tag{2.3}$$

introduced in [19] for mappings between finite-dimensional spaces. As we know from [21], yet another limiting coderivative construction

$$\begin{aligned} D_M^*F(\bar{x},\bar{y})(y^*) := \Big\{ x^* \in X^* \Big| \, &\exists\, \varepsilon_k \downarrow 0,\ (x_k,y_k) \xrightarrow{\mathrm{gph}\, F} (\bar{x},\bar{y}),\ x_k^* \xrightarrow{w^*} x^*, \text{ and} \\ &\|y_k^* - y^*\| \to 0 \ \text{ with } \ (x_k^*, -y_k^*) \in \widehat{N}_{\varepsilon_k}\big((x_k,y_k); \mathrm{gph}\, F\big) \Big\}, \end{aligned} \tag{2.4}$$

called the *mixed coderivative* of F at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$, has been proved to be very instrumental in infinite dimensions. We can omit ε_k in (2.4) when both spaces X and Y are Asplund and when the graph of F is locally closed around $(\bar{x}, \bar{y})$. As follows from the definitions above, $D_M^* F(\bar{x}, \bar{y})(y^*) = D_N^* F(\bar{x}, \bar{y})(y^*)$ if $\dim Y < \infty$ while in general we have the inclusions

$$D_M^* F(\bar{x}, \bar{y})(y^*) \subset D_N^* F(\bar{x}, \bar{y})(y^*) \text{ for all } y^* \in Y^*.$$

When $F = f \colon X \to Y$ is single-valued at $\bar{x}$, we drop $\bar{y} = f(\bar{x})$ in the coderivative notation and observe that both coderivatives (2.3) and (2.4) reduce to the *adjoint derivative* operator

$$D_M^* f(\bar{x})(y^*) = D_N^* f(\bar{x})(y^*) = \big\{\nabla f(\bar{x})^* y^*\big\}, \quad y^* \in Y^*, \tag{2.5}$$

provided that f is *strictly differentiable* at $\bar{x}$ in the sense that

$$\lim_{x, x' \to \bar{x}} \frac{f(x) - f(x') - \langle \nabla f(\bar{x}), x - x' \rangle}{\|x - x'\|} = 0,$$

which is automatic when f is $\mathcal{C}^1$ around this point. Note also that, being nonconvex, the normal and mixed coderivatives are *not dual* to any tangentially generated derivatives of F in primal spaces. Nevertheless both these coderivatives and related normal and (first-order) subdifferential constructions enjoy *full calculi* based on the *extremal/variational principles* of variational analysis; see [21].

Before considering the first-order and second-order subdifferential constructions associated with the above coderivatives, recall some "normal compactness" properties of sets and set-valued mappings used in this paper. These properties are automatic in finite-dimensions while playing a crucial role in infinite-dimensional variational analysis.

A set $\Omega \subset X$ is *sequentially normally compact* (SNC) at $\bar{x} \in \Omega$ if for any sequences $\varepsilon_k \downarrow 0$, $x_k \stackrel{\Omega}{\to} \bar{x}$, and $x_k^* \stackrel{w^*}{\to} 0$ with $x_k^* \in \widehat{N}_{\varepsilon_k}(x_k; \Omega)$ for $k \in I\!\!N$ it follows that $\|x_k^*\| \to 0$ as $k \to \infty$. A set-valued mapping $F \colon X \rightrightarrows Y$ is *SNC* at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ if its graph is SNC at this point. We also need the following two partial modifications of the SNC property for mappings. Namely, F is *partially SNC* (PSNC) at $(\bar{x}, \bar{y})$ if if for any sequences $\varepsilon_k \downarrow 0$, $(x_k, y_k) \to (\bar{x}, \bar{y})$ with $(x_k, y_k) \in \operatorname{gph} F$, $(x_k^*, y_k^*) \in \widehat{N}_{\varepsilon_k}((x_k, y_k); \operatorname{gph} F)$ we have

$$\big[x_k^* \stackrel{w^*}{\to} 0,\ \|y_k^*\| \to 0\big] \Longrightarrow \|x_k^*\| \to 0 \text{ as } k \to \infty. \tag{2.6}$$

Furthermore F is said to be *strongly PSNC* at $(\bar{x}, \bar{y})$ if implication (2.6) is replaced by

$$\big[x_k^* \stackrel{w^*}{\to} 0,\ y_k^* \stackrel{w^*}{\to} 0\big] \Longrightarrow \|x_k^*\| \to 0 \text{ as } k \to \infty.$$

As always, we can equivalently remove ε_k from the above definitions for closed sets and closed-graph mappings in Asplund spaces. Note that, besides finite-dimensional settings, these properties hold for sets and mappings with some Lipschitzian structures and also are preserved under various operations; see [21] for precise results. Here we only mention that F is PSNC at $(\bar{x}, \bar{y})$ if it is *Lipschitz-like* (or satisfies the Aubin property) around this point if there exist neighborhoods U of $\bar{x}$ and V of $\bar{y}$ as well as a constant $\ell \geq 0$ such that

$$F(u) \cap V \subset F(v) + \ell \|u - v\| I\!\!B \text{ whenever } u, v \in U, \tag{2.7}$$

where $I\!B$ stands as usual for closed unit ball of the space in question.

Next we consider an extended-real-valued functions $\varphi\colon X \to \overline{I\!R}$ finite at $\bar{x}$ and associate with it the *epigraphical multifunction* $E_\varphi\colon X \rightrightarrows I\!R$ defined by

$$E_\varphi(x) := \big\{\alpha \in I\!R \big|\ \alpha \geq \varphi(x)\big\} \ \text{ with } \ \operatorname{epi}\varphi := \operatorname{gph} E_\varphi.$$

For brevity of the presentation we provide just geometric definitions of the first-order subdifferential constructions used in this paper, referring the reader to [2, 21, 38, 39] for analytic descriptions. In fact, we need below only the following two limiting subdifferential constructions generated by the normal cone (2.2) to the epigraph of φ or, equivalently, by the corresponding coderivative (2.3) of the epigraphical multifunction. Namely, the *basic subdifferential* $\partial\varphi(\bar{x})$ and the *singular subdifferential* $\partial^\infty\varphi(\bar{x})$ of φ at $\bar{x}$ are defined by

$$\partial\varphi(\bar{x}) := D^*E_\varphi\big(\bar{x},\varphi(\bar{x})\big)(1) = \big\{x^* \in X^* \big|\ (x^*,-1) \in N\big((\bar{x},\varphi(\bar{x}));\operatorname{epi}\varphi\big)\big\}, \tag{2.8}$$

$$\partial^\infty\varphi(\bar{x}) := D^*E_\varphi\big(\bar{x},\varphi(\bar{x})\big)(0) = \big\{x^* \in X^* \big|\ (x^*,0) \in N\big((\bar{x},\varphi(\bar{x}));\operatorname{epi}\varphi\big)\big\}, \tag{2.9}$$

respectively. Observe that, since the mapping E_φ takes values in $I\!R$, there is no difference between its normal and mixed coderivatives; so we use the common coderivative symbol D^* in (2.8) and (2.9) as everywhere in the paper. Note also that

$$\partial\delta_\Omega(\bar{x}) = \partial^\infty\delta_\Omega(\bar{x}) = N(\bar{x};\Omega), \quad \bar{x} \in \Omega,$$

for the indicator function $\delta_\Omega(x)$ of the set $\Omega \subset X$ equal to 0 for $x \in \Omega$ to ∞ otherwise.

It is well known that $\partial^\infty\varphi(\bar{x}) = \{0\}$ if φ is locally Lipschitzian around $\bar{x}$. In what follows we employ the singular subdifferential (2.9) only for formulations of qualification conditions, which thus hold automatically for Lipschitz continuous functions. Recall also that $\varphi\colon X \to \overline{I\!R}$ is *sequentially normally epi-compact* (SNEC) at $\bar{x}$ if E_φ is SNC at $(\bar{x},\varphi(\bar{x}))$, which is surely the case of locally Lipschitzian functions.

Now we are ready to define the second-order subdifferentials of extended-real-valued functions, which are the main constructions of our study. By the scheme described in Section 1 they are obtained by applying coderivatives to first-order subdifferential mappings. The following definition, employing the normal (2.3) and mixed (2.4) coderivatives of the basic subdifferential mapping (2.8), concerns *partial* subdifferentials for functions of two variables, which of course contain full subdifferentials as a special case. We use the notation

$$\partial_x\varphi(x,y) := \big\{\text{ set of subgradients } \ u \in X^* \ \text{ of } \ \varphi_y := \varphi(\cdot,y) \ \text{ at } \ x\big\} = \partial\varphi_y(x)$$

for the *partial first-order* subdifferential mapping $\partial_x\varphi\colon X \times Y \rightrightarrows X^*$.

Definition 2.1 (partial second-order subdifferentials). *Let $\varphi\colon X \times Y \to \overline{I\!R}$ be finite at $(\bar{x},\bar{y}) \in X \times Y$, and let $\bar{u} \in \partial_x\varphi(\bar{x},\bar{y})$. Then the* NORMAL PARTIAL SECOND-ORDER SUBDIFFERENTIAL *$\partial^2_{N,x}\varphi(\bar{x},\bar{y},\bar{u})\colon X^{**} \rightrightarrows X^* \times Y^*$ and the* MIXED PARTIAL SECOND-ORDER SUBDIFFERENTIAL *$\partial^2_{M,x}\varphi(\bar{x},\bar{y},\bar{u})\colon X^{**} \rightrightarrows X^* \times Y^*$ of φ with respect to x at $(\bar{x},\bar{y})$ relative to $\bar{u}$ are defined, respectively, by*

$$\partial^2_{N,x}\varphi(\bar{x},\bar{y},\bar{u})(w) := \big(D^*_N\partial_x\varphi\big)(\bar{x},\bar{y},\bar{u})(w), \quad w \in X^{**}, \tag{2.10}$$

$$\partial^2_{M,x}\varphi(\bar{x},\bar{y},\bar{u})(w) := \big(D^*_M\partial_x\varphi\big)(\bar{x},\bar{y},\bar{u})(w), \quad w \in X^{**}. \tag{2.11}$$

It follows from Definition 2.1 that the normal and mixed second-order constructions (2.10) and (2.11) agree when $\dim X < \infty$; if in addition $\dim Y < \infty$, they both reduce to the "extended" partial second-order subdifferential studied in [15, 29]. However, we do not use the "tilde" notation for (2.10) and (2.11), since we do not consider their "standard" versions similar to [15, 29] in finite dimensions; see the discussions therein.

The next proposition shows that for $\mathcal{C}^2$ (and a bit more general) functions the partial second-order subdifferential constructions (2.10) and (2.11) go back to the the partial Hessians involving *both* variables of φ.

Proposition 2.2 (partial second-order subdifferentials of $\mathcal{C}^2$ type functions). *Let $\varphi \in \mathcal{C}^1$ around $(\bar{x}, \bar{y})$, and let its partial derivative operator $\nabla_x \varphi \colon X \times Y \to X^*$ be strictly differentiable at $(\bar{x}, \bar{y})$ with the partial derivatives denoted by $\nabla^2_{xx}\varphi(\bar{x}, \bar{y})$ and $\nabla^2_{xy}\varphi(\bar{x}, \bar{y})$. Then for all $w \in X^{**}$ we have the representation*

$$\partial^2_{N,x}\varphi(\bar{x}, \bar{y})(w) = \partial^2_{M,x}\varphi(\bar{x}, \bar{y})(w) = \Big\{\Big(\nabla^2_{xx}\varphi(\bar{x}, \bar{y})^* w, \nabla^2_{xy}\varphi(\bar{x}, \bar{y})^* w\Big)\Big\}.$$

Proof. It follows from (2.5) that there is a neighborhood U of $(\bar{x}, \bar{y})$ such that

$$\partial_x \varphi(x, y) = \big\{\nabla_x \varphi(x, y)\big\} \text{ for all } (x, y) \in U.$$

Taking this into account and applying [21, Theorem 1.38] to mapping $\nabla_x \varphi \colon X \times Y \to X^*$ strictly differentiable at $(\bar{x}, \bar{y})$, we get

$$\partial^2_x \varphi(\bar{x}, \bar{y})(w) = \big(D^* \nabla_x \varphi\big)(\bar{x}, \bar{y})(w) = \Big\{\big(\nabla^2_{xx}\varphi(\bar{x}, \bar{y})^* w, \nabla^2_{xy}\varphi(\bar{x}, \bar{y})^* w\big)\Big\}$$

for all $w \in X^{**}$, where D^* stands for either normal or mixed coderivative. $\triangle$

Note that in general the second-order constructions of Definition 2.1 are positively homogeneous mappings from X^{**} to $X^* \times Y^*$; we use the common notation $\partial^2_x \varphi(\bar{x}, \bar{y}, \bar{u})$ for them when they agree with each other as in finite dimensions. The following example illustrates the calculation of partial second-order subdifferentials for nonsmooth functions directly based on the definition. Much more in this direction comes from the second-order calculus rules developed in the subsequent sections.

