Public Facility Location Using Dispersion, Population, and Equity Criteria

Rajan Batta *

Miguel Lejeune and Srinivas Prasad †

January 2013

Abstract

Administrators/Decision Makers (DMs) responsible for making locational decisions for public facilities have many other overriding factors to consider that dominate traditional OR/MS objectives that relate to response time. We propose that an appropriate role for the OR/MS analyst is to help the DMs identify a *good* set of solutions rather than an optimal solution that may not be practical. In this paper, good solutions can be generated/prescribed assuming that the DMs have (i) a dispersion criterion that ensures a minimum distance between every pair of facilities, and (ii) a population criterion which stipulates that the distance from a demand point to its closest facility is inversely proportional to its population, and (iii) an equity criterion which stipulates that no demand point is further than a specified distance from its closest facility. We define parameters capturing these three criteria and specify values for them based on the p-median solution. Sensitivity analysis with respect to the parameters is performed and computational results for both real and simulated networks are reported. Our results show close collaboration with the p-median solution when decision makers restrict location to demand points, and use parameter values for the population, dispersion, and equity criteria as implied by the p-median solution. The significance of our work is twofold. For practitioners, it is comforting to know that using common-sense measures such as the above criteria result in fairly good solutions. For researchers it suggests the need for developing techniques for finding the k best solutions of the p median problem.

Keywords: *Location, OR in Societal Problem Analysis, Health care, Public Policy, Operations Philosophy*

*Department of Industrial and Systems Engineