

SECOND-ORDER VARIATIONAL ANALYSIS IN CONIC PROGRAMMING WITH APPLICATIONS TO OPTIMALITY AND STABILITY

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Abstract. This paper is devoted to the study of a broad class of problems in conic programming modeled via parameter-dependent generalized equations. In this framework we develop a second-order generalized differential approach of variational analysis to calculate appropriate derivatives and coderivatives of the corresponding solution maps. These developments allow us to resolve some important issues related to conic programming. They include: verifiable conditions for isolated calmness of the considered solution maps, sharp necessary optimality conditions for a class of mathematical programs with equilibrium constraints, and characterizations of tilt-stable local minimizers for cone-constrained problems. The main results obtained in the general conic programming setting are specified for and illustrated by the second-order cone programming.

Key words. variational analysis, second-order theory, conic programming, generalized differentiation, optimality conditions, isolated calmness, tilt stability.

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1. Introduction. The major motivation for this paper comes from considering the following parametric problem of *conic programming* in finite-dimensional spaces:

$$\begin{aligned} & \text{minimize} && \varphi(q, y) - \langle p, y \rangle \\ & \text{subject to} && \\ & && g(y) \in \Theta, \end{aligned} \tag{1.1}$$

where $y \in \mathbb{R}^m$ is the *decision* variable, $x = (q, p) \in \mathbb{R}^s \times \mathbb{R}^m$ is the two-component *perturbation parameter* (with q signifying the *basic* perturbations and p the *tilt* ones), $\Theta \subset \mathbb{R}^l$ is a *closed convex cone*, and $\varphi: \mathbb{R}^s \times \mathbb{R}^m \rightarrow \mathbb{R}$ and $g: \mathbb{R}^m \rightarrow \mathbb{R}^l$ are *twice continuously differentiable*, i.e., of class \mathcal{C}^2 . These are our *standing assumptions* in this paper unless otherwise stated.

The characteristic feature of the optimization problem (1.1) is the *cone constraint* given by $g(y) \in \Theta$, which unifies remarkable subclasses of conic programs when the cone Θ is given in a particular form. Among well-recognized theoretically and most important for applications subclasses in conic programming we mention problems of second-order cone programming, semidefinite programming, and copositive programming; see, e.g., [1, 2, 4, 5, 6, 7, 8, 32, 38, 39] and the reference therein. Note that the cone-constrained form (1.1) accommodates also the class of semi-infinite programs

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provided that Θ is a closed convex subcone of the corresponding infinite-dimensional space; see the recent paper [24] containing the study of nonsmooth conic programs in both finite and infinite dimensions.

It is well known from elementary variational analysis (see, e.g., [22, Proposition 5.1] and [37, Theorem 6.12]) that first-order necessary optimality conditions for problem (1.1) are described by the parameterized *generalized equation* (GE)

$$0 \in f(x, y) + \widehat{N}_\Gamma(y) \quad \text{with } \Gamma := g^{-1}(\Theta) \quad (1.2)$$

in the sense of Robinson [35], where $\widehat{N}_\Gamma(y)$ stands for the *regular normal cone* to Γ at $y \in \Gamma$ (see Section 2 below for the precise definition), and where $f(x, y) := \nabla_y \varphi(p, y) - q$ with the symbol ∇ used for the (partial) gradient of scalar functions as well as for the Jacobian matrix in the case of vector functions. Besides being associated with conic programs (1.1), model (1.2) is of its independent interest and deserves the study for its own sake. When the set Γ is convex, (1.2) encompasses classical variational inequalities, which have been widely studied in the literature together with similar models generated by other normal cone mappings replacing $\widehat{N}_\Gamma(y)$; see, e.g., [9, 20, 22, 31, 35, 36] and the references therein. In this paper we mainly concentrate on model (1.2) generated by the regular normal cone to Γ . Note that most of the results obtained below are new even when the set Γ is convex.

In what follows we consider the variational system (1.2) with adding to our standing assumptions on g and Θ that $f: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ is an arbitrary *continuously differentiable* mapping. This surely covers the original model (1.1) while having a much broader range of applications; some of them are given in Section 6. Observe here another important case concerning perturbed conic programming that can be described by the generalized equation (1.2). Let $x = (u, v) \in \mathbb{R}^s \times \mathbb{R}^l$, $y = (z, \lambda) \in \mathbb{R}^s \times \mathbb{R}^l$,

$$f(x, y) := \begin{bmatrix} \nabla_y \varphi(z) - u + \nabla g(z)^* \lambda \\ -g(z) + v \end{bmatrix},$$

and $\Theta = \mathbb{R}^s \times \Xi^*$, where the notation A^* for a matrix A signifies the matrix *transposition/adjoint operator* while that of Ξ^* for the cone Ξ denotes the *dual/polar cone* of Ξ given by $\Xi^* := \{a \in \mathbb{R}^l \mid \langle a, b \rangle \leq 0 \text{ for all } b \in \Xi\}$. Then (1.2) amounts (under some constraint qualification) to the *Karush-Kuhn-Tucker (KKT) system* associated with the following *canonically* perturbed conic program:

$$\begin{aligned} & \text{minimize} && \varphi(y) - \langle u, y \rangle \\ & \text{subject to} && g(y) - v \in \Xi. \end{aligned}$$

Consider the (generally set-valued) *solution map* $S: \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ to the parametric variational system (1.2) defined by

$$S(x) := \{y \in \mathbb{R}^m \mid 0 \in f(x, y) + \widehat{N}_\Gamma(y)\} \quad \text{for all } x \in \mathbb{R}^n \quad (1.3)$$

with the *graph* $\text{gph } S := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y \in S(x)\}$, and let $(\bar{x}, \bar{y}) \in \text{gph } S$ be our *reference point*. Throughout the whole paper we impose the following assumptions on the set Θ and the mapping g in (1.2) standard in conic programming; see, e.g., [4]:

- (A1)** The set Θ is \mathcal{C}^2 -*reducible* to a closed convex set Ξ at $\bar{z} := g(\bar{y})$, and the reduction is pointed. This means that there exist a neighborhood V of \bar{z} and a \mathcal{C}^2 mapping $h: V \rightarrow \mathbb{R}^k$ such that: (i) for all $z \in V$ we have $z \in \Theta$ if and only if $h(z) \in \Xi$, where the cone $T_\Xi(h(\bar{z}))$ is pointed; (ii) $h(\bar{z}) = 0$ and the derivative mapping $\nabla h(\bar{z}): \mathbb{R}^l \rightarrow \mathbb{R}^k$ is surjective/onto.

(A2) The point $\bar{y} \in \mathbb{R}^m$ is *nondegenerate* for g with respect to Θ , i.e.,

$$\nabla g(\bar{y})\mathbb{R}^m + \text{lin}(T_{\Theta}(\bar{z})) = \mathbb{R}^l. \quad (1.4)$$

In (A1), (A2), $T_{\Omega}(a)$ stands for the classical tangent cone to Ω at $a \in \Omega$, and $\text{lin}(L)$ denotes the largest linear subspace of \mathbb{R}^l contained in $L \subset \mathbb{R}^l$.

Note that all the assumptions in (A1) are automatically fulfilled at any point $g(\bar{y}) \in \Theta$ in the following two important settings of cone programming: when Θ is either the *SDP cone*, i.e., the positive cone in semidefinite programming (cf. [3, Corollary 4.6]), or it is the *Lorentz cone* known also as the *second-order cone* and as the *ice-cream cone*; cf. [5, Lemma 15].

To proceed further, associate with (1.2) the *Lagrangian* function

$$\mathcal{L}(x, y, \lambda) := f(x, y) + \nabla g(y)^* \lambda \quad \text{with } \lambda \in \mathbb{R}^l. \quad (1.5)$$

It has been well recognized in conic programming that under the assumptions imposed in (A1) and (A2) there is a *unique* Lagrange multiplier λ satisfying the KKT conditions

$$\begin{aligned} 0 &= \mathcal{L}(x, y, \lambda), \\ 0 &\in -g(y) + N_{\Theta^*}(\lambda). \end{aligned} \quad (1.6)$$

The intention of this paper is to study two important stability properties of parametric equilibrium problems involving the solution map (1.3) and also to derive necessary optimality conditions for mathematical programs with equilibrium constraints (MPECs) generated by GE (1.2). It has been well recognized in basic variational analysis that all these three issues for general set-valued mappings are closely related to certain *generalized derivatives/coderivatives* of the mappings in question. Observe to this end that the set-valued part of the variational system (1.2) is given by a *normal cone* mapping, i.e., by the corresponding *first-order subdifferential* of the indicator function of the underlying set $\Gamma = g^{-1}(\Theta)$. Therefore, applying yet another generalized differentiation, we enter *second-order variational analysis* of the initial cone-constrained systems. Our main tool will be *second-order* generalized derivatives; see Section 2 for the precise definitions of and more discussions on these and related constructions in variational analysis and the subsequent sections for their calculations and applications in the settings under consideration. Note that the major results obtained in the paper in the *general conic programming* setting are specified for the case when Θ amounts to the Lorentz cone and illustrated by examples.

The rest of the paper is organized as follows. In Section 2 we present *basic definitions* and discussions of the first-order and second-order generalized differential constructions used in the formulations and proofs of the main results below.

Section 3 is devoted to calculating the second-order generalized differential construction defined by *graphical derivative* of the *normal cone mapping* from (1.2) under the convexity assumption imposed on Γ . The principal result obtained here represents this generalized derivative in terms of the problem data involving the directional derivative of the metric *projection* onto the polar of the cone Θ .

Section 4 concerns calculating the *regular coderivative* of the *regular normal cone* mapping $\widehat{N}_{\Gamma}(y)$, which can be treated as the *regular second-order subdifferential* of the indicator function of Γ . The main result of this section gives in fact a new *second-order subdifferential chain rule* for the regular constructions under consideration in the case of *nondegeneracy* imposed in (A2).

In Section 5 we unify the results of calculating the second-order constructions with known calculus rules to express the graphical derivatives and both regular and limiting coderivatives of the *solution map* (1.3) in terms of the initial data. The derivative/coderivative formulas derived in this way and combined with basic characterizations in general frameworks of variational analysis allow us to establish in Section 6 new criteria for *isolated calmness* of the solution map (1.3), necessary optimality conditions for *MPECs* defined via (1.3), and to characterize *tilt stability* of local minimizers in conic programming. The concluding Section 7 describes some lines of our future research concerning, in particular, the characterization of *full stability* of local solutions to cone-constrained optimization problems and related topics.

Throughout the paper we use standard notation of variational analysis and optimization; see, e.g., the books [4, 22, 37]. Recall that the (Painlevé-Kuratowski) *outer limit* of the set-valued mapping/multifunction $F: \mathbb{R}^d \rightrightarrows \mathbb{R}^s$ as $z \rightarrow \bar{z}$ is defined by

$$\text{Lim sup}_{z \rightarrow \bar{z}} F(z) := \left\{ v \in \mathbb{R}^s \mid \exists z_k \rightarrow \bar{z}, v_k \rightarrow v \text{ with } v_k \in F(z_k) \text{ as } k \in \mathbb{N} \right\} \quad (1.7)$$

and that, given a set $\Omega \subset \mathbb{R}^d$, the symbol $z \xrightarrow{\Omega} \bar{z}$ signifies that $z \rightarrow \bar{z}$ with $z \in \Omega$. As usual, the notation a^T indicates the vector transposition, I stands for the identity matrix, and \mathbb{B} denotes the closed unit ball of the space in question.

2. Tools of Variational Analysis. Generalized differentiation of nonsmooth and set-valued mappings, as well as generalized normals and tangents to sets, play a crucial role in modern variational analysis and optimization; see, e.g., the books [22, 37] and the references therein. In this section we briefly review some first-order and second-order generalized differential constructions employed in the paper, confining ourselves only to the settings that appear below. The reader can find more details and extended frameworks in the aforementioned books and in the papers we refer to.

Let us start with geometric objects. Given a set $\Omega \subset \mathbb{R}^d$ and a point $\bar{z} \in \Omega$, define the (Bouligand-Severi) *tangent/contingent cone* to Ω at \bar{z} via (1.7) by

$$T_{\Omega}(\bar{z}) := \text{Lim sup}_{t \downarrow 0} \frac{\Omega - \bar{z}}{t} = \left\{ u \in \mathbb{R}^d \mid \exists t_k \downarrow 0, u_k \rightarrow u \text{ with } \bar{z} + t_k u_k \in \Omega \right\}. \quad (2.1)$$

The (Fréchet) *regular normal cone* to Ω at $\bar{z} \in \Omega$ can be equivalently defined by

$$\widehat{N}_{\Omega}(\bar{z}) := \left\{ v \in \mathbb{R}^d \mid \limsup_{z \xrightarrow{\Omega} \bar{z}} \frac{\langle v, z - \bar{z} \rangle}{\|z - \bar{z}\|} \leq 0 \right\} = T_{\Omega}^*(\bar{z}). \quad (2.2)$$

The (Mordukhovich) *limiting normal cone* can be also defined in the two equivalent ways: via the outer limit (1.7) of the regular normal cone (2.2) and via the (Euclidean) *metric projection* operator $P_{\Omega}(z) := \{y \in \Omega \text{ with } \|z - y\| = \text{dist}(z; \Omega)\}$ onto Ω by

$$N_{\Omega}(\bar{z}) := \text{Lim sup}_{z \xrightarrow{\Omega} \bar{z}} \widehat{N}_{\Omega}(z) = \text{Lim sup}_{z \rightarrow \bar{z}} \left\{ \text{cone}[z - P_{\Omega}(z)] \right\}, \quad (2.3)$$

where Ω is assumed to be locally closed around \bar{z} in the second representation, and where the symbol ‘cone’ signifies the (nonconvex) conic hull of a set. Note that both regular and limiting normal cones reduce to the classical normal cone of convex analysis when the set Ω is convex and when the common notation $N_{\Omega}(\bar{z})$ is used. For general sets Ω we have the inclusion

$$\widehat{N}_{\Omega}(\bar{z}) \subset N_{\Omega}(\bar{z}) \text{ as } \bar{z} \in \Omega,$$

where the regular normal cone values are always convex while it is not often the case for the limiting normal cone; see, e.g., $\Omega = \text{gph}|z| \subset \mathbb{R}^2$ at $\bar{z} = (0, 0)$. At the same time, in contrast to (2.2), we have $N_\Omega(\bar{z}) \neq 0$ for boundary points. Furthermore, the limiting normal cone (2.3) and the corresponding generalized differential constructions for functions and multifunctions (in particular, those discussed below) possess *full calculi* based on *variational/extremal principles* of variational analysis; see [22, 37]. Although it is not generally the case for the regular normal cone, in this paper we derive new results in this direction for the *second-order* objects induced by (2.2).