Example 2.3 (calculation of partial second-order subdifferentials for nonsmooth functions). Consider the function $\varphi(x, y) := |x| + |y|$ on $I\!R^2$ for which we easily get that

$$\partial_x \varphi(x, y) = \begin{cases} \{1\} & \text{if } x > 0, \\ [-1, 1] & \text{if } x = 0, \\ \{-1\} & \text{if } x < 0. \end{cases}$$

Fix $(\bar{x}, \bar{y}) = (0, 0)$ with $\bar{u} = 1 \in \partial_x \varphi(0, 0)$ and have for $(x, y, u) \in \operatorname{gph} \partial_x \varphi$ near $(0, 0, 1)$ that

$$\widehat{N}\big((x, y, u); \operatorname{gph} \partial_x \varphi\big) = \begin{cases} \big\{(0, 0)\big\} \times I\!R & \text{if } x > 0,\ u = 1, \\ (-\infty, 0] \times \{0\} \times [0, \infty) & \text{if } x = 0,\ u = 1, \\ I\!R \times \big\{(0, 0)\big\} & \text{if } x = 0,\ u < 1. \end{cases}$$

Then applying (2.2) *with* $\varepsilon = 0$ *to the set* $\mathrm{gph}\,\partial_x\varphi$ *gives us the normal cone expression*

$$N\big((0,0,1);\mathrm{gph}\,\partial_x\varphi\big) = \big[(-\infty,0]\times\{0\}\times[0,\infty)\big]\cup\big[\{(0,0)\}\times(-\infty,0)\big]\cup\big[(0,\infty)\times\{(0,0)\}\big].$$

Thus we arrive by Definition 2.1 at the complete calculation

$$\partial_x^2\varphi(0,0,1)(w) = \begin{cases} \{(0,0)\} & \text{if } w>0,\\ (-\infty,\infty)\times\{0\} & \text{if } w=0,\\ (-\infty,0]\times\{0\} & \text{if } w<0, \end{cases}$$

which illustrates a typical shape of $\partial_x^2\varphi$ *for this kind of nonsmooth functions.*

Next we provide convenient representations of both second-order constructions $\partial^2_{M,x}\varphi(\bar{x},\bar{y},\bar{u})$ and $\partial^2_{N,x}\varphi(\bar{x},\bar{y},\bar{u})$ for special classes of smooth mappings.

Proposition 2.4 (partial second-order subdifferentials for smooth functions with Lipschitzian derivatives). *Let* $\varphi\colon X\times Y\to I\!R$ *be* $\mathcal{C}^1$ *around* $(\bar{x},\bar{y})$*, and let its partial derivative* $\nabla_x\varphi$ *be Lipschitz continuous around* $(\bar{x},\bar{y})$*. Then we have*

$$\partial^2_{M,x}\varphi(\bar{x},\bar{y})(w) = \partial\langle w,\nabla_x\varphi\rangle(\bar{x},\bar{y}) \ \text{ for all } \ w\in X^{**}.$$

If in addition X *is Asplund and the partial derivative* $\nabla_x\varphi$ *is* w^**-strictly Lipschitzian at* $(\bar{x},\bar{y})$ *in the sense of* [21, Definition 3.25]*, then*

$$\partial^2_{N,x}\varphi(\bar{x},\bar{y})(w) = \partial\langle w,\nabla_x\varphi\rangle(\bar{x},\bar{y}) \neq \emptyset \ \text{ for all } \ w\in X^{**}.$$

Proof. Since φ is $\mathcal{C}^1$ around $(\bar{x},\bar{y})$, there exists a neighborhood U of $(\bar{x},\bar{y})$ such that

$$\partial_x\varphi(x,y) = \nabla_x\varphi(x,y) \ \text{ for all } \ (x,y)\in U.$$

Then using [21, Theorem 1.90] on scalarizing the mixed coderivative of local Lipschitzian mappings gives us the representation

$$\big(D^*_M\nabla_x\varphi\big)(\bar{x},\bar{y})(w) = \partial\langle w,\nabla_x\varphi\rangle(\bar{x},\bar{y}) \ \text{ for all } \ w\in X^{**},$$

which justifies the formula for $\partial^2_{M,x}\varphi(\bar{x},\bar{y})$ by definition (2.11). To justify the claimed formula for $\partial^2_{N,x}\varphi(\bar{x},\bar{y})$, we employ the scalarization result from [21, Theorem 3.28], which also ensures that $\partial^2_{N,x}\varphi(\bar{x},\bar{y})(w)\neq\emptyset$ for all $w\in X^{**}$ in the Asplund space X. $\triangle$

3 Second-Order Calculus in Banach Spaces

In this section we derive calculus rules for both partial second-order subdifferentials (2.10) and (2.11) held in general Banach spaces. To derive sum and chain rules for these constructions, we proceed via Definition 2.1 applying calculus rules for the normal and mixed coderivatives to the set-valued mapping generated by the basic first-order partial subdifferential. In this way we have to restrict ourselves to favorable classes of functions for which the corresponding first-order partial subdifferential calculus rules hold as *equalities*, since neither normal nor mixed coderivative enjoys monotonicity properties that may allow us to use well-developed partial first-order subdifferential calculus rules of the inclusion type.

We begin with simple while useful second-order sum rules identical for (2.10) and (2.11). Note that considering *partial* smoothness for functions and mappings in what follows we use the standard notation $\mathcal{C}^1$ and $\mathcal{C}^2$ having in mind just the variable in question.

Proposition 3.1 (partial second-order sum rules with smooth additions). *Let $\bar{u} \in \partial_x(\varphi_1 + \varphi_2)(\bar{x}, \bar{y})$, where φ_1 is $\mathcal{C}^1$ around $(\bar{x}, \bar{y})$ with $\nabla_x\varphi_1$ strictly differentiable at $(\bar{x}, \bar{y})$ while φ_2 is finite at $(\bar{x}, \bar{y})$ with $\bar{u}_2 := \bar{u} - \nabla_x\varphi_1(\bar{x}, \bar{y}) \in \partial_x\varphi_2(\bar{x}, \bar{y})$. Then we have*

$$\partial^2_x(\varphi_1+\varphi_2)(\bar{x},\bar{y},\bar{u})(w) = \big(\nabla^2_{xx}\varphi_1(\bar{x},\bar{y})^*w, \nabla^2_{xy}\varphi_1(\bar{x},\bar{y},\bar{u})^*w\big)+\partial^2_x\varphi_2(\bar{x},\bar{y},\bar{u}_2)(w), \quad w \in X^{**},$$

for both normal ($\partial^2_x = \partial^2_{N,x}$) and mixed ($\partial^2_x = \partial^2_{M,x}$) partial second-order subdifferentials.

Proof. Let U be a neighborhood of $(\bar{x}, \bar{y})$ on which φ_1 is $\mathcal{C}^1$. Applying [21, Proposition 1.107] gives us the first-order subdifferential sum rule in the equality form

$$\partial_x(\varphi_1 + \varphi_2)(x, y) = \nabla_x\varphi_1(x, y) + \partial_x\varphi_2(x, y) \text{ for all } (x, y) \in U.$$

Since $\nabla_x\varphi_1$ is strictly differentiable at $(\bar{x}, \bar{y})$, it follows from the coderivative sum rules of [21, Therem 1.62(ii)] that

$$\big(D^*\partial_x(\varphi_1 + \varphi_2)\big)(\bar{x}, \bar{y}, \bar{u})(w) = \big(D^*\nabla_x\varphi_1\big)(\bar{x}, \bar{y})(w) + \big(D^*\partial\varphi_2\big)(\bar{x}, \bar{y}, \bar{u}_2)(w), \quad w \in X^{**},$$

for both normal ($D^* = D^*_N$) and mixed ($D^* = D^*_M$) coderivatives. The obtained representation implies the claimed sum rules for both normal and mixed partial second-order subdifferentials due to their definitions and Proposition 2.2. $\triangle$

The next theorem establishes important chain rules for the mixed and normal partial second-order subdifferentials under the *surjectivity* assumption on partial derivatives of inner mappings in compositions. On one hand, it extends to partial constructions the chain rules from [21, Theorem 1.127] obtained for the full second-order subdifferential; see also the references therein for the previous results in this direction concerning the latter construction under the surjectivity condition in finite and infinite dimensions. On the other hand, the theorem below extends to general Banach spaces the recent finite-dimensional result of [29, Theorem 3.1] for the partial second-order subdifferential of compositions under the full rank condition imposed on the partial Jacobian matrix.

In this theorem we obtain the *exact* (equality type) chain rule for the mixed construction (2.11) in *arbitrary* Banach spaces, while the corresponding result for the normal one (2.10) requires the *weak* extensibility* of the range of inner mapping derivative $\nabla_x g(\bar{x}, \bar{y})^* \subset X^*$ in the sense of [21, Definition 1.122], which holds automatically when either $\nabla_x g(\bar{x}, \bar{y})^*$ is complemented in X^* (in particular, when the kernel of $\nabla_x g(\bar{x}, \bar{y})$ is complemented in X), or the closed unit ball of X^{**} is weak* sequentially compact, which if the case if either X is reflexive or X^* is separable; see [21, Proposition 1.123].

Theorem 3.2 (partial second-order chain rules with surjective derivatives). *Let $\bar{u} \in \partial_x(\varphi \circ g)(\bar{x}, \bar{y})$ with $g\colon X \times Y \to Z$ and $\varphi : Z \to \overline{I\!R}$. Assume that g is $\mathcal{C}^1$ around $(\bar{x}, \bar{y})$*

with the surjective partial derivative $\nabla_x g(\bar{x},\bar{y}) : X \to Z$ *and that the derivative mapping* $\nabla_x g : X \times Y \to \mathcal{L}(X,Z)$ *is strictly differentiable at* $(\bar{x},\bar{y})$*. Denoting by* $\bar{p} \in Z^*$ *a unique element satisfying the relationships*

$$\bar{u} = \nabla_x g(\bar{x},\bar{y})^*\bar{p} \ \ \textit{and} \ \ \bar{p} \in \partial\varphi(\bar{z}) \ \ \textit{with} \ \ \bar{z} := g(\bar{x},\bar{y}),$$

we have the following chain rules for all $w \in X^{**}$*:*

$$\begin{aligned} \partial^2_{M,x}(\varphi\circ g)(\bar{x},\bar{y},\bar{u})(w) = \ & \big(\nabla^2_{xx}\langle\bar{p},g\rangle(\bar{x},\bar{y})^*w, \nabla^2_{xy}\langle\bar{p},g\rangle(\bar{x},\bar{y})^*w\big) \\ & +\nabla g(\bar{x},\bar{y})^*\partial^2_M\varphi(\bar{z},\bar{p})\big(\nabla_x g(\bar{x},\bar{y})^{**}w\big), \end{aligned} \tag{3.1}$$

$$\begin{aligned} \partial^2_{N,x}(\varphi\circ g)(\bar{x},\bar{y},\bar{u})(w) \subset \ & \big(\nabla^2_{xx}\langle\bar{p},g\rangle(\bar{x},\bar{y})^*w, \nabla^2_{xy}\langle\bar{p},g\rangle(\bar{x},\bar{y})^*w\big) \\ & +\nabla g(\bar{x},\bar{y})^*\partial^2_N\varphi(\bar{z},\bar{p})\big(\nabla_x g(\bar{x},\bar{y})^{**}w\big). \end{aligned} \tag{3.2}$$

Furthermore, inclusion (3.2) *holds as equality provided that the range of the operator* $\nabla_x g(\bar{x},\bar{y})^*$ *is weak** *extensible in the space* X^**.*

Proof. It follows from the first-order calculus rule in [21, Proposition 1.112] that

$$\partial_x(\varphi\circ g)(x,y) = \nabla_x g(x,y)^*\partial\varphi\big(g(x,y)\big)$$

for all (x,y) around $(\bar{x},\bar{y})$. This allows us to represent $\partial_x(\varphi\circ g)$ as the composition

$$\partial_x(\varphi\circ g)(x,y) = (f\circ G)(x,y) := f\big(x,y,G(x,y)\big) \tag{3.3}$$

with the mappings $f\colon X\times Y\times Z^* \to X^*$ and $G\colon X\times Y \rightrightarrows Z^*$ defined by

$$f(x,y,p) := \nabla_x g(x,y)^*p \ \text{ and } \ G(x,y) := \big(\partial\varphi\circ g\big)(x,y).$$

Let us check that the assumptions of [21, Lemma 1.126] are satisfied in our framework (3.3) under those imposed in the theorem. Actually the only one from [21, Lemma 1.126] that needs to be to verified is the *injectivity* of the operator $\nabla_x g(\bar{x},\bar{y})^*\colon Z^* \to X^*$, which readily follows from the assumed surjectivity of $\nabla_x g(\bar{x},\bar{y})$ due to [21, Lemma 1.18]. Employing now the coderivative chain rules from [21, Lemma 1.126] gives us

$$D^*_M(f\circ G)(\bar{x},\bar{y},\bar{u})(w) = \big(\nabla_x f(\bar{x},\bar{y},\bar{p})^*w, \nabla_y f(\bar{x},\bar{y},\bar{p})^*w\big) + D^*_M G(\bar{x},\bar{y},\bar{p})\big(f(\bar{x},\bar{y},\cdot)^*w\big),$$

$$D^*_N(f\circ G)(\bar{x},\bar{y},\bar{u})(w) \subset \big(\nabla_x f(\bar{x},\bar{y},\bar{p})^*w, \nabla_y f(\bar{x},\bar{y},\bar{p})^*w\big) + D^*_N G(\bar{x},\bar{y},\bar{p})\big(f(\bar{x},\bar{y},\cdot)^*w\big),$$

for all $w \in X^{**}$, where the second inclusion holds as equality under the additional assumption on the weak* extensibility of the range of the operator $\nabla_x g(\bar{x},\bar{y})^*$ in X^*. On the other hand, it follows from the constructions of f and G that

$$\big(\nabla_x f(\bar{x},\bar{y},\bar{p})^*w, \nabla_y f(\bar{x},\bar{y},\bar{p})^*w\big) = \big(\nabla^2_{xx}\langle\bar{p},g\rangle(\bar{x},\bar{y})^*w, \nabla^2_{xy}\langle\bar{p},g\rangle(\bar{x},\bar{y})^*w\big),$$

$$\begin{aligned} D^*G(\bar{x},\bar{y},\bar{p})\big(f(\bar{x},\bar{y},\cdot)^*w\big) =& D^*(\partial\varphi\circ g)(\bar{x},\bar{y},\bar{p})\big(\nabla_x g(\bar{x},\bar{y})^{**}w\big) \\ =& \nabla g(\bar{x},\bar{y})^*\partial^2\varphi(\bar{z},\bar{p})\big(\nabla_x g(\bar{x},\bar{y})^{**}w\big), \end{aligned}$$

where the latter equality holds due to the coderivative chain rule from [21, Theorem 1.66] valid under the assumed surjectivity of $\nabla_x g(\bar{x},\bar{y})$, which implies the surjectivity of $\nabla g(\bar{x},\bar{y})$, for both coderivatives $D^* = D^*_M, D^*_N$ and the corresponding second-order subdifferentials $\partial^2 = \partial^2_M, \partial^2_N$. Combining all the above completes the proof of the theorem. △

Now we employ Theorem 3.2 to derive new and more general chain rules for both partial second-order subdifferentials and their full counterparts under certain reduction and nondegeneracy conditions in the vein of [1]. In what follows we proceed in the framework of two-variable mappings in the definition below and derive calculus rules for partial second-order subdifferentials while the constructions and results obtained seem to be equally new even in one-variable/full subdifferential settings in finite dimensions.

