New Fractional Error Bounds for Polynomial Systems with Applications to Hölderian Stability in Optimization and Spectral Theory of Tensors*

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Abstract

In this paper we derive new fractional error bounds for polynomial systems with exponents explicitly determined by the dimension of the underlying space and the number/degree of the involved polynomials. Our major result extends the existing error bounds from the system involving only a single polynomial to a general polynomial system and do not require any regularity assumptions. In this way we resolve, in particular, some open questions posed in the literature. The developed techniques are largely based on variational analysis and generalized differentiation, which allow us to establish, e.g., a nonsmooth extension of the seminal Łojasiewicz's gradient inequality to maxima of polynomials with explicitly determined exponents. Our major applications concern quantitative Hölderian stability of solution maps for parameterized polynomial optimization problems and nonlinear complementarity systems with polynomial data as well as high-order semismooth properties of the eigenvalues of symmetric tensors.

Keywords: Error Bounds, polynomials, Variational Analysis, Generalized Differentiation, Łojasiewicz's Inequality, Hölderian Stability, Polynomial Optimization and Complementarity

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1 Introduction

Constraint sets in many optimization problems can be described by systems of inequalities and equalities

$$g_i(x) \le 0,\ i = 1, \ldots, r, \ \text{ and } \ h_j(x) = 0,\ j = 1, \ldots, s, \tag{1.1}$$

where $g_i, h_j : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, r$ and $j = 1, \ldots, s$ are real-valued functions on $\mathbb{R}^n$. One of the most important issues for (1.1) is the so-called *error bounds*. Denoting by S the set of solutions to (1.1), recall

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that this system has a (local) *error bound* with *exponent* $\tau > 0$ at $\overline{x} \in \mathbb{R}^n$ if there exist a constant $c > 0$ and an neighborhood U of $\overline{x}$ such that

$$d(x, S) \le c\bigg(\sum_{i=1}^{r}[g_i(x)]_+ + \sum_{j=1}^{s}|h_j(x)|\bigg)^{\tau} \quad \text{for all} \quad x \in U, \tag{1.2}$$

where $d(x, S)$ signifies the Euclidean distance between x and the set S, and where $[\alpha]_+ := \max\{\alpha, 0\}$. This estimates bounds the distance from an arbitrary point x around the reference one $\overline{x}$ to the solution set S via a constant multiple of a computable *residual function*, which measures the violation of the constraint $S := \{x \in \mathbb{R}^n \,|\, g_i(x) \le 0,\ h_j(x) = 0\}$. The study of error bounds has attracted a lot of attention of many researchers over the years and has found numerous applications to, in particular, sensitivity analysis for various problems of mathematical programming, termination criteria for descent algorithms, etc. We refer the reader to [25, 48, 60] for excellent surveys in these directions. It is worth noting relationships between error bounds and *metric regularity/subregularity* issues in basic variational analysis [41, 56], where the main attention has been paid to the case of "linear rate" ($r = 1$); see also [1, 12, 29, 16] and their references for certain "fractional/root" versions.

One of the most important and celebrated error bound/metric regularity result is due to Hoffman [18] who proved, in the case of linear functions g_i and h_j and solvability of system (1.1), the existence of $c > 0$ such that the error bound (1.2) holds with $U = \mathbb{R}^n$ and $\tau = 1$. Extensions of Hoffman's error bound result to convex inequalities have been well established in the literature; see, e.g. [10, 19, 20, 22, 27, 30, 54, 59] and the references therein. Quite recently [26] various extensions of these results have been obtained for convex inequality systems on finite-dimensional Riemannian and Hadamard manifolds. For nonconvex inequality and equality systems some local error bound results have been established in [13, 44, 45, 46, 58] under certain regularity conditions, which bound the size of a suitable subdifferential of the function in question via its values around the reference point. On the other hand, it is proved in [35, 36] by using the cerebrated Łojasiewicz's inequality [33] that (1.2) holds with some *unknown* fractional exponent τ when all g_i and h_j are polynomials or analytic functions. Furthermore, it is stated by the authors of [36] in their concluding remarks that "we have not been able to obtain explicit formulas for the multiplier or the exponent in the error bound. We feel that such formulas would be useful for computational and other purposes." Note to this end that local error bound results with *explicit* exponents are indeed important for both theory and applications since they can be used, e.g., to establish explicit *convergence rates* of the proximal point algorithm as demonstrated in [5, 29, 32]. We also refer the reader to [37] for relevant discussions on other algorithms and to Section 5 below for new applications to quantitative Hölderian stability of polynomial optimization problems and nonlinear complementarity systems with polynomial data. There are some important progress along this direction for special polynomial systems. For example, as shown in [38], regularity assumptions are not needed to obtain (1.2) with $\tau = \frac{1}{2}$ if system (1.1) involves only one quadratic function; see also [45, 46] for infinite-dimensional extensions. Moreover, error bound results for system (1.1) that involves only one single polynomial has also been established in [8] without regularity assumptions.

Among major goals of this paper are extending the results in [8] from a single polynomial to general polynomial systems and establishing *error bound* results (1.1) with *explicit exponents* τ in (1.2). Employing advanced techniques of variational analysis and generalized differentiation allows us to derive error bounds for such systems with exponents explicitly determined by the dimension of the underlying space and the number/degree of the involved polynomials *without any regularity conditions.* Besides meeting the aforementioned general goals formulated in [36], in this way we resolve, in particular, a long-standing *open question* raised in [38] about Hölderian error bounds with explicit exponents for *nonconvex quadratic systems.* Furthermore, we apply our error bound results to deriving verifiable conditions for *Hölderian stability* of general polynomial optimization problems as well as nonlinear complementarity problems with polynomial data. As a by-product of our analysis, we give a positive answer to another *open question* raised in [31] about the *ρth-order semismoothness* of the maximum eigenvalue for a *symmetric tensor* with explicit estimating the exponent ρ. Since the concept of symmetric tensors has been well recognized as a high-order extension of symmetric matrices with various applications to automatic control and image science [51, 47, 53], the result obtained is of undoubted importance for further applications to these areas. Note that much of our study on error bounds is in the spirit of [35, 36, 27, 28] being largely motivated by the recent work on nonsmooth extensions of Łojasiewicz's inequality initiated in [3]. It is worth emphasizing that, as demonstrated in this paper, *generalized differential* techniques can be very instrumental for revolving applied quantitative issues even for *smooth*/polynomial systems. We also refer the reader to [6, 7, 22, 23, 25, 49, 30, 48, 58] and the bibliographies therein for other approaches to error bounds and their numerous applications.

The rest of the paper is organized as follows. In Section 2 we present some constructions and statements from generalized differentiation of variational analysis and polynomial theory, which are widely used in the formulations and proofs of the main results below. Section 3 is devoted to establishing major error bounds for polynomial systems with explicitly calculated exponents. In Section 4 we consider some special settings for which the exponents in error bounds obtained in Section 3 can be significantly sharpen. Section 5 concerns applications of the error bounds established in the previous sections to deriving new results on quantitative Hölderian stability for polynomial optimization problems and as well as for nonlinear complementarity problems with polynomial data. Finally, in Section 6 we present concluding remarks and discuss some directions of the future research.

Our notation is basically standard in variational analysis and optimization theory. All the spaces under consideration are finite-dimensional with the inner product $\langle x,y\rangle := x^Ty$ and the Euclidean norm $(\|x\| := (x^Tx)^{1/2}$ for any $x,y\in\mathbb{R}^n$, where x^T signifies the vector (as well as matrix) transposition. We use the symbols $\mathbb{B}(x,\epsilon)$ and $\overline{\mathbb{B}}(x,\epsilon)$ to denote the open and closed balls, respectively, of the space in question with center x and radius $\epsilon>0$. Given a set $\Omega\subset\mathbb{R}^n$, its interior (resp. closure, boundary and convex hull) is denoted by $\operatorname{int}\Omega$ (resp. $\mathrm{cl}\Omega, \operatorname{bd}\Omega$, and $\operatorname{co}\Omega$). Recall also that $\mathbb{N} := \{1,2,\ldots\}$.

2 Preliminaries

This section contains the necessary preliminaries needed in the paper. We start with generalized differentiation of variational analysis referring the reader to the books [41, 56] for more details and commentaries.

Given a proper extended-real-valued function $f\colon\mathbb{R}^n\to\overline{\mathbb{R}} := (\infty,\infty]$, we use the symbol $z\xrightarrow{f}x$ to indicate that $z\to x$ and $f(z)\to f(x)$. Our basic *subdifferential* of f at $x\in\operatorname{dom}f$ (known also as the general, or limiting, or Mordukhovich subdifferential) is defined by

$$\partial f(x) := \Big\{v\in\mathbb{R}^n\Big|\ \exists x_k\xrightarrow{f}x,\ v_k\to v\ \text{ with }\ \liminf_{z\to x_k}\frac{f(z)-f(x_k)-\langle v_k,z-x_k\rangle}{\|z-x_k\|}\ge 0\Big\}. \tag{2.1}$$

For convex functions f the subdifferential (2.1) reduces to the classical subdifferential of convex analysis

$$\partial f(x) = \big\{v\in\mathbb{R}^n\big|\ \langle v,z-x\rangle\le f(z)-f(x)\ \text{ whenever }\ z\in\mathbb{R}^n\big\},\quad x\in\operatorname{dom}f. \tag{2.2}$$

In the general case the subdifferential set (2.1) is often nonconvex (e.g., for $f(x)=-|x|$ at $0\in\mathbb{R}$) while ∂f enjoys comprehensive calculus rules based on *variational/extremal principles* of variational analysis [41, 56]. Note also that $\partial f(x)\neq\emptyset$ if f is locally Lipschitzian around x.

Definition 2.1 (subdifferential slope). *Given* $f\colon\mathbb{R}^n\to\overline{\mathbb{R}}$ *and using* (2.1)*, the* SUBDIFFERENTIAL SLOPE *of* f *at* $x\in\operatorname{dom}f$ *is defined by*

$$\mathfrak{m}_f(x) := \inf\big\{\|v\|\big|\ v\in\partial f(x)\big\}.$$

We can see directly from the definition that $\mathfrak{m}_f(x) = \infty$ whenever $\partial f(\bar{x}) = \emptyset$. Observe also that for $f \in \mathcal{C}^1$ around x we have $\partial f(x) = \{\nabla f(x)\}$ and hence $\mathfrak{m}_f(x) = \|\nabla f(x)\|$.

The following useful result is a consequence of [41, Theorem 3.46(ii)]; cf. also [56, Exercise 8.31].

Lemma 2.2 (subdifferential slope for maximum functions). *Let $g_1, \ldots, g_l \colon \mathbb{R}^n \to \mathbb{R}$ be functions of class $\mathcal{C}^1$, and let $f(x) := \max_{i=1,\ldots,l} g_i(x)$. Then f is a locally Lipschitz function, and we have*

$$\mathfrak{m}_f(x) = \min\Big\{ \sum_{i\in I(x)} \lambda_i \nabla g_i(x) \Big| \lambda_i \geq 0, \sum_{i \in I(x)} \lambda_i = 1 \Big\},$$

where $I(x)$ is the active index set at x defined by $I(x) := \{i \mid g_i(x) = f(x)\}$.