Considering next set-valued (in particular, single-valued) mappings $F: \mathbb{R}^d \rightrightarrows \mathbb{R}^s$, define for them the corresponding derivative and coderivative constructions generated by the tangent cone (2.1) and the normal cones (2.2) and (2.3), respectively. Given $(\bar{z}, \bar{w}) \in \text{gph } F$, the *graphical derivative* $DF(\bar{z}, \bar{w}): \mathbb{R}^d \rightrightarrows \mathbb{R}^s$ of F at (\bar{z}, \bar{w}) is

$$DF(\bar{z}, \bar{w})(u) := \{q \in \mathbb{R}^s \mid (u, q) \in T_{\text{gph } F}(\bar{z}, \bar{w})\}, \quad u \in \mathbb{R}^d. \quad (2.4)$$

From the dual perspectives we define the *regular coderivative* $\widehat{D}^*F(\bar{z}, \bar{w}): \mathbb{R}^s \rightrightarrows \mathbb{R}^d$ of F at $(\bar{z}, \bar{w}) \in \text{gph } F$ generated by the regular normal cone (2.2) as

$$\widehat{D}^*F(\bar{z}, \bar{w})(v) := \{p \in \mathbb{R}^d \mid (p, -v) \in \widehat{N}_{\text{gph } F}(\bar{z}, \bar{w})\}, \quad v \in \mathbb{R}^s, \quad (2.5)$$

and the corresponding *limiting coderivative* $D^*F(\bar{x}, \bar{y})$ generated by (2.3) as

$$D^*F(\bar{z}, \bar{w})(v) := \{p \in \mathbb{R}^d \mid (p, -v) \in N_{\text{gph } F}(\bar{x}, \bar{y})\}, \quad v \in \mathbb{R}^s. \quad (2.6)$$

If F is single-valued at \bar{z} , we drop \bar{w} in the notation of (2.4)–(2.6). Note that, while the regular coderivative (2.5) is indeed dual to the graphical derivative (2.4) due to the duality correspondence in (2.2), the limiting coderivative (2.6) is *dual to none*, since the nonconvex normal cone (2.3) cannot be tangentially generated. In the case of smooth single-valued mappings, for all $u \in \mathbb{R}^d$ and $v \in \mathbb{R}^s$ we have the representation

$$DF(\bar{z})(u) = \{\nabla F(\bar{z})u\} \quad \text{and} \quad \widehat{D}^*F(\bar{x})(v) = D^*F(\bar{x})(v) = \{\nabla F(\bar{z})^*v\}.$$

As discussed in Section 1, the main emphasis of this paper is on the *second-order* generalized differential constructions appropriate for the aforementioned applications. Among several approaches to second-order generalized differentiation in variational analysis (cf. [4, 22, 37]), we choose the set-valued version of the “derivative-of-derivative” approach initiated in [21], which is applied to arbitrary extended-real-valued functions and treats second-order differentiation for them as a certain generalized derivative of a (set-valued) first-order subdifferential mapping that reduces to the corresponding normal cone mapping in the case of the indicator functions of sets. In this way the (limiting) *second-order subdifferential/generalized Hessian*

$$\partial^2\varphi(\bar{z}, \bar{w})(v) := (D^*\partial\varphi)(\bar{z}, \bar{w})(v), \quad v \in \mathbb{R}^d, \quad (2.7)$$

of $\varphi: \mathbb{R}^d \rightarrow \overline{\mathbb{R}} := (-\infty, \infty]$ at $\bar{w} \in \partial\varphi(\bar{z})$ has been introduced in [21] and then widely applied to various stability and optimality issues in mathematical programming; see, e.g., [8, 10, 14, 15, 18, 19, 21, 22, 24, 25, 26, 27, 28, 29, 30, 32, 33, 34, 40] and the references therein. In (2.7), $\partial\varphi$ stands for the (first-order) *limiting subdifferential* mapping $\partial\varphi: \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ defined, via $\text{epi } \varphi := \{(z, \alpha) \in \mathbb{R}^{d+1} \mid \alpha \geq \varphi(z)\}$, by

$$\partial\varphi(z) := \{w \in \mathbb{R}^d \mid (w, -1) \in N_{\text{epi } \varphi}(z, \varphi(z))\}. \quad (2.8)$$

if $\varphi(x) < \infty$ and $\partial\varphi(x) := \emptyset$ otherwise. In this paper we use it only in the case of the set *indicator function* $\varphi(z) = \delta_\Omega(z)$, equal to 0 when $z \in \Omega$ and to ∞ otherwise, when

$$\partial\varphi(z) = N_\Omega(z) \quad \text{and} \quad \partial^2\varphi(\bar{z}, \bar{w})(v) = D^*N(\bar{z}, \bar{w})(v) \quad \text{for } \bar{z} \in \Omega \quad \text{and} \quad \bar{w} \in N_\Omega(\bar{z}).$$

Similarly to (2.7) we can define the *regular second-order subdifferential/generalized Hessian* $\widehat{\partial}^2\varphi(\bar{z}, \bar{w}) := (\widehat{D}^*\widehat{\partial}\varphi)(\bar{z}, \bar{w})$, where $\widehat{\partial}\varphi$ comes from (2.8) with the replacement of (2.3) by (2.2), and its indicator function specification $\widehat{D}^*\widehat{N}(\bar{z}, \bar{w})$, which has already been studied and applied in [14, 15] in the settings different from this paper.

Yet another second-order construction for φ of our interest here has been considered in [37] under the name of the “subgradient graphical derivative” defined in scheme (2.7) as $(D\partial\varphi)(\bar{z}, \bar{w})(v)$ for $\bar{w} \in \partial\varphi(\bar{z})$, $v \in \mathbb{R}^d$. Note that for $\varphi \in \mathcal{C}^2$ we have

$$(D\partial\varphi)(\bar{z})(v) = \widehat{\partial}^2\varphi(\bar{z})(v) = \partial^2\varphi(\bar{z})(v) = \{\nabla^2\varphi(\bar{z})v\} \quad \text{for all } v \in \mathbb{R}^d$$

due to the symmetry of the classical Hessian. As for $\partial^2\varphi$ and $\widehat{\partial}^2\varphi$, we draw our attention to calculating $D\partial\varphi$ for $\varphi = \delta_\Gamma$, i.e., of the *graphical derivative of the normal cone mapping* DN_Γ with $\Gamma = g^{-1}(\Theta)$ arising from conic programming.

3. Graphical Derivatives of Normal Cone Mappings. Throughout this section the set $\Gamma \subset \mathbb{R}^m$ from (1.2) is *convex*, i.e., $\widehat{N}_\Gamma(y)$ reduces to the normal cone $N_\Gamma(y)$ of convex analysis. The main result of the section provides a complete calculation of the graphical derivative of the normal cone mapping N_Γ . Besides deriving the main result used in the subsequent applications, the methods developed and other results obtained in this section seem to be of their independent interest.

The convexity of $\Gamma = g^{-1}(\Theta)$ is obviously ensured by the Θ -convexity of the mapping $g: \mathbb{R}^m \rightarrow \mathbb{R}^l$, in the sense that the set

$$\{(y, z) \in \mathbb{R}^m \times \mathbb{R}^l \mid g(y) - z \in \Theta\}$$

is convex. This definition corresponds to the “ $-\Theta$ -convexity” of g in the terminology of [4, Definition 2.103], while it is more convenient for us to use the term “ Θ -convexity” in the sense defined above. Since $g \in \mathcal{C}^2$, its Θ -convexity is equivalent to the condition

$$\langle \nabla^2 g(y)(h, h), \nu \rangle \geq 0 \quad \text{for all } \nu \in \Theta^* \quad \text{and} \quad y, h \in \mathbb{R}^m, \quad (3.1)$$

which is assumed in the rest of this section.

Consider first the auxiliary *linear GE*

$$0 \in y - u + N_\Gamma(y) \quad (3.2)$$

associated with the (unique) *metric projection* $P_\Gamma(u)$ of $u \in \mathbb{R}^m$ onto Γ . We clearly have that $y = P_\Gamma(u)$ if and only if

$$\begin{bmatrix} y \\ u - y \end{bmatrix} \in \text{gph } N_\Gamma. \quad (3.3)$$

Take next a vector $\bar{u} \in \mathbb{R}^m$ with $P_\Gamma(\bar{u}) = \bar{y}$. It follows from (A2) that there is a unique Lagrange multiplier $\nu \in \mathbb{R}^l$ such that the triple $(\bar{u}, \bar{y}, \bar{\nu})$ satisfies the GE (partially perturbed KKT system)

$$\begin{aligned} 0 &= y - u + \nabla g(y)^*\nu, \\ 0 &\in -g(y) + N_{\Theta^*}(\nu), \end{aligned} \quad (3.4)$$

where we treat (y, ν) as the decision variable and u as a perturbation parameter.

Fix $u = \bar{u}$ and recall that system (3.4) is *strongly regular*, in the sense of Robinson [36], at $(\bar{y}, \bar{\nu})$ if for all $t = (t_1, t_2)$ near $0 \in \mathbb{R}^m \times \mathbb{R}^l$ the partial linearization

$$\begin{aligned} t_1 &= y - \bar{u} + \nabla g(\bar{y})^* \nu + \bar{\nu}^T \nabla^2 g(\bar{y})(y - \bar{y}), \\ t_2 &\in -g(\bar{y}) - \nabla g(\bar{y})(y - \bar{y}) + N_{\Theta^*}(\nu) \end{aligned}$$

admits a single-valued Lipschitzian solution $(y(t), \nu(t))$ near $(\bar{y}, \bar{\nu})$.

The following results justifies strong regularity of the KKT system (3.4) under the standing assumptions of this section.

LEMMA 3.1 (strong regularity). *Given $\bar{y} \in P_\Gamma(\bar{u})$ and the corresponding Lagrange multiplier $\bar{\nu}$, the perturbed KKT system (3.4) is strongly regular at $(\bar{y}, \bar{\nu})$.*

Proof. It is shown in [4, Theorem 5.24] that the strong regularity of (3.4) at $(\bar{y}, \bar{\nu})$ is equivalent to the validity of the *uniform second-order growth condition* from [4, Definition 5.16] for the problem under consideration at \bar{y} . Arguing by contradiction, suppose that this condition does not hold and consider the function

$$\vartheta(u, y) := \frac{1}{2} \|y - u\|^2.$$

Then there are sequences $u_k \rightarrow \bar{u}$, $y_k \rightarrow \bar{y}$, $\nu_k \rightarrow \bar{\nu}$, and $h_k \rightarrow 0$ such that the pair (y_k, ν_k) solves the generalized equation (3.4) with $u = u_k$, $g(y_k + h_k) \in \Theta$, and

$$\vartheta(u_k, y_k + h_k) \leq \vartheta(u_k, y_k) + o(\|h_k\|^2) \text{ for all } k \in \mathbb{N}. \quad (3.5)$$

By passing to a subsequence if necessary, we suppose without loss of generality that the sequence $\{h_k/\|h_k\|\}$ itself converges to some vector $\bar{h} \neq 0$ as $k \rightarrow \infty$. On the other hand, since the inclusion $g(y_k) \in N_{\Theta^*}(\nu_k)$ in (3.4) readily implies that

$$\nu_k \in \Theta^* \text{ and } \langle \nu_k, g(y_k) \rangle = 0 \text{ for any } k \in \mathbb{N},$$

we deduce from it as well as from $g(y_k) \in \Theta$ and $g(y_k + h_k) \in \Theta$ that $\langle \nu_k, g(y_k + h_k) - g(y_k) \rangle \leq 0$, which together with (3.5) ensure that

$$L(u_k, y_k + h_k, \nu_k) - L(u_k, y_k, \nu_k) \leq \vartheta(u_k, y_k + h_k) - \vartheta(u_k, y_k) \leq o(\|h_k\|^2) \quad (3.6)$$

via the Lagrangian $L(u, y, \nu) := \vartheta(u, y) + \langle \nu, g(y) \rangle$ associated with the *convex* optimization problem $\min_{y \in \Gamma} \vartheta(u, y)$. Considering the other Lagrangian \mathcal{L} from (1.5) in the case of GE (3.2), observe the relationship $\mathcal{L} = \nabla_y L$, which implies therefore that $\nabla_y L(u_k, y_k, \nu_k) = 0$. Taking now (3.6) into account, we get

$$\frac{1}{2} \nabla_{yy}^2 L(u_k, y_k, \nu_k)(h_k, h_k) = L(u_k, y_k + h_k, \nu_k) - L(u_k, y_k, \nu_k) + o(\|h_k\|^2) \leq o(\|h_k\|^2)$$

for all $k \in \mathbb{N}$. This gives us by passing to the limit as $k \rightarrow \infty$ that

$$\nabla_{yy}^2 L(\bar{u}, \bar{y}, \bar{\nu})(\bar{h}, \bar{h}) \leq 0.$$

Since $\bar{h} \neq 0$, we get therefore that

$$\langle \nabla^2 g(\bar{y})(\bar{h}, \bar{h}), \bar{\nu} \rangle < \|\bar{h}\|^2 + \langle \nabla^2 g(\bar{y})(\bar{h}, \bar{h}), \bar{\nu} \rangle = \nabla_{yy}^2 L(\bar{u}, \bar{y}, \bar{\nu})(\bar{h}, \bar{h}) \leq 0,$$

which clearly contradicts (3.1) and thus completes the proof of the lemma. \square

The next lemma gives a workable representation of the classical directional derivative of the (single-valued) metric projection operator onto the convex set Γ in terms of the initial data of (1.2) and plays a crucial role in deriving the main result of this section in what follows. It contains an assumption on the directional differentiability of the projection operator onto the polar of the original constraint cone Θ , which is not restrictive and holds for the vast majority of conic programs important for optimization theory and applications (in particular, for semidefinite and second-order cone programming). On the other hand, the required directional differentiability of the projection operator is not always available for an arbitrary convex cone Θ in finite dimensions; see a rather involved counterexample [16] of a solid cone in \mathbb{R}^3 .