Definition 3.3 ($\mathcal{C}^2$-reduction condition). *Let $\varphi\colon Z \to \overline{I\!R}$ be an extended-real-valued function finite at $\bar{z}$. We say that φ is $\mathcal{C}^2$-REDUCIBLE at $\bar{z}$ to an extended-real-valued function $\psi\colon W \to \overline{I\!R}$ if there exists a $\mathcal{C}^2$ mapping $h\colon Z \to W$ with the surjective derivative $\nabla h(\bar{z})$ such that $\varphi(z) = (\psi \circ h)(z)$ for all $z \in Z$ around $\bar{z}$.*

When $\varphi = \delta_\Theta$ and $\psi = \delta_\Xi$ are the indicator functions of convex sets, the reduction condition of Definition 3.3 amounts to saying that the set Θ is $\mathcal{C}^2$-reducible to the set Ξ at $\bar{z} \in \Theta$ in the sense of [1, Definition 3.135].

The next definition is a partial version of the nondegeneracy condition for mappings from [1, Definition 4.70] with respect to sets formulated now via another smooth mapping.

Definition 3.4 (nondegeneracy condition). *Let $g\colon X\times Y \to Z$ be Fréchet differentiable in x at $(\bar{x},\bar{y})$, and let $h\colon Z \to W$ be Fréchet differentiable at $\bar{z} := g(\bar{x},\bar{y})$. We say that $(\bar{x},\bar{y})$ is a PARTIAL NONDEGENERATE POINT of g in x relative to h if*

$$\nabla_x g(\bar{x},\bar{y})X + \ker \nabla h(\bar{z}) = Z. \tag{3.4}$$

Before deriving the main result of this section, we present the following special second-order chain rule involving smooth mappings, which is used in the proof of the main result.

Lemma 3.5 (special second-order chain rule for compositions of smooth mappings). *Let $g : X \to Y$ and $h : Y \to Z$ be $\mathcal{C}^2$ mappings, and let $\bar{v} \in Z^*$, $\bar{x} \in X$, and $\bar{y} := g(\bar{x})$. Denoting $\bar{p} := \nabla h(\bar{y})^*\bar{v}$, we have*

$$\nabla^2\langle \bar{v}, h \circ g\rangle(\bar{x})^*u = \nabla^2\langle \bar{p}, g\rangle(\bar{x})^*u + \nabla g(\bar{x})^*\nabla^2\langle \bar{v}, h\rangle(\bar{y})^*\nabla g(\bar{x})^{**}u, \quad u \in X^{**}. \tag{3.5}$$

Proof. Define $\psi(y) := \langle \bar{v}, h(y)\rangle$ and observe that $\nabla\psi(\bar{y}) = \nabla h(\bar{y})^*\bar{v} = \bar{p}$ and that

$$(\psi \circ g)(x) = \psi\big(g(x)\big) = \big\langle \bar{v}, h\big(g(x)\big)\big\rangle = \langle \bar{v}, h \circ g\rangle(x).$$

Then it follows from [21, Theorem 1.128] that

$$\partial^2(\psi \circ g)(\bar{x})(u) = \nabla^2\langle \bar{p}, g\rangle(\bar{x})^*u + \nabla g(\bar{x})^*\nabla^2\psi(\bar{y})^*\nabla g(\bar{x})^{**}u, \quad u \in X^{**},$$

which readily implies (3.5) and completes the proof. △

Now we are ready to establish the second-order chain rules for (2.11) and (2.10) under the coordinated combination of the reduction and nondegeneracy conditions.

Theorem 3.6 (second-order chain rules under reduction and nondegeneracy conditions). *Consider the composition $\varphi\circ g$ of an extended-real-value function $\varphi\colon Z\to\overline{I\!R}$ and a locally $\mathcal{C}^2$ mapping $g\colon X\times Y\to Z$, and let $\bar u\in\partial_x(\varphi\circ g)(\bar x,\bar y)$. Assume that φ satisfies the $\mathcal{C}^2$-reduction condition at $\bar z:=g(\bar x,\bar y)$ with some $\psi\colon W\to\overline{I\!R}$ and $h\colon Z\to W$ from Definition* 3.3 *and that $(\bar x,\bar y)$ is a partially nondegenerate point of g in x relative to h. Suppose furthermore that $\bar u$ admits the first-order subdifferential representation:*

$$\bar u=\nabla_x g(\bar x,\bar y)^*\bar p\ \text{ with some }\ \bar p\in\partial\varphi(\bar z). \tag{3.6}$$

Then for all $w\in X^{**}$ we have the partial second-order chain rule (3.1). *We also have the partial second-order chain rule* (3.2) *if the closed unit ball of Z^{**} is weak* sequentially compact, which holds in particular when either Z is reflexive or Z^* is separable. The partial second-order chain rule* (3.2) *is fulfilled as equality if we assume additionally that the closed unit ball of X^{**} is weak* sequentially compact.*

Proof. It follows from the classical chain rule that

$$\nabla_x(h\circ g)(\bar x,\bar y)X=\nabla h(\bar z)\big[\nabla_x g(\bar x,\bar y)X\big]=\nabla h(\bar z)\big[\nabla_x g(\bar x,\bar y)X+\ker\nabla h(\bar z)\big]=\nabla h(\bar z)(Z)=W,$$

i.e., the simultaneous validity of the nondegeneracy condition (3.4) and the surjectivity of $\nabla h(\bar z)$ implies the surjectivity of $\nabla_x(h\circ g)(\bar x,\bar y)$. Denote $f:=h\circ g$ and by the $\mathcal{C}^2$-reduction condition of Definition 3.3 find a neighborhood U of $(\bar x,\bar y)$ such that

$$(\psi\circ f)(x,y)=(\varphi\circ g)(x,y)\ \text{ for all }\ (x,y)\in U. \tag{3.7}$$

Since $\bar u\in\partial_x(\varphi\circ g)(\bar x,\bar y)=\partial_x(\psi\circ f)(\bar x,\bar y)$ and $\nabla_x f(\bar x,\bar y)$ is surjective, we find, by applying the first-order subdifferential chain rule from [21, Proposition 1.112] to the composition $\psi\circ f$, a unique (by [21, Lemma 1.18]) subgradient $\bar v\in\partial\psi(\bar z)$ satisfying the relationships

$$\bar u=\nabla_x f(\bar x,\bar y)^*\bar v=\nabla_x g(\bar x,\bar y)^*\nabla h(\bar z)^*\bar v, \tag{3.8}$$

where the latter is due to the classical chain rule. Apply further the second-order subdifferential chain rules from Theorem 3.2 to the composition $\psi\circ f$ with the surjective partial derivative $\nabla_x f(\bar x,\bar y)$. For brevity we consider here and in the rest of the proof of this theorem only the case of the mixed second-order subdifferential in (3.1) by taking into account that the other case (with the equality therein) is completely similar. Thus we get

$$\begin{aligned}\partial^2_{M,x}(\psi\circ f)(\bar x,\bar y,\bar u)(w)=\ &\big(\nabla^2_{xx}\langle\bar v,f\rangle(\bar x,\bar y)^*w,\nabla^2_{xy}\langle\bar v,f\rangle(\bar x,\bar y)^*w\big)\\ &+\nabla f(\bar x,\bar y)^*\partial^2_M\psi\big(h(\bar z),\bar v\big)\big(\nabla_x f(\bar x,\bar y)^{**}w\big)\end{aligned} \tag{3.9}$$

for all $w\in X^{**}$. Having $\bar p\in\partial\varphi(\bar z)=\partial(\psi\circ h)(\bar z)$ from (3.6) and using the assumed surjectivity of $\nabla h(\bar z)$, we find by [21, Proposition 1.112] a unique element $\bar q\in\partial\psi(h(\bar z))$ such that $\nabla h(\bar z)^*\bar q=\bar p$. Applying the operator $\nabla_x g(\bar x,\bar y)^*$ to both parts of this equality and employing the chain rule from (3.8) shows that

$$\nabla_x f(\bar x,\bar y)^*\bar q=\nabla_x g(\bar x,\bar y)^*\bar p=\bar u$$

by (3.6). Combining it with the first equality in (3.8) yields $\nabla_x f(\bar x,\bar y)^*\bar q=\nabla_x f(\bar x,\bar y)^*\bar v$, which ensures that $\bar q=\bar v$ due to the surjectivity of $\nabla_x f(\bar x,\bar y)$ in the reduction condition.

Now let us apply the full subdifferential version of Theorem 3.2 to the composition $\psi \circ h$ with taking into account that $\bar{q} = \bar{v}$. This gives us the representation

$$\partial^2_M(\psi \circ h)(\bar{z}, \bar{p})(s) = \nabla^2\langle \bar{v}, h\rangle(\bar{z})^* s + \nabla h(\bar{z})^* \partial^2_M \psi\big(h(\bar{z}), \bar{v}\big)\big(\nabla h(\bar{z})^{**} s\big) \tag{3.10}$$

for all $s \in Z^*$. Substituting $s = \nabla_x g(\bar{x}, \bar{y})^{**} w$ into (3.10) with an arbitrary element $w \in X^{**}$ and then employing the operator $\nabla g(\bar{x}, \bar{y})^*$ in the both sides of (3.10), we arrive by (3.8) at

$$\begin{aligned}\nabla g(\bar{x}, \bar{y})^* \partial^2_M \varphi(\bar{z}, \bar{p})\big(\nabla_x g(\bar{x}, \bar{y})^{**} w\big) = \; & \nabla g(\bar{x}, \bar{y})^* \nabla^2 \langle \bar{v}, h\rangle(\bar{z})^* \nabla_x g(\bar{x}, \bar{y})^{**} w \\ & + \nabla f(\bar{x}, \bar{y})^* \partial^2_M \psi\big(h(\bar{z}), \bar{v}\big)\big(\nabla_x f(\bar{x}, \bar{y})^{**} w\big)\end{aligned}$$

for all $w \in X^{**}$. Combining the latter with (3.7) and (3.9) shows that

$$\begin{aligned}&\partial^2_{M,x}(\varphi \circ g)(\bar{x}, \bar{y}, \bar{u})(w) = \big(\nabla^2_{xx}\langle \bar{v}, f\rangle(\bar{x}, \bar{y})^* w, \nabla^2_{xy}\langle \bar{v}, f\rangle(\bar{x}, \bar{y})^* w\big) \\ &+ \nabla g(\bar{x}, \bar{y})^* \partial^2_M \varphi(\bar{z}, \bar{p})\big(\nabla_x g(\bar{x}, \bar{y})^{**} w\big) - \nabla g(\bar{x}, \bar{y})^* \nabla^2\langle \bar{v}, h\rangle(\bar{z})^* \nabla_x g(\bar{x}, \bar{y})^{**} w\end{aligned} \tag{3.11}$$

whenever $w \in X^{**}$. Furthermore, the first term on the right-hand side of (3.11) with $f = h \circ g$ admits the representation

$$\begin{aligned}&\big(\nabla^2_{xx}\langle \bar{v}, h \circ g\rangle(\bar{x}, \bar{y})^* w, \nabla^2_{xy}\langle \bar{v}, h \circ g\rangle(\bar{x}, \bar{y})^* w\big) \\ &= \big(\nabla^2_{xx}\langle \bar{p}, g\rangle(\bar{x}, \bar{y})^* w, \nabla^2_{xy}\langle \bar{p}, g\rangle(\bar{x}, \bar{y})^* w\big) + \nabla g(\bar{x}, \bar{y}) \nabla^2\langle \bar{v}, h\rangle(\bar{z})^* \nabla_x g(\bar{x}, \bar{y})^{**} w,\end{aligned} \tag{3.12}$$

which is is a partial version of Lemma 3.5; see also Theorem 3.8(ii). Substituting (3.12) into (3.11), we arrive at the chain rule (3.1) for the mixed partial second-order subdifferential under the assumptions made. To complete the proof of the theorem in the normal second-order subdifferential case, it remains to observe that the weak* extensibility of the ranges of both $\nabla f(\bar{x})^*$ and $\nabla h(\bar{z})^*$ holds under the assumption that the closed units balls of X^{**} and Z^{**} are weak* sequentially compact. $\triangle$

Note that Theorem 3.6 reduces to Theorem 3.2 if the operator $\nabla_x g(\bar{x}, \bar{y})$ is surjective. In Section 4 we develop further elaborations of the second-order chain rules from Theorem 3.6 in the case of *Asplund* spaces, giving verifiable conditions for the validity of the first-order subdifferential representation (3.6) and also for the surjectivity of $\nabla_x(h \circ g)(\bar{x}, \bar{y})$ in the relative reduction condition of Definition 3.3. The latter relates to the notion of *nondegeneracy* explored in [1, 33] in finite dimensions.