Next let us recall some facts concerning real polynomials (or polynomials with real coefficients). As usual, we say that $f \colon \mathbb{R}^n \to \mathbb{R}$ is a *polynomial* if there is a number $r \in \mathbb{N}$ such that

$$f(x) = \sum_{0 \leq |\alpha| \leq r} \lambda_\alpha x^\alpha,$$

where $\lambda_\alpha \in \mathbb{R}$, $x = (x_1, \ldots, x_n)$, $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $\alpha_i \in \mathbb{N} \cup \{0\}$, and $|\alpha| := \sum_{j=1}^n \alpha_j$. The corresponding constant r is called the *degree* of f. Recall further that $f \colon \mathbb{R}^n \to \mathbb{R}$ is (real) *analytic* if it can be locally represented on $\mathbb{R}^n$ by a convergent infinite power series, i.e., for all vectors $\overline{x} = (\overline{x}_1, \ldots, \overline{x}_n) \in \mathbb{R}^n$ there is a neighborhood U of $\overline{x}$ such that for every $x = (x_1, \ldots, x_n) \in U$ we have

$$f(x) = \sum_{|\alpha|=0}^{\infty} \lambda_\alpha (x - \overline{x})^\alpha.$$

A major property of analytic functions that is most important for this paper is given by the following classical result by Łojasiewicz [33]:

- **(Łojasiewicz's gradient inequality)** If f is an analytic function with $f(0) = 0$ and $\nabla f(0) = 0$, then there exist positive constants c, τ, and ϵ such that

$$\|\nabla f(x)\| \geq c|f(x)|^\tau \text{ for all } \|x\| \leq \epsilon.$$

As pointed out in [37], it is often difficult to determine the corresponding exponents τ in Łojasiewicz's gradient inequality, and they are typically unknown. Some estimates of the exponent τ in the gradient inequality were derived in [8, 15] in the case when f is a polynomial. To formulate these results, for each $n, d \in \mathbb{N}$ define the following two constants:

$$\kappa(n,d) := (d-1)^n + 1 \quad \text{and} \quad R(n,d) := \begin{cases} 1 & \text{if } d = 1, \\ d(3d-3)^{n-1} & \text{if } d \geq 2. \end{cases} \tag{2.3}$$

It is not hard to verify that $R(n,d) \geq \kappa(n,d)$ for any natural numbers n and d and that this inequality is strict when $n \geq 2$ and $d \geq 2$.

Lemma 2.3 (exponent estimates in Łojasiewicz's gradient inequality for polynomials). *Let f be a real polynomial on $\mathbb{R}^n$ with degree $d \in \mathbb{N}$. The following hold:*

(i) (cf. [8, Theorem 4.2]) *Suppose that $f(0) = 0$ and $\nabla f(0) = 0$. Then there exist constants $c, \epsilon > 0$ such that for all $\|x\| \leq \epsilon$ we have*

$$\|\nabla f(x)\| \geq c|f(x)|^\tau \quad \textit{with} \quad \tau = 1 - R(n,d)^{-1}.$$

(ii) (cf. [15, 21]) *Suppose that $\bar{x} = 0$ is an isolated zero of f in the sense that $f(0) = 0$ and there is $\delta > 0$ with $f(x) > 0$ for all $x \in \mathbb{B}(0,\delta) \backslash \{0\}$. Then there exist positive constants c, ϵ such that for all $\|x\| \leq \epsilon$ we have*

$$\|\nabla f(x)\| \geq c|f(x)|^\tau \quad \textit{with} \quad \tau = 1 - \kappa(n,d)^{-1}.$$

3 Error Bounds for Polynomial Systems

In this section we establish new error bound results for polynomial system *without any regularity conditions.* Let us begin with specifying the definition of local error bounds.

Definition 3.1 (local error bounds). *We say that system* (1.1) *has a* LOCAL HÖLDERIAN (OR HÖLDER TYPE) ERROR BOUND WITH EXPONENT $\tau > 0$ *at* $\overline{x} \in \mathbb{R}^n$ *if there are positive constants* c *and* ϵ *such that*

$$d(x,S) \le c\bigg(\sum_{i=1}^{r}[g_i(x)]_+ + \sum_{j=1}^{s}|h_j(x)|\bigg)^{\tau} \quad \text{for all } x \text{ with } \|x-\overline{x}\| \le \epsilon, \tag{3.1}$$

where S *is the solution set for the system* (1.1) *given by*

$$S := \big\{x \in \mathbb{R}^n \big|\ g_i(x) \le 0,\ i = 1,\ldots,r, \text{ and } h_j(x) = 0,\ j = 0,\ldots,s\big\}. \tag{3.2}$$

Throughout this paper, to avoid triviality, we always assume that $\emptyset \ne S \ne \mathbb{R}^n$.

Prior to deriving the main results of this section we present an example illustrating the dependence of error bounds for polynomial systems on the degree of the polynomials involved and on the dimension of the problem/space in question. Note that for $d = 2$ this example is given in [36] (see also [21]).

Example 3.2 (dependence of error bounds on polynomial degrees and space dimensions). Let $d \in \mathbb{N}$, and let $h_j(x_1,\ldots,x_n) := x_{j+1} - x_j^d$ for $i = 1,\ldots,n-1$, $h_n(x_1,\ldots,x_n) := x_n^d$, and $g_i(x) \equiv 0$ for all $i = 1,\ldots,r$ in (3.1). Then the solution set S for (3.1) is $S = \{x \in \mathbb{R}^n |\ x = 0\}$. Take further $\overline{x} = 0$ and consider the family of vectors $x(\epsilon) := (\epsilon, \epsilon^d, \ldots, \epsilon^{d^{n-1}}) \in \mathbb{R}^n$ with $\epsilon \in (0,1]$. It is easy to see that $d(x(\epsilon),S) = \sqrt{\sum_{i=1}^n \epsilon^{2d^{i-1}}} = O(\epsilon)$, $\sum_{j=1}^n |h_j(x)| = \epsilon^{d^n}$, and thus we have

$$d\big(x(\epsilon),S\big) = O\bigg(\bigg[\sum_{j=1}^n |h_j(x(\epsilon))|\bigg]^{\frac{1}{d^n}}\bigg),$$

which shows that the exponent τ in (3.1) for this system at $\overline{x}$ does not exceed d^{-n}.

Our first goal in this section is employing Lemma 2.2 and Lemma 2.3(i) to obtain a *nonsmooth* version of Łojasiewicz's gradient inequality for *maximum* functions over finitely many polynomials with an *explicit exponent.* It is certainly of its own interest while being applied in what follows to deriving error bounds for polynomial systems with explicit fractional exponents.

Theorem 3.3 (nonsmooth Łojasiewicz's inequality with explicit exponent for maximum functions). *Let* $f(x) := \max\{g_1(x),\ldots,g_l(x)\}$*, where* g_i *for* $i = 1,\ldots,l$ *are real polynomials on* $\mathbb{R}^n$ *with their degrees not exceeded* d*, and let* $\overline{x} \in \mathbb{R}^n$ *with* $f(\overline{x}) = 0$*. Then there are numbers* $c, \epsilon > 0$ *such that*

$$\mathfrak{m}_f(x) \ge c\,|f(x)|^{1-\frac{1}{R(n+l-1,\,d+1)}} \quad \text{for all } x \text{ with } \|x-\overline{x}\| \le \epsilon,$$

where $\mathfrak{m}_f(x)$ *is the subdifferential slope from Definition* 2.1*, and where the constant* R *is defined in* (2.3).

Proof. Without loss of generality, assume that $g_i(\overline{x}) = 0$ for all $i = 1,\ldots,l$. Then for each subset $I := \{i_1,\ldots,i_q\} \subset \{1,\ldots,l\}$, we define the function $F_I : \mathbb{R}^n \times \mathbb{R}^{q-1} \to \mathbb{R}$ by

$$F_I(x,\lambda) := \begin{cases} \displaystyle\sum_{j=1}^{q-1}\lambda_j g_{i_j}(x) + \Big(1 - \sum_{j=1}^{q-1}\lambda_j\Big) g_{i_q}(x) & \text{if } q \ge 2, \\ g_{i_1}(x) & \text{if } q = 1, \end{cases}$$

which is clearly a polynomial on $\mathbb{R}^{n+q-1}$ with degree at most $d+1$ and $F(\bar{x}, \lambda) = 0$ for all $\lambda \in \mathbb{R}^{q-1}$. Define further the set $\mathbf{P} \subset \mathbb{R}^{q-1}$ by

$$\mathbf{P} := \Big\{\lambda \in \mathbb{R}^{q-1} \Big| \ \lambda_j \geq 0, \ \sum_{j=1}^{q-1} \lambda_j \leq 1\Big\}.$$

Then, there exist numbers $c_I > 0$ and $\epsilon_I > 0$ for which we have

$$\|\nabla F_I(x, \lambda)\| \geq c_I \, |F_I(x, \lambda)|^{1 - \frac{1}{R(n+q-1, \, d+1)}} \quad \text{whenever} \quad \|x - \overline{x}\| \leq \epsilon_I \quad \text{and} \quad \lambda \in \mathbf{P}. \tag{3.3}$$

To verify (3.3), by standard compactness arguments, we only need to check that for each $\bar{\lambda} \in \mathbf{P}$ there are numbers $c(\bar{\lambda}) > 0$ and $\epsilon(\bar{\lambda}) > 0$ such that

$$\|\nabla F(x, \lambda)\| \geq c(\bar{\lambda}) \, |F(x, \lambda)|^{1 - \frac{1}{R(n+q-1, \, d+1)}} \qquad \text{for all} \quad \|x - \overline{x}\| \leq c(\bar{\lambda}), \ \|\lambda - \bar{\lambda}\| \leq \epsilon(\bar{\lambda}).$$

Indeed, since $F(\bar{x}, \lambda) = 0$ for all $\lambda \in \mathbf{P}$, it is obvious if $\|\nabla F(\overline{x}, \bar{\lambda})\| \neq 0$, while in the remaining case of $\|\nabla F(\overline{x}, \bar{\lambda})\| = 0$ this inequality follows from Lemma 2.3(i).

Let $c := \min\big\{c_I | \ I \subset \{1, \ldots, l\}, \ I \neq \emptyset\big\} > 0$ and $\epsilon := \min\big\{\epsilon_I | \ I \subset \{1, \ldots, l\}, \ I \neq \emptyset\big\} > 0$. Pick an arbitrary point x in $\mathbb{R}^n$ with $\|x - \overline{x}\| \leq \epsilon$ and denote $\bar{I} = I(x) := \{i \mid g_i(x) = f(x)\}$. Lemma 2.2 tells us that there are numbers $\lambda_i \geq 0$ for $i \in \bar{I}$ such that $\sum_{i \in \bar{I}} \lambda_i = 1$ and

$$\mathfrak{m}_f(x) = \left\|\sum_{i \in \bar{I}} \lambda_i \nabla g_i(x)\right\|.$$

Let us renumerate the index set $\bar{I}$ as $\bar{I} = \{i_1, \ldots, i_{q_0}\}$, where q_0 signifies its cardinality. Then

$$F_{\bar{I}}(x, \lambda_{i_1}, \ldots, \lambda_{i_{q_0-1}}) \ = \ \sum_{j=1}^{q_0} \lambda_{i_j} g_{i_j}(x) = \sum_{i \in \bar{I}} \lambda_i g_i(x) = \sum_{i \in I(x)} \lambda_i g_i(x) = f(x).$$

Furthermore, we have the representations

$$\begin{aligned} \|\nabla F_{\bar{I}}(x, \lambda_{i_1}, \ldots, \lambda_{i_{q_0-1}})\| \ &= \ \left\|\left(\sum_{j=1}^{q_0} \lambda_{i_j} \nabla g_{i_j}(x), g_{i_1}(x) - g_{i_{q_0}}(x), \ldots, g_{i_{q_0-1}}(x) - g_{i_{q_0}}(x)\right)\right\| \\ &= \ \left\|\sum_{j=1}^{q_0} \lambda_{i_j} \nabla g_{i_j}(x)\right\| = \left\|\sum_{i \in I(x)} \lambda_i \nabla g_i(x)\right\| = \mathfrak{m}_f(x), \end{aligned}$$

which, being combined with inequality (3.3), allow us to conclude that

$$\begin{aligned} \mathfrak{m}_f(x) \ &= \ \|\nabla F_{\bar{I}}(x, \lambda_{i_1}, \ldots, \lambda_{i_{q_0-1}})\| \\ &\geq \ c_{\bar{I}} \, |F_{\bar{I}}(x, \lambda_{i_1}, \ldots, \lambda_{i_{q_0-1}})|^{1 - \frac{1}{R(n+q_0-1, \, d+1)}} \\ &= \ c_{\bar{I}} \, |f(x)|^{1 - \frac{1}{R(n+q_0-1, \, d+1)}}, \\ &\geq \ c \, |f(x)|^{1 - \frac{1}{R(n+l-1, \, d+1)}} \end{aligned}$$

and thus to complete the proof of the theorem. □

Employing further the nonsmooth Łojasiewicz's inequality of Theorem 3.3 leads us to effective error bounds of polynomial systems with explicit exponents. To proceed, we need the following lemma on error bounds for locally Lipschitz functions taken from [46, Corollary 2].

Lemma 3.4 (sufficient condition for error bounds of Lipschitz functions). *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitzian around $\overline{x} \in \operatorname{bd} S_f$, where $S_f = \{x|\ f(x) \le 0\}$. Assume that there are numbers $c, \epsilon > 0$ such that $\mathfrak{m}_f(x) \ge c\,|f(x)|^{1-\tau}$ for all x with $\|x-\overline{x}\| \le \epsilon$ and $x \notin S_f$. Then we have*

$$d(x, S_f) \le \frac{1}{c}\big[f(x)\big]_+^{\tau} \quad \textit{whenever} \quad \|x-\overline{x}\| \le \frac{\epsilon}{2}.$$

Now we are ready to derive the first error bound result of this paper.

Theorem 3.5 (local error bounds with explicit fractional exponents for polynomial systems, type I). *Let g_i as $i = 1, \ldots, r$ and h_j as $j = 1, \ldots, s$ be real polynomials on $\mathbb{R}^n$ with degree at most d, and let S be the solution set* (3.2). *Then there are numbers $c, \epsilon > 0$ such that*

$$d(x, S) \le c\Bigg(\sum_{i=1}^{r}[g_i(x)]_+ + \sum_{j=1}^{s}|h_j(x)|\Bigg)^{\frac{1}{R(n+r+s,d+1)}} \quad \textit{whenever} \quad \|x-\overline{x}\| \le \epsilon,$$

where the quantity R is defined in (2.3).

Proof. The conclusion is rather straightforward if either $\overline{x} \in \operatorname{int} S$ or $\overline{x} \notin S$. To proceed with the remaining case of $\overline{x} \in \operatorname{bd} S$, define the function $f\colon \mathbb{R}^n \to \mathbb{R}_+$ by

$$f(x) := \max\big\{[g_1(x)]_+, \ldots, [g_r(x)]_+, |h_1(x)|, \ldots, |h_s(x)|\big\}$$

and easily verify the representations

$$\begin{aligned} f(x) &= \max\big\{[g_1(x)]_+, \ldots, [g_r(x)]_+, |h_1(x)|, \ldots, |h_s(x)|\big\} \\ &= \max\big\{0, g_1(x), \ldots, g_r(x), h_1(x), \ldots, h_s(x), -h_1(x), \ldots, -h_s(x)\big\} \end{aligned}$$

with $f(\overline{x}) = 0$. Form further the vector $e := (e_1, \ldots, e_s) \in \{-1, 1\}^s$ and define the function

$$f_e(x) := \max\big\{0, g_1(x), \ldots, g_r(x), e_1 h_1(x), \ldots, e_s h_s(x)\big\}, \quad x \in \mathbb{R}^n,$$

which is the maximum of $r+s+1$ polynomials with degree at most d and with $f_e(\overline{x}) = 0$. Employing Theorem 3.3 gives us numbers $c(e) > 0$ and $\epsilon(e) > 0$ such that

$$\mathfrak{m}_{f_e}(x) \ge c(e)|f_e(x)|^{1-\frac{1}{R(n+r+s,d+1)}} \quad \text{whenever} \quad \|x-\overline{x}\| \le \epsilon(e).$$

Let $c := \min_{e\in\{-1,1\}^s} c(e) > 0$ and $\epsilon := \min_{e\in\{-1,1\}^s} \epsilon(e) > 0$. Take any x with $\|x-\overline{x}\| \le \epsilon$ and $f(x) > 0$. Then for each $j = 1, \ldots, s$ we have that either $h_j(x) \ne -h_j(x)$ or $h_j(x) < f(x)$. It allows us to find $\bar{e} \in \{-1, 1\}^s$ so that $f(x) = f_{\bar{e}}(x)$ and $\mathfrak{m}_f(x) = \mathfrak{m}_{f_{\bar{e}}}(x)$. This gives us the estimate

$$\mathfrak{m}_f(x) = \mathfrak{m}_{f_{\bar{e}}}(x) \ge c(\bar{e})\,|f_{\bar{e}}(x)|^{1-\frac{1}{R(n+r+s,d+1)}} \ge c\,|f(x)|^{1-\frac{1}{R(n+r+s,d+1)}},$$

which completes the proof of the theorem by applying Lemma 3.4. $\square$

Employing another technique (somewhat similar to [35, 36]) and Lemma 2.3(i), the next theorem provides a local error bound with an explicit exponent for polynomial systems, which is different from that in Theorem 3.5. The idea of the proof is to use certain *slack variables* to convert the polynomial system (1.1) into a single polynomial and then apply Lemma 2.3(i).

Theorem 3.6 (local error bounds with explicit fractional exponents for polynomial systems, type II). *Let g_i as $i = 1, \ldots, r$ and h_j as $j = 1, \ldots, s$ be real polynomials on $\mathbb{R}^n$ with degree at most d, and let S be given in* (3.2). *Then there are numbers $c, \epsilon > 0$ such that*

$$d(x, S) \le c\Bigg(\sum_{i=1}^{r}[g_i(x)]_+ + \sum_{j=1}^{s}|h_j(x)|\Bigg)^{\frac{2}{R(n+r,2d)}} \quad \textit{whenever} \quad \|x-\overline{x}\| \le \epsilon,$$

where the quantity R is defined in (2.3).

Proof. Similarly to the proof of Theorem 3.5, we only need to examine the case of $\overline{x} \in \operatorname{bd} S$. Define the polynomial $\theta : \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}$ by

$$\theta(x,z) := \sum_{i=1}^{r} \big(g_i(x) + z_i^2\big)^2 + \sum_{j=1}^{s} h_j(x)^2$$

and note that its degree does not exceed $2d$. Consider the set $\widetilde{S} := \{(x,z) \in \mathbb{R}^n \times \mathbb{R}^r \mid \theta(x,z) = 0\}$ and the continuous mapping $\phi(x) := (\sqrt{[-g_1(x)]_+}, \ldots, \sqrt{[-g_r(x)]_+})$ on $\mathbb{R}^n$.

Since $\overline{x} \in S$, we have $\theta(\overline{x}, \phi(\overline{x})) = 0$ and $\nabla\theta(\overline{x}, \phi(\overline{x})) = 0$. Applying Lemma 2.3(i) to θ gives us positive numbers ϵ_0 and c_0 such that $\|\nabla\theta(x,z)\| \geq c_0\, \theta(x,z)^{1-\frac{1}{R(n+r,2d)}}$ for all $\|(x,z) - (\overline{x}, \phi(\overline{x}))\| \leq \epsilon_0$. Let $c := c_0^{-1}$. Then Lemma 3.4 ensures the estimate

$$d\big((x,z), \widetilde{S}\big) \leq c\, \theta(x,z)^{\frac{1}{R(n+r,2d)}} \text{ for all } (x,z) \text{ with } \|(x,z) - (\overline{x}, \phi(\overline{x}))\| \leq \frac{\epsilon_0}{2}. \tag{3.4}$$

By continuity of ϕ we find $0 < \epsilon < \frac{\epsilon_0}{4}$ such that $\|\phi(x) - \phi(\overline{x})\| \leq \frac{\epsilon_0}{4}$ whenever $\|x - \overline{x}\| \leq \epsilon$, which clearly implies the inequality

$$\|(x, \phi(x)) - (\overline{x}, \phi(\overline{x}))\| \leq \frac{\epsilon_0}{2} \text{ whenever } \|x - \overline{x}\| \leq \epsilon. \tag{3.5}$$

Now let x be an arbitrary vector satisfying $\|x - \overline{x}\| \leq \epsilon$. There exists a point $(\widetilde{x}, \widetilde{z}) \in \widetilde{S}$ such that $\|(x, \phi(x)) - (\widetilde{x}, \widetilde{z})\| = d\big((x, \phi(x)), \widetilde{S}\big)$. By definition of $\widetilde{S}$ we have $\theta(\widetilde{x}, \widetilde{z}) = 0$, and hence $g_i(\widetilde{x}) = -\widetilde{z}_i^2 \leq 0,\ i = 1, \ldots, r,$ and $h_j(\widetilde{x}) = 0,\ j = 1, \ldots, s$. This implies that $\widetilde{x} \in S$. Therefore

$$\begin{aligned} d(x,S) &\leq \|x - \widetilde{x}\| \leq \|(x, \phi(x)) - (\widetilde{x}, \widetilde{z})\| = d\big((x, \phi(x)), \widetilde{S}\big) \\ &\leq c\, \theta(x, \phi(x))^{\frac{1}{R(n+r,2d)}} \\ &= c\bigg(\sum_{i=1}^{r} [g_i(x)]_+^2 + \sum_{j=1}^{s} h_j(x)^2\bigg)^{\frac{1}{R(n+r,2d)}} \\ &\leq c\bigg(\sum_{i=1}^{r} [g_i(x)]_+ + \sum_{j=1}^{s} |h_j(x)|\bigg)^{\frac{2}{R(n+r,2d)}}, \end{aligned}$$

where the third inequality follows from (3.4) and (3.5) while the last equality follows from the fact that $g_i(x) + [-g_i(x)]_+ = [g_i(x)]_+$. This justifies the claimed error bound. ☐

Remark 3.7 (comparing the two types of local error bounds). It is worth noting that the two types of local error bounds obtained in Theorem 3.5 and Theorem 3.6 are generally *independent* from each other. Recall that $R(p,q) = q(3q-3)^{p-1}$ in the setting of Theorem 3.6. Consider, e.g., the case of $n = 3$, $r = 4$, $s = 1$, and $d = 2$. Then $\frac{2}{R(n+r,2d)} = \frac{2}{R(7,4)} = \frac{1}{4\cdot 9^6}$ and $\frac{1}{R(n+r+s,d+1)} = \frac{1}{R(8,3)} = \frac{1}{3\cdot 6^7}$; thus we have in this case that $\frac{2}{R(n+r,2d)} < \frac{1}{R(n+r+s,d+1)}$. On the other hand, letting $n = r = 1$, $s = 2$, and $d = 2$, we get that $\frac{2}{R(n+r,2d)} = \frac{2}{R(2,4)} = \frac{1}{18}$ and $\frac{1}{R(n+r+s,d+1)} = \frac{1}{R(4,3)} = \frac{1}{4\cdot 9^2}$; so it gives $\frac{2}{R(n+r,2d)} > \frac{1}{R(n+r+s,d+1)}$.

As a consequence of the theorem, we now establish some *globalized* error bound results with explicit exponents for polynomial systems of type (1.1) over *compact* sets.