LEMMA 3.2 (directional derivative of the projection operator). *In addition to the standing assumptions, suppose that the projection operator P_{Θ^*} onto the polar of Θ is directionally differentiable on \mathbb{R}^l . Then for any $\bar{u} \in \mathbb{R}^m$ there is a neighborhood \mathcal{U} of \bar{u} such that the projection operator P_Γ onto Γ is directionally differentiable at each $u \in \mathcal{U}$ in every direction $h \in \mathbb{R}^m$. Furthermore, this directional derivative is calculated by $P'_\Gamma(u; h) = v_1$, where v_1 is the first component of the unique solution $v = (v_1, v_2) \in \mathbb{R}^m \times \mathbb{R}^l$ to the system of equations*

$$\begin{aligned} h &= \left(I + \sum_{i=1}^l \nu_i \nabla^2 g_i(y) \right) v_1 + \nabla g(y)^* v_2, \\ 0 &= v_2 - P'_{\Theta^*}(g(y) + \nu; \nabla g(y) v_1 + v_2) \end{aligned} \quad (3.7)$$

with $y = P_\Gamma(u)$ and $\nu = (\nu_1, \dots, \nu_l) \in \mathbb{R}^l$ being the unique Lagrange multiplier corresponding to the pair (u, y) in the KKT system (3.4).

Proof. Define the mapping $\Phi: \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^l \rightarrow \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^l$ by

$$\Phi(w, y, \nu) := \begin{bmatrix} w \\ y - w + \nabla g(y)^* \nu \\ \nu - P_{\Theta^*}(g(y) + \nu) \end{bmatrix} \quad (3.8)$$

and observe that the standard nonlinear equation

$$\begin{bmatrix} u \\ t_1 \\ t_2 \end{bmatrix} = \Phi(w, y, \nu)$$

amounts exactly to the perturbed GE

$$\begin{aligned} t_1 &= y - u + \nabla g(y)^* \nu, \\ t_2 &\in -g(y) + N_{\Theta^*}(\nu - t_2) \end{aligned} \quad (3.9)$$

with the perturbations t_1, t_2, u and the decision variables y, ν . Let us now introduce the new decision variable $\mu := \nu - t_2$ to obtain the new perturbed generalized equation

$$\begin{aligned} t_1 &= y - u + \nabla g(y)^*(\mu + t_2), \\ t_2 &\in -g(y) + N_{\Theta^*}(\mu) \end{aligned} \quad (3.10)$$

with the same perturbations t_1, t_2, u and the decision variables y, μ .

First we claim that the strong regularity of system (3.4) at $(\bar{y}, \bar{\nu})$ with $u = \bar{u}$, ensured by Lemma 3.1, implies the *strong regularity* of the generalized equation (3.10) at the point $(\bar{y}, \bar{\nu})$ with $(t_1, t_2, u) = (0, 0, \bar{u})$. Indeed, since $\bar{\mu} = \bar{\nu}$, it follows from

the observation that the partial linearization of the single-valued term in (3.10) with respect to (y, μ) at $(0, 0, \bar{u}, \bar{y}, \bar{\mu})$ reduces to the partial linearization of the single-valued term in (3.4) with respect to (y, ν) at $(\bar{u}, \bar{y}, \bar{\nu})$. Since the set-valued parts in (3.4) and (3.10) are the same, our claim is justified.

In the second step we prove that Φ is a *Lipschitzian homeomorphism* near $(\bar{u}, \bar{y}, \bar{\nu})$. To proceed, denote by $\Sigma: (u, t_1, t_2) \mapsto (y, \mu)$ the solution map associated with the generalized equation (3.10). The justified strong regularity of (3.10) tells us that there are neighborhoods \mathcal{O}_1 of \bar{u} , \mathcal{O}_2 of $0_{\mathbb{R}^m}$, \mathcal{O}_3 of $0_{\mathbb{R}^l}$, \mathcal{M} of \bar{y} , and \mathcal{N} of $\bar{\nu}$ as well as a single-valued Lipschitzian mapping $\sigma: \mathcal{O}_1 \times \mathcal{O}_2 \times \mathcal{O}_3 \rightarrow \mathcal{M} \times \mathcal{N}$ such that

$$\sigma(\bar{u}, 0, 0) = (\bar{y}, \bar{\nu}) \quad \text{and}$$

$$\Sigma(u, t_1, t_2) \cap (\mathcal{M} \times \mathcal{N}) = \{\sigma(u, t_1, t_2)\} \quad \text{for all } (u, t_1, t_2) \in \mathcal{O}_1 \times \mathcal{O}_2 \times \mathcal{O}_3.$$

Let now $\varrho: \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^l \rightarrow \mathbb{R}^m \times \mathbb{R}^l$ be a Lipschitzian mapping defined by

$$\varrho(u, t_1, t_2) := \begin{bmatrix} \sigma_1(u, t_1, t_2) \\ \sigma_2(u, t_1, t_2) + t_2 \end{bmatrix}, \quad (3.11)$$

where σ_1 and σ_2 are the components of σ mapping the triple (u, t_1, t_2) to the variables u and μ , respectively. It follows that the second component of ϱ assigns (u, t_1, t_2) the original variable ν from (3.9). Replacing if necessary the above neighborhoods $\mathcal{O}_1 \times \mathcal{O}_2 \times \mathcal{O}_3$ by some smaller ones $\widetilde{\mathcal{O}}_1, \widetilde{\mathcal{O}}_2$, and $\widetilde{\mathcal{O}}_3$ satisfying the condition

$$\Phi^{-1}(u, t_1, t_2) \cap (\widetilde{\mathcal{O}}_1 \times \mathcal{M} \times \mathcal{N}) = \{u\} \times \{\varrho(u, t_1, t_2)\} \quad \text{for all } (u, t_1, t_2) \in \widetilde{\mathcal{O}}_1 \times \widetilde{\mathcal{O}}_2 \times \widetilde{\mathcal{O}}_3,$$

we conclude that Φ is indeed a Lipschitzian homeomorphism near $(\bar{u}, \bar{y}, \bar{\nu})$.

In the third step of the proof, observe that the mapping Φ from (3.8) is *directionally differentiable* due to the assumption on P_{Θ^*} . Invoking now the inverse mapping theorem due to Kummer (see, e.g., [31, Lemma 6.1] and the references therein), we conclude that the single-valued and Lipschitz continuous local inverse mapping

$$\Psi := \Phi^{-1} \cap (\widetilde{\mathcal{O}}_1 \times \mathcal{M} \times \mathcal{N})$$

is also directionally differentiable at $\Phi(w, y, \nu) \in \widetilde{\mathcal{O}}_1 \times \widetilde{\mathcal{O}}_2 \times \widetilde{\mathcal{O}}_3$ and its directional derivative is related to the directional derivative of Φ by

$$\Psi'(\Phi(w, y, \nu); (h, 0, 0)) = \begin{bmatrix} h \\ v_1 \\ v_2 \end{bmatrix}, \quad \text{where} \quad \begin{bmatrix} h \\ 0 \\ 0 \end{bmatrix} = \Phi'((w, y, \nu); (h, v_1, v_2)).$$

This yields our conclusion on the directional differentiability of P_Γ with $P'_\Gamma(u; h) = v_1$ along the solution to system (3.7) by using standard calculus rules in the expression for Φ in (3.8). Observe finally that, since the mapping ϱ in (3.11) is locally Lipschitzian, the required neighborhood \mathcal{U} of \bar{u} can be found in such a way that $\mathcal{U} \subset \widetilde{\mathcal{O}}_1$ and

$$\varrho(\mathcal{U}, 0, 0) \subset \mathcal{M} \times \mathcal{N},$$

which is claimed in the lemma. \square

Lemma 3.2 established above is a crucial ingredient in deriving the main result of this section given in the following theorem.

THEOREM 3.3 (calculating the graphical derivative of the normal cone mapping). *Let the assumptions of Lemma 3.2 hold, let $(\bar{u}, \bar{y}) \in \text{gph } P_\Gamma$, $\bar{w} := \bar{u} - \bar{y}$, and $\bar{\nu} = (\bar{\nu}_1, \dots, \bar{\nu}_l)$ be a unique Lagrange multiplier associated with (\bar{u}, \bar{y}) via (3.4). Then the tangent cone (2.1) to the graph of N_Γ at (\bar{y}, \bar{w}) admits the representation*

$$T_{\text{gph}N_\Gamma}(\bar{y}, \bar{w}) = \left\{ (v, p) \mid \exists d \in \mathbb{R}^l \text{ with } p = \left(\sum_{i=1}^l \bar{\nu}_i \nabla^2 g_i(\bar{y}) \right) v + \nabla g(\bar{y})^* d, \right. \\ \left. d = P'_{\Theta^*}(g(\bar{y}) + \bar{\nu}; \nabla g(\bar{y})v + d) \right\}. \quad (3.12)$$

Consequently, for any direction $v \in \mathbb{R}^m$ the graphical derivative of N_Γ at (\bar{y}, \bar{w}) is

$$DN_\Gamma(\bar{y}, \bar{w})(v) = \left\{ p \in \mathbb{R}^l \mid \exists d \in \mathbb{R}^l \text{ with } p = \left(\sum_{i=1}^l \bar{\nu}_i \nabla^2 g_i(\bar{y}) \right) v + \nabla g(\bar{y})^* d, \right. \\ \left. d = P'_{\Theta^*}(g(\bar{y}) + \bar{\nu}; \nabla g(\bar{y})v + d) \right\}. \quad (3.13)$$

Proof. By [15, Proposition 1] we have the relationship

$$T_{\text{gph}P_\Gamma}(\bar{u}, \bar{y}) = \text{gph } P'_\Gamma(\bar{u}; \cdot).$$

Using this and Lemma 3.2 gives us the tangent cone representation

$$T_{\text{gph}P_\Gamma}(\bar{u}, \bar{y}) = \left\{ (h, v_1) \in \mathbb{R}^m \times \mathbb{R}^m \mid \exists v_2 \in \mathbb{R}^l \text{ such that} \right. \\ \left. h = \left(I + \sum_{i=1}^l \bar{\nu}_i \nabla^2 g_i(\bar{y}) \right) v_1 + \nabla g(\bar{y})^* v_2, v_2 = P'_{\Theta^*}(g(\bar{y}) + \bar{\nu}; \nabla g(\bar{y})v_1 + v_2) \right\}.$$

Since $(y, w) \in \text{gph } N_\Gamma$ if and only if $(y + w, y) \in \text{gph } P_\Gamma$, it follows by the elementary calculus rule from [37, Exccercise 6.7] that

$$T_{\text{gph}N_\Gamma}(\bar{y}, \bar{w}) = \left\{ (v, p) \in \mathbb{R}^m \times \mathbb{R}^m \mid \begin{pmatrix} v + p \\ v \end{pmatrix} \in T_{\text{gph}P_\Gamma}(\bar{u}, \bar{y}) \right\},$$

which justifies the tangent cone formula (3.12). The graphical derivative result (3.13) follows immediately from (3.12) and definition (2.4). \square

We can see that formulas (3.12) and (3.13) for calculating the tangent cone to $\text{gph } N_\Gamma$ and the graphical derivative of N_Γ are valid for general conic programs while involving the directional derivative of the projection operator onto the polar Θ^* of the underlying cone Θ . In particular, this directional derivative was calculated entirely in terms of the initial data for the following two remarkable subclasses of conic programming: *second-order cone programs* (SOCPs) defined by the *Lorentz cone*

$$\Theta = \mathcal{K}^l := \{(\theta_1, \dots, \theta_l) \in \mathbb{R}^l \mid \theta_1 \geq \|(\theta_2, \dots, \theta_l)\|\} \quad (3.14)$$

with the Euclidean norm $\|\cdot\|$ (see [33, Lemma 2]) and *semidefinite programs* (SDPs) with Θ being the cone of *symmetric positive semidefinite matrices*

$$\Theta := \mathcal{S}_+^l = \{A \in \mathbb{R}^{l \times l} \mid z^T A z \geq 0 \text{ for all } z \in \mathbb{R}^l\}; \quad (3.15)$$

see [38, Theorem 4.7]. Let us present the calculation results for the case of $\Theta = \mathcal{K}^l$, which are used in what follows. Note that similar calculations can be also done in the case when Θ is the *Cartesian product* of finitely many Lorentz cones.

Prior to presenting the corresponding results from [33] for $\Theta = \mathcal{K}^l$, let us recall the relevant notation and facts from the theory of symmetric cones needed below; see, e.g., [12]. Given any vector $u = (u_1, \tilde{u}) \in \mathbb{R} \times \mathbb{R}^{l-1}$, we have its *spectral decomposition*

$$u = \lambda_1(u)c_1(u) + \lambda_2(u)c_2(u), \quad (3.16)$$

where $\lambda_1(u), \lambda_2(u)$ and $c_1(u), c_2(u)$ are the *spectral values* and *vectors* of u given by

$$\lambda_i(u) = u_1 + (-1)^i \|\tilde{u}\| \quad \text{and} \quad (3.17)$$

$$c_i(u) = \begin{cases} \frac{1}{2} \begin{pmatrix} 1, (-1)^i \frac{\tilde{u}}{\|\tilde{u}\|} \end{pmatrix} & \text{if } \tilde{u} \neq 0, \\ \frac{1}{2} \begin{pmatrix} 1, (-1)^i v \end{pmatrix} & \text{if } \tilde{u} = 0 \end{cases} \quad (3.18)$$

for $i = 1, 2$, with v being any unit vector in \mathbb{R}^{l-1} . The following proposition taken from [33, Lemma 2] describes the directional derivative of the metric projection onto the polar of the Lorentz cone $\Theta = \mathcal{K}^l$. Note that in this case we have $\Theta^* = -\mathcal{K}^l$.