Finally in this section, we derive yet another chain rules for both normal and mixed partial second-order subdifferentials in general Banach spaces for compositions $\varphi \circ g$, where the outer functions φ but not the inner mappings g are twice differentiable. To proceed, we first define the following notions of normal and mixed *partial second-order coderivatives*, which are partial counterparts of those given in [21, (1.69)].

Definition 3.7 (partial second-order coderivatives). *Given a Lipschitzian mapping $g : X \times Y \to Z$ between Banach spaces, the* NORMAL *and* MIXED PARTIAL SECOND-ORDER CODERIVATIVES *of g in x at $(\bar{x}, \bar{y}, \bar{p}, \bar{u}) \in X \times Y \times Z^* \times X^*$ with $\bar{u} \in \partial_x\langle \bar{p}, g\rangle(\bar{x}, \bar{y})$ are*

$$D^2_x g(\bar{x}, \bar{y}, \bar{p}, \bar{u})(w) := \big(D^* \partial_x\langle \cdot, g\rangle\big)(\bar{x}, \bar{y}, \bar{p}, \bar{u})(w), \quad w \in X^{**},$$

where D^* *stands for the normal* ($D^* = D^*_N$) *and mixed* ($D^* = D^*_M$) *coderivatives of the mapping* $(x,y,p) \rightrightarrows \partial_x\langle p,g\rangle(x,y)$*, respectively. If* g *is strictly differentiable at* $(\bar x,\bar y)$*, then*

$$\partial_x\langle\bar p,g\rangle(\bar x,\bar y) = \nabla_x g(\bar x,\bar y)^*\bar p$$

and we omit $\bar u$ *in the arguments for* $D^2_x g$*.*

Now we are ready to establish the aforementioned second-order chain rules of the *equality type* one of which has been already used in the proof of Theorem 3.6.

Theorem 3.8 (partial second-order chain rules with twice differentiable outer mappings). *Let* g *be strictly differentiable at* $(\bar x,\bar y)$*, let* $\varphi \in \mathcal{C}^1$ *around* $\bar z := g(\bar x,\bar y)$ *with* $\nabla\varphi$ *strictly differentiable at this point, and let* $\bar p := \nabla\varphi(\bar z)$*. The following assertions hold.*

(i) *Assume that the operator* $\nabla^2\varphi(\bar z)\nabla g(\bar x,\bar y)\colon X\times Y \to Z^*$ *is surjective. Then*

$$\partial^2_x(\varphi\circ g)(\bar x,\bar y)(w) = \bigcup_{(x^*,y^*,q)\in D^2_x g(\bar x,\bar y,\bar p)(w)} \big[(x^*,y^*) + \nabla g(\bar x,\bar y)^*\nabla^2\varphi(\bar z)^* q\big] \tag{3.13}$$

for all $w \in X^{**}$*, where* ∂^2_x *and* D^2_x *stand for the corresponding normal and mixed partial second-order constructions.*

(ii) *Assume that the partial derivative operator* $\nabla_x g$ *is strictly differentiable at* $(\bar x,\bar y)$*. Then the chain rules in* (3.13) *are identical for* $\partial^2_{N,x}(\varphi\circ g)(\bar x,\bar y)$ *and* $\partial^2_{M,x}(\varphi\circ g)(\bar x,\bar y)$ *with*

$$D^2_x g(\bar x,\bar y,\bar p)(w) = \big(\nabla^2_{xx}\langle\bar p,g\rangle(\bar x,\bar y)^* w, \nabla^2_{xy}\langle\bar p,g\rangle(\bar x,\bar y)^* w, \nabla_x g(\bar x,\bar y)^{**} w\big), \quad w\in X^{**},$$

for both partial second-order coderivatives $D^2_x = D^2_{N,x}$ *and* $D^2_x = D^2_{M,x}$*.*

Proof. Since φ is $\mathcal{C}^1$ around $\bar z$ and g is Lipschitz continuous around $(\bar x,\bar y)$ due to its strict differentiability at this point, it implies by [21, Theorem 1.110] that there is a neighborhood U of $(\bar x,\bar y)$ such that for all $(x,y)\in U$ we have

$$\partial_x(\varphi\circ g)(x,y) = \partial_x\big\langle\nabla\varphi\big(g(x,y)\big),g\big\rangle(x,y) =: (F\circ h)(x,y)$$

for $F\colon X\times Y\times Z^* \rightrightarrows X^*$ and $h\colon X\times Y \to X\times Y\times Z^*$ defined by

$$F(x,y,p) := \partial_x\langle p,g\rangle(x,y), \;\text{ and }\; h(x,y) := \Big(x,y,\nabla\varphi\big(g(x,y)\big)\Big).$$

To proceed further with the proof of (i), observe that the surjectivity of $\nabla^2\varphi(\bar z)\nabla g(\bar x,\bar y)$ implies that of $\nabla h(\bar x,\bar y)$. Thus it follows from [21, Theorem 1.66] that

$$D^*(F\circ h)(\bar x,\bar y,\bar p)(w) = \nabla h(\bar x,\bar y)^* D^*F(\bar x,\bar y,\bar p,\bar u)(w), \quad w\in X^{**},$$

for both normal and mixed coderivatives with $\bar u \in \partial_x\langle\bar p,g\rangle(\bar x,\bar y) = \{\nabla_x g(\bar x,\bar y)^*\bar p\}$. On the other hand, the construction of F and Definition 3.7 give us

$$D^*F(\bar x,\bar y,\bar p)(w) = D^*\partial_x\langle\cdot,g\rangle(\bar x,\bar y,\bar p)(w) = D^2_x g(\bar x,\bar y,\bar p)(w),$$

which ensures by the construction of h that

$$D^*(F\circ h)(\bar x,\bar y,\bar p)(w) = \bigcup_{(x^*,y^*,q)\in D^2_x g(\bar x,\bar y,\bar p)(w)} \big[(x^*,y^*) + \nabla g(\bar x,\bar y)^*\nabla^2\varphi(\bar z)^* q\big]$$

for all $w\in X^{**}$ and thus justifies (i). Assertion (ii) is proved in the same way by employing the chain rule from [21, Theorem 1.65(iii)] under the assumed strict differentiability of $\nabla_x g$ instead of using [21, Theorem 1.66] in the case of surjectivity in (i). △

4 Second-Order Calculus in Asplund Spaces

In this section we continue developing the second-order partial subdifferential calculus started in the preceding section in the framework of general Banach spaces. Here we follow the same scheme in *Asplund* space settings that leads us to second-order partial subdifferential sum and chain rules by using coderivative calculus applied to *equality-type* sum and chain rule for first-order partial subgradient mappings. The results obtained below are generally independent from those derived in Section 3 while being addressed to broader classes of sums and compositions. Except the second-order chain rules of Theorems 4.2 and 4.3 derived by the reduction approach, which are new even in finite dimensions for both full and partial second-order constructions (see the discussion below), the other calculus rules obtained in this section are appropriate partial extensions of those results established in [21, Subsection 3.2.5] and the references therein for full second-order subdifferential mappings. Note that in some results below we use the notion of *lower regularity* of extended-real-valued functions from [21, Definition 1.91] and its obvious partial subdifferential counterpart.

Recall [21, Definition 1.63] that a mapping $S\colon X \rightrightarrows Z$ is *inner semicontinuous* at $(\bar{x},\bar{z}) \in \operatorname{gph} S$ if for every sequence $x_k \to \bar{x}$ with $S(x_k) \neq \emptyset$ there is a sequence $z_k \in S(x_k)$ converging to $\bar{z}$. The mapping S is *inner semicompact* at $\bar{x}$ if for every sequence $x_k \to \bar{x}$ with $S(x_k) \neq \emptyset$ there is a sequence $z_k \in S(x_k)$, which contains a convergent subsequence.

The latter property is clearly less restrictive than the former and holds, in particular, for *locally bounded* mappings with values in finite-dimensional spaces. On the other hand, the results obtained in [21, Subsection 3.2.5] for full second-order subdifferentials are completely parallel under the inner semicontinuity and inner semicompactness assumptions on the corresponding mappings while those under inner semicontinuity admit simpler formulations. We have the same situation for partial second-order subdifferentials.

Let us begin with *sum rules* for partial second-order subdifferentials restricting ourselves, for brevity and simplicity, just to the case of inner semicontinuity.

Theorem 4.1 (sum rules for partial second-order subdifferentials). *Consider the functions $\varphi_1,\varphi_2\colon X\times Y \to \overline{I\!R}$ with $\bar{u} \in \partial_x(\varphi_1+\varphi_2)(\bar{x},\bar{y})$, where the spaces X, Y, X^*, Y^* are Asplund. The following assertions hold for both normal ($\partial^2_x = \partial^2_{N,x}$) and mixed ($\partial^2_x = \partial^2_{M,x}$) partial second-order subdifferentials.*

(i) *Let $\varphi_1 \in \mathcal{C}^1$ around $(\bar{x},\bar{y})$ with $\bar{u}_1 := \nabla_x\varphi_1(\bar{x},\bar{y})$, and let the graph of $\partial_x\varphi_2$ be norm-closed around $(\bar{x},\bar{y},\bar{u}_2)$ with $\bar{u}_2 := \bar{u}-\bar{u}_1$. Assume that either $\nabla_x\varphi_1$ is locally Lipschitzian around $(\bar{x},\bar{y})$, or $\partial_x\varphi_2$ is PSNC at $(\bar{x},\bar{y},\bar{u}_2)$ and*

$$\partial^2_{M,x}\varphi_1(\bar{x},\bar{y},\bar{u}_1)(0) \cap \big(-\partial^2_{M,x}\varphi_2(\bar{x},\bar{y},\bar{u}_2)(0)\big) = \{0\}. \tag{4.1}$$

Then for all $w \in X^{**}$ we have the inclusion

$$\partial^2_x(\varphi_1+\varphi_2)(\bar{x},\bar{y},\bar{u})(w) \subset \partial^2_x\varphi_1(\bar{x},\bar{y},\bar{u}_1)(w) + \partial^2_x\varphi_2(\bar{x},\bar{y},\bar{u}_2)(w). \tag{4.2}$$

(ii) *Let both φ_i be l.s.c. around $(\bar{x},\bar{y})$, and let $S\colon X\times Y\times X^* \rightrightarrows X^*\times X^*$ defined by*

$$S(x,y,u) := \big\{(u_1,u_2) \in X^*\times X^* \big|\ u_i \in \partial_x\varphi_i(x,y),\ i=1,2,\ \text{ and }\ u_1+u_2=u\big\}$$

be inner semicontinuous at $(\bar{x}, \bar{y}, \bar{u}, \bar{u}_1, \bar{u}_2)$ *for a given* $(\bar{u}_1, \bar{u}_2) \in S(\bar{x}, \bar{y}, \bar{u})$*. Assume that the graph of each* $\partial_x \varphi_i$ *is norm-closed around* $(\bar{x}, \bar{y}, \bar{u}_i)$*, that one of* $\partial_x \varphi_i$ *is PSNC at the corresponding* $(\bar{x}, \bar{y}, \bar{u}_i)$*, and that the qualification condition* (4.1) *is satisfied. Suppose also that there is a neighborhood* U *of* $(\bar{x}, \bar{y})$ *such that*

$$\partial_x^\infty \varphi_1(x, y) \cap \big(-\partial_x^\infty \varphi_2(x, y) \big) = \{0\}$$

for all $(x, y) \in U$*, that one of* φ_i *is SNEC at every* $(x, y) \in U$ *(both assumptions are fulfilled when one of* φ_i *is Lipschitz continuous around* $(\bar{x}, \bar{y})$*), and that each* φ_i *are partially lower regular in* x *at every* $(x, y) \in U$*. Then the sum rule* (4.2) *holds for all* $w \in X^{**}$*.*

Proof. Since φ_1 is $\mathcal{C}^1$ around $(\bar{x}, \bar{y})$, we get from [21, Proposition 1.107] that there is a neighborhood U of $(\bar{x}, \bar{y})$ such that

$$\partial_x(\varphi_1 + \varphi_2)(\bar{x}, \bar{y}, \bar{u})(w) = \nabla_x \varphi_1(x, y) + \partial_x \varphi_2(x, y) \ \text{ for all } \ (x, y) \in U.$$

Apply then to this equality the coderivative sum rule in Asplund spaces from [21, Theorem 3.10(i)] with $F_1 := \nabla_x \varphi_1$ and $F_2 := \partial_x \varphi_2$. It yields

$$\big(D^* \partial_x(\varphi_1 + \varphi_2)\big)(\bar{x}, \bar{y}, \bar{u})(w) \subset \big(D^* \nabla_x \varphi_1\big)(\bar{x}, \bar{y}, \bar{u}_1)(w) + \big(D^* \partial_x \varphi_2\big)(\bar{x}, \bar{y}, \bar{u}_2)(w)$$

for all $w \in X^{**}$ and thus justifies assertion (i). We prove (ii) in the same way by applying [21, Theorem 3.10(i)] to the first-order subdifferential equality

$$\partial_x(\varphi_1 + \varphi_2)(x, y) = \partial_x \varphi_1(x, y) + \partial_x \varphi_2(x, y), \quad (x, y) \in U,$$

which follows from [21, Theorem 3.36] under the assumptions imposed in (ii). △

Next we proceed with deriving various *chain rules* for partial second-order subdifferentials (2.10) and (2.11) and consider first the following Asplund space version of Theorem 3.6 that provides verifiable conditions for the validity the first-order subdifferential representation (3.6) and thus leads us to advanced second-order results via the reduction approach.