Corollary 3.8 (Hölderian error bounds with explicit exponents for polynomial systems over compact sets). *Let g_i, h_j, and S be as in Theorem 3.6. Then for any compact set $K \subset \mathbb{R}^n$ there is a number $c > 0$ such that*

$$d(x,S) \leq c\bigg(\sum_{i=1}^{r} [g_i(x)]_+ + \sum_{j=1}^{s} |h_j(x)|\bigg)^{\tau} \quad \textit{for all} \quad x \in K, \tag{3.6}$$

where the exponent τ is calculated as

$$\tau := \max\left\{\frac{1}{R(n+r+s,d+1)}, \frac{2}{R(n+r,2d)}\right\} = \max\left\{\frac{1}{(d+1)(3d)^{n+r+s-1}}, \frac{1}{d(6d-3)^{n+r-1}}\right\}. \quad (3.7)$$

In particular, the local Hölderian error bound (3.1) *holds with τ given by* (3.7)*.*

Proof. Combining the results of Theorem 3.5 and Theorem 3.6, for every $\overline{x} \in \mathbb{R}^n$ we can find numbers $\epsilon(\overline{x}) > 0$ and $c(\overline{x}) > 0$ such that

$$d(x,S) \le c(\overline{x})\left(\sum_{i=1}^{r}[g_i(x)]_+ + \sum_{j=1}^{s}|h_j(x)|\right)^{\tau} \text{ whenever } \|x-\overline{x}\| \le \epsilon(\overline{x}).$$

Then the conclusion follows by using standard compactness arguments. □

Let us mention that the authors of [38] established a Hölder error bound with exponent $\tau = \frac{1}{2}$ over compact sets for a single quadratic function and then *raised the question* about the possibility to extend this result to *finitely many quadratic functions.* They actually conjectured that a Hölder error bound would hold over compact sets for nonconvex quadratic systems with exponent $\tau = \frac{1}{2^p}$ with p denoting the number of quadratic functions involved in the system. Now we provide a *partial answer* for their conjecture by showing that such an error bound holds with a *larger* while *exactly calculated exponent.*

Corollary 3.9 (Hölderian error bounds over compact sets for nonconvex quadratic systems). *Let $r, s \in \mathbb{N}$, let g_i as $i = 1, \ldots, r$ and h_j as $j = 1, \ldots, s$ be quadratic functions on $\mathbb{R}^n$, and let S be defined in* (3.2)*. Then for any compact set $K \subset \mathbb{R}^n$ there is a number $c > 0$ such that the error bound inequality* (3.6) *holds with the explicit exponent τ calculated by*

$$\tau = \max\left\{\frac{1}{R(n+r+s,3)}, \frac{2}{R(n+r,4)}\right\} = \max\left\{\frac{1}{3\cdot 6^{n+r+s-1}}, \frac{1}{2\cdot 9^{n+r-1}}\right\}.$$

Proof. It follows from Corollary 3.8 with $d = 2$ and formula (3.7) for calculating τ. □

Next we show that the globalized version of the Hölderian error bound result from Corollary 3.8 over compact set *cannot* be generally extended to the *global* one over the whole space $\mathbb{R}^n$. The following example was used in [11] in the case of $d = 2$,

Example 3.10 (failure of global error bounds for polynomial systems). Let d be any even number. Define the polynomial function $h\colon \mathbb{R}^2 \to \mathbb{R}$ by $h(x) := (x_1x_2-1)^d + (x_1-1)^d$. The solution set here is $S = \{x \in \mathbb{R}^2 |\ h(x) = 0\} = \{(1,1)\}$. The global version of the error bound in Corollary 3.8 is as follows: there are numbers $c, \tau > 0$ such that

$$d(x,S) \le c\,|h(x)|^{\tau} \text{ for all } x \in \mathbb{R}^n. \quad (3.8)$$

To show that (3.8) fails, consider a sequence $x_k = (\frac{1}{k}, k)$ for $h(x_k) = (1-\frac{1}{k})^d \to 1$ and $d(x_k, S) \to \infty$ as $k \to \infty$. Then the global error bound (3.8) is obviously violated along this sequence.

We conclude this section by establishing (as yet another consequence of the main results above) a Hölder-type regularity property for two *nonconvex semi-algebraic* sets, i.e., subsets of $\mathbb{R}^n$ that can be described by finitely many equality and inequality constraints given by polynomials. We say that the pair of sets $\{Q, T\}$ has the *bounded Hölderian regularity property* with exponent $\tau > 0$ if for each compact set K there is a constant $c > 0$ such that

$$d(x, Q\cap T) \le c\big(d(x,Q) + d(x,T)\big)^{\tau} \text{ whenever } x \in K. \quad (3.9)$$

For convex sets with $\tau = 1$ in (3.9) this property reduces to the so-called *bounded linear regularity* of [2], which is an important concept of convex analysis and optimization with various applications; in particular, to convergence rates of alternative projection algorithms [2, 5]. Observe that in real algebraic geometry properties of this type are referred to as *separation* of semi-algebraic sets and go back to Łojasiewicz [34]. The following corollary ensures the bounded Hölderian regularity of nonconvex semi-algebraic sets with explicit calculating the Hölder exponent τ in (3.9).

Corollary 3.11 (bounded Hölderian regularity of semi-algebraic sets). *Let $g_i^{(m)}$ for $i = 1, \ldots, r_m$ and $h_j^{(m)}$ for $j = 1, \ldots, s_m$, $m = 1, 2$, be real polynomials on $\mathbb{R}^n$ with degree at most $d \geq 2$. Consider the two semi-algebraic sets in $\mathbb{R}^n$ defined by*

$$\begin{aligned} Q &:= \big\{x \in \mathbb{R}^n \big|\ g_i^{(1)}(x) \leq 0,\ i = 1, \ldots, r_1,\ h_j^{(1)}(x) = 0,\ j = 1, \ldots, s_1\big\}, \\ T &:= \big\{x \in \mathbb{R}^n \big|\ g_i^{(2)}(x) \leq 0,\ i = 1, \ldots, r_2,\ h_j^{(2)}(x) = 0,\ j = 1, \ldots, s_2\big\}. \end{aligned}$$

Then for any compact set $K \subset \mathbb{R}^n$ there is a constant $c > 0$ such that the bounded Hölder regularity property (3.9) *holds with the exponent τ calculated in* (3.7)*, where $r := r_1 + r_2$ and $s := s_1 + s_2$.*

Proof. Define the real-valued functions

$$\begin{aligned} f_Q(x) &:= \sum_{i=1}^{r_1} \big[g_i^{(1)}(x)\big]_+ + \sum_{j=1}^{s_1} |h_j^{(1)}(x)|, \\ f_T(x) &:= \sum_{i=1}^{r_2} \big[g_i^{(2)}(x)\big]_+ + \sum_{j=1}^{s_2} |h_j^{(2)}(x)|, \end{aligned}$$

and observe that $f_Q^{-1}(0) = Q$, $f_T^{-1}(0) = T$, and $(f_Q + f_T)^{-1}(0) = Q \cap T$. Since K is compact, we have that $M := \max\{\max_{x \in K} d(x, Q), \max_{x \in K} d(x, Q)\} < \infty$ and that the set $K_0 = K + M\,\overline{\mathbb{B}(0,1)}$ is also compact. It follows from Corollary 3.8 that there is a constant $c > 0$ such that

$$d(x, Q \cap T) \leq c\,\big(f_Q(x) + f_T(x)\big)^\tau \quad \text{for all} \quad x \in K_0. \tag{3.10}$$

On the other hand, it is easy to see that the functions f_Q, f_T are locally Lipschitzian, and so they are Lipschitz continuous on the compact set K_0, i.e., there is a constant $L > 0$ for which $x, y \in K_0$,

$$|f_Q(x) - f_Q(y)| \leq L\|x - y\|, \quad |f_T(x) - f_T(y)| \leq L\|x - y\| \ \text{ whenever } \ x, y \in K_0.$$

Now pick any $x \in K$ and find $y \in Q, z \in T$ such that $d(x, Q) = \|x - y\|$ and $d(x, T) = \|x - z\|$. Since $y, z \in K_0$, we get the estimates

$$|f_Q(x)| \ = \ |f_Q(x) - f_Q(y)| \leq L\|x - y\| = Ld(x, Q), \quad |f_T(x)| \ = \ |f_T(x) - f_T(z)| \leq L\|x - z\| = Ld(x, T).$$

Combining them with (3.10) completes the proof of the corollary. □

4 Hölderian Error Bounds with Sharper Exponents

In this section we study two particular classes of polynomial systems and derive for them Hölderian error bounds with sharper explicit exponents in comparison with general results of Section 3.

4.1 Polynomial Systems with Finitely Many Solutions

This subsection deals with polynomial systems (1.1) whose solution sets (3.2) consists of only *finitely many points*. We now show that the fractional exponent τ in Corollary 3.8 on the Hölderian error bound over compact sets can be significantly sharpen for such systems.

Theorem 4.1 (sharper error bounds over compact sets for systems with finitely many solutions). *Let g_i as $i = 1, \ldots, r$ and h_j as $j = 1, \ldots, s$ be real polynomials on $\mathbb{R}^n$ with degree at most d, and let the solution set* (3.2) *consist of finitely many points. Then for any compact set $K \subset \mathbb{R}^n$ there is a constant $c > 0$ such that we have the error bound*

$$d(x, S) \le c \left(\sum_{i=1}^{r} [g_i(x)]_+ + \sum_{j=1}^{s} |h_j(x)| \right)^{\frac{2}{\kappa(n+r,2d)}} \quad \text{for all} \;\; x \in K,$$

where the quantity $\kappa > 0$ is defined in (2.3).

Proof. The proof follows on the same lines as that of Theorem 3.6, by using Lemma 2.3(ii) instead of Lemma 2.3(i) and by employing a standard compactness argument. We omit the details. □

4.2 Polynomial Systems with Simple Equalities

In this subsection we sharpen exponents in error bounds for another type of polynomial systems. Recall that a polynomial f with degree d is *simple* if it can be written as

$$f(x) = \gamma \prod_{i \in I} (x_i - a_i)^{\alpha_i}, \tag{4.1}$$

where $I \subset \{1, \ldots, n\}$, $\gamma \neq 0$, $a_i \in \mathbb{R}$, and $\alpha_i \in \mathbb{N}$ for $i \in I$ with $\sum_{i \in I} \alpha_i = d$. Note that a simple polynomial system may have infinitely many solutions. Consider, e.g., the function $f(x_1, x_2) = x_1^3$, which is a simple polynomial with the solution set $\{(x_1, x_2) \in \mathbb{R}^2 |\; f(x_1, x_2) = 0\} = \{0\} \times \mathbb{R}$.

We begin with a particular case when the polynomial system involves one simple polynomial equality.

Lemma 4.2 (global error bound for one simple polynomial). *Let $h \colon \mathbb{R}^n \to \mathbb{R}$ be a real simple polynomial of degree d, and let $S := \{x \in \mathbb{R}^n \mid h(x) = 0\}$. Then there is a constant $c > 0$ such that*

$$d(x, S) \le c\, |h(x)|^{\frac{1}{d}} \quad \text{for all} \;\; x \in \mathbb{R}^n.$$

Proof. Representing h in form (4.1), we have $S = \bigcup_{i \in I} \{x \in \mathbb{R}^n \big|\; x_i = a_i\}$ and arrive at

$$|h(x)| = |\gamma| \prod_{i \in I} |x_i - a_i|^{\alpha_i} \ge |\gamma| (\min_{i \in I} |x_i - a_i|)^d = |\gamma| \big(d(x, S)\big)^d, \quad x \in \mathbb{R}^n.$$

This readily ensures the claimed error bound. □

It is worth noting that simple polynomial assumption is essential in Lemma 4.2. Indeed, consider the function $h(x) := (x_1 x_2 - 1)^d + (x_1 - 1)^d$, which is not a simple polynomial. Then it follows from Example 3.10 that the global error bound of Lemma 4.2 fails. The next example shows that this global error bound can also fail for simple polynomial systems involving more than one simple polynomial.

Example 4.3 (failure of global error bound for general simple polynomial systems). Consider the two polynomials $h_1(x_1, x_2) := x_1^2$ and $h_2(x_1, x_2) := (x_1 - 2)x_2$ with degree $d = 2$. Then we have

$$S = \big\{x = (x_1, x_2) \in \mathbb{R}^2 \big|\; h_1(x) = 0,\; h_2(x) = 0\big\} = \{(0, 0)\}$$

for the solution set. If the global error bound of Lemma 4.2 holds, then there is $c > 0$ such that

$$d(x, S) \le c\big(|h_1(x)| + |h_2(x)|\big)^{\frac{1}{2}} \quad \text{for all} \;\; x \in \mathbb{R}^2. \tag{4.2}$$

Consider the sequence of $x^k := (1, k)$ as $k \in \mathbb{N}$ and observe that

$$d(x^k, S) = \sqrt{1 + k^2}, \quad h_1(x^k) = 1, \;\; \text{and} \;\; h_2(x^k) = -k.$$

Then it follows from the error bound (4.2) that $\sqrt{1+k^2} \le c(1+|-k|)^{\frac{1}{2}} = c(1+k)^{\frac{1}{2}}$ for all $k \in \mathbb{N}$, which is a contradiction. It is worth noting in this example we have the following local error bound:

$$d(x,S) = \sqrt{x_1^2 + x_2^2} \le \left(x_1^2 + |x_1 - 2| \cdot |x_2|\right)^{\frac{1}{2}} \text{ for all } (x_1,x_2) \in \overline{\mathbb{B}}(0,1).$$

The next theorem establishes a sharpen error bound over compact sets for simple polynomial systems.