PROPOSITION 3.4 (directional derivative of the projection onto the polar of the Lorentz cone). *Let $\Theta = \mathcal{K}^l$. Then the projection operator P_{Θ^*} is directionally differentiable at u and for any direction $h \in \mathbb{R}^l$ we have the relationships:*

(i) *If $u \in \text{int } \Theta \cup \text{int } \Theta^*$, then $P'_{\Theta^*}(u; h) = \nabla P_{\Theta^*}(u)h$ with*

$$\nabla P_{\Theta^*}(u) = -2 \sum_{i=1}^2 [\beta^{[1]}(\lambda(u))]_{ii} c_i(u) c_i(u)^T - [\beta^{[1]}(\lambda(u))]_{12} A(u),$$

$$\text{where we have } A(u) := \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} - (c_2(u) - c_1(u))(c_2(u) - c_1(u))^T,$$

$$\lambda(u) := (\lambda_1(u), \lambda_2(u)), \text{ and}$$

$$[\beta^{[1]}(\lambda)]_{ij} := \begin{cases} \frac{\beta(\lambda_i) - \beta(\lambda_j)}{\lambda_i - \lambda_j} & \text{if } \lambda_i \neq \lambda_j, \\ \beta'(\lambda_i) & \text{if } \lambda_i = \lambda_j, \end{cases} \quad i, j = 1, 2,$$

for the first divided difference matrix of the real-valued function $\beta(x) = (x)_- := \min\{0, x\}$ at λ , with $\lambda := (\lambda_1, \lambda_2) \in \mathbb{R}^2$ and $\lambda_1 \lambda_2 \neq 0$.

(ii) *If $u \in \text{bd } \Theta \setminus \{0\}$, then $P'_{\Theta^*}(u; h) = -2 \langle (c_1(u), h) \rangle_- c_1(u)$ with the notation $(x)_- := \min\{0, x\}$ as in (i).*

(iii) *If $u \in \text{bd } (\Theta^*) \setminus \{0\}$, then $P'_{\Theta^*}(u; h) = -h + 2 \langle (c_2(u), h) \rangle_+ c_2(u)$ with the notation $(x)_+ := \max\{0, x\}$.*

(iv) *If $u = 0$, then $P'_{\Theta^*}(u; h) = P_{\Theta^*}(h)$.*

4. Regular Coderivatives of Normal Cone Mappings. This section is devoted to deriving a formula for calculating the regular coderivative (2.5) of the regular normal cone mapping (2.2) to the set $\Gamma = g^{-1}(\Theta)$ from (1.2) via the initial data g and Θ under the standing assumptions formulated in Section 1. Note that due the

duality relationship between the regular normal and tangent cones in (2.2) we have, for an arbitrary set-valued mapping $F: \mathbb{R}^d \rightrightarrows \mathbb{R}^s$, that

$$\widehat{D}^*F(\bar{y}, \bar{v})(w) = \{p \in \mathbb{R}^d \mid \langle p, u \rangle \leq \langle w, q \rangle \text{ for all } (u, q) \in \text{gph } DF(\bar{y}, \bar{v})\} \quad (4.1)$$

whenever $(\bar{y}, \bar{v}) \in \text{gph } F$ and $w \in \mathbb{R}^s$. This implies that formula (3.13) of Theorem 3.3 induces, under the assumptions of Section 3, the corresponding representation of the regular coderivative of N_Γ by the duality relation (4.1).

However, in this section we explore another route to deal with the regular normal and coderivative constructions, which treats them directly, with no appeal to the tangential counterparts and duality correspondences. In this way we arrive at a new *second-order chain rule* to calculate the regular coderivative of the regular normal cone mapping \widehat{N}_Γ in the setting under consideration *without* imposing the convexity assumption on the set Γ as well as assuming the directional differentiability of the projection operator P_{Θ^*} . In the case of the limiting second-order subdifferential (2.7) such a calculus approach has been developed in the series of previous publications (see, e.g., [22, 26, 27, 29, 32]), with the result for D^*N_Γ closest to our setting appeared in [32, Theorem 7] under the nondegeneracy assumption. We are not familiar with any previous developments in this direction for the case of $\widehat{D}^*\widehat{N}_\Gamma$.

To proceed, consider an $[l \times k]$ -matrix $A(\cdot)$ whose entries are continuously differentiable functions of $x \in \mathbb{R}^n$, and let the components of $b(\cdot) \in \mathbb{R}^p$ be continuously differentiable functions of x . Then we have by the classical product rule that

$$\nabla(A(\cdot)b(\cdot))|_{x=\bar{x}} = \nabla(A(\cdot)b(\bar{x}))|_{x=\bar{x}} + \nabla(A(\bar{x})b(\cdot))|_{x=\bar{x}}. \quad (4.2)$$

THEOREM 4.1 (second-order chain rule for the regular coderivative of the regular normal cone mapping). *Let $\bar{v} \in \widehat{N}_\Gamma(\bar{y})$ under the standing assumptions from Section 1. Then for all $w \in \mathbb{R}^m$ we have the representation*

$$\widehat{D}^*\widehat{N}_\Gamma(\bar{y}, \bar{v})(w) = \sum_{i=1}^l \bar{\lambda}_i \nabla^2 g_i(\bar{y})w + \nabla g(\bar{y})^* \widehat{D}^*N_\Theta(g(\bar{y}), \bar{\lambda})(\nabla g(\bar{y})w), \quad (4.3)$$

where $\bar{\lambda} = (\bar{\lambda}_1, \dots, \bar{\lambda}_l) \in \mathbb{R}^l$ is a unique solution to the system

$$\nabla g(\bar{y})^* \lambda = \bar{v}, \quad \lambda \in N_\Theta(g(\bar{y})). \quad (4.4)$$

Proof. First we justify the chain rule (4.3) in the case when the derivative operator $\nabla g(\bar{y}): \mathbb{R}^m \rightarrow \mathbb{R}^l$ is *surjective*, i.e., the associated Jacobian matrix $\nabla g(\bar{y})$ with the same notation is of *full rank*. Consider the set

$$Q := \{(y, \lambda, v) \in \mathbb{R}^m \times \mathbb{R}^l \times \mathbb{R}^m \mid v = \nabla g(y)^* \lambda \text{ as } \lambda \in N_\Theta(g(y))\}$$

and observe that $Q = \Phi^{-1}(\{0\} \times \text{gph } N_\Theta)$ with

$$\Phi(y, \lambda, v) := \begin{bmatrix} v - \nabla g(y)^* \lambda \\ g(y) \\ \lambda \end{bmatrix}.$$

It easy follows from the surjectivity of $\nabla g(\bar{y})$ that the operator $\nabla \Phi(\bar{y}, \bar{\lambda}, \bar{v})$ is also

surjective, and then invoking [22, Corollary 1.15] gives us

$$\widehat{N}_Q(\bar{y}, \bar{\lambda}, \bar{v}) = \left\{ \left[\begin{array}{c} \sum_{i=1}^l \bar{\lambda}_i \nabla^2 g_i(\bar{y}) p + \nabla g(\bar{y})^* q \\ \nabla g(\bar{y}) p + r \\ -p \end{array} \right] \mid p \in \mathbb{R}^m, (q, r) \in \widehat{N}_{\text{gph}N_\Theta}(g(\bar{y}), \bar{\lambda}) \right\}.$$

Since $\text{gph} \widehat{N}_\Gamma$ is the canonical projection of the set Q onto the space generated by the first and the third components, we get from [37, Theorem 6.43] that

$$\widehat{N}_{\text{gph}\widehat{N}_\Gamma}(\bar{y}, \bar{v}) \subset \left\{ (z, u) \in \mathbb{R}^l \times \mathbb{R}^m \mid \begin{array}{l} z = \sum_{i=1}^l \bar{\lambda}_i \nabla^2 g_i(\bar{y})(-u) + \nabla g(\bar{y})^* q, \\ (q, \nabla g(\bar{y})u) \in \widehat{N}_{\text{gph}N_\Theta}(g(\bar{y}), \bar{\lambda}) \end{array} \right\}. \quad (4.5)$$

By (2.5) this implies in turn with putting $w := -u$ that

$$\widehat{D}^* \widehat{N}_\Gamma(\bar{y}, \bar{v})(w) \subset \sum_{i=1}^l \bar{\lambda}_i \nabla^2 g_i(\bar{y}) w + \nabla g(\bar{y})^* \widehat{D}^* N_\Theta(g(\bar{y}), \bar{\lambda})(\nabla g(\bar{y})w). \quad (4.6)$$

To justify the opposite inclusion in (4.3), we express according to [22] the set \widehat{N}_Γ locally around \bar{y} as follows:

$$\widehat{N}_\Gamma(y) = \psi(y, N_\Theta(g(y))) = \bigcup \{ \psi(y, z) \mid z \in N_\Theta(g(y)) \}$$

with the function $\psi: \mathbb{R}^m \times \mathbb{R}^l \rightarrow \mathbb{R}^m$ defined by

$$\varphi(y, u) := \nabla g(y)^* u \text{ for all } y \in \mathbb{R}^m \text{ and } u \in \mathbb{R}^l.$$

This representation enables us to invoke the arguments used in the proof of [22, Lemma 1.126]. Proceeding in this way, we pick an arbitrary pair (z, w) satisfying

$$(z, -\nabla g(\bar{y})w) \in N_{\text{gph}N_\Theta}(g(\bar{y}), \bar{\lambda})$$

and arrive at the inclusion

$$\left(\sum_{i=1}^l \bar{\lambda}_i \nabla^2 g_i(\bar{y}) w + \nabla g(\bar{y})^* z, w \right) \in \widehat{N}_{\text{gph}\widehat{N}_\Gamma}(\bar{y}, \bar{v}),$$

which ensures together with (4.6) the validity of (4.3) in the case of surjectivity.

It remains now to replace the subjectivity of $\nabla g(\bar{y})$ by the weaker *nondegeneracy* assumption from (A2). To proceed, we employ the local representation of Θ provided by its *reducibility* at $g(\bar{y})$; see the assumptions in (A1) with the notation used therein.

Denote $f := h \circ g$ and observe that its derivative $\nabla f(\bar{y})$ is surjective due to the nondegeneracy assumption. Applying (4.3) to the mapping f allows us to find a unique multiplier $\bar{\mu} = (\bar{\mu}_1, \dots, \bar{\mu}_k) \in N_\Xi(f(\bar{x}))$ satisfying the relationships $\nabla f(\bar{x})^* \bar{\mu} = \bar{v}$ and

$$\widehat{D}^* \widehat{N}_\Gamma(\bar{y}, \bar{v})(w) = \left(\sum_{i=1}^k \bar{\mu}_i \nabla^2 f_i(\bar{y}) \right) w + \nabla f(\bar{y})^* \widehat{D}^* N_\Xi(f(\bar{y}), \bar{\mu})(\nabla f(\bar{y})w). \quad (4.7)$$

By the classical chain rule we have the equalities

$$\nabla f(\bar{y}) = \nabla(h \circ g)(\bar{y}) = \nabla h(g(\bar{y}))\nabla g(\bar{y}),$$

$$\nabla f(\bar{y})^* = \nabla g(\bar{y})^* \nabla h(g(\bar{y}))^*. \quad (4.8)$$

Furthermore, it follows from the surjectivity of $\nabla h(g(\bar{y}))$ and applying (4.3) to h that

$$\widehat{D}^* N_{\Theta}(g(\bar{y}), \bar{\lambda})(\nu) = \left(\sum_{i=1}^k \bar{\vartheta}_i \nabla^2 h_i(g(\bar{y})) \right) \nu + \nabla h(g(\bar{y}))^* \widehat{D}^* N_{\Xi}(h(g(\bar{y})), \bar{\vartheta})(\nabla h(g(\bar{y}))\nu),$$

for all $\nu \in \mathbb{R}^k$, where $\bar{\vartheta} = (\bar{\vartheta}_1, \dots, \bar{\vartheta}_k)$ is a unique vector from \mathbb{R}^k such that

$$\bar{\vartheta} \in N_{\Xi}(h(g(\bar{y}))) \quad \text{and} \quad \nabla h(g(\bar{y}))^* \bar{\vartheta} = \bar{\lambda}.$$

Since $h(g(\bar{y})) = f(\bar{y})$, it follows from the uniqueness of the multiplier $\bar{\mu}$ in (4.7) that $\bar{\vartheta} = \bar{\mu}$. Indeed, both multipliers $\bar{\vartheta}$ and $\bar{\mu}$ belong to $N_{\Xi}(f(\bar{y}))$, and by (4.8) we have

$$\nabla f(\bar{y})^* \bar{\vartheta} = \nabla g(\bar{y})^* \nabla h(g(\bar{y}))^* \bar{\vartheta} = \nabla g(\bar{y})^* \bar{\lambda} = \bar{v}.$$

Taking this into account, observe the equalities

$$\begin{aligned} \nabla f(\bar{y})^* \widehat{D}^* N_{\Xi}(f(\bar{y}), \bar{\mu})(\nabla f(\bar{y})w) &= \nabla g(\bar{y})^* \nabla h(g(\bar{y}))^* \widehat{D}^* N_{\Xi}(h(g(\bar{y})), \bar{\vartheta})(\nabla h(g(\bar{y}))\nabla g(\bar{y})w) \\ &= \nabla g(\bar{y})^* \left[\widehat{D}^* N_{\Theta}(g(\bar{y}), \bar{\lambda})(\nabla g(\bar{y})w) - \left(\sum_{i=1}^k \bar{\mu}_i \nabla^2 h_i(g(\bar{y})) \right) \nabla g(\bar{y})w \right]. \end{aligned}$$

To complete the proof of the theorem, it remains to show that

$$\left(\sum_{i=1}^k \bar{\mu}_i \nabla^2 f_i(\bar{y}) \right) w = \left(\sum_{i=1}^l \bar{\lambda}_i \nabla^2 g_i(\bar{y}) \right) w + \nabla g(\bar{y})^* \left(\sum_{i=1}^k \bar{\mu}_i \nabla^2 h_i(g(\bar{y})) \right) \nabla g(\bar{y})w.$$

To proceed, we invoke the product rule (4.2) and get the equality

$$\nabla(\nabla f_i(\bar{y})) = \nabla(\nabla g(\cdot)^* \nabla h_i(g(\cdot)))|_{y=\bar{y}} = \nabla(\nabla g(\cdot)^* \nabla h_i(g(\bar{y})))|_{y=\bar{y}} + \nabla g(\bar{y})^* \nabla(\nabla h_i(g(\cdot)))|_{y=\bar{y}}.$$

This allows us to conclude that

$$\begin{aligned} \left(\sum_{i=1}^k \bar{\mu}_i \nabla^2 f_i(\bar{y}) \right) w &= \left[\nabla(\nabla g(\cdot)^* \left(\sum_{i=1}^k \bar{\mu}_i \nabla^2 h_i(g(\bar{y})) \right)) \right]_{y=\bar{y}} w \\ &\quad + \nabla g(\bar{y})^* \left(\sum_{i=1}^k \bar{\mu}_i \nabla^2 h_i(g(\bar{y})) \right) \nabla g(\bar{y})w. \end{aligned} \quad (4.9)$$

Since $\sum_{i=1}^k \bar{\mu}_i \nabla^2 h_i(g(\bar{y})) = \bar{\lambda}$, the first term on the right-hand side of (4.9) amounts to

$$(\nabla([g_1(\cdot), \dots, g_l(\cdot)]\bar{\lambda})|_{y=\bar{y}})w = \left(\sum_{i=1}^l \bar{\lambda}_i \nabla^2 g_i(\bar{y}) \right) w,$$

which justifies (4.3) and thus completes the proof of the theorem. \square

To facilitate the usage of formula (4.3), e.g., in the case of $\Theta = \mathcal{K}^l$, we restate it now in terms of the regular coderivative of the metric projection mapping P_Θ .