Theorem 4.2 (refined second-order chain rules under reduction and nondegeneracy conditions). *Let all the assumptions of Theorem* 3.6 *be satisfied, except that on the first-order subdifferential representation* (3.6)*. Suppose in addition that the spaces* X, Y*, and* Z *are Asplund, that the qualification condition*

$$\ker \nabla_x g(\bar{x}, \bar{y})^* \cap \partial^\infty \varphi(\bar{z}) = \{0\} \tag{4.3}$$

is fulfilled, and that either φ *is SNEC at* $\bar{z}$ *or* $g^{-1}(\cdot, \bar{y})$ *is PSNC at* $(\bar{z}, \bar{x})$*. Given now* $\bar{u} \in \partial_x(\varphi \circ g)(\bar{x}, \bar{y})$*, then the conclusions of Theorem* 3.6 *hold with* $\bar{p}$ *from* (3.6)*. The equality holds in* (3.2) *if the closed unit ball of* X^{**} *is weak* sequentially compact; in particular when either* X *is reflexive or* X^* *is separable.*

Proof. To justify the conclusions of the theorem, it is sufficient to show that the additional assumptions ensure the fulfillment of the first-order representation (3.6) in Theorem 3.6. But this follows from the partial counterpart of the first-order chain rule in [21, Theorem 3.41(i)] applied to $\bar{u} \in \partial_x(\varphi \circ g)(\bar{x}, \bar{y})$, which holds under the assumptions made. $\triangle$

As mentioned above, that the SNEC property of φ and the qualification condition (4.3) are automatic if φ is locally Lipschitzian around $\bar{z}$. Likewise, (4.3) and the PSNC property of $g^{-1}(\cdot, \bar{y})$ in Theorem 4.2 hold if the mapping $\nabla_x g(\bar{x}, \bar{y})$ is surjective (see [21, Corollary 1.69]) while the latter case has been covered by Theorem 3.2 in more generality.

Observe further that in the case of the set indicator functions $\varphi(z) := \delta_\Theta(z)$ and $\psi(w) := \delta_\Xi(w)$ for $\Theta \subset Z$ and $\Xi \subset W$ the reduction representation $\varphi = \psi \circ h$ in Definition 3.3 reads as $\Theta \cap O = h^{-1}(\Xi) \cap O$ for some neighborhood O of $\bar{z} \in \Theta$ with $h(\bar{z}) \in \Xi$. The qualification condition (4.3) reduces in this case to

$$\ker \nabla_x g(\bar{x}, \bar{y})^* \cap N(\bar{z}; \Theta) = \{0\}. \tag{4.4}$$

The chain rules obtained in Theorems 3.6 and 4.2 readily apply to the reduction setting for sets, where the second-order subdifferential of the indicator functions are the corresponding coderivatives of the normal cone mappings under the assumptions imposed in these theorems. However, the next theorem provides more information in this direction: it describes a rather broad setting when the qualification condition (4.4) and hence the first-order subdifferential representation (3.6) follow *directly from nondegeneracy* (3.4). For the notational simplicity we formulate this theorem just in the case of $g = g(x)$ in the above composition and denote by $\mathcal{N}_\Omega(x) := N(x; \Omega)$ the normal cone mapping (2.2).

Theorem 4.3 (second-order rules for inverse images under nondegeneracy). *Let $\Theta \subset Z$ be a closed set that is normally regular at $\bar{z} \in \Theta$ and $\mathcal{C}^2$-reducible at this point to some closed set $\Xi \subset W$ by a mapping $h\colon Z \to W$ with $h(\bar{z}) := \bar{\vartheta} \in \Xi$, and let $g\colon X \to Z$ be a $\mathcal{C}^2$ mapping with $g(\bar{x}) = \bar{z}$. Assume that $\bar{x}$ is a nondegenerate point of g relative to h, that the spaces X and Z are Asplund, and that either Θ is SNC at $\bar{z}$ or g^{-1} is PSNC at $(\bar{z}, \bar{x})$. Then given a normal $\bar{u} \in N(\bar{x}; g^{-1}(\Theta))$, we have for all $w \in X^{**}$ the relationship*

$$D_M^* \mathcal{N}_{g^{-1}(\Theta)}(\bar{x}, \bar{u})(w) = \nabla^2 \langle \bar{p}, g \rangle (\bar{x})^* w + \nabla g(\bar{x})^* D_M^* \mathcal{N}_\Theta(\bar{z}, \bar{p})\big(\nabla g(\bar{x})^{**} w\big), \tag{4.5}$$

where $\bar{p} \in N(\bar{z}; \Theta)$ is any element satisfying the equation $\bar{u} = \nabla g(\bar{x})^* \bar{p}$. We also have the inclusion

$$D_N^* \mathcal{N}_{g^{-1}(\Theta)}(\bar{x}, \bar{u})(w) \subset \nabla^2 \langle \bar{p}, g \rangle (\bar{x})^* w + \nabla g(\bar{x})^* D_N^* \mathcal{N}_\Theta(\bar{z}, \bar{p})\big(\nabla g(\bar{x})^{**} w\big), \tag{4.6}$$

where $\bar{p} \in N(\bar{z}; \Theta)$ is any element satisfying the equation $\bar{u} = \nabla g(\bar{x})^* \bar{p}$ if the closed unit ball of Z^{**} is weak sequentially compact. The inclusion becomes equality if we assume additionally that the closed unit ball of X^{**} is weak* sequentially compact.*

Proof. To derive the results of this theorem from those in Theorem 4.2, it remains to show that the normal regularity of the set Θ in the structure under consideration implies, together with the other assumptions of the theorem, the validity of the qualification condition

$$\ker \nabla g(\bar{x})^* \cap N(\bar{z}; \Theta) = \{0\}, \tag{4.7}$$

which obviously yields the existence of $\bar{p} \in N(\bar{z};\Theta)$ satisfying $\bar{u} = \nabla g(\bar{x})^*\bar{p}$. To proceed, we deduce from [23, Proposition 4.3] that

$$\ker \nabla h(\bar{z}) = T\big(\bar{z}; h^{-1}(\bar{\vartheta})\big) \subset T(\bar{z};\Theta), \tag{4.8}$$

for the (Bouligand-Severi) contingent cone T, where the inclusion holds due to $h^{-1}(\bar{\vartheta}) \cap O \subset \Theta$. Take now any $z^* \in \ker \nabla g(\bar{x})^* \cap N(\bar{z};\Theta)$ and fix $z \in Z$. By the nondegeneracy condition (3.4) we find $v \in X$ and $y \in \ker \nabla h(\bar{z})$ such that $z = \nabla g(\bar{x})v + y$. Therefore

$$\langle z^*, z\rangle = \langle z^*, \nabla g(\bar{x})v + y\rangle = \langle z^*, \nabla g(\bar{x})v\rangle + \langle z^*, y\rangle.$$

Since $z^* \in N(\bar{z};\Theta)$ and $y \in \ker \nabla h(\bar{z}) \subset T(\bar{z};\Theta)$ by (4.8), it follows from the assumed normal regularity of Θ at $\bar{z}$ and the duality between $T(\bar{z};\Theta)$ and $N(\bar{z};\Theta)$ in this case that $\langle z^*, y\rangle \le 0$. By $z^* \in \ker \nabla g(\bar{x})^*$, we get furthermore that

$$\langle z^*, \nabla g(\bar{x})v\rangle = \langle \nabla g(\bar{x})^* z^*, v\rangle = 0,$$

and thus $\langle z^*, z\rangle \le 0$. Since $z \in Z$ was chosen arbitrarily, it gives us that $z^* = 0$, which justifies (4.7) and completes the proof of the theorem. $\triangle$

Note that if the qualification condition (4.9) is replaced by

$$\ker \nabla_x g(\bar{x},\bar{y})^* \cap \operatorname{span} N(\bar{z};\Theta) = \{0\}, \tag{4.9}$$

the element $\bar{p}$ in Theorem 4.3 is *unique*. Moreover, when all the spaces in questions are finite-dimensional, Theorem 4.3 extends the result of [33, Theorem 7], where both sets Θ and Ξ are assumed to be convex and the tangent cone $T(h(\bar{z});\Xi)$ is assumed to be pointed.

Our next step is to establish chain rules for partial second-order subdifferentials of compositions $\varphi \circ g = \varphi(g(x,y))$ without imposing the surjectivity or nondegeneracy assumptions on inner mappings g as before. The results obtained below concern lower regular outer functions φ and hold as *inclusions* even in finite dimensions. For brevity we present the results only under the inner semicompactness assumption on the corresponding mapping.

Theorem 4.4 (second-order chain rules without nondegeneracy). *Consider the composition $\varphi \circ g$, where $g\colon X \times Y \to Z$ is $\mathcal{C}^1$ around some point $(\bar{x},\bar{y})$ with the partial derivative $\nabla_x g$ strictly differentiable at this point, and where $\varphi\colon Z \to \overline{I\!R}$ is l.s.c. and lower regular around $\bar{z} := g(\bar{x},\bar{y})$. Suppose that the inverse mapping g^{-1} is PSNC at $(\bar{z},\bar{x},\bar{y})$, that φ is SNEC around $\bar{z}$, that the first-order qualification condition*

$$\partial^\infty \varphi\big(g(x,y)\big) \cap \ker \nabla_x g(x,y)^* = \{0\} \tag{4.10}$$

is satisfied around $(\bar{x},\bar{y})$, and that the spaces X, Y, Z, and Z^* are Asplund. Define the set-valued mapping $S : X \times Y \times X^* \rightrightarrows Z^*$ by

$$S(x,y,u) := \big\{p \in Z^* \big|\, p \in \partial\varphi\big(g(x,y)\big),\ \nabla_x g(x,y)^* p = u\big\} \tag{4.11}$$

and, given $\bar{u} \in \partial_x(\varphi \circ g)(\bar{x}, \bar{y})$, assume that it is inner semicontinuous at $(\bar{x}, \bar{y}, \bar{u})$, that the graph of the subdifferential mapping $\partial\varphi$ is norm-closed in $Z \times Z^*$ for all z is near $\bar{z}$, and that the following mixed second-order qualification condition

$$\partial^2_M \varphi(\bar{z}, \bar{p})(0) \cap \ker \nabla g(\bar{x}, \bar{y})^* = \{0\} \ \textit{ for all } \ \bar{p} \in S(\bar{x}, \bar{y}, \bar{u}) \tag{4.12}$$

holds. Then for all $w \in X^{**}$ we have the inclusion

$$\begin{aligned}\partial^2_x(\varphi \circ g)(\bar{x}, \bar{y}, \bar{u})(w) \subset \bigcup_{\bar{p} \in S(\bar{x}, \bar{y}, \bar{u})} & \Big\{ \big(\nabla^2_{xx} \langle \bar{p}, g \rangle (\bar{x}, \bar{y})^* w, \nabla^2_{xy} \langle \bar{p}, g \rangle (\bar{x}, \bar{y})^* w \big) \\ & + \nabla g(\bar{x}, \bar{y})^* \partial^2_N \varphi(\bar{z}, \bar{p}) \big(\nabla g(\bar{x})^{**} w \big) \Big\}\end{aligned} \tag{4.13}$$

for both normal ($\partial^2_x = \partial^2_{N,x}$) and mixed ($\partial^2_x = \partial^2_{M,x}$) partial second-order subdifferentials.

Proof. Observe first that it suffices to prove (4.13) for the normal second-order subdifferential of $\varphi \circ g$, since the other one is smaller. To proceed, let us begin with using the *exact* first-order chain rule from [21, Theorem 3.41(iii)]. Since g is $\mathcal{C}^1$ around $(\bar{x}, \bar{y})$, we have

$$D^*_{N,x} g^{-1}(z, x, y)(0) = \ker \nabla_x g(x, y)^*$$

for all (x, y) from some neighborhood U of $(\bar{x}, \bar{y})$ and $z := g(x, y)$. Thus the assumed first-order qualification condition (4.10) is equivalent to the one needed for the partial version of [21, Theorem 3.41(iii)], which implies therefore (under the lower regularity and SNC assumptions imposed here) that

$$\partial_x(\varphi \circ g)(x, y) = \bigcup_{p \in \partial\varphi(g(x,y))} \nabla_x g(x, y)^* p \ \text{ whenever } \ (x, y) \in U.$$

This can be rewritten in the composition form $\partial_x(\varphi \circ g)(x, y) = (f \circ G)(x, y)$ on U with

$$G(x, y) := \Big(x, y, \partial\varphi\big(g(x, y)\big) \Big) \ \text{ and } \ f(x, y, p) := \nabla_x g(x, y)^* p \ \text{ as } \ p \in Z^*.$$

Since f is smooth and $G \circ f^{-1}$ is inner semicompact at $(\bar{x}, \bar{y}, \bar{u})$ under the assumptions made, we get from the construction of $\partial^2_x(\varphi \circ g) = \partial^2_{N,x}(\varphi \circ g)$ and the coderivative chain rule of [21, Theorem 1.65(ii)] that

$$\partial^2_x(\varphi \circ g)(\bar{x}, \bar{y}, \bar{u})(w) \subset \bigcup_{\bar{p} \in G(\bar{x}, \bar{y}) \cap f^{-1}(\bar{u})} \Big(D^*_N G(\bar{x}, \bar{y}, \bar{p}) \circ \nabla f(\bar{x}, \bar{y}, \bar{p})^* \Big)(w)$$

for all $w \in X^{**}$. On the other hand, it is easy to observe from the structure of G that

$$D^*_N G(\bar{x}, \bar{y}, \bar{p})(x^*, y^*, q) = (x^*, y^*) + D^*_N(\partial\varphi \circ g)(\bar{x}, \bar{y}, \bar{p})(q)$$

for all $(x^*, y^*, q) \in X^* \times Y^* \times Z^{**}$. Substituting this formula into the above formula for $\partial^2_x(\varphi \circ g)(\bar{x}, \bar{y}, \bar{u})$, applying Proposition 2.2 to f, and using the construction of S give us

$$\begin{aligned}\partial^2_x(\varphi \circ g)(\bar{x}, \bar{y}, \bar{u})(w) \subset \bigcup_{\bar{p} \in S(\bar{x}, \bar{y}, \bar{u})} & \Big\{ \big(\nabla^2_{xx} \langle \bar{p}, g \rangle (\bar{x}, \bar{y})^* w, \nabla^2_{xy} \langle \bar{p}, g \rangle (\bar{x}, \bar{y})^* w \big) \\ & + D^*_N(\partial\varphi \circ g)(\bar{x}, \bar{y}, \bar{p}) \big(\nabla_x g(\bar{x}, \bar{y})^{**} w \big) \Big\}.\end{aligned}$$