Theorem 4.4 (sharper error bounds over compact sets for systems of simple polynomials). *Let $h_j \colon \mathbb{R}^n \to \mathbb{R}$ as $j = 1,\ldots,s$ be simple real polynomials of degree at most d, let $S := \{x \in \mathbb{R}^n |\ h_j(x) = 0,\ j = 1,\ldots,s\} \neq \emptyset$, and let K be a compact set in $\mathbb{R}^n$. Then there is a constant $c > 0$ such that*

$$d(x,S) \le c\left(\sum_{j=1}^{s} |h_j(x)|\right)^{\frac{1}{d}} \text{ for all } x \in K.$$

Proof. By (4.1) we represent each simple polynomial h_j by $h_j(x) = \gamma_j \prod_{i \in I_j}(x_i - a_{ij})^{\alpha_{ij}},\ j = 1,\ldots,s$, where $I_j \subset \{1,\ldots,n\}$, $\gamma_j \neq 0$, $a_{ij} \in \mathbb{R}$, and $\alpha_{ij} \in \mathbb{N}$ for $i \in I_j$ with $\sum_{i \in I_j} \alpha_{ij} = d$. Since K is compact, it suffices to show that for each point $\overline{x} \in K$, there are constants $c, \epsilon > 0$ such that

$$d(x,S) \le c\left(\sum_{j=1}^{s} |h_j(x)|\right)^{\frac{1}{d}} \text{ for all } x \in \overline{\mathbb{B}}(\overline{x},\epsilon).$$

Without loss of generality we suppose that $\overline{x} \in S$. Then for each $j = 1,\ldots,s$ consider the index set $I_j(\overline{x}) := \{i \in I_j |\ \overline{x}_i - a_{ij} = 0\}$ and define the polynomial

$$\widetilde{h}_j(x) := \gamma_j \prod_{i \in I_j(\overline{x})} (x_i - a_{ij})^{\alpha_{ij}}.$$

Let $\epsilon > 0$ be such that for all $x \in \overline{\mathbb{B}}(\overline{x},\epsilon)$ we have

$$|x_i - a_{ij}| > 3\epsilon \text{ whenever } i \notin I_j(\overline{x}),\ j = 1,\ldots,s. \tag{4.3}$$

It follows from the above relationships that

$$M := \min_{j=1,\ldots,s} \min_{x \in \overline{\mathbb{B}}(\overline{x},\epsilon)} |\gamma_j| \prod_{i \in I_j \setminus I_j(\overline{x})} |x_i - a_{ij}|^{\alpha_{ij}} > 0.$$

By further shrinking ϵ if necessary, we can assume that $|\widetilde{h}_j(x)| \le 1$ for all $x \in \overline{\mathbb{B}}(\overline{x},\epsilon),\ j = 1,\ldots,s$. Taking any $j = 1,\ldots,s$, consider the sets $S_j := \{x \in \mathbb{R}^n |\ \widetilde{h}_j(x) = 0\}$ and find by Lemma 4.2 positive constants $c_j > 0$ ensuring the error bounds

$$d(x,S_j) \le c_j\, |\widetilde{h}_j(x)|^{\frac{1}{d_j}} \text{ whenever } x \in \mathbb{R}^n \text{ with } d_j := \deg \widetilde{h}_j \le d. \tag{4.4}$$

Given now an arbitrary vector $x^0 \in \overline{\mathbb{B}}(\overline{x},\epsilon)$, we get by the constructions above that for each $j = 1,\ldots,s$ there exists $i(j) \in I_j(\overline{x})$ such that the linear function $x \mapsto x_{i(j)} - a_{i(j)j}$ divides the polynomial $\widetilde{h}_j$ and $d(x^0,S_j) = |x^0_{i(j)} - a_{i(j)j}|$. Denote $Z := \{x \in \overline{\mathbb{B}}(\overline{x},\epsilon) |\ x_{i(j)} = a_{i(j)j}$ for all $j = 1,\ldots,s\}$. By the definition of Z and by (4.3) it is not hard to see that $\overline{x} \in Z \subset S$ and

$$d(x^0,Z) \le \sum_{j=1}^{s} |x^0_{i(j)} - a_{i(j)j}|.$$

These imply together with (4.4) the following estimates:

$$\begin{aligned}
d(x^0,S) &\leq d(x^0,Z) \leq \sum_{j=1}^{s} |x^0_{i(j)} - a_{i(j)j}| = \sum_{j=1}^{s} d(x^0,S_j) \\
&\leq c_j\, |\widetilde{h}_j(x)|^{\frac{1}{d_j}} \leq \sum_{j=1}^{s} c_j |\widetilde{h}_j(x^0)|^{\frac{1}{d}} \leq \sum_{j=1}^{s} c_j \left|\frac{h_j(x^0)}{M}\right|^{\frac{1}{d}} \\
&\leq \left(\max_{j=1,\ldots,s} c_j\right)\left(\frac{1}{M}\right)^{\frac{1}{d}} \sum_{j=1}^{s} |h_j(x^0)|^{\frac{1}{d}},
\end{aligned}$$

where the fourth inequality follows due to $|\widetilde{h}_j(x)| \leq 1$ for all $x \in \overline{\mathbb{B}}(\overline{x},\epsilon)$ as $j = 1,\ldots,s$, and the fifth one follows by the definition of M. Since the function $t \mapsto t^{\frac{1}{d}}$ is concave on $\mathbb{R}_+$, we get for each $t_j \geq 0$ that $\frac{1}{s}\sum_{j=1}^{s} t_j^{\frac{1}{d}} \leq \left(\sum_{j=1}^{s}\frac{1}{s}t_j\right)^{\frac{1}{d}}$. Consequently, it gives us for all $x^0 \in \overline{\mathbb{B}}(\overline{x},\epsilon)$ the desired estimate

$$d(x^0,S) \leq \left(\max_{j=1,\ldots,s} c_j\right)\left(\frac{1}{M}\right)^{\frac{1}{d}} s^{1-\frac{1}{d}} \left(\sum_{j=1}^{s} |h_j(x^0)|\right)^{\frac{1}{d}},$$

which thus completes the proof of the theorem. □

5 Applications: Higher-Order Stability Analysis

The main aim of this section is to apply the error bound results derived above to *quantitative stability* of two important classes of parametric variational systems playing a crucial role in optimization theory and applications, namely problems of polynomial optimization and nonlinear complementarity with polynomial data. In contrast to first-order stability results related to *Lipschitzian stability*, we concentrate here on *higher-order* issues unified around *Hölderian stability* with fractional exponents. Based on our error bound analysis and advanced tools of generalized differentiation, we establish general results in this direction and their specifications with *explicit calculations* of Hölder exponents.

Let us begin with Hölderian stability of optimal solution maps in polynomial optimization.

5.1 Hölderian Stability in Polynomial Optimization

Consider the following parameterized polynomial optimization problem:

$$\begin{array}{lll}
(POP)_u & \max_{x\in\mathbb{R}^n} & f(x,u) \\
& \text{subject to} & g_i(x) \leq 0,\ i = 1,\ldots,r, \\
& & h_j(x) = 0,\ j = 1,\ldots,s,
\end{array}$$

where $u \in \mathbb{R}^l$ is the perturbation parameter, where $x \mapsto f(x,u)$ is a real polynomial on $\mathbb{R}^n$ with degree at most d for each fixed $u \in \mathbb{R}^l$, and where g_i as $i = 1,\ldots,r$ and h_j as $j = 1,\ldots,s$ are all real polynomials on $\mathbb{R}^n$ with degree at most d. For each $u \in \mathbb{R}^l$ denote the solution set of $(POP)_u$ by $S(u)$.

Let $\overline{u} \in \mathbb{R}^l$. We are interested in behavior of the solution map $S\colon \mathbb{R}^l \rightrightarrows \mathbb{R}^n$ when its argument u changes around the reference point $\overline{u}$. The following assumptions are imposed:

- **Assumption 1:** The set $K := \{x \in \mathbb{R}^n |\ g_i(x) \leq 0 \ \text{ as } \ i = 1,\ldots\ r \text{ and } \ h_j(x) = 0 \ \text{ as } j = 1,\ldots,s\}$ is compact and the function f is continuous on $\mathbb{R}^n \times \mathbb{R}^l$.

- **Assumption 2:** There are constants $L, \delta > 0$ such that

$$\|f(x,u) - f(x,\overline{u})\| \leq L\|u - \overline{u}\| \tag{5.1}$$

for all $x \in K$ and for all u with $\|u - \overline{u}\| \leq \delta$.

The class of polynomial optimization problems (POP_u) satisfying Assumptions 1 and 2 covers a number of remarkable models. To illustrate, we mention the two important subclasses as follows.

Subclass 1: Polynomial Optimization with Tilt/Canonical Perturbations

Consider the parametric polynomial optimization problems with tilt/canonical perturbations defined by

$$\begin{array}{ll} \max_{x \in \mathbb{R}^n} & p(x) + u^T x \\ \text{subject to} & \|x\|^2 = 1, \end{array}$$

where p is a polynomial of degree $d \geq 2$ on $\mathbb{R}^n$, and where $u \in \mathbb{R}^n$. Denoting $f(x,u) := p(x) + u^T x$, it is easy to see that both Assumptions 1 and 2 are satisfied.

Subclass 2: Maximum Eigenvalues of Symmetric Tensors

Recall that an mth-order n-dimensional *tensor* $\mathcal{A}$ consists of n^m real entries given by

$$\mathcal{A} = (\mathcal{A}_{i_1 i_2 \ldots i_m}), \quad \mathcal{A}_{i_1 i_2 \ldots i_m} \in \mathbb{R}, \quad 1 \leq i_1, i_2, \ldots, i_m \leq n.$$

We say that the tensor $\mathcal{A}$ is *symmetric* if the values of $\mathcal{A}_{i_1 i_2 \ldots i_m}$ are invariant under any permutation of the indices $\{i_1, i_2, \ldots, i_m\}$. When $m = 2$, a symmetric tensor is nothing but a symmetric matrix. The concept of symmetric tensor is a multilinear extension of symmetric matrices and has recently found numerous applications in polynomial optimization, automatic control, image science, etc.; see, e.g., [47, 51, 53].