COROLLARY 4.2 (regular coderivative of the regular normal cone and projection mappings). *In the setting of Theorem 4.1 we have*

$$\widehat{D}^* \widehat{N}_\Gamma(\bar{y}, \bar{v})(w) = \left\{ \sum_{i=1}^l \bar{\lambda}_i \nabla^2 g_i(\bar{y})w + (\nabla g(\bar{y}))^* p \mid -\nabla g(\bar{y})w \in \widehat{D}^* P_\Theta(g(\bar{y}) + \bar{\lambda})(-\nabla g(\bar{y})w - p) \right\}.$$

Proof. It follows from to the well-known relationship

$$P_\Theta = (I + N_\Theta)^{-1}$$

between the projection and normal cone operators for convex sets (see, e.g., [37, Proposition 6.17]) that $(y, w) \in \text{gph } N_\Theta$ if and only if $(y + w, y) \in \text{gph } P_\Theta$. Hence we get from [37, Exercise 6.7] that

$$\widehat{N}_{\text{gph } N_\Theta}(g(\bar{y}), \bar{\lambda}) = \{(p, r) \mid p = u + w, r = u, (u, w) \in \widehat{N}_{\text{gph } P_\Theta}(g(\bar{y}) + \bar{\lambda}g(\bar{y}))\},$$

and it suffices to apply the definition of the regular coderivative. \square

Note that a similar relationship to that in Corollary 4.2 holds for the *limiting coderivatives* of N_Γ and P_Θ . In the Lorentz cone case $\Theta = \mathcal{K}^l$ the corresponding formulas for the regular and limiting coderivatives of the projection operator P_Θ can be found [33, Theorem 1] and [33, Theorem 2 and Theorem 3], respectively.

To conclude this section, let us discuss further specifications and extensions of the second-order chain rule obtained in Theorem 4.1.

REMARK 4.3 (calculating coderivatives of the normal cone mapping defined by the SDP cone). Besides the Lorentz cone case discussed above, the second-order calculus rule in Theorem 4.1 as well as its limiting counterpart from [32, Theorem 7] allow us to fully calculate the corresponding coderivatives $\widehat{D}^* \widehat{N}_\Gamma(\bar{y}, \bar{v})$ and $D^* N_\Gamma(\bar{y}, \bar{v})$ entirely via the problem data for the SDP cone $\Theta = \mathcal{S}_+^l$ from (3.15). This is based on the calculations of the corresponding coderivative constructions for $P_{\mathcal{S}_+^l}$ given recently in [8, Proposition 3.2 and Theorem 3.1].

REMARK 4.4 (infinite-dimensional extensions). Observe that Theorem 4.1 holds as formulated in arbitrary *Banach spaces*, where the notion of nondegeneracy is taken from [4, Definition 4.70] without assuming the finite dimensionality of the spaces in question. Indeed, the only change in the proof given above is to replace the application of the finite-dimensional result from [37, Theorem 6.43] by its Banach space counterpart from [23, Theorem 4.2] ensuring the *equality* in (4.5) and hence in (4.1) under the surjectivity assumption imposed on $\nabla g(\bar{y})$. In this way we can also derive the Banach space version of second-order chain rule from [32, Theorem 7] for the limiting constructions. Furthermore, the developed approach based on [22, Lemma 1.126] and its proof in the case of surjectivity and on the nondegeneracy reduction to the surjectivity case employed in [32, Theorem 7] and in Theorem 4.1 above allows us to establish—in the general case of *nondegeneracy*—the *exact/equality* type second-order chain rules for *various combinations* of coderivatives and first-order subdifferentials defined via the dual “derivative-of-derivative” scheme of (2.7) (see, e.g., [22, 25, 26]) in arbitrary Banach spaces.

REMARK 4.5 (**relations to tangential constructions**). Having in hands the calculations of Theorem 4.1, it is appealing to employ them in deriving workable formulas for the corresponding representations of the graphical derivative of \widehat{N}_Γ by *reversing* the duality scheme (4.1) between DF and \widehat{D}^*F . However, the realization of this scheme requires the *graphical regularity* of F (cf. [37, Corollary 6.29]), which in fact reduces to a certain *smoothness* of F in the case of *graphically Lipschitzian* (in the sense of [37, Definition 9.66] and [22, Definition 1.45]) mappings; see [22, Theorem 1.46] with the references and discussions therein. These observations show that the reversed duality scheme *cannot* be applied to the normal cone mappings under consideration as well as to a large class of subdifferential mappings, which exhibit the graphical Lipschitzian property; see [37, Proposition 13.46].

5. Generalized Derivatives of Solution Maps. On the basis of the results in Section 3 and 4 we are now able to derive the corresponding representations of the generalized derivatives and coderivatives of the solution map S from (1.3), crucial for the subsequent applications in Section 6. Let us start with the graphical derivative.

THEOREM 5.1 (**graphical derivative of solution maps**). *Let $(\bar{x}, \bar{y}) \in \text{gph } S$ for the solution map S in (1.3), let all the assumptions of Theorem 3.3 be fulfilled, and let $\bar{\lambda} \in \mathbb{R}^l$ be a unique Lagrange multiplier satisfying the KKT system*

$$\begin{aligned} \mathcal{L}(\bar{x}, \bar{y}, \bar{\lambda}) &= 0, \\ \bar{\lambda} &\in N_{\Theta}(g(\bar{y})). \end{aligned} \quad (5.1)$$

Then we have the inclusion for the graphical derivative of S at (\bar{x}, \bar{y}) :

$$\begin{aligned} DS(\bar{x}, \bar{y})(v) \subset \{ &u \in \mathbb{R}^m \mid 0 = \nabla_x f(\bar{x}, \bar{y})v + \nabla_y \mathcal{L}(\bar{x}, \bar{y}, \bar{\lambda})u + \nabla g(\bar{y})^*d, \\ &d = P'_{\Theta^*}(g(\bar{y}) + \bar{\lambda}; \nabla g(\bar{y})u + d)\}, \quad v \in \mathbb{R}^n. \end{aligned} \quad (5.2)$$

If in addition the partial Jacobian $\nabla_x f(\bar{x}, \bar{y})$ is surjective, then (5.2) holds as equality.

Proof. It is easy to observe that

$$\text{gph } S = \left\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid h(x, y) := \begin{bmatrix} y \\ -f(x, y) \end{bmatrix} \in \text{gph } \widehat{N}_\Gamma \right\}, \quad (5.3)$$

i.e., $\text{gph } S = h^{-1}(\text{gph } \widehat{N}_\Gamma)$. Thus we can deduce from [37, Theorem 6.31] that

$$T_{\text{gph } S}(\bar{x}, \bar{y}) \subset \left\{ (v, u) \in \mathbb{R}^n \times \mathbb{R}^m \mid \begin{bmatrix} u \\ -\nabla_x f(\bar{x}, \bar{y})v - \nabla_y f(\bar{x}, \bar{y})u \end{bmatrix} \in T_{\text{gph } N_\Gamma}(\bar{y}, -f(\bar{x}, \bar{y})) \right\}$$

which holds as equality provided that $\nabla_x f(\bar{x}, \bar{y})$ is surjective; see [37, Exercise 6.7]. Employing now formula (3.12) for calculating the tangent cone $T_{\text{gph } N_\Gamma}$ under the assumptions made in Section 3 and then recalling the definitions of Lagrangian (1.5) and graphical derivative (2.4), we ensure the validity of inclusion (5.2) and then the equality therein under the additional surjectivity assumption on $\nabla_x f(\bar{x}, \bar{y})$. \square

Employing the complete calculations from Proposition 3.4 of the directional derivative of the projection operator P_{Θ^*} for the case of the Lorentz cone $\Theta = \mathcal{K}^l$ would allow us to express the results of Theorem 5.1 entirely via the initial data of the corresponding second-order cone program. This is illustrated by the following example.

EXAMPLE 5.2 (**calculating graphical derivative of solutions maps for SOCPs**). Consider the following GE of type (1.2):

$$x \in y + \widehat{N}_{g^{-1}(\Theta)}(y),$$

where $g(y) := (1 - \frac{1}{2}y_1^2, y_2, y_3 + 1)$, $\Theta = \mathcal{K}^3$, and $(\bar{x}, \bar{y}) = (0, 0)$. Then $g(\bar{y}) = (1, 0, 1)$, $\bar{\lambda} = (0, 0, 0)$, $\text{Im} \nabla g(\bar{y}) = \{0\} \times \mathbb{R} \times \mathbb{R}$, and $\text{lin } T_{\mathcal{K}^3}(g(\bar{y})) = \{(\alpha, \beta, \alpha) : \alpha, \beta \in \mathbb{R}\}$. This ensures that \bar{y} is a nondegenerate point of g with respect to \mathcal{K}^3 . It is also easy to check (3.1), which ensures that g is Θ -convex. Moreover, on the basis of Proposition 3.4(ii), for $u := g(\bar{y}) + \bar{\lambda} = (1, 0, 1)$ and for any direction h , we have that

$$P'_{\Theta^*}(u; h) = -2(\langle c_1(u), h \rangle)_- c_1(u) = -(h_1 - h_3)_- \begin{pmatrix} 1/2 \\ 0 \\ -1/2 \end{pmatrix},$$

with $c_1(u) = (1/2, 0, -1/2)$ and the notation $(x)_- := \min\{0, x\}$. Thus, the surjectivity of $\nabla_x f(\bar{x}, \bar{y}) = -I$ and the equality in (5.2) lead us to the precise formula

$$DS(\bar{x}, \bar{y})(v) = \left\{ u \mid u_1 = v_1, u_2 = v_2, u_3 = \begin{cases} v_3 & \text{if } v_3 \leq 0 \\ \frac{1}{3}v_3 & \text{if } v_3 > 0 \end{cases} \right\}$$

for calculating the graphical derivative of the solution map in question.

Next we proceed with calculating the regular coderivative of the solution map.

THEOREM 5.3 (regular coderivative of solution maps). *Let $(\bar{x}, \bar{y}) \in \text{gph } S$ under the standing assumptions of Section 1, and let $\bar{\lambda} = (\bar{\lambda}_1, \dots, \bar{\lambda}_l) \in \mathbb{R}^l$ be a unique solution to system (4.4) with $\bar{v} = -f(\bar{x}, \bar{y})$. Suppose in addition that the partial Jacobian $\nabla_x f(\bar{x}, \bar{y})$ is surjective. Then for any $v \in \mathbb{R}^m$ we have the regular coderivative representation of the solution map (1.3):*

$$\widehat{D}^*S(\bar{x}, \bar{y})(v) = \left\{ \nabla_x f(\bar{x}, \bar{y})^*w \mid \begin{aligned} &0 \in v + \nabla_y f(\bar{x}, \bar{y})^*w + \sum_{i=1}^l \bar{\lambda}_i \nabla^2 g_i(\bar{y})w \\ &+ \nabla g(\bar{y})^* \widehat{D}^*N_{\Theta}(g(\bar{y}), \bar{\lambda})(\nabla g(\bar{y})w) \end{aligned} \right\}. \quad (5.4)$$

It can be equivalently rewritten in the Lagrangian form:

$$\widehat{D}^*S(\bar{x}, \bar{y})(v) = \left\{ \nabla_x f(\bar{x}, \bar{y})^*w \mid \begin{aligned} &0 \in v + \nabla_y \mathcal{L}(\bar{x}, \bar{y}, \bar{\lambda})^*w \\ &+ \nabla g(\bar{y})^* \widehat{D}^*N_{\Theta}(g(\bar{y}), \bar{\lambda})(\nabla g(\bar{y})w) \end{aligned} \right\}. \quad (5.5)$$

Proof. Taking into account the surjectivity of $\nabla h(\bar{x}, \bar{y})$ from (5.3) due to the surjectivity of $\nabla_x f(\bar{x}, \bar{y})$ and using the well-known formula for the normal cone of the inverse image of sets under smooth mappings (see, e.g., [22, Corollary 1.15] and [37, Exercise 6.7]) as well as definition (2.5) of the regular coderivative, we arrive at

$$\widehat{D}^*S(\bar{x}, \bar{y})(v) = \left\{ u \in \mathbb{R}^n \mid \begin{aligned} &\text{there is } w \in \mathbb{R}^m \text{ with } u = \nabla_x f(\bar{x}, \bar{y})^*w, \\ &0 \in v + \nabla_y f(\bar{x}, \bar{y})^*w + \widehat{D}^*N_{\Gamma}(\bar{y}, -f(\bar{x}, \bar{y}))(w) \end{aligned} \right\}.$$

Then formula (5.4) and its Lagrangian version (5.5) are implied by Theorem 4.1. \square

On the basis of Corollary 4.2 formula (5.5) can be reformulated in terms of the regular coderivative of the *projection operator* as follows:

$$\widehat{D}^*S(\bar{x}, \bar{y})(v) = \left\{ \nabla_x f(\bar{x}, \bar{y})^*w \mid \begin{aligned} &\text{there is } p \in \mathbb{R}^l \text{ such that} \\ &0 = v + \nabla_y \mathcal{L}(\bar{x}, \bar{y}, \bar{\lambda})^*w + (\nabla g(\bar{y}))^*p, \\ &-\nabla g(\bar{y})w \in \widehat{D}^*P_{\Theta}(g(\bar{y}) + \bar{\lambda})(-p - \nabla g(\bar{y})w) \end{aligned} \right\}. \quad (5.6)$$

This variant will be used in Example 5.5 for $\Theta = \mathcal{K}^3$ given below.