To arrive from here at (4.13), it remains to calculate $D^*_N(\partial\varphi\circ g)$. For this we employ the coderivative chain rule from [21, Theorem 3.13(i)] ensuring that

$$\begin{aligned} D^*_N(\partial\varphi\circ g)(\bar{x},\bar{y},\bar{p})(q) &\subset \big(D^*_N g(\bar{x},\bar{y})\circ\big(D^*_N\partial\varphi\big)(\bar{z},\bar{p})\big)(q)\\ &= \nabla g(\bar{x},\bar{y})^*\partial^2_N\varphi(\bar{z},\bar{p})(q),\quad q\in Z^{**}, \end{aligned}$$

under the PSNC assumption on g^{-1} and the mixed qualification condition

$$(D^*_M\partial\varphi)(\bar{z},\bar{p})(0)\cap\ker\nabla g(\bar{x},\bar{y})^*=\{0\},\quad \bar{p}\in S(\bar{x},\bar{y},\bar{u}),$$

which is exactly the second-order qualification condition (4.12) of the theorem. △

The result of in Theorem 4.4 can be significantly simplified when the intermediate space Z in the composition is finite dimensional while X and Y may be not.

Corollary 4.5 (partial second-order chain rules for compositions with finite-dimensional intermediate spaces). *Let $\bar{u}\in\partial_x(\varphi\circ g)(\bar{x},\bar{y})$, where $\varphi:I\!R^m\to\overline{I\!R}$ and $g\colon X\times Y\to I\!R^m$ with Asplund spaces X,Y. Assume that $g\in\mathcal{C}^1$ around some $(\bar{x},\bar{y})$ with the partial derivative $\nabla_x g$ strictly differentiable at this point, that φ is l.s.c. and lower regular around $\bar{z}:=g(\bar{x},\bar{y})$ with closed graphs of $\partial\varphi$ and $\partial^\infty\varphi$ near $\bar{z}$. Suppose also that the first-order qualification condition* (4.10) *is satisfied at $(\bar{x},\bar{y})$ and that the second-order one* (4.12) *also holds with mapping* (4.11) *inner semicompact at $(\bar{x},\bar{y},\bar{u})$. Then we have the partial second-order partial chain rule* (4.13) *for both partial second-order subdifferentials.*

Proof. It is similar to that of [21, Corollary 3.75] for full second-order subdifferentials. Notice that the additional assumption on the closedness of the graph of $\partial^\infty\varphi$ near $\bar{z}$ along with other assumptions made allow us to prove that the first-order qualification condition (4.10) in fact holds around the reference point. △

The next corollary justifies the partial second-order subdifferential chain rules for an important class of extended-real-valued functions that automatically satisfy all the first-order assumptions in Corollary 4.5. Recall that $\psi\colon X\times Y\to\overline{I\!R}$ is *strongly amenable* in x at $\bar{x}$ with *compatible parameterization* in y at $\bar{y}$ if there is a neighborhood U of $(\bar{x},\bar{y})$ on which ψ is represented in the composite form $\psi=\varphi\circ g$ with a $\mathcal{C}^2$ mapping $g\colon U\to I\!R^m$ and a proper l.s.c. convex function $\varphi\colon I\!R^m\to\overline{I\!R}$ such that the first-order qualification condition (4.10) holds at $(\bar{x},\bar{y})$. It has been well recognized that amenable functions play a major role in second-order variational analysis and optimization; see, e.g., [38] and more recent publications [14, 15, 16, 29, 30, 32, 33] with the references therein.

Corollary 4.6 (partial second-order chain rules for strongly amenable functions). *Let $\psi:X\times Y\to\overline{I\!R}$ be strongly amenable in x at $\bar{x}$ with compatible parameterization in y at $\bar{y}$, let $\varphi:I\!R^m\to\overline{I\!R}$ and $g:U\to I\!R^m$ be taken from its composite representation, and let $\bar{u}\in\partial_x\psi(\bar{x},\bar{y})$. Assume that X and Y are Asplund spaces and that the second-order qualification condition* (4.12) *holds with $\bar{z}=g(\bar{x},\bar{y})$. Then for all $w\in X^{**}$ we have the second-order chain rule* (4.13) *for both partial subdifferentials $\partial^2_x=\partial^2_{N,x}=\partial^2_{M,x}$.*

Proof. It follows from Corollary 4.5, since the lower regularity of φ, and the closedness of the graphs of $\partial\varphi$ and $\partial^\infty\varphi$ are implied by the convexity of φ. $\triangle$

Note that Corollary 4.6 extends, with a completely different proof, the recent result of [29, Theorem 3.3] not only to the infinite-dimensional setting of X and Y, but also replaces the second-order qualification condition

$$\partial^2\varphi(\bar{z},\bar{p})(0)\cap\ker\nabla_x g(\bar{x},\bar{y})^*=\{0\}\ \text{ for all }\ \bar{p}\in S(\bar{x},\bar{y},\bar{u})$$

of [29, Theorem 3.3] by the less restrictive one (4.12). It is worth mentioning that there are *exact* chain rules for full and partial second-order subdifferentials derived in [29, 30] for some classes of *fully amenable* functions that seem to be essentially finite-dimensional.

We conclude this section with chain rules for both normal and mixed partial second-order subdifferentials involving *Lipschitzian inner mappings* between Asplund spaces with the usage of the *partial second-order coderivatives* from Definition 3.7. The following theorem is in the direction of Theorem 3.8 with avoiding the smoothness and surjectivity assumptions therein while providing only upper estimates instead of equalities. Recall that the notation $\mathcal{C}^{1,1}$ stands for $\mathcal{C}^1$ functions with Lipschitzian derivatives.

Theorem 4.7 (partial second-order chain rules with Lipschitzian inner mappings). *$\bar{u}\in\nabla_x(\varphi\circ g)(\bar{x},\bar{y})$, where $g:X\times Y\to Z$ is Lipschitz continuous around $(\bar{x},\bar{y})$, where $\varphi:Z\to\overline{I\!R}$ is $C^{1,1}$ around $\bar{z}:=g(\bar{x},\bar{y})$ with $\bar{p}:=\nabla\varphi(\bar{z})$, and where the spaces X,Y,Z,X^*,Y^*, and Z^* are Asplund. Assume also that the graph of the set-valued mapping $(x,y,p)\rightrightarrows\partial_x\langle p,g\rangle(x,y)$ is norm-closed in $X\times Y\times Z^*\times X^*$ whenever (x,y,p) are near $(\bar{x},\bar{y},\bar{p})$. Then we have the second-order partial chain rule*

$$\partial_x^2(\varphi\circ g)(\bar{x},\bar{y},\bar{u})(w)\subset\bigcup_{(u,v,q)\in D_x^2 g(\bar{x},\bar{y},\bar{p},\bar{u})(w)}\big[(u,v)+D_N^*g(\bar{x},\bar{y})\circ\partial_N^2\varphi(\bar{z})(q)\big]\tag{4.14}$$

for all $w\in X^{**}$, where ∂^2 and D^2 stand for the corresponding normal and mixed partial second-order constructions. Furthermore, both second-order inclusions in (4.14) *hold for an arbitrary Banach space X of $\nabla\varphi$ is strictly differentiable at $\bar{z}$.*

Proof. Similarly to the proof of Theorem 3.8 we find a neighborhood U of $(\bar{x},\bar{y})$ such that

$$\partial_x(\varphi\circ g)(x,y)=(F\circ h)(x,y)\ \text{ for all }\ (x,y)\in U,\tag{4.15}$$

where the mappings $F:X\times Y\times Z^*\rightrightarrows X^*$ and $h:X\times Y\to X\times Y\times Z^*$ are defined by

$$F(x,y,s):=\partial_x\langle s,g\rangle(x,y)\ \text{ and }\ h(x,y):=\Big(x,y,\nabla\varphi\big(g(x,y)\big)\Big).$$

Applying to (4.15) the coderivative chain rule from [21, Theorem 3.13(i)] gives us

$$D^*(F\circ h)(\bar{x},\bar{y},\bar{p},\bar{u})(w)\subset D_N^*h(\bar{x},\bar{y})\circ D^*F(\bar{x},\bar{y},\bar{p},\bar{u})(w),\quad w\in X^{**},$$

for both normal and mixed coderivatives under the assumptions made. We also have

$$D_N^*(\nabla\varphi\circ g)(\bar{x},\bar{y})(q)\subset D_N^*g(\bar{x},\bar{y})\circ\partial_N^2\varphi(\bar{z})(q),\quad q\in Z^{**},\tag{4.16}$$

from the same chain rule of [21, Theorem 3.13(i)] with the assumption that $\nabla\varphi$ is Lipschitz around $\bar{z}$. Combining these two inclusions and taking into account the that $D^*F(\bar{x},\bar{y},\bar{p},\bar{u}) = D^2_x g(\bar{x},\bar{y},\bar{p},\bar{u})$ in the notation above, we arrive at (4.14). Observe finally that (4.16) holds as equality in any Banach space Z provides that $\nabla\varphi$ is strictly differentiable at $\bar{z}$; see [21, Theorem 1.65(iii)]. This justifies the last statement of the theorem. △

5 Coderivatives of Perturbed Normal Cone Mappings

This section is devoted to some applications of second-order calculus results obtained in Sections 3 and 4 to calculating and estimating the normal and mixed coderivatives of the (limiting) *normal cone mappings* defined by

$$\mathcal{N}(x,y) := N\big(x; G(y)\big) \ \text{ with } \ G(y) := \big\{x \in X \big|\ g(x,y) \in \Theta\big\}, \tag{5.1}$$

where $g\colon X \times Y \to Z$ and $\emptyset \neq \Theta \subset Z$. Set-valued mappings of the normal cone type (5.1) as well as their specifications and subdifferential counterparts are of strong interest for numerous aspects of variational analysis and optimization, since they describe, in particular, perturbed KKT systems in optimization, solutions maps to parameterized variational inequalities and equilibrium problems, dynamics of the sweeping and other mechanical processes, etc. Due to a well-recognized role of coderivatives in characterizing quantitative/Lipschitzian stability and optimality conditions in variational systems of this type (see, e.g., [21, 38]), it is appealing to express the coderivatives of (5.1) in terms of the initial data of problems under consideration. A number of results in this direction with their various applications can be found in [3, 6, 9, 10, 11, 12, 13, 21, 26, 31, 32, 33, 35, 36, 40, 41] and the references therein for particular systems of type (5.1), mostly in finite dimensions. Furthermore, the vast majority of these papers concern the normal cone mappings (5.1) with G independent of the parameter vector y while special cases of parameter-dependent mappings $G(y)$ have been considered in [26] and more recent developments [3, 9, 31, 35, 36].

It is easy to see from the structures of (5.1) that the normal cone mapping $\mathcal{N}$ is represented in the partial first-order *subdifferential composite form*

$$\mathcal{N}(x,y) = \partial_x\psi(x,y) \ \text{ with } \ \psi(x,y) := (\delta_\Theta \circ g)(x,y), \quad (x,y) \in X \times Y, \tag{5.2}$$

which yields that the (normal and mixed) coderivatives of $\mathcal{N}$ are *partial second-order subdifferentials* (2.10) and (2.11) of the composition $\delta_\Theta \circ g$. This allows us to evaluate the coderivatives of $\mathcal{N}$ by using the partial second-order chain obtained in Sections 3 and 4.

We begin with the following Banach space results under reduction and nondegeneracy conditions, which are straightforward consequences of those in Theorem 3.6.

Proposition 5.1 (coderivatives of normal cone mappings under reduction and nondegeneracy conditions). *Let $\bar{u} \in \mathcal{N}(\bar{x},\bar{y})$ for the normal cone mapping* (5.1)*, where $g\colon X \times Y \to Z$ is a $\mathcal{C}^2$ mapping around $(\bar{x},\bar{y})$.*

(i) *Assume that $\nabla_x g(\bar{x},\bar{y})$ is surjective. Then $\bar{u}$ admits the representation*

$$\bar{u} = \nabla_x g(\bar{x},\bar{y})^*\bar{p} \ \textit{ for a unique } \ \bar{p} \in N\big(g(\bar{x},\bar{y});\Theta\big), \tag{5.3}$$

and we have the following relations:

$$
\begin{aligned}
D_M^*\mathcal{N}(\bar{x},\bar{y},\bar{u})(w) \;&= \big(\nabla_{xx}^2\langle\bar{p},g\rangle(\bar{x},\bar{y})^*w,\nabla_{xy}^2\langle\bar{p},g\rangle(\bar{x},\bar{y})^*w\big)\\
&+\nabla g(\bar{x},\bar{y})^*D_M^*N\big((g(\bar{x},\bar{y}),\bar{p});\Theta\big)\big(\nabla_x g(\bar{x},\bar{y})^{**}w\big),
\end{aligned}
\tag{5.4}
$$

$$
\begin{aligned}
D_N^*\mathcal{N}(\bar{x},\bar{y},\bar{u})(w) \;&\subset \big(\nabla_{xx}^2\langle\bar{p},g\rangle(\bar{x},\bar{y})^*w,\nabla_{xy}^2\langle\bar{p},g\rangle(\bar{x},\bar{y})^*w\big)\\
&+\nabla g(\bar{x},\bar{y})^*D_N^*N\big((g(\bar{x},\bar{y}),\bar{p});\Theta\big)\big(\nabla_x g(\bar{x},\bar{y})^{**}w\big).
\end{aligned}
\tag{5.5}
$$

The inclusion (5.5) *becomes an equality if the range of* $\nabla_x f(\bar{x},\bar{y})$ *is weak* extensible in the space* X^*.