Let m be an even number and let $\mathcal{T} := \{\mathcal{A} |\ \mathcal{A} \text{ is an } m\text{th-order } n\text{-dimensional symmetric tensor}\}$, which is a vector space under the *addition* and *multiplication* defined as follows: for any $t \in \mathbb{R}$ and any tensors $\mathcal{A} = (\mathcal{A}_{i_1,\ldots,i_m})_{1 \leq i_1,\ldots,i_m \leq n}$ and $\mathcal{B} = (\mathcal{B}_{i_1,\ldots,i_m})_{1 \leq i_1,\ldots,i_m \leq n}$ we have

$$\mathcal{A} + \mathcal{B} := (\mathcal{A}_{i_1,\ldots,i_m} + \mathcal{B}_{i_1,\ldots,i_m})_{1 \leq i_1,\ldots,i_m \leq n} \text{ and } t\mathcal{A} := (t\mathcal{A}_{i_1,\ldots,i_m})_{1 \leq i_1,\ldots,i_m \leq n}.$$

Further, for each $\mathcal{A}, \mathcal{B} \in \mathcal{T}$ define the *inner product* and *norm* by $\langle \mathcal{A}, \mathcal{B} \rangle_{\mathcal{T}} := \sum_{i_1,\ldots,i_m=1}^n \mathcal{A}_{i_1,\ldots,i_m} \mathcal{B}_{i_1,\ldots,i_m}$ and $\|\mathcal{A}\|_{\mathcal{T}} := (\langle \mathcal{A}, \mathcal{A} \rangle_{\mathcal{T}})^{\frac{1}{2}}$, respectively. We say that $\lambda \in \mathbb{R}$ is an *eigenvalue* of $\mathcal{A}$ and that $x \in \mathbb{R}^n \backslash \{0\}$ is the *eigenvector* corresponding to λ if the pair (x, λ) satisfies

$$\sum_{i_2,\ldots,i_m=1}^n \mathcal{A}_{i\, i_2 \ldots i_m} x_{i_2} \ldots x_{i_m} = \lambda x_i^{m-1} \text{ for all } i = 1, \ldots, n,$$

where $x^{\otimes m}$ is the mth-order n-dimensional symmetric *rank one* tensor induced by x, i.e.,

$$(x^{\otimes m})_{i_1 \ldots i_m} = x_{i_1} \ldots x_{i_m} \text{ for all } i_1, \ldots, i_m \in \{1, \ldots, n\}.$$

Observe that a symmetric tensor always has finitely many eigenvalues [31], and we may consider the *maximum eigenvalue* of $\mathcal{A}$ defined by $\lambda_1(\mathcal{A}) := \max\{\lambda \in \mathbb{R} \mid \lambda \text{ is an eigenvalue of } \mathcal{A}\}$. Note also that a symmetric tensor uniquely determines a real mth degree homogeneous polynomial function by

$$\langle \mathcal{A}, x^{\otimes m} \rangle_{\mathcal{T}} := \sum_{i_1,\ldots,i_m=1}^n \mathcal{A}_{i_1 i_2 \ldots i_m} x_{i_1} \ldots x_{i_m}$$

for all $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. It can be verified (see, e.g., [31, 51]) that the maximum eigenvalue $\lambda_1(\mathcal{A})$ is the optimal value of the following *polynomial optimization problem*:

$$(P)_{\mathcal{A}} \qquad \max_{x\in\mathbb{R}^n} \quad \langle \mathcal{A}, x^{\otimes m}\rangle_{\mathcal{T}}$$
$$\text{subject to} \quad \sum_{i=1}^n x_i^m = 1.$$

Letting now $f(x,\mathcal{A}) := \langle \mathcal{A}, x^{\otimes m}\rangle_{\mathcal{T}}$ with $(x,\mathcal{A}) \in \mathbb{R}^n \times \mathcal{T}$, it is not hard to check that both Assumption 1 and Assumption 2 are satisfied.

To derive next our major sensitivity result for polynomial optimization problems, we denote by

$$\phi(u) := \max_{x\in K} f(x,u), \quad u \in \mathbb{R}^l, \tag{5.2}$$

the optimal value function in (POP_u).

Lemma 5.1 (optimal value functions in polynomial optimization). *Under the validity of Assumptions* 1 *and* 2*, the optimal value function* (5.2) *in* (POP_u) *is continuous on* $\mathbb{R}^l$.

Proof. Let u_k be a sequence in $\mathbb{R}^l$ such that $u_k \to \bar{u}$. To show that $\lim_{k\to\infty}\phi(u_k) = \phi(\bar{u})$, choose $x_k \in K$ so that $\phi(u_k) = \max_{x\in K} f(x,u_k) = f(x_k,u_k)$. Since K is compact, we may assume that there exists $\bar{x} \in K$ such that $x_k \to \bar{x}$. It follows from the continuity of f that $\lim_{k\to\infty}\phi(u_k) = \lim_{k\to\infty} f(x_k,u_k) = f(\bar{x},\bar{u})$. Picking an arbitrary vector $x \in K$, we get that $\phi(u_k) = f(x_k,u_k) \ge f(x,u_k)$. This implies in turn that

$$f(\bar{x},\bar{u}) = \lim_{k\to\infty} f(x_k,u_k) \ge \lim_{k\to\infty} f(x,u_k) = f(x,\bar{u}) \ \text{ for all } \ x \in K.$$

Thus $f(\bar{x},\bar{u}) = \phi(\bar{u})$, which completes the proof of the lemma. □

Now we are ready to establish the quantitative Hölderian stability of polynomial optimization.

Theorem 5.2 (Hölder continuity of solution maps in polynomial optimization). *Under the validity of Assumptions* 1 *and* 2*, for any fixed* $\overline{u} \in \mathbb{R}^l$ *the solution map* $S\colon \mathbb{R}^l \rightrightarrows \mathbb{R}^n$ *in* (POP_u) *satisfies the following Hölderian stability property at* $\bar{u}$*: there are constants* $c, \delta > 0$ *such that we have*

$$S(u) \subset S(\overline{u}) + c\,\|u-\overline{u}\|^{\tau}\,\overline{\mathbb{B}}(0,1) \ \ \textit{whenever} \ \ \|u-\overline{u}\| \le \delta \tag{5.3}$$

with the explicit exponent

$$\tau = \max\left\{\frac{1}{R(n+r+s+1,d+1)}, \frac{2}{R(n+r,2d)}\right\}$$

Proof. Note that for any fixed $u \in \mathbb{R}^l$ the solution set to (POP_u) is represented as

$$S(u) := \big\{x \in \mathbb{R}^n \big|\ g_i(x) \le 0 \ \text{ as } \ i = 1,\ldots,r, \ h_j(x) = 0 \ \text{ as } \ j = 1,\ldots,s, \ \text{ and } \ \phi(u) - f(x,u) = 0\big\}.$$

It is easy to see that g_i, h_j, and $\phi(u) - f(\cdot,u)$ are all polynomials on $\mathbb{R}^n$ with degree at most d. Define

$$\Phi_u(x) := \left(\sum_{i=1}^r \big[g_i(x)\big]_+\right) + \left(\sum_{j=1}^s |h_j(x)|\right) + |\phi(u) - f(x,u)|$$

and observe that $S(u) = \{x \in \mathbb{R}^n |\ \Phi_u(x) = 0\}$. Let $\bar{u}$ be an arbitrary point in $\mathbb{R}^l$. Since K is compact (Assumption 1), it follows from Corollary 3.8 that there is a constant $c_0 > 0$ such that

$$d\big(x, S(\overline{u})\big) \le c_0\,\Phi_{\overline{u}}(x)^{\tau} \ \text{ for all } x \in K.$$

Next we employ Assumption 2 and find numbers $L > 0$ and $\delta > 0$ such that the estimate (5.1) holds for all $x \in K$ and $u \in \mathbb{R}^l$ with $\|u - \overline{u}\| \le \delta$. Denote $c := (2\beta^{-1}L)^\tau$ with $\beta := c_0^{-\frac{1}{\tau}} > 0$. For any $y \in S(u)$ we select now $z \in S(\overline{u})$ satisfying $\|y - z\| = d(y, S(\overline{u}))$. To finish the proof, it suffices to show that

$$\|y - z\| \le c\,\|u - \overline{u}\|^\tau. \tag{5.4}$$

To see this, note that $|\phi(\overline{u}) - f(y, \overline{u})| = \Phi_{\overline{u}}(y) \ge \beta d\big(y, S(\overline{u})\big)^{\frac{1}{\tau}} = \beta\|y - z\|^{\frac{1}{\tau}}$. Since $z \in S(\overline{u})$, we have that $f(z, \overline{u}) = \phi(\overline{u}) \ge f(y, \overline{u})$, and hence

$$\|y - z\|^{\frac{1}{\tau}} \le \beta^{-1}|\phi(\overline{u}) - f(y, \overline{u})| = \beta^{-1}\big(f(z, \overline{u}) - f(y, \overline{u})\big). \tag{5.5}$$

Furthermore, it follows from $y \in S(u)$ that $f(z, u) \le f(y, u)$, and therefore (5.1) gives us the relationships

$$\begin{aligned} f(z, \overline{u}) - f(y, \overline{u}) &= \big(f(z, u) - f(y, u)\big) + \big(f(z, \overline{u}) - f(z, u)\big) + \big(f(y, u) - f(y, \overline{u})\big) \\ &\le \big(f(z, \overline{u}) - f(z, u)\big) + \big(f(y, u) - f(y, \overline{u})\big) \\ &\le 2L\|u - \overline{u}\| \ \text{ as } \ y, z \in K \end{aligned}$$

implying together with (5.5) that $\|y - z\|^{\frac{1}{\tau}} \le \beta^{-1}\big(f(z, \overline{u}) - f(y, \overline{u})\big) \le 2\beta^{-1}L\|u - \overline{u}\|$. Thus

$$d\big(y, S(\overline{u})\big) = \|y - z\| \le c\,\|u - \overline{u}\|^\tau,$$

which justifies (5.4) and completes the proof of the theorem. □

Remark 5.3 (comparison with Lipschitzian stability). If $\tau = 1$ in (5.3), then we get the *upper Lipschitz property* of S *at* $\bar{u}$ in the sense of Robinson [55], which more restrictive than the Hölderian/fractional one established in Theorem 5.2 for general problems of polynomial optimization. We refer the reader to the books [41, 56] and the bibliographies therein for such a Lipschitzian stability, its *robust* (*around* the reference point) version, and their further Lipschitzian type extensions.

As a consequence of the Hölderian stability in Theorem 5.2 we now show that the *maximum eigenvalue* function over the mth-order n-dimensional symmetric tensor space $\mathcal{T}$ is at least *ρth-order semismooth* with the fractional quantity ρ calculated by

$$\rho := \max\Big\{\frac{1}{R(n+2, m+1)}, \frac{2}{R(n, 2m)}\Big\}, \tag{5.6}$$

where R is taken from (2.3). This answers the following *open question* raised in [31], where the authors showed that the maximum eigenvalue of an mth-order n-dimensional symmetric tensor is always ρth-order semismooth for *some* $\rho > 0$ and posed the question about the possibility to give an estimate for the constant ρ. Note that the *order* of semismoothness plays an important role in establishing convergence rates of nonsmooth Newton methods in solving nonsmooth equations; see [4, 31, 52] for more details.

To proceed, recall the definition of semismoothness, which goes back to [39] for scalar functions; see [52] for its extension to the vector case and important applications to the generalized Newton method. We also present the ρth-order version of semismoothness on tensor spaces, which is used in what follows.

Definition 5.4 (semismoothness). *Let $f\colon \mathcal{T} \to \mathbb{R}$ be locally Lipschitzian around and directionally differentiable at the point in question. Then it is* SEMISMOOTH *at $\mathcal{A} \in \mathcal{T}$ if*

$$f(\mathcal{A} + \Delta\mathcal{A}) - f(\mathcal{A}) - \langle V(\Delta\mathcal{A}), \Delta\mathcal{A}\rangle_{\mathcal{T}} = o(\|\Delta\mathcal{A}\|_{\mathcal{T}}) \ \textit{ for all } \ V(\Delta\mathcal{A}) \in \operatorname{co}\partial f(\mathcal{A} + \Delta\mathcal{A}).$$

*Furthermore, $f\colon \mathcal{T} \to \mathbb{R}$ is ρth-*ORDER SEMISMOOTH *at $\mathcal{A} \in \mathcal{T}$ with some $\rho \in (0, 1]$ if*

$$f(\mathcal{A} + \Delta\mathcal{A}) - f(\mathcal{A}) - \langle V(\Delta\mathcal{A}), \Delta\mathcal{A}\rangle_{\mathcal{T}} = O(\|\Delta\mathcal{A}\|_{\mathcal{T}}^{1+\rho}) \ \textit{ for all } \ , V(\Delta\mathcal{A}) \in \operatorname{co}\partial f(\mathcal{A} + \Delta\mathcal{A}).$$

When $\rho = 1$, f is called STRONGLY SEMISMOOTH *at $\mathcal{A}$. We also say that f* SEMISMOOTH (*resp. ρth-*ORDER SEMISMOOTH) ON *$\mathcal{T}$ if it is semismooth (resp. ρth-order semismooth) at every $\mathcal{A} \in \mathcal{T}$.*

It easily follows from Definition 5.4 that the classes of semismooth and ρth-order semismooth functions is closed with respect to summation. The next result taken from [57, Theorem 3.7] provides a convenient tool for dealing with ρth-order semismoothness.