Next we present a representation of the *limiting coderivative* of the solution map S which readily follows from [32, Theorem 7] and [13, Theorem 4.1]. Recall [37, p. 399] that a set-valued mapping $M: \mathbb{R}^d \rightrightarrows \mathbb{R}^s$ is *calm* at $(\bar{p}, \bar{z}) \in \text{gph } M$ if there are neighborhoods U of \bar{p} and V of \bar{z} together with a constant $\kappa \in \mathbb{R}_+$ such that

$$M(p) \cap V \subset M(\bar{p}) + \kappa \|p - \bar{p}\| \mathbb{B} \quad \text{for all } p \in U.$$

THEOREM 5.4 (limiting coderivative of solution maps). *Let $(\bar{x}, \bar{y}) \in \text{gph } S$ under the standing assumptions of Section 1, and let $\bar{\lambda} = (\bar{\lambda}_1, \dots, \bar{\lambda}_l) \in \mathbb{R}^l$ be a unique solution to (4.4) with $\bar{v} = -f(\bar{x}, \bar{y})$. The following assertions hold.*

(i) *Suppose that $\nabla_x f(\bar{x}, \bar{y})$ is surjective. Then for any $v \in \mathbb{R}^m$ we have*

$$D^*S(\bar{x}, \bar{y})(v) = \left\{ \nabla_x f(\bar{x}, \bar{y})^* w \mid \begin{array}{l} 0 \in v + \nabla_y \mathcal{L}(\bar{x}, \bar{y}, \bar{\lambda})^* w \\ + \nabla g(\bar{y})^* D^*N_{\Theta}(g(\bar{y}), \bar{\lambda})(\nabla g(\bar{y})w) \end{array} \right\}. \quad (5.7)$$

(ii) *Suppose that the perturbation mapping $M: \mathbb{R}^m \rightrightarrows \mathbb{R}^n \times \mathbb{R}^m$ defined by*

$$M(p) := \left\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid p \in f(x, y) + \nabla g(y)^* N_{\Theta}(g(y)) \right\}$$

is calm at $(0, \bar{x}, \bar{y})$. Then equality (5.7) is replaced by the inclusion “ \subset ”.

Proof. Under the surjectivity of the operator $\nabla_x f(\bar{x}, \bar{y})$ we can proceed similarly to the proof of Theorem 5.3 with replacing the second-order chain rule from Theorem 4.1 by the one obtained in [32, Theorem 7].

To justify assertion (ii), observe by [40, Theorem 3.1] and [13, Theorem 4.1] that the calmness of M at $(0, \bar{x}, \bar{y})$ implies the inclusion

$$D^*S(\bar{x}, \bar{y})(v) \subset \left\{ \nabla_x f(\bar{x}, \bar{y})^* v \mid 0 \in v + \nabla_y f(\bar{x}, \bar{y})^* w + D^*N_{\Gamma}(\bar{y}, -f(\bar{x}, \bar{y}))(w) \right\}. \quad (5.8)$$

Employing again [32, Theorem 7] completes the proof of the theorem. \square

Note that the corresponding counterpart of (5.6) attains the form

$$\begin{aligned} D^*S(\bar{x}, \bar{y})(v) &= \left\{ \nabla_x f(\bar{x}, \bar{y})^* w \mid \text{there is } p \in \mathbb{R}^l \text{ such that} \right. \\ &\quad 0 = v + \nabla_y \mathcal{L}(\bar{x}, \bar{y}, \bar{\lambda})^* w + (\nabla g(\bar{y}))^* p, \\ &\quad \left. -\nabla g(\bar{y})w \in D^*P_{\Theta}(g(\bar{y}) + \bar{\lambda})(-p - \nabla g(\bar{y})w) \right\}. \end{aligned} \quad (5.9)$$

Let us now illustrate the statements of Theorem 5.3 and Theorem 5.4 by the following example from second-order cone programming.

EXAMPLE 5.5 (calculating coderivatives of solution maps in SOCPs). Consider the GE as in Example 5.2, i.e.,

$$x \in y + \widehat{N}_{g^{-1}(\Theta)}(y),$$

where $g(y) := (1 + y_3, y_2^2, 1 + y_2)$, $\Theta = \mathcal{K}^3$, and $(\bar{x}, \bar{y}) = (0, 0)$. Then $g(\bar{y}) = (1, 0, 1)$, $\bar{\lambda} = (0, 0, 0)$, $\text{Im } \nabla g(\bar{y}) = \mathbb{R} \times \{0\} \times \mathbb{R}$, and $\text{lin } T_{\mathcal{K}^3}(g(\bar{y})) = \{(\alpha, \beta, \alpha) \mid \alpha, \beta \in \mathbb{R}\}$. This ensures that \bar{y} is a nondegenerate point of g with respect to \mathcal{K}^3 . Employing further [33, Theorem 1(ii)], we calculate the regular coderivative $\widehat{D}^*P_{\Theta}(g(\bar{y}) + \bar{\lambda})$ by

$$\widehat{D}^*P_{\Theta}(g(\bar{y}) + \bar{\lambda})(u) = \left\{ z \in \mathbb{R}^3 \mid u - z \in \mathbb{R}_+ c_1, \langle z, c_1 \rangle \geq 0 \right\}, \quad (5.10)$$

where the spectral vector $c_1 := c_1(g(\bar{y}) + \bar{\lambda})$ in this case is

$$c_1 = \frac{1}{2}(1, 0, -1)^{\top}.$$

Using (5.6) gives us the regular coderivative expression in the two equivalent forms:

$$\widehat{D}^*S(\bar{x}, \bar{y})(v) = \begin{cases} \left\{ v + \alpha \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \mid \alpha \in \left[\frac{1}{2}(v_2 - v_3), 0 \right] \right\} & \text{if } v_3 \geq v_2, \\ \emptyset & \text{otherwise;} \end{cases}$$

$$\widehat{D}^*S(\bar{x}, \bar{y})(v) = \left\{ -w \mid \begin{array}{l} w_1 = -v_1, \\ \begin{bmatrix} w_2 \\ w_3 \end{bmatrix} \in \begin{cases} \text{co} \left\{ - \begin{bmatrix} v_2 \\ v_3 \end{bmatrix}, - \begin{bmatrix} \frac{v_2 + v_3}{2} \\ \frac{v_2 + v_3}{2} \end{bmatrix} \right\} & \text{if } v_3 \geq v_2, \\ \emptyset & \text{otherwise} \end{cases} \end{array} \right\}.$$

Likewise, on the basis of [33, Theorem 3(i)] we obtain that

$$D^*P_\Theta(z)(u) = \begin{cases} \text{co} \{u, A(z)u\} & \text{if } \langle u, c_1 \rangle \geq 0, \\ \{u, A(z)u\} & \text{otherwise,} \end{cases}$$

where $z := g(\bar{y}) + \bar{\lambda}$, and where

$$A(z) := P_{(c_1)^\perp}(z) = I + \frac{1}{2} \begin{bmatrix} -1 & \frac{\bar{z}^T}{\|\bar{z}\|_2} \\ \frac{\bar{z}}{\|\bar{z}\|_2} & -\frac{\bar{z}(\bar{z})^T}{\|\bar{z}\|_2^2} \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

via the spectral decomposition of the vector $z = g(\bar{y}) + \bar{\lambda}$; cf. (3.16) and (3.17) above. It follows now from (5.9) that

$$D^*S(\bar{x}, \bar{y})(v) = \left\{ -w \mid \begin{array}{l} w_1 = -v_1, \\ \begin{bmatrix} w_2 \\ w_3 \end{bmatrix} \in \begin{cases} \text{co} \left\{ - \begin{bmatrix} v_2 \\ v_3 \end{bmatrix}, - \begin{bmatrix} \frac{v_2 + v_3}{2} \\ \frac{v_2 + v_3}{2} \end{bmatrix} \right\} & \text{if } v_3 \geq v_2 \\ \left\{ - \begin{bmatrix} v_2 \\ v_3 \end{bmatrix}, - \begin{bmatrix} \frac{v_2 + v_3}{2} \\ \frac{v_2 + v_3}{2} \end{bmatrix} \right\} & \text{otherwise} \end{cases} \end{array} \right\}.$$

REMARK 5.6 (modified solution map). Along with S we can define the *modified solution map*

$$\widetilde{S}(x) := \{y \in \mathbb{R}^m \mid 0 \in f(x, y) + N_\Gamma(y)\} \text{ for } \Gamma = g^{-1}(\Theta) \text{ and } x \in \mathbb{R}^n,$$

with the replacement of the regular normal cone $\widehat{N}_\Gamma(y)$ by the limiting one $N_\Gamma(y)$ in (1.3). Observe, however, that under the imposed assumptions the set Γ is *normally regular* on a neighborhood of \bar{y} , i.e., $\widehat{N}_\Gamma(y) = N_\Gamma(y)$ for y near \bar{y} , and so both solution

maps coincide for such y . This happens even if nondegeneracy is replaced by the weaker qualification condition

$$N_{\Theta}(\bar{y}) \cap \ker \nabla g(\bar{y})^* = \{0\}, \quad (5.11)$$

ensuring the *strong amenability* property of Γ near \bar{y} ; see [37, Exercise 10.25]. Note in this connection that the nondegeneracy condition in (A2) is mainly needed for deriving the second-order chain rule of the *equality* type in Theorem 4.1 (as well as in [32, Theorem 7]) while calculus rule of the *inclusion* type can be obtained with replacing nondegeneracy by some other assumptions; see, e.g. [27]. Normal regularity of the set Γ around \bar{y} is lost especially when Θ is nonconvex, and then the solution maps $S(y)$ and $\tilde{S}(y)$ would be essentially different.

6. Applications. This section contains some applications of the results obtained above on computing the generalized derivative/coderivative constructions for solution maps to the following three important issues in variational analysis: *isolated calmness*, *optimality conditions* for *MPECs*, and *tilt stability* in conic programming. Accordingly, we split this section into three subsections.

6.1. Isolated calmness of solution maps. Given $F: \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, recall that it has the *isolated calmness* property at $(\bar{x}, \bar{y}) \in \text{gph } F$ if there are neighborhoods U of \bar{x} and V of \bar{y} and a constant $\kappa \geq 0$ such that

$$F(x) \cap V \subset \{\bar{y}\} + \kappa \|x - \bar{x}\| \mathbb{B} \quad \text{for all } x \in U. \quad (6.1)$$

This well-posedness/stability property and its equivalent description as *strong metric subregularity* of the inverse F^{-1} play a significant role in variational analysis and optimization; see, e.g., [9] and the references therein.

It is known from [17] (cf. also [9, Theorem 4C.1]) that isolated calmness of an arbitrary closed-graph multifunction F between finite-dimensional spaces at any point $(\bar{x}, \bar{y}) \in \text{gph } F$ can be fully characterized via the graphical derivative (2.4) as follows:

$$DF(\bar{x}, \bar{y})(0) = \{0\}. \quad (6.2)$$

Note that this form is similar to the limiting *coderivative characterization* (known also as the Mordukhovich criterion [37, Theorem 9.40])

$$D^*F(\bar{x}, \bar{y})(0) = \{0\} \quad (6.3)$$

of the well-recognized *Aubin/Lipschitz-like* property of F around (\bar{x}, \bar{y}) , which is equivalent to the fundamental *metric regularity* property of the inverse F^{-1} . The latter criterion (6.3) has been widely used in numerous aspects of variational analysis, optimization, and their applications; see, e.g., [22, 37] with the references and commentaries therein. It has been recently implemented in [32] to obtain a complete characterization, entirely via the initial data, of the Aubin property of solution maps to perturbed second-order cone programs on the basis of calculating the limiting coderivative of the normal cone mapping D^*N_{Θ} generated by the Lorentz cone.

Following this line and utilizing the graphical derivative calculations for the solution map (1.3) in Theorem 5.1 allow us to derive, on the basis of (6.2), verifiable characterizations of the isolated calmness property of S for the general convex cone Θ in (1.3) and provide its implementation for $\Theta = \mathcal{K}^l$. To the best of our knowledge, these results are first in the literature for isolated calmness in conic programming.

THEOREM 6.1 (isolated calmness of solution maps in perturbed conic programming). *Let $(\bar{x}, \bar{y}) \in \text{gph } S$ for the solution map (1.3) under the assumptions of Theorem 3.3, and let $\bar{\lambda} \in \mathbb{R}^l$ be a unique Lagrange multiplier satisfying (5.1). Then S enjoys the isolated calmness property at (\bar{x}, \bar{y}) provided that $u = 0$ for any solution $(u, d) \in \mathbb{R}^m \times \mathbb{R}^l$ of the system*

$$\begin{aligned} 0 &= \nabla_y \mathcal{L}(\bar{x}, \bar{y}, \bar{\lambda})u + \nabla g(\bar{y})^* d, \\ d &= P'_{\Theta^*}(g(\bar{y}) + \bar{\lambda}; \nabla g(\bar{y})u + d). \end{aligned} \tag{6.4}$$

If in addition the partial Jacobian $\nabla_x f(\bar{x}, \bar{y})$ is surjective, then the above condition is also necessary for the isolated calmness of S at (\bar{x}, \bar{y}) .

Proof. It follows directly by substituting the graphical derivative calculations from Theorem 5.1 into the isolated calmness criterion (6.2). \square

In the case of the Lorentz cone $\Theta = \mathcal{K}^l$ the complete calculation of the directional derivative P'_{Θ^*} provided in Proposition 3.4 leads us to the isolated calmness characterization of Theorem 6.1 entirely via the initial data of SOCPs. Let us illustrate this by the following example.