(ii) *Assume that* $\Theta\subset Z$ *satisfies the reduction condition of Definition* 3.3 *at* $\bar{z}:=g(\bar{x},\bar{y})\in\Theta$ *with some mapping* $h\colon Z\to W$. *Assume that* $(\bar{x},\bar{y})$ *is a partial nondegenerate point of* g *in* x *relative to* h *and that* $\bar{u}$ *admits the representation*

$$
\bar{u}=\nabla_x g(\bar{x},\bar{y})^*\bar{p}\;\text{ with some }\;\bar{p}\in N\big(g(\bar{x},\bar{y});\Theta\big). \tag{5.6}
$$

Then for any $w\in X^{**}$ *we have the relations* (5.4) *and* (5.5) *provided that the closed unit ball of* Z^{**} *is weak* sequentially compact. The inclusion* (5.5) *if we assume additionally that the closed unit ball of* X^{**} *is weak* sequentially compact.*

Proof. It follows directly from Theorem 3.2 and Theorem 3.6 for $\varphi=\delta_\Theta$ by taking into account the composite representation (5.2) and Definition 2.1 of the partial second-order subdifferentials. △

We can see right away that both reduction and nondegeneracy conditions are satisfied and that representation (5.6) holds with a unique element $\bar{p}$ if the partial derivative $\nabla_x g(\bar{x},\bar{y})$ is *surjective*. Furthermore, it follows from Theorem 4.2 and the discussion after its proof that the first-order representation (5.6) holds automatically with a unique normal $\bar{p}$ under the validity of the *qualification condition* (4.9) provided that the spaces in question are Asplund and that the SNC assumptions of Theorem 4.2 are fulfilled. Finally, the qualification condition (4.9) itself *follows from the imposed nondegeneracy* if the additional assumptions of Theorem 4.3 are satisfied.

If the spaces X and Y are finite-dimensional and if the mapping G in (5.1) is defined by

$$
G(y):=\big\{x\in I\!R^n\big|\;g_i(x,y)\le 0\;\text{ for all }\;i=1,\ldots,m\big\} \tag{5.7}
$$

with the linear independent partial gradient vectors $\nabla_x g_1(\bar{x},\bar{y}),\ldots,\nabla_x g_m(\bar{x},\bar{y})$ corresponding to constraints (5.7) active at $(\bar{x},\bar{y})$, then the coderivative representation for the normal cone mapping in Proposition 5.1 with $D_M^*=D_N^*$ has been recently presented in [9]. Other recent results in this direction for the normal coderivative of (5.1) have been derived by direct calculations in [36, Theorem 3.3] for the case of (5.7) with $m=1$ in general Banach spaces X and Y. Let us show that these results of [36], and the identical ones for the mixed coderivative, can be deduced from Proposition 5.1.

Corollary 5.2 (coderivatives of the normal cone mapping defined by a single inequality constraint). *Consider the normal cone mapping* (5.1) *defined by* G *in* (5.7) *with* $m=1$, *where* $g:=g_1$ *is a scalar function that is* $\mathcal{C}^2$ *around* $(\bar{x},\bar{y})$ *with* $g(\bar{x},\bar{y})=0$.

Assume that $\nabla_x g(\bar x,\bar y)\neq 0$ and pick any $\bar u\in\mathcal N(\bar x,\bar y)$. The following assertions hold for both coderivatives $D^*=D^*_N,D^*_M$ of $\mathcal N$ at $(\bar x,\bar y,\bar u)$.

(i) *Let $\bar u\neq 0$. Then $\bar u=\mu\nabla_x g(\bar x,\bar y)$ for some $\mu>0$ and, whenever $w\in X^{**}$ with $\langle w,\nabla_x g(\bar x,\bar y)=0$, we have the formula*

$$D^*\mathcal N(\bar x,\bar y,\bar u)(w)=\big\{(x^*,y^*)\in X^*\times Y^*\big|\;x^*=\gamma\nabla_x g(\bar x,\bar y)+\mu\nabla^2_{xx}g(\bar x,\bar y)^*w\\ y^*=\gamma\nabla_y g(\bar x,\bar y)+\mu\nabla^2_{xy}g(\bar x,\bar y)^*w\;\text{ for some }\;\gamma\in I\!R\big\}$$

with $\mu=\|\bar u\|\cdot\|\nabla_x g(\bar x,\bar y)\|^{-1}$. Furthermore $D^*\mathcal N(\bar x,\bar y,\bar u)=\emptyset$ if $\langle w,\nabla_x g(\bar x,\bar y)\rangle\neq 0$.

(ii) *Let $\bar u=0$. Then we have*

$$D^*\mathcal N(\bar x,\bar y,\bar u)(w)=\begin{cases} I\!R_+\nabla g(\bar x,\bar y) & \text{if }\;\langle w,\nabla_x g(\bar x,\bar y)\rangle>0,\\ I\!R\nabla g(\bar x,\bar y) & \text{if }\;\langle w,\nabla_x g(\bar x,\bar y)\rangle=0,\\ \{0\} & \text{if }\;\langle w,\nabla_x g(\bar x,\bar y)\rangle<0.\end{cases}$$

Proof. Since $\nabla_x g(\bar x,\bar y)\neq 0$, we can apply Proposition 5.1 under the surjectivity assumption on $\nabla_x g(\bar x,\bar y)$ with $\bar p$ denoted now by $\mu>0$. To justify (i), it remains to calculate $D^*N(0,\mu)(\nabla_x g(\bar x,\bar y)^{**}w)$. Observe that

$$F(x):=N(x;I\!R_-)=\begin{cases}\{0\}, & x<0,\\ I\!R_+, & x=0,\\ \emptyset, & x>0,\end{cases}\tag{5.8}$$

and hence, given $a\in F(0)$ with $a>0$, we have

$$D^*F(0,a)(v)=\big\{u\in I\!R\big|\;(u,v)\in N\big((0,a);\operatorname{gph}F\big)\big\}=\begin{cases} I\!R, & v=0,\\ \emptyset, & v\neq 0.\end{cases}\tag{5.9}$$

It follows from these calculations that $\langle\nabla_x g(\bar x,\bar y),w\rangle=0$ whenever $(x^*,y^*)\in D^*\mathcal N(\bar x,\bar y,\bar u)(w)$ and furthermore $D^*N(g(\bar x,\bar y),\mu)(0)=I\!R$. Thus there exists $\gamma\in I\!R$ such that

$$(x^*,y^*)=\big(\mu\nabla^2_{xx}g(\bar x,\bar y)^*w,\mu\nabla^2_{xy}g(\bar x,\bar y)^*w\big)+\gamma\nabla g(\bar x,\bar y),$$

which justifies the coderivative representation in (i) when $\langle\nabla_x g(\bar x,\bar y),w\rangle=0$ for both mixed and normal coderivatives because $D^*_M\mathcal N(\bar x,\bar y,\bar u)(w)\subset D^*_N\mathcal N(\bar x,\bar y,\bar u)(w)$. It also follows from (5.9) that $D^*\mathcal N(\bar x,\bar y,\bar u)(w)=\emptyset$ if $\langle\nabla_x g(\bar x,\bar y),w\rangle\neq 0$, which completes the proof of (i).

To prove (ii), we proceed as before and get in this case that $\mu:=\bar p=0$. Thus it remains to calculate $D^*N(0,0)(\nabla_x g(\bar x,\bar y)^{**}w)$. It follows from (5.9) for F defined in (5.8) that

$$D^*F(0,0)(v)=\big\{u\in I\!R\big|\;(u,-v)\in N\big((0,0);\operatorname{gph}F\big)\big\}=\begin{cases}\{0\}, & v<0.\\ I\!R, & v=0,\\ I\!R_+, & v>0,\end{cases}$$

which completes the proof of the proposition. $\triangle$

Finally, it is easy to observe directly from the above structures that

$$D^*\mathcal N(\bar x,\bar y,\bar u)(w)=\big\{(0,0)\big\}\subset X^*\times Y^*\tag{5.10}$$

if $g(\bar{x},\bar{y})<0$ in the frameworks of (5.1) and (5.7) with $m=1$.

Let us further present an application of the obtained results to precise calculating the coderivatives of the normal cone mapping for a particular setting frequently arising in the theory of *trust-region methods* in nonlinear programming; see, e.g., [4]. The usage of this approach in the stability analysis and numerical solution of the so-called *trust-region subproblem* has been recently discovered in [13], where the reader can find the next corollary proved by direct calculations in finite dimensions.

Corollary 5.3 (coderivatives of the normal cone mapping generated by the trust-region subproblem). *Consider the mapping* (5.1) *with* $G\colon I\!R_+ \rightrightarrows X$ *given by*

$$G(y) := \big\{x\in X\big|\ \|x\|\le y\big\} \tag{5.11}$$

with values in the Hilbert space X. *Given* $\bar{u}\in\mathcal{N}(\bar{x},\bar{y})$ *with* $\bar{y}>0$, *the following hold for both coderivatives* $D^*=D^*_M, D^*_M$.

(i) *Let* $\|\bar{x}\|=\bar{y}$, *and let* $\bar{u}=\mu\bar{x}$ *with* $\mu>0$. *Then for any* $w\in X$ *we have*

$$D^*\mathcal{N}(\bar{x},\bar{y},\bar{u})(w)=\begin{cases}\Big\{(x^*,y^*)\in X\times I\!R\Big|\ x^*=\mu y^*-\dfrac{\bar{x}}{\bar{y}}y^*\Big\} & \text{if } \langle w,\bar{x}\rangle=0,\\ \emptyset & \text{if } \langle w,\bar{x}\rangle\ne 0.\end{cases}$$

(ii) *Let* $\|\bar{x}\|=\bar{y}$, *and let* $\bar{u}=0$. *Then for any* $w\in X$ *we have*

$$D^*\mathcal{N}(\bar{x},\bar{y},\bar{u})(w)=\begin{cases}\big\{(x^*,y^*)\in X\times I\!R\big|\ x^*=2\lambda\bar{x},\ y^*=-2\lambda\bar{y},\ \lambda\in I\!R\big\} & \text{if } \langle w,\bar{x}\rangle=0,\\ \big\{(x^*,y^*)\in X\times I\!R\big|\ x^*=2\lambda\bar{x},\ y^*=-2\lambda\bar{y},\ \lambda\in I\!R_+\big\} & \text{if } \langle w,\bar{x}\rangle>0,\\ \big\{(0,0)\big\}\subset X^*\times I\!R & \text{if } \langle w,\bar{x}\rangle<0.\end{cases}$$

(iii) *Let* $\|\bar{x}\|<\bar{y}$. *Then* $\bar{u}=0$ *and for any* $w\in X$ *we have*

$$D^*\mathcal{N}(\bar{x},\bar{y},\bar{u})(w)=\big\{(0,0)\big\}\subset X^*\times I\!R.$$

Proof. It follows from Corollary 5.2 and formula (5.10) with the function $g(x,y):=\|x\|^2-y^2$ equivalently describing (5.11). △

To conclude this section, we present the following application of Theorem 4.4 to the normal cone mapping (5.1), where the nondegeneracy condition is not imposed while the result is given not in the equality but in the inclusion form. For brevity and simplicity we suppose that the space Z in (5.1) is finite-dimensional.

Proposition 5.4 (coderivatives of normal cone mappings without nondegeneracy). *Let* $\bar{u}\in\mathcal{N}(\bar{x},\bar{y})$ *for the normal cone mapping* (5.1), *where* $g\colon X\times Y\to I\!R^m$ *is a* $\mathcal{C}^1$ *mapping around* $(\bar{x},\bar{y})$ *with strict derivative* $\nabla g(\bar{x},\bar{y})$, *and where* $\Theta\subset I\!R^m$ *is locally closed around* $\bar{z}:=g(\bar{x},\bar{y})$. *Assume that the spaces* X *and* Y *are Asplund, that the mapping* $S\colon X\times Y\times X^*\rightrightarrows I\!R^m$ *is inner semicompact at* $(\bar{x},\bar{y},\bar{u})$, *and that the following first-order and second-order qualification conditions*

$$N(\bar{z};\Theta)\cap\nabla_x g(\bar{x},\bar{y})^*=\{0\},$$

$$D^*N(\bar z,\bar p)(0)\cap\ker g(\bar x,\bar y)^*=\{0\}\ \textit{ for all }\ \bar p\in S(\bar x,\bar y,\bar u)$$

are satisfied. Then we have the inclusion

$$\begin{aligned}D^*\mathcal{N}(\bar x,\bar y,\bar u)(w)\quad\subset\bigcup_{\bar p\in S(\bar x,\bar y,\bar u)}&\big(\nabla^2_{xx}\langle\bar p,g\rangle(\bar x,\bar y)^*w,\nabla^2_{xy}\langle\bar p,g\rangle(\bar x,\bar y)^*w\big)\\&+\nabla g(\bar x,\bar y)^*D^*N\big((g(\bar x,\bar y),\bar p);\Theta\big)\big(\nabla_x g(\bar x,\bar y)^{**}w\big)\end{aligned}$$

for all $w\in X^{**}$ *and both coderivatives* $D^*=D^*_N,D^*_M$ *of the normal cone mapping* $\mathcal{N}$.