Lemma 5.5 (equivalent description of ρth-order semismoothness). *Let $f\colon \mathcal{T} \to \mathbb{R}$ be locally Lipschitzian and directionally differentiable on a neighborhood of $\mathcal{A}$. Then f is ρth-order semismooth at $\mathcal{A}$ with $\rho \in (0,1]$ if and only if for any point $\mathcal{A}+\Delta\mathcal{A}$ of differentiability of f we have*

$$f(\mathcal{A}+\Delta\mathcal{A}) - f(\mathcal{A}) - \nabla f(\mathcal{A}+\Delta\mathcal{A})\Delta\mathcal{A} = O(\|\Delta\mathcal{A}\|^{1+\rho}).$$

Now we are ready to derive the aforementioned result on the ρth-order semismoothness of the maximum eigenvalue function λ_1 with the explicit calculation of ρ.

Theorem 5.6 (ρth-order semismoothness of maximum eigenvalue functions). *Let $\mathcal{A}$ be an mth-order n-dimensional symmetric tensor with an even number m. Then the maximum eigenvalue function λ_1 is at least ρth-order semismooth at $\mathcal{A}$, where the exponent ρ is explicitly calculated in* (5.6).

Proof. Recall that $\lambda_1(\mathcal{A})$ is the optimal value of the problem $(P_T)_{\mathcal{A}}$ defined above, i.e.,

$$\lambda_1(A) = \max\Big\{\langle \mathcal{A}, x^{\otimes m}\rangle_{\mathcal{T}} \Big|\ \sum_{i=1}^n x_i^m = 1\Big\},$$

where $x^{\otimes m}$ is the rank one tensor induced by x. Observe that the function λ_1 is Lipschitz continuous, convex, and thus directionally differentiable at the points in question. Denote by $E_1(\mathcal{A})$ the solution set of $(P_T)_{\mathcal{A}}$, i.e., $E_1(\mathcal{A}) = \{\mathcal{A} : \langle \mathcal{A}, x^{\otimes m}\rangle_{\mathcal{T}} = \lambda_1(\mathcal{A}),\ \sum_{i=1}^n x_i^m = 1\}$. By Danskin's Theorem [9] we get

$$\partial\lambda_1(\mathcal{A}) = \mathrm{co}\big\{x^{\otimes m} \big|\ x \in E_1(\mathcal{A})\big\}. \tag{5.7}$$

It follows from Theorem 5.2 the existence of constants $c, \delta > 0$ such that

$$E_1(\mathcal{B}) \subset E_1(\mathcal{A}) + c\,\|\mathcal{B}-\mathcal{A}\|_{\mathcal{T}}^{\rho}\,\overline{\mathbb{B}}(0,1) \tag{5.8}$$

for all $\mathcal{B} \in \mathcal{T}$ with $\|\mathcal{B}-\mathcal{A}\|_{\mathcal{T}} \le \delta$. Consider further an mth-order n-dimensional symmetric tensor $\Delta\mathcal{A}$ such that $0 < \|\Delta\mathcal{A}\|_{\mathcal{T}} \le \epsilon$ and that λ_1 is differentiable at $\mathcal{A}+\Delta\mathcal{A}$; the existence of such a tensor follows from the classical Rademacher theorem due to the Lipschitz continuity of λ_1; see, e.g., [56]. This implies that $\partial\lambda_1(\mathcal{A}+\Delta\mathcal{A})$ is a singleton. Then we get from (5.7) that

$$\partial\lambda_1(\mathcal{A}+\Delta\mathcal{A}) = \big\{\nabla\lambda_1(\mathcal{A}+\Delta\mathcal{A})\big\} = \big\{(w_{\Delta\mathcal{A}})^{\otimes m}\big\} \text{ for some } w_{\Delta\mathcal{A}} \in E_1(\mathcal{A}+\Delta\mathcal{A}).$$

To complete the proof of the theorem by employing Lemma 5.5, it remains to show that

$$\lambda_1(\mathcal{A}+\Delta\mathcal{A}) - \lambda_1(\mathcal{A}) - \langle (w_{\Delta\mathcal{A}})^{\otimes m}, \Delta\mathcal{A}\rangle_{\mathcal{T}} = O(\|\Delta\mathcal{A}\|_{\mathcal{T}}^{1+\rho}). \tag{5.9}$$

Since the mapping $x \mapsto x^{\otimes m}$ from $\mathbb{R}^n$ to $\mathcal{T}$ is local Lipschitz, there is $L > 0$ with

$$\|x^{\otimes m} - y^{\otimes m}\|_{\mathcal{T}} \le L\|x-y\| \text{ for all } x, y \in \Big\{x \in \mathbb{R}^n \Big|\ \sum_{i=1}^n x_i^m = 1\Big\}. \tag{5.10}$$

Select $v \in E_1(\mathcal{A})$ so that $\|w_{\Delta\mathcal{A}} - v\| = d(w_{\Delta\mathcal{A}}, E_1(\mathcal{A}))$. Then inclusion (5.8) implies that $\|w_{\Delta\mathcal{A}} - v\| \le c\|\Delta\mathcal{A}\|_{\mathcal{T}}^{\rho}$. It follows from (5.7) that $v^{\otimes m} \in \partial\lambda_1(\mathcal{A})$, which gives us by (2.2) the estimate

$$\lambda_1(\mathcal{A}+\Delta\mathcal{A}) - \lambda_1(\mathcal{A}) \ge \langle v^{\otimes m}, \Delta\mathcal{A}\rangle_{\mathcal{T}}.$$

Then by using (5.10) we get the relationships

$$\begin{aligned}
\lambda_1(\mathcal{A}+\Delta\mathcal{A})-\lambda_1(\mathcal{A})-\langle (w_{\Delta\mathcal{A}})^m,\Delta\mathcal{A}\rangle_{\mathcal{T}} &\geq \langle v^{\otimes m},\Delta\mathcal{A}\rangle_{\mathcal{T}}-\langle (w_{\Delta\mathcal{A}})^m,\Delta\mathcal{A}\rangle_{\mathcal{T}}\\
&\geq -\|v^{\otimes m}-(w_{\Delta\mathcal{A}})^{\otimes m}\|_{\mathcal{T}}\,\|\Delta\mathcal{A}\|_{\mathcal{T}}\\
&\geq -L\|v-w_{\Delta\mathcal{A}}\|\,\|\Delta\mathcal{A}\|_{\mathcal{T}}\\
&\geq -Lc\|\Delta\mathcal{A}\|_{\mathcal{T}}^{1+\rho}. \qquad (5.11)
\end{aligned}$$

On the other hand, it follows from $\nabla\lambda_1(\mathcal{A}+\Delta\mathcal{A})=\{(w_{\Delta\mathcal{A}})^m\}$ and the convexity of λ_1 that

$$\langle (w_{\Delta\mathcal{A}})^m,-\Delta\mathcal{A}\rangle_{\mathcal{T}}=\langle (w_{\Delta\mathcal{A}})^m,\mathcal{A}-(\mathcal{A}+\Delta\mathcal{A})\rangle_{\mathcal{T}}\leq\lambda_1(\mathcal{A})-\lambda_1(\mathcal{A}+\Delta\mathcal{A}),$$

which implies that $\lambda_1(\mathcal{A}+\Delta\mathcal{A})-\lambda_1(\mathcal{A})-\langle (w_{\Delta\mathcal{A}})^m,\Delta\mathcal{A}\rangle_{\mathcal{T}}\leq 0$. Combining this with (5.11), we arrive at (5.9) and complete the proof of the theorem. □

Remark 5.7 (matrix case). In the special case of $m=2$, a symmetric tensor $\mathcal{A}$ is nothing but an $(n\times n)$ symmetric matrix. It follows from [57] that in this case the maximum eigenvalue function is strongly semismooth at $\mathcal{A}$, i.e., it is ρth-order semismooth with $\rho=1$. However, our general result in Theorem 5.6 shows that $\mathcal{A}$ is merely ρth-order semismooth with $\rho=\max\left\{\frac{1}{R(n+2,3)},\frac{2}{R(n,4)}\right\}$. Thus, although our order estimate works for general tensors, it may not be tight in particular settings. This calls for further improvements of the order semismoothness result obtained in the general tensor case.

5.2 Hölderian Stability of Complementarity Systems with Polynomial Data

This subsection is devoted to the study of Hölderian stability with explicit exponent for the class of parameterized *nonlinear complementarity problems* described by

$$(NCP)\quad x^TF(x,u)=0,\ x\geq 0,\ \text{ and }\ F(x,u)\geq 0,$$

where $x\in\mathbb{R}^n$ and $u\in\mathbb{R}^p$. In what follows we assume that each component of the mapping $F(x,u)=(F_1(x,u),\ldots,F_n(x,u))$ is a *polynomial* on $\mathbb{R}^{n+p}$ with degree d. It has been well recognized that nonlinear complementarity systems under consideration constitute an important class of optimization-related problems with numerous practical applications to, e.g., economics and engineering; see [14, 37].

For each $u\in\mathbb{R}^p$ we define the solution set $S(u)$ to (NCP) by

$$S(u):=\left\{x\in\mathbb{R}^n\,\middle|\, x^TF(x,u)=0,\ x\geq 0,\ \text{ and }\ F(x,u)\geq 0\right\} \qquad (5.12)$$

and say that the set-valued mapping $S\colon\mathbb{R}^p\rightrightarrows\mathbb{R}^n$ is *Hölder calm* with exponent $\tau>0$ at $(\overline{u},\overline{x})\in\operatorname{gph}S$ if there are positive numbers c,ϵ, and δ such that

$$S(u)\cap\mathbb{B}(\overline{x},\epsilon)\subset S(\overline{u})+c\|u-\overline{u}\|^{\tau}\overline{\mathbb{B}}(0,1)\ \text{ whenever }\ \|u-\overline{u}\|\leq\delta. \qquad (5.13)$$

Note that for $\tau=1$ this property reduces to the (Lipschitz) calmness of multifunctions (a graphical localization of Robinson's upper Lipschitz property in (5.3) with $\tau=1$) and has been widely studied in the literature; see, e.g., [56] and the references therein.

Theorem 5.8 (Hölder calmness of solution maps for NCP). *Let $S\colon\mathbb{R}^p\rightrightarrows\mathbb{R}^n$ be the solution map (5.12) for (NCP), and let $(\bar{u},\bar{x})\in\operatorname{gph}S$. Then S is Hölder calm at $(\bar{u},\bar{x})$ with the explicit exponent $\tau=\max\left\{\frac{1}{R(3n+1,d+1)},\frac{2}{R(3n,2d)}\right\}$.*

Proof. Since $S(\bar{u})$ is the solution set for a polynomial system, we apply to it the local bound estimate from Corollary 3.8 with $r=2n$ and $s=1$ finding in this way constants $c_0,\epsilon>0$ such that

$$d\big(x,S(\overline{u})\big)\leq c_0\bigg(\sum_{i=1}^n\big[-x_i\big]_+ +\sum_{i=1}^n\big[-F_i(x,\overline{u})\big]_+ +\bigg|\sum_{i=1}^n x_iF_i(x,\overline{u})\bigg|\bigg)^{\tau} \qquad (5.14)$$

whenever $\|x - \overline{x}\| \le \epsilon$. Considering the function

$$h(x,u) := \sum_{i=1}^{n} \big[- x_i \big]_+ + \sum_{i=1}^{n} \big[- F_i(x,u) \big]_+ + \Big| \sum_{i=1}^{n} x_i F_i(x,u) \Big|,$$

we see that h is locally Lipschit around $(\bar{x}, \bar{u})$ with nonnegative values and that the representation $S(u) = \{x \in \mathbb{R}^n |\ h(x,u) = 0\}$ holds. Fix $\delta > 0$ and denote by $L > 0$ some Lipschitz constant of the function h on the set $\overline{\mathbb{B}}(\overline{x},\epsilon) \times \overline{\mathbb{B}}(\overline{u},\delta)$, i.e.,

$$|h(x,u) - h(x',u')| \le L(\|x - x'\| + \|u - u'\|) \ \text{ for all } \ (x,u),(x',u') \in \overline{\mathbb{B}}(\overline{x},\epsilon) \times \overline{\mathbb{B}}(\overline{u},\delta). \tag{5.15}$$