EXAMPLE 6.2 (verifying isolated calmness for second-order cone programs with no Aubin property). Consider the solution map $S: \mathbb{R} \rightrightarrows \mathbb{R}^2$ of the generalized equation (1.2) with the following initial data:

$$f(x, y) = (y_2, -x + y_1), \quad \Gamma = \mathcal{K}^2, \quad \text{and} \quad (\bar{x}, \bar{y}) = (0, 0) \in \mathbb{R}^3.$$

In this case $\Theta = \Gamma = \mathcal{K}^2$ and g is the identity map; thus all assumptions of Theorem 3.3, used in Theorems 5.1 and 6.1, are satisfied. It is easy to see that the only solution to the corresponding KKT system (5.1) is $\bar{\lambda} = 0 \in \mathbb{R}^2$. Observe that $(\bar{x}, \bar{y}) \in \text{gph } S$ and that $S(x) = \emptyset$ for all $x < 0$, which shows that the Aubin property is violated for the solution map S around (\bar{x}, \bar{y}) ; this is obviously confirmed by (6.3). On the other hand, we get from (5.2) that

$$DS(\bar{x}, \bar{y})(0) \subset \left\{ u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathbb{R}^2 \mid - \begin{pmatrix} u_2 \\ u_1 \end{pmatrix} = P'_{\Theta^*} \left(0; \begin{bmatrix} u_1 - u_2 \\ u_2 - u_1 \end{bmatrix} \right) = P_{\Theta^*} \left(\begin{bmatrix} u_1 - u_2 \\ u_2 - u_1 \end{bmatrix} \right) \right\},$$

where the last equality comes from Proposition 3.4. It is not hard to check that right-hand side of the last equation reduces to $\{0\}$. Indeed, it follows from the classical characterization of projections onto convex cones that any $u \in \mathbb{R}^2$ from the aforementioned set satisfies the equality

$$0 = \left\langle \begin{pmatrix} u_1 - u_2 \\ u_2 - u_1 \end{pmatrix} + \begin{pmatrix} u_2 \\ u_1 \end{pmatrix}, - \begin{pmatrix} u_2 \\ u_1 \end{pmatrix} \right\rangle = -2u_1 u_2.$$

Splitting into the two possible cases of $u_1 = 0$ and $u_2 = 0$, we deduce from the equality

$$- \begin{pmatrix} u_2 \\ u_1 \end{pmatrix} = P_{\Theta^*} \left(\begin{bmatrix} u_1 - u_2 \\ u_2 - u_1 \end{bmatrix} \right)$$

that $u = (0, 0)$ in both cases. This ensures by the graphical derivative criterion (6.2) that S possesses the isolated calmness property at (\bar{x}, \bar{y}) .

6.2. Mathematical programs with equilibrium constraints. The main concern of this subsection is the following general optimization problem, which belongs to the class of *mathematical programs with equilibrium constraints* (MPECs):

$$\begin{aligned} & \text{minimize} && \varphi(x, y) \\ & \text{subject to} && \\ & && 0 \in f(x, y) + \widehat{N}_\Gamma(y) \end{aligned} \tag{6.5}$$

with the cost function $\varphi: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ and the constraints described by the generalized equation (1.2) under the standing assumptions formulated in Section 1. Optimization problems of this type have drawn a strong attention in the literature, mainly in the case of convex sets Γ when the generalized equation in (6.5) reduces to the classical parameterized variational inequality; see the books [20, 22, 31] and the references therein. When the set Γ is nonconvex, the vast majority of MPECs models studied and applied in optimization are described in form (6.5) with replacing $\widehat{N}_\Gamma(y)$ by either the limiting normal cone (2.3) or its convex closure, which is the normal cone of Clarke. To the best of our knowledge, a systematic study of MPEC models with the regular normal cone in (6.5) has started quite recently in [15] and [14]. These papers are devoted to the study of the parameterized solution maps (1.3) and applications to necessary optimality conditions in MPECs (6.5) in the case of Γ being a set of feasible solutions to problems of nonlinear programming with \mathcal{C}^2 inequality constraints. The next theorem seems to be the first result in this direction for MPECs described via the *regular coderivative* of the normal cone mapping to feasible sets in *conic programming*. Note that it immediately implies a similar result with replacing \widehat{D}^*N_Θ by the limiting coderivative D^*N_Θ , being *significantly better* than the latter one due to the intrinsic *irregularity* of the normal cone mappings; see Remark 4.5.

THEOREM 6.3 (necessary optimality conditions for MPECs governed by regular normal cone mappings in conic programming). *Let $(\bar{x}, \bar{y}) \in \text{gph } S$ be a local optimal solution to MPEC (6.5), and let $\bar{\lambda} \in \mathbb{R}^l$ be a unique Lagrange multiplier satisfying (5.1). In addition to the standing assumptions on f and Γ formulated in Section 1, suppose that φ is (Fréchet) differentiable at (\bar{x}, \bar{y}) and that the operator $\nabla_x f(\bar{x}, \bar{y})$ is surjective. Then there exists an MPEC multiplier $\bar{\mu} \in \mathbb{R}^m$ such that*

$$\begin{aligned} 0 &= \nabla_x \varphi(\bar{x}, \bar{y}) + \nabla_x f(\bar{x}, \bar{y})^* \bar{\mu}, \\ 0 &\in \nabla_y \varphi(\bar{x}, \bar{y}) + \nabla_y \mathcal{L}(\bar{x}, \bar{y}, \bar{\lambda})^* \bar{\mu} + \nabla g(\bar{y})^* \widehat{D}^* \widehat{N}_\Theta(g(\bar{y}), \bar{\lambda})(\nabla g(\bar{y}) \bar{\mu}). \end{aligned} \tag{6.6}$$

Proof. MPEC (6.5) can be obviously rewritten in the unconstrained form

$$\text{minimize } \phi(x, y) := \varphi(x, y) + \delta_{\text{gph } S}(x, y) \text{ over all } (x, y) \in \mathbb{R}^n \times \mathbb{R}^m \tag{6.7}$$

via the indicator function $\delta_{\text{gph } S}$ of the graph of the solution map (1.3). Applying to the optimal solution (\bar{x}, \bar{y}) of (6.7) the generalized Fermat rule $0 \in \widehat{\partial} \phi(\bar{x}, \bar{y})$ via the regular subdifferential and then the regular subdifferential sum rule for $\varphi + \delta_{\text{gph } S}$ with the differentiable function φ (see, e.g., [22, Propositions 1.107 and 1.114]), we get

$$0 \in \nabla \varphi(\bar{x}, \bar{y}) + \widehat{N}_{\text{gph } S}(\bar{x}, \bar{y}),$$

which amounts by virtue of (2.5) to the regular coderivative condition

$$0 \in \nabla_x \varphi(\bar{x}, \bar{y}) + \widehat{D}^* S(\bar{x}, \bar{y})(\nabla_y \varphi(\bar{x}, \bar{y})). \tag{6.8}$$

To complete the proof of the theorem, it remains to employ Theorem 5.3 on the calculation of the regular coderivative of the solution map in (6.8). \square

REMARK 6.4 (necessary optimality conditions for nonsmooth MPECs associated with conic programming). Following the proof of Theorem 6.3 and using more involved results from nonsmooth optimization together with Theorem 5.3, we can consider MPECs (6.5) for nonsmooth cost functions and derive for them both *upper subdifferential* necessary optimality conditions via the upper version $\widehat{\partial}^+ \varphi(\bar{z}) := -\widehat{\partial}(-\varphi)(\bar{z})$ of the regular subdifferential and *lower subdifferential* necessary optimality conditions via the limiting subdifferential (2.8); cf. [22, Propositions 5.2 and 5.3] in the general constrained framework.

For some applications it is more convenient to present necessary optimality conditions of Theorem 6.3 via the regular coderivative of the metric *projection operator* P_Θ onto the underlying cone Θ .

COROLLARY 6.5 (necessary optimality conditions for MPECs via the regular coderivative of the projection operator). *In the setting of Theorem 6.3 there are multipliers $\bar{\mu} \in \mathbb{R}^m$ and $\bar{\nu} \in \mathbb{R}^l$ such that we have the inclusion*

$$-\nabla g(\bar{y})\bar{\mu} \in \widehat{D}^*P_\Theta(g(\bar{y}) + \bar{\lambda})(-\nabla g(\bar{y})\bar{\mu} - \bar{\nu}) \quad (6.9)$$

along with the equality system

$$\begin{aligned} 0 &= \nabla_x \varphi(\bar{x}, \bar{y}) + \nabla_x f(\bar{x}, \bar{y})^* \bar{\mu}, \\ 0 &= \nabla_y \varphi(\bar{x}, \bar{y}) + \nabla_y \mathcal{L}(\bar{x}, \bar{y}, \bar{\lambda})^* \bar{\mu} + \nabla g(\bar{y})^* \bar{\nu}. \end{aligned} \quad (6.10)$$

Proof. It follows immediately from the relationships between the projection and normal cone operators for convex sets used in the proof of Corollary 4.2. \square

As discussed in Remark 4.3, the results obtained in [33] and [8] allow us to calculate the regular coderivative of the projection mapping in (6.9) entirely via the initial data for the cases of the Lorentz and SDP cones Θ , respectively, and hence effectively implement the MPEC necessary optimality conditions of Corollary 6.5 in these settings. Let us illustrate it by the following example for the MPEC generated by the Lorentz cone $\Theta = \mathcal{K}^3$ in (6.5).

EXAMPLE 6.6 (illustrating the MPEC optimality conditions for the Lorentz cone). Consider MPEC (6.5) with $x \in \mathbb{R}^3$, $y \in \mathbb{R}^3$, and

$$\varphi(x, y) := y_1 - y_2 + \frac{1}{2}y_3^2,$$

in which the equilibrium is governed by the GE from Example 5.5. It is easy to see that the pair $(\bar{x}, \bar{y}) = (0, 0)$ is a local minimizer of this MPEC. Since all assumptions of Theorem 6.3 are fulfilled, we can invoke Corollary 6.5 and conclude from the equations (6.10) that $\bar{\mu} = 0$, $\bar{\nu}_1 = -1$, and $\bar{\nu}_2 = 1$. Thus it remains to find $\bar{\nu}_3 \in \mathbb{R}$ such that the vector $\bar{\nu} = (-1, 1, \bar{\nu}_3)$ satisfies relation (6.9), which reads in this case as

$$0 \in \widehat{D}^*P_\Theta(g(\bar{y}) + \bar{\lambda})(-\bar{\nu}).$$

It holds by (5.10) for $\bar{\nu}_3 = 0$, which confirms therefore that the solution (\bar{x}, \bar{y}) indeed fulfills the optimality conditions of Corollary 6.5.

6.3. Tilt stability in conic programming. This part of the paper is devoted to the application of the results obtained above as well as related developments in second-order variational analysis to *tilt stability* of conic programs written as

$$\text{minimize } \varphi(y) \text{ subject to } y \in \Gamma = g^{-1}(\Theta) \quad (6.11)$$

under the standing assumptions on the initial data imposed in Section 1.

Recall that the concept of tilt stability of local minimizers was introduced by Poliquin and Rockafellar [34] in the extended-real-valued format of unconstrained optimization as in Definition 6.7 and characterized therein in terms of the second-order subdifferential (2.7) of the extended-real-valued objective. As discussed in [34], the major motivation to consider tilt stability came from the requirement to characterize strong manifestations of optimality that support computational work via the study of how optimal solutions react to linear shifts (tilt perturbations) of the problem data.

DEFINITION 6.7 (tilt stability of local minimizers for extended-real-valued functions). *Let $\phi: \mathbb{R}^m \rightarrow \mathbb{R}$ be an extended-real-valued function finite at \bar{y} . We say that \bar{y} is a tilt-stable local minimizer of ϕ if there is $\gamma > 0$ such that the mapping*

$$M: p \mapsto \operatorname{argmin} \{ \phi(y) - \phi(\bar{y}) - \langle p, y - \bar{y} \rangle \mid \|y - \bar{y}\| \leq \gamma \}$$

is single-valued and Lipschitzian on some neighborhood of $p = 0$ with $M(0) = \{\bar{y}\}$.

It is clear that constrained optimization problems can be written in the extended-real-valued unconstrained format of Definition 6.7 as, e.g., in (6.7). On the other hand, deducing results for constrained problems from those obtained in [34] in the unconstrained case requires second-order generalized differential calculus as well as calculating the corresponding second-order constructions in particular situations under consideration, which was not previously available. Note also that yet another type of tilt stability characterizations was obtained in [4, Theorem 5.36] in the framework of conic programming via the so-called “uniform second-order growth conditions with respect to tilt perturbation” without employing generalized differentiation.

The recent years have witnessed strong interest in tilt stability and its applications from several viewpoints of new developments in variational analysis and generalized differentiation; see, e.g., [10, 11, 19, 25, 26, 29, 30]. The closest to our developments in this paper are those presented in [25, 26, 29, 30], where a number of necessary conditions, sufficient conditions, and complete characterizations of tilt stability were obtained on the basis of second-order generalized differential calculus for various classes of constrained optimization problems including classical nonlinear programs with equality and inequality constraints, mathematical programs with polyhedral constraints, and the so-called extended nonlinear programs with \mathcal{C}^2 data. We are not familiar with any results in this direction for general or special classes of problems of conic programming of type (6.11), which are under consideration here.

Note that in conic programming the tilt stability of a local optimal solution $\bar{y} \in \Gamma$ to (6.11) amounts to the single-valuedness and Lipschitz continuity of the mapping

$$\mathcal{M}: p \mapsto \operatorname{argmin} \{ \varphi(y) - \varphi(\bar{y}) - \langle p, y - \bar{y} \rangle \mid y \in \Gamma, \|y - \bar{y}\| \leq \gamma \} \quad (6.12)$$

around the nominal parameter value $p = 0$ with some $\gamma > 0$ and $\mathcal{M}(0) = \{\bar{y}\}$. Prior to deriving second-order characterizations of tilt stability in (6.11), let us give an example showing that isolated local minimizers in the standard sense may not be tilt-stable for simple cone programs described by the Lorentz cone $\Theta = \mathcal{K}^3$ in (6.11).