Proof. It follows from the application of Corollary 4.5 to the composite representation (5.2) of $\mathcal{N}(x,y)$ by taking into account the closedness of the graph of the limiting normal cone mapping $N(\cdot;\Theta)=\partial\delta_\Theta(\cdot)$ in the finite-dimensional space $Z=I\!R^m$. △

6 Lipschitzian Stability of Stationary Point Mappings

In this section we study the *stationary point mapping* $S\colon Y\rightrightarrows X$ whose values

$$S(y):=\big\{x\in X\big|\ 0\in\partial_x\psi_0(x,y)+\partial_x\psi(x,y)\big\}\ \text{ for all }\ y\in Y\tag{6.12}$$

are stationary points of the parametric optimization problem (1.1). Our main goal here is to establish verifiable conditions ensuring *robust Lipschitzian stability* of S with respect to parameter perturbations. Developing the *coderivative analysis* of variational systems in the vein of [21, Chapter 4], we first obtain an upper estimate of the normal (and hence mixed) coderivative of S via the initial data of (6.12). Observe to this end that the generalized equation (6.12) is different from those considered in [21] and the vast majority of publications, since its both "base" and "field" parts are set-valued and parameter-dependent. Such models have been addressed in [22] in the general set-valued framework.

To proceed with the study the stationary point mapping (6.12) from this viewpoint, let us introduce the following *partial second-order qualification condition* (PSOQC), which is formulated in terms of the normal partial second-order subdifferential (2.10) and plays a crucial role in the results of this section.

Definition 6.1 (partial second-order qualification condition). *Let* $(\bar y,\bar x)\in\operatorname{gph}S$ *for the mapping* S *in* (6.12), *and let* $\bar u\in\partial_x\psi(\bar x,\bar y)\cap(-\partial_x\psi(\bar x,\bar y))$. *We say that the* PARTIAL SECOND-ORDER QUALIFICATION CONDITION (PSOQC) *holds at* $(\bar x,\bar y,\bar u)$ *if*

$$\Big[(x^*,y^*)\in\partial^2_{N,x}\psi_0(\bar x,\bar y,\bar u)(w)\bigcap\big(-\partial^2_{N,x}\psi(\bar x,\bar y,-\bar u)(w)\big)\Big]\Longrightarrow(x^*,y^*,w)=(0,0,0).$$

Note that for $\mathcal{C}^2$ functions ψ_0 (i.e., for twice continuously differentiable cost functions in parametric constrained optimization) the introduced PSOQC amounts to saying that the *partial adjoint second-order generalized equation*

$$0\in\nabla^2_{xx}\psi_0(\bar x,\bar y)^*w+\partial^2_{N,x}\psi\big(\bar x,\bar y,-\nabla_x\psi_0(\bar x,\bar y)\big)(w)\tag{6.13}$$

admits only the trivial solution $w=0$.

Now we are ready to provide an efficient upper estimate of the normal coderivative of (6.12) involving the normal partial second-order subdifferentials (2.10) of the functions

ψ_9 and ψ therein under the PSOQC from Definition 6.1. This estimate follows from the corresponding result of [22, Corollary 4.3] in the abstract generalized equation framework. For brevity we confine ourselves to the case of inner semicompactness of the intersection mapping $\partial_x\psi_0 \cap (-\partial_x\psi)$; the case of inner semicontinuity is similar.

Proposition 6.2 (coderivative upper estimate for stationary point mappings). *Let $S\colon X \rightrightarrows Y$ be the stationary point mapping (6.12) between Asplund spaces, and let its graph be locally closed around $(\bar{y}, \bar{x}) \in \text{gph}\, S$. Assume that the intersection mapping $\partial_x\psi_0 \cap (-\partial_x\psi)$ is inner semicompact at $(\bar{x}, \bar{y})$ and that for any $\bar{u} \in \partial_x\psi_0(\bar{x}, \bar{y}) \cap (-\partial_x\psi(\bar{x}, \bar{y}))$ the following hold: the graphs of $\partial_x\psi_0$ and $\partial_x\psi$ are locally norm-closed around $(\bar{x}, \bar{y}, \bar{u})$ and $(\bar{x}, \bar{y}, -\bar{u})$, respectively; the PSOQC of Definition 6.1 is satisfied at $(\bar{x}, \bar{y}, \bar{u})$; and that either $\partial_x\psi_0$ is PSNC at $(\bar{x}, \bar{y}, \bar{u})$ and $(\partial_x\psi)^{-1}$ is strongly PSNC at $(-\bar{u}, \bar{x}, \bar{y})$, or $(\partial_x\psi_0)^{-1}$ is PSNC at $(\bar{u}, \bar{x}, \bar{y})$ and $\partial_x\psi$ is strongly PSNC at $(\bar{x}, \bar{y}, -\bar{u})$, or the similar PSNC conditions hold with changing places of ψ_0 and ψ. Then for all $x^* \in X^*$ we have*

$$D_N^*S(\bar{y}, \bar{x})(x^*) \subset \Big\{y^* \in Y^* \Big| \exists \bar{u} \in \partial_x\psi_0(\bar{x}, \bar{y}) \cap \big(-\partial_x\psi(\bar{x}, \bar{y})\big),\ w \in X^{**} \text{ with } (-x^*, y^*) \in \partial^2_{N,x}\psi_0(\bar{x}, \bar{y}, \bar{u})(w) + \partial^2_{N,x}\psi(\bar{x}, \bar{y}, -\bar{u})(w)\Big\}.$$

Proof. It follows from [22, Corollary 4.3] by taking into account the first-order subdifferential form of the generalized equation (6.12) and construction (2.10) of the normal second-order partial subdifferential for ψ_0 and ψ. △

Observe that the PSNC assumptions of Proposition 6.2 holds automatically if the spaces X and Y are finite-dimensional. Furthermore, in this case the imposed inner semicompactness of the mapping $\partial_x\psi_0 \cap (-\partial_x\psi)$ follows from its boundedness the subdifferential closed-graph assumptions are surely satisfied for continuous functions ψ_0, ψ as well as broad classes of extended-real-valued l.s.c. functions, e.g., for amenable ones; see [21, 38].

The next corollary examines the case of infinite-dimensional spaces, where the cost function ψ_0 in (6.12) is $\mathcal{C}^2$ while the constraint one ψ is still extended-real-valued.

Corollary 6.3 (coderivative estimate for stationary point mappings with $\mathcal{C}^2$ costs). *Let ψ_0 in (6.12) be $\mathcal{C}^2$ around $(\bar{x}, \bar{y}) \in X \times Y$ in Asplund spaces with $(\bar{y}, \bar{x}) \in \text{gph}\, S$, and let $\psi\colon X \times Y \to \overline{I\!R}$ be such that the graph of $\partial_x\psi$ is norm-closed around $(\bar{x}, \bar{y}, -\nabla_x\psi_0(\bar{x}, \bar{y}))$. Assume that the partial adjoint second-order generalized equation (6.13) has only the trivial solutions $w = 0$, and either $\dim X < \infty$ or the mapping $(\partial_x\psi)^{-1}$ is strongly PSNC at $(-\nabla_x\psi_0(\bar{x}, \bar{y}), \bar{x}, \bar{y})$. Then for all $x^* \in X^*$ we have the normal coderivative estimate*

$$D_N^*S(\bar{y}, \bar{x})(x^*) \subset \bigcup_{w \in X^{**}} \Big\{y^* \in Y^* \Big| \big(-x^* - \nabla^2_{xx}\psi_0(\bar{x}, \bar{y})^*w, y^*\big) \in \partial^2_{N,x}\psi\big(\bar{x}, \bar{y}, -\nabla_x\psi_0(\bar{x}, \bar{y})\big)(w)\Big\}.$$

Proof. Since the cost function ψ_0 is $\mathcal{C}^2$, the stationary point mapping S in (6.12) becomes

$$S(y) = \big\{x \in X \big|\ 0 \in \nabla_x\psi_0(x, y) + \partial_x\psi(x, y)\big\}$$

and the closedness of its graph follows that of $\text{gph}\partial_x\psi$. Furthermore, the intersection mapping $\nabla_x\psi_0 \cap (-\partial_x\psi)$ is single-valued around $(\bar{x}, \bar{y})$ and hence inner semicompact at this point. The Lipschitz continuity of $\nabla_x\psi_0$ ensures its PSNC property, and so the validity of either the strong PSNC property of $(\partial_x\psi)^{-1}$ or the finite dimensionality of X guarantees the corresponding PSNC properties of Proposition 6.2. Recalling that the triviality of solutions to (6.13) reduces to the PSOQC in this case completes the proof of the corollary. $\triangle$

Finally in this section, we derive verifiable characterizations of the Lipschitz-like property (2.7) for (6.12) that reduces to the classical local Lipschitz continuity in the single-valued case and presents the most natural manifestation of robust Lipschitzian stability for set-valued mappings. Considering for simplicity only the case of $\psi_0 \in \mathcal{C}^2$, we obtain the result in terms of the partial second-order subdifferential of the constraint function ψ.

Theorem 6.4 (Lipschitz-like property of stationary point mappings). *Let $\bar{x} \in S(\bar{y})$ for the stationary point mapping* (6.12) *between Asplund spaces, where $\psi_0 \in \mathcal{C}^2$ around $(\bar{x}, \bar{y})$, and where the graph of $\partial_x\psi$ is norm-closed around $\big(\bar{x}, \bar{y}, -\nabla_x\psi_0(\bar{x}, \bar{y})\big)$ and SNC at this point. Assume that the partial adjoint second-order generalized equation* (6.13) *has only the trivial solution $w = 0$ and that*

$$\Big[(0, y^*) - \nabla^2\psi_0(\bar{x}, \bar{y})^*(w, 0) \in \partial^2_{N,x}\psi\big(\bar{x}, \bar{y}, -\nabla_x\psi_0(\bar{x}, \bar{y})\big)(w)\Big] \Longrightarrow y^* = 0 \tag{6.14}$$

Then the stationary point mapping S is Lipschitz-like around $(\bar{y}, \bar{x})$.

Proof. It is clear that all the assumptions of Corollary 6.3 are satisfied in the setting of the theorem. Thus it implies by (6.14) that

$$D^*_M S(\bar{y}, \bar{x})(0) \subset D^*_N S(\bar{y}, \bar{x})(0) = \big\{0\big\}.$$

To justify the Lipschitz-like property of S around $(\bar{y}, \bar{x})$, it remains to verify by the *coderivative criterion* from [21, Theorem 4.10] that S is PSNC at $(\bar{y}, \bar{x})$. Let us show that S is actually SNC at this point under the assumptions made. Observe that

$$(y, x) \in \operatorname{gph} S \iff (x, y) \in \ker\left(\nabla_x\psi_0 + \partial_x\psi\right) = \operatorname{dom}\big(\nabla_x\psi_0 \cap (-\partial_x\psi)\big).$$

Thus it follows from [22, Proposition 4.1] that the set $\operatorname{gph} S$ is SNC at $(\bar{y}, \bar{x})$ if the intersection mapping $\nabla_x\psi_0 \cap (-\partial_x\psi)$ is PSNC at $(\bar{x}, \bar{y})$. The latter means that the set

$$\operatorname{gph}\big(\nabla_x\psi_0 \cap (-\partial_x\psi)\big) = \operatorname{gph}\left(\nabla_x\psi\right) \cap \operatorname{gph}\left(-\partial_x\psi\right) \subset X \times Y \times X^*$$

is PSNC at $(\bar{x}, \bar{y}, \nabla_x\psi_0(\bar{x}, \bar{y}))$ with respect to $X \times Y$; see [21, Definition 3.3]. Since ψ_0 is C^2 around $(\bar{x}, \bar{y})$, employing now the result of [21, Corollary 3.80] on the PSNC property of set intersections in product spaces, we conclude that the mapping $\nabla_x\psi_0 \cap (-\partial_x\psi)$ is PSNC at $(\bar{x}, \bar{y})$ if $\partial_x\psi$ is SNC around $(\bar{x}, \bar{y}, -\nabla\psi_0(\bar{x}, \bar{y}))$ and the pair $\big\{\operatorname{gph}\nabla_x\psi_0, \operatorname{gph}(-\partial_x\psi)\big\}$ satisfies the mixed qualification condition at $(\bar{x}, \bar{y}, \nabla_x\psi_0(\bar{x}, \bar{y}))$ relative to X^* in the sense of [21, Definition 3.78]. It is easy to derive directly from the definition that this condition is implied by the assumed solution triviality of (6.13), which this completes the proof of the theorem. $\triangle$

Note that in the case of finite-dimensional spaces X and Y the result of Theorem 6.4 improves that of [14, Corollary 2.3], where the Lipschitz-like property of S is justified under the additional assumptions via the *total* second-order subdifferential of ψ in both variables.

7 Concluding Remarks

The paper develops general calculus rules for partial (as well as total) second-order subdifferentials of extended-real-valued functions that are new in both finite-dimensional and infinite-dimensional spaces. These developments constitute the main results of the paper from the viewpoints of second-order variational analysis and generalized differentiation. Besides being of undoubted interest for their own sake, the calculus rules obtained have strong potentials for applications to various problems of nonlinear analysis and optimization. Some of these applications are presented in the paper while more should come in the near future. Among them we mention *tilt* and *full stability* in remarkable classes of constrained optimization problems, where second-order calculus rules could allow us to make a bridge between unconstrained and constrained problems; see Section 1 with the references therein.

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