Taking further any $y = (y_1, \ldots, y_n) \in S(u) \cap \overline{\mathbb{B}}(\overline{x},\epsilon)$ and using (5.14), we get the relationships

$$\begin{aligned} d\big(y, S(\overline{u})\big) &\le c_0 \Big(\sum_{i=1}^{n} [-y_i]_+ + \sum_{i=1}^{n} \big[- F_i(y,\overline{u}) \big]_+ + \Big| \sum_{i=1}^{n} x_i F_i(a,\overline{u}) \Big| \Big)^{\tau} \\ &= c_0 h(y,\overline{u})^{\tau} \\ &\le c_0 \Big(h(y,u) + L\|u - \overline{u}\| \Big)^{\tau} \\ &= c_0 L^{\tau} \|u - \overline{u}\|^{\tau}, \end{aligned}$$

where the second inequality follows by (5.15) while the last equality is due to $y \in S(u)$ and so $h(y,u) = 0$. Thus justifies the Hölder calmness (5.13) of map (5.12) and completes the proof of the theorem. $\square$

In the same way as the classical local Lipschitzian behavior of set-valued mappings *around* the reference point is a *robust* version of Robinson's upper Lipschitz property (see Remark 5.3), the robust counterpart of calmness in (5.13) with $\tau = 1$ is known as the *Lipschitz-like* (also as pseudo-Lipschitz or Aubin) property of S around $(\bar{u}, \bar{x})$, which corresponds to the case of $\tau = 1$ in the relationship

$$S(u_1) \cap \mathbb{B}(\overline{x},\epsilon) \subset S(u_2) + c\|u_1 - u_2\|^{\tau} \overline{\mathbb{B}}(0,1) \ \text{ whenever } \ \|u_i - \overline{u}\| \le \delta \ \text{ as } \ i = 1,2 \tag{5.16}$$

with some positive constants ϵ, δ, and c. The Lipschitz-like property of general multifunctions has been extensively studied and applied in variational analysis and optimization; see, e.g., the books [41, 56] and their commentaries. We particularly refer the reader to the recent paper [17] and the bibliography therein, where constructive characterizations of the Lipschitz-like property are obtained in terms of the initial data for solution maps to parameterized variational systems, including the complementarity ones from (5.12), on the basis of the coderivative/Mordukhovich criterion from [40] and [56, Theorem 9.40]. Now we derive, for the first time in the literature, a verifiable condition ensuring the *robust Hölderian stability* (5.16) with *any exponent* $\tau > 0$ for polynomial complementarity systems as in (NCP).

To proceed, define via the initial data of (NCP) the function

$$f(x,u) := \max\Big\{ \max_{1 \le i \le n} \big\{ - x_i \big\},\ \max_{1 \le i \le n} \big\{ - F_i(x,u) \big\},\ \Big| \sum_{i=1}^{n} x_i F_i(x,u) \Big| \Big\} \tag{5.17}$$

and consider the two index subsets given by

$$I_0(x,u) := \big\{ i \in \{1, \ldots, n\} \big|\ - x_i = f(x,u) \big\} \ \text{ and } \ I_<(x,u) := \big\{ i \in \{1, \ldots, n\} \big|\ - F_i(x,u) = f(x,u) \big\}.$$

By e_i we denote an element of $\mathbb{R}^n$ whose ith coordinate is 1 and all the other coordinates are 0.

Theorem 5.9 (robust Hölderian stability of solution maps for NCP). *Let* $(\bar{u}, \bar{x}) \in \operatorname{gph} S$ *for the solution map* (5.12), *and let* $\tau > 0$. *Suppose that there exist positive numbers* c, δ, *and* ϵ *such that for all*

$x \in \mathbb{R}^n$ and $u \in \mathbb{R}^p$ with $\|x-\overline{x}\| \le \epsilon$, $\|u-\overline{u}\| \le \delta$, and $f(x,u) > 0$ we have

$$\inf\Big\{\Big\| \sum_{i\in I_<(x,u)} \alpha_i + \sum_{i\in I_<(x,u)} \beta_i \nabla_x F_i(x,u) + \gamma\Big(\sum_{i=1}^n x_i \nabla_x F_i(x,u) + \sum_{i=1}^n F_i(x,u)e_i\Big)\Big\| \;\Big|\; \sum_{i\in I_0(x,u)} \alpha_i + \sum_{i\in I_<(x,u)} \beta_i + |\gamma| = 1,\ \alpha_i \ge 0,\ \beta_i \ge 0,\ \gamma\in\mathbb{R}\Big\} \ge c|f(x,u)|^{1-\tau}.$$

Then the solution map (5.12) *has the robust Hölder stability property* (5.16) *with exponent τ.*

Proof. Let $f_i(x,u) := -x_i$ and $f_{i+n}(x,u) := -F_i(x,u)$ for $i = 1,\ldots,n$, $f_{2n+1}(x,u) := \sum_{i=1}^n x_iF_i(x,u)$, and $f_{2n+2}(x,u) := -\sum_{i=1}^n x_iF_i(x,u)$. Then each f_i is a real polynomial on $\mathbb{R}^{n+p}$ with degree at most $d+1$. It follows from the definitions that the function f from (5.17) is represented as

$$f(x,u) = \max_{1\le i\le 2n+2} f_i(x,u).$$

For any fixed $u\in\mathbb{R}^p$ with $\|u-\overline{u}\|\le\delta$, write for convenience ${f_i}^u(x) := f_i(x,u)$ as $i = 1,\ldots,2n+2$ and $f^u(x) := f(x,u)$ whenever $x\in\mathbb{R}^n$. Then $f^u(x) = \max_{1\le i\le 2n+2} f_i^u(x)$. Considering further the index set $I(x) := \{i|\ f_i^u(x) = f^u(x)\}$, we deduce from the assumption made and Lemma 2.2 that

$$\begin{aligned}
\mathfrak{m}_{f^u}(x) &= \inf\Big\{\Big\|\sum_{i\in I(x)} \lambda_i \nabla f_i^u(x)\Big\| \;\Big|\; \sum_{i\in I(x)} \lambda_i = 1,\ \lambda_i \ge 0\Big\} \\
&\ge \inf\Big\{\Big\| -\sum_{i\in I_<(x,u)} \alpha_i - \sum_{i\in I_<(x,u)} \beta_i\nabla_x F_i(x,u) + (\gamma_1-\gamma_2)\Big(\sum_{i=1}^n x_i\nabla_x F_i(x,u) + \sum_{i=1}^n F_i(x,u)e_i\Big)\Big\| \\
&\qquad \Big|\; \sum_{i\in I_0(x,u)} \alpha_i + \sum_{i\in I_<(x,u)} \beta_i + \gamma_1+\gamma_2 = 1,\ \alpha_i\ge 0,\ \beta_i\ge 0,\ \gamma_i\ge 0\Big\} \\
&= \inf\Big\{\Big\| \sum_{i\in I_<(x,u)} \alpha_i \sum_{i\in I_<(x,u)} \beta_i\nabla_x F_i(x,u) + \gamma\Big(\sum_{i=1}^n x_i\nabla_x F_i(x,u) + \sum_{i=1}^n F_i(x,u)e_i\Big)\Big\| \\
&\qquad \Big|\; \sum_{i\in I_0(x,u)} \alpha_i + \sum_{i\in I_<(x,u)} \beta_i + |\gamma| = 1,\ \alpha_i\ge 0,\ \beta_i\ge 0,\ \gamma\in\mathbb{R}\Big\} \\
&\ge c\,|f(x,u)|^{1-\tau} = c\,|f^u(x)|^{1-\tau} \ \text{ whenever }\ \|x-\overline{x}\|\le\epsilon \ \text{ and }\ f(x,u)>0,
\end{aligned}$$

where the third equality follows from the fact that $f(x,u) > 0$ and hence $\gamma_1\gamma_2 = 0$. Employing now Lemma 3.4 ensures that for each u with $\|u-\overline{u}\|\le\delta$ and for all x with $\|x-\overline{x}\|\le\epsilon/2$ we have

$$\begin{aligned}
d\big(x,S(u)\big) &= d\big(x,\{x \mid f^u(x)\le 0\}\big) \\
&\le \frac{1}{c}\big[f(x,u)\big]_+^\tau \\
&\le \frac{1}{c}\Big(\sum_{i=1}^n\big[-x_i\big]_+ + \sum_{i=1}^n\big[-F_i(x,u)\big]_+ + \Big|\sum_{i=1}^n x_iF_i(x,u)\Big|\Big)^\tau. \qquad (5.18)
\end{aligned}$$

Consider the function $h(x,u) := \sum_{i=1}^n[-x_i]_+ + \sum_{i=1}^n[-F_i(x,u)]_+ + |\sum_{i=1}^n x_iF_i(x,u)|$ and note that it is nonnegative and Lipschitz continuous on $\overline{\mathbb{B}}(\overline{x},\frac{\epsilon}{2})\times\overline{\mathbb{B}}(\overline{u},\delta)$ with some constant L, i.e.,

$$|h(x,u)-h(x',u')| \le L(\|x-x'\|+\|u-u'\|) \ \text{ for all }\ (x,u),(x',u')\in\overline{\mathbb{B}}(\overline{x},\frac{\epsilon}{2})\times\overline{\mathbb{B}}(\overline{u},\delta). \qquad (5.19)$$

Observing that $S(u) = \{x\in\mathbb{R}^n|\ h(x,u) = 0\}$ and picking any $u_1,u_2\in\mathbb{R}^p$ with $\|u_i-\overline{u}\|\le\delta$ as well as any $y \in S(u_1) \cap \overline{\mathbb{B}}(\overline{x}, \frac{\epsilon}{2})$, we deduce from (5.18) that

$$\begin{aligned} d\big(y, S(u_2)\big) &\le \frac{1}{c}\Big(\sum_{i=1}^n \big[-y_i\big]_+ + \sum_{i=1}^n \big[-F_i(y,u_2)\big]_+ + \Big|\sum_{i=1}^n x_i F_i(y,u_2)\Big|\Big)^\tau \\ &= \frac{1}{c} h(y,u_2)^\tau \\ &\le \frac{1}{c}\Big(h(y,u_1) + L\|u_2 - u_1\|\Big)^\tau \\ &= \frac{1}{c} L^\tau \|u_2 - u_1\|^\tau, \end{aligned}$$

where the second inequality holds by (5.19) while the last equality is a consequence of $y \in S(u_1)$ and hence $h(y,u_1) = 0$. This justifies the claimed Hölder continuity of the solution map (5.12). □

6 Concluding Remarks

In this paper we employ advanced techniques of variational analysis and generalized differentiation to extended, in particular, the local and global error bounds in [8] from a single polynomial to general *polynomial systems* with *explicitly calculated exponents*. Besides being of their own interest, these results are important for convergence rates of numerical algorithms. The obtained error bounds are applied to *Hölderian stability* of solution maps for *polynomial optimization problems* and their tensor *eigenvalue* specifications as well as for parameterized *nonlinear complementarity systems* with polynomial data. In this way we resolve, in particular, some *open questions* posted in the literature.

Nevertheless, many significant issues in these directions still needs further investigation. Some of them are indicated in the text; see, e.g., Remark 5.7. It would be also important to identify remarkable classes of polynomial systems for which the general local and global error bounds can be sharpened. On the other hand, it is appealing to extend the proposed techniques and the results obtained on Hölderian stability to polynomial optimization problems with perturbations not only in the cost function but also in the constraint functions as well.

Furthermore, in contrast to Lipschitzian stability, its higher-order Hölderian counterpart seems to be largely uninvestigated in variational analysis and optimization; in particular, for polynomial systems considered in the paper. Among the most important and challenging issues of further research related to the context of our Section 5 we mention the desired developments of *Hölderian tilt* and *full stability* of optimal solutions to extend the original Lipschitzian frameworks proposed in [50] and [24], respectively; see [42, 43] and the references therein for recent Lipschitzian type results in these directions.

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