EXAMPLE 6.8 (**not tilt-stable isolated local minimizers in second-order cone programming**). Consider the following second-order cone program:

$$\text{minimize } \varphi(y) := y_1^3 + y_1 - y_3 \text{ subject to } y = (y_1, y_2, y_3) \in \mathcal{K}^3.$$

Thus in this case $\Theta = \Gamma = \mathcal{K}^3$ and g is the identity map. It is easy to check that $\bar{y} = (0, 0, 0)$ is the only optimal solution of the problem, which is an isolated minimizer. Fix any number $\gamma > 0$ and choose the sequence $p_k = (0, 0, \frac{3}{k}) \in \mathbb{R}^3$ as $k \in \mathbb{N}$. Then for any $y \in \mathcal{K}^3$ with $\|y - \bar{y}\| \leq \gamma$ and for any $k \in \mathbb{N}$ we have the inequalities

$$\varphi(y) - \langle p_k, y \rangle = y_1^3 + y_1 - \left(1 + \frac{3}{k}\right)y_3 \geq y_1^3 - \frac{3}{k}y_1 \geq -\frac{2}{k\sqrt{k}},$$

which become equalities for $y_k = (\frac{1}{\sqrt{k}}, 0, \frac{1}{\sqrt{k}})$. Note furthermore that

$$\|y_k - \bar{y}\| = \frac{\sqrt{2}}{\sqrt{k}} > \frac{\sqrt{k}}{3} \|p_k - 0\| \text{ for all } k \in \mathbb{N},$$

which shows that the argminimum mapping (6.12) is not Lipschitz continuous. Thus the minimizer \bar{y} is not tilt-stable for this cone-constrained program.

The next theorem provides a characterization of tilt-stable minimizers for general conic programs (6.11) via the limiting coderivative of the normal cone mapping to Γ .

THEOREM 6.9 (**tilt stability in the second-order framework of conic programming**). *Let $\bar{y} \in \Gamma$ be a feasible solution to conic program (6.11) under our standing assumptions. Then \bar{y} a tilt-stable local minimizer of (6.11) if and only if*

$$\langle w, \nabla^2 \varphi(\bar{y})w \rangle > -\langle u, w \rangle \text{ whenever } u \in D^*N_\Gamma(\bar{y}, -\nabla \varphi(\bar{y}))(w) \text{ and } w \neq 0. \quad (6.13)$$

Proof. Rewriting the conic program (6.11) in the unconstrained format

$$\text{minimize } \phi(y) := \varphi(y) + \delta_\Gamma(y) \text{ for all } y \in \mathbb{R}^m \quad (6.14)$$

with the extended-real-valued objective ϕ , we get from [34, Theorem 1.3] that $\bar{y} \in \Gamma$ is a tilt-stable local minimizer to (6.14), and hence to (6.11), if and only if

$$\langle u, w \rangle > 0 \text{ whenever } u \in \partial^2 \phi(\bar{y}, 0) \text{ with } w \neq 0 \quad (6.15)$$

in terms of the limiting second-order subdifferential (2.7), provided that $0 \in \partial \phi(\bar{y})$ and the function ϕ is prox-regular and subdifferentially continuous at $(\bar{y}, 0)$ in the sense of [37, Definition 13.27 and Definition 13.28], respectively. It follows from the elementary sum rule for the limiting subdifferential (see, e.g., [22, Proposition 1.107(ii)]) that the stationary condition $0 \in \partial \phi(\bar{y})$ for the function ϕ in (6.15) is equivalent to $0 \in \nabla \varphi(\bar{y}) + N_\Gamma(\bar{y})$. Furthermore, it is easy to see that the \mathcal{C}^2 -reducibility and nondegeneracy assumptions in (A1) and (A2) ensure that the set Γ is *strongly amenable* at \bar{y} in the sense of [37, Definition 10.23(b)]. This implies, by invoking [37, Example 10.24 and Proposition 13.32], that the function ϕ is both prox-regular and subdifferentially continuous at $(\bar{y}, 0)$. To complete the proof of the theorem, it remains to use the second-order subdifferential sum rule from [22, Proposition 1.121] telling us that

$$\partial^2(\varphi + \delta_\Gamma)(\bar{y}, 0)(w) = \nabla^2 \varphi(\bar{y})w + D^*N_\Gamma(\bar{y}, -\nabla \varphi(\bar{y}))(w) \text{ for all } w \in \mathbb{R}^m \quad (6.16)$$

and therefore (6.15) is equivalent to the claimed condition (6.13). \square

Employing next the second-order chain rule for the limiting constructions in (6.13) obtained in [32, Theorem 7] under the standing assumptions made (including the crucial nondegeneracy condition), we arrive at the following characterization of tilt-stable minimizers in the general problem (6.11) of conic programming expressed in terms of the initial problem data.

THEOREM 6.10 (characterization of tilt-stable local minimizers for general conic programs). *Let $\bar{y} \in g^{-1}(\Theta)$ satisfy all the assumptions of Theorem 6.9, and let $\bar{\lambda} \in \Theta^* \subset \mathbb{R}^l$ be a unique Lagrange multiplier satisfying the KKT system*

$$\nabla_y L(\bar{y}, \lambda) = 0, \quad \langle \lambda, g(\bar{y}) \rangle = 0 \quad (6.17)$$

with the Lagrangian $L(y, \lambda) := \varphi(y) + \langle \lambda, g(y) \rangle$. Then \bar{y} is a tilt-stable local minimizer of (6.11) if and only if for all $w \in \mathbb{R}^m \setminus \{0\}$ we have the relationship

$$\langle w, \nabla_{yy}^2 L(\bar{y}, \bar{\lambda})w \rangle + \langle \nabla g(\bar{y})w, D^*N_{\Theta}(g(\bar{y}), \bar{\lambda})(\nabla g(\bar{y})w) \rangle > 0. \quad (6.18)$$

Proof. It follows from the representation of the limiting coderivative D^*N_{Γ} via the second-order chain rule established in [32, Theorem 7] which can be combined with condition (6.13) and expressed via the Lagrangian L arising in (6.17). \square

If the constraint mapping g in the conic program (6.11) is Θ -convex as defined at the beginning of Section 3, we get the following sufficient condition for tilt stability.

COROLLARY 6.11 (sufficient condition for tilt-stable minimizers of conic programs with Θ -convex constraints). *In addition to the assumptions of Theorem 6.10, suppose that the mapping g is Θ -convex. Then the condition*

$$\langle w, \nabla^2 \varphi(\bar{y})w \rangle > 0 \quad \text{whenever} \quad \nabla g(\bar{y})w \in \text{dom } D^*N_{\Theta}(g(\bar{y}), \bar{\lambda}) \quad \text{and} \quad w \neq 0 \quad (6.19)$$

is sufficient for \bar{y} to be a tilt-stable local minimizer of the conic program (6.11).

Proof. It follows from the *monotonicity* result of [34, Theorem 2.1] and the maximal monotonicity of the normal cone mapping in convex analysis that

$$\langle \nabla g(\bar{y})w, D^*N_{\Theta}(g(\bar{y}), \bar{\lambda})(\nabla g(\bar{y})w) \rangle \geq 0 \quad \text{for all } w \in \mathbb{R}^m.$$

Furthermore, it follows from the inclusion $\bar{\lambda} = (\bar{\lambda}_1, \dots, \bar{\lambda}_l) \in \Theta^*$ and the Θ -convexity description (3.1) for \mathcal{C}^2 mappings that

$$\langle w, \nabla_{yy}^2 L(\bar{y}, \bar{\lambda})w \rangle = \langle w, \nabla^2 \varphi(\bar{y})w \rangle + \sum_{i=1}^l \bar{\lambda}_i \langle w, \nabla^2 g_i(\bar{y})w \rangle \geq \langle w, \nabla^2 \varphi(\bar{y})w \rangle.$$

Thus condition (6.19) implies (6.18), and we complete the proof of the corollary. \square

The tilt stability theory developed above can be related to stability analysis of GE (1.2) as follows. With program (6.11) we associate its *canonically perturbed* optimality condition and the corresponding solution map

$$S(x) = \{y \in \mathbb{R}^m \mid x \in \nabla \varphi(y) + \widehat{N}_{\Gamma}(y)\}, \quad (6.20)$$

which is in form (1.3) with $f(x, y) = \nabla \varphi(y) - x$. On the basis of the obtained characterization of tilt stability and a result from the recent paper [19] we can now characterize another important stability property of S .

Recall that a multifunction $F: \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ has a *single-valued Lipschitzian localization* at $(\bar{u}, \bar{v}) \in \text{gph } F$ if there exist neighborhoods \mathcal{U} of \bar{u} , \mathcal{V} of \bar{v} , and a single-valued Lipschitzian mapping $\rho: \mathcal{U} \rightarrow \mathbb{R}^m$ such that $\rho(\bar{u}) = \bar{v}$ and

$$F(u) \cap \mathcal{V} = \{\rho(u)\} \text{ for all } u \in \mathcal{U}.$$

By [9, Section 3G] the respective inverse mapping $F^{-1}: \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is then called *strongly metrically regular* at (\bar{v}, \bar{u}) .

THEOREM 6.12 (existence of single-valued Lipschitzian localization of S). *Let $\bar{y} \in g^{-1}(\Theta)$ be a local minimizer of (6.11) under the validity of the standing assumptions of this section, and let $\bar{\lambda} \in \Theta^* \subset \mathbb{R}^l$ be a unique Lagrange multiplier satisfying the KKT system (6.17). Then the solution map S given by (6.20) has a single-valued Lipschitzian localization at (\bar{x}, \bar{y}) with $\bar{x} = 0$ if and only if the second-order condition (6.18) is satisfied.*

Proof. It follows from Theorem 6.10 under the assumptions made that condition (6.18) is equivalent to \bar{y} being a tilt-stable local minimizer of (6.11). Moreover, as mentioned in the proof of Theorem 6.9, the extended-real-valued objective $\phi = \varphi + \delta_\Gamma$ is prox-regular and subdifferentially continuous at $(\bar{y}, 0)$. We can thus invoke [19, Proposition 7.2], which states in this case the equivalence between tilt stability of \bar{y} in (6.11) and strong metric regularity of the subgradient mapping $\partial\phi$ at $(\bar{y}, 0)$. Since our assumptions ensure the equalities

$$\partial\phi(y) = \nabla\varphi(y) + N_\Gamma(y) = \nabla\varphi(y) + \widehat{N}_\Gamma(y)$$

for all y close to \bar{y} , it follows that for these vectors y we have the equivalence

$$x \in \partial\phi(y) \iff y \in S(x),$$

which thus completes the proof of the theorem. \square

Having in hands the more detailed calculations of the limiting coderivative D^*N_Θ in the cases of the Lorentz cone $\Theta = \mathcal{K}^l$ and of the SDP cone $\Theta = \mathcal{S}_+^l$ discussed after Corollary 4.2 and in Remark 4.3, respectively, we can obtain further specifications of the tilt stability characterization from Theorem 6.10 for second-order cone programs and semidefinite programs expressed entirely via their initial data.

REMARK 6.13 (tilt stability in conic programming via composite optimization). In [29, 30] the so-called *composite optimization approach* was suggested to analyze tilt stability in constrained optimization and was applied there to special classes of problems in mathematical programming. In this approach the constrained problem (6.11) can be equivalently represented in the unconstrained composite format

$$\text{minimize } \phi(y) := \varphi(y) + (\delta_\Theta \circ g)(y), \quad y \in \mathbb{R}^m, \quad (6.21)$$

via the composition of the extended-real-valued indicator function $\delta_\Theta: \mathbb{R}^l \rightarrow \overline{\mathbb{R}}$ and the constraint mapping $g: \mathbb{R}^m \rightarrow \mathbb{R}^l$. In fact, (6.21) is yet another form of (6.14), which emphasizes the composite structure of $\Gamma = g^{-1}(\Theta)$ in (6.11). On the basis of this approach and the second-order calculus rules developed in [29, 30], verifiable characterizations of tilt stability were established in these papers for mathematical programs with a certain *polyhedral structure* of constraints such as classical nonlinear programs (NLPs), extended nonlinear programs (ENLPs), and mathematical programs with polyhedral constraints (MPPCs) for which a polyhedral version of the *nondegeneracy* condition played a crucial role. However, such a polyhedrality is *not*

the case for the general conic constraint under consideration, and thus we cannot make a full use of these results. Combining them with [32, Theorem 7] allows us to recover the second-order characterizations of tilt stability in conic programming presented in Theorem 6.10 and Corollary 6.11 above.

7. Concluding Remarks. This paper presents calculations of the major derivative and coderivative constructions of variational analysis for set-valued solution maps to parameterized generalized equations/KKT systems associated with conic constraints. The results established in this direction are based on new second-order calculus rules of generalized differentiation derived in the paper, which are of their own interest. The obtained derivative and coderivative formulas are applied to deriving sharp necessary optimality conditions for a class of MPECs with conic constraints and to characterizing two important stability properties, namely: isolated calmness of solution maps and tilt stability of local optimal solutions. These general results are specified for an important class of equilibria with second-order cone constraints and illustrated by examples. Moreover, they open a possibility to deal efficiently also with SDPs.

The developed approaches and results can be extended to more general calculus and application settings in both finite and infinite dimensions, which has been discussed in the remarks given above. In these concluding remarks we would like to point out two lines of a possible future research work associated with conic constraints.

The first one concerns the study of *full stability* [18] of local optimal solutions to conic programs. Recall that the full stability concept is an extension of tilt stability in the sense that the perturbed problem

$$\text{minimize } \phi(q, y) - \langle p, y \rangle \text{ over } y \in \mathbb{R}^m$$

depends on the parameter pair (q, p) . In the vein of [34], characterizations of full stability were established in [18] in the unconstrained framework via a certain *partial version* of the second-order subdifferential (2.7) and then were developed in [30] for some classes of constrained optimization problems by the composite approach discussed in Remark 6.13. The results obtained in [30] require a certain *polyhedral* structure of constraints, which is unfortunately not available in major classes of equilibria with conic constraints like, e.g., SOCPs and SDPs. The derivation of a suitable counterpart of Theorem 6.9 is thus definitely not straightforward.

Another challenging issue concerns *relaxing* the *nondegeneracy* assumption. Some results in this direction were obtained in [27] under condition (5.11) and a special second-order qualification condition and also in [14, 15] for classical nonlinear programs under the combination of the Mangasarian-Fromovitz and the constant rank constraint qualifications. We plan to proceed further in both of these directions.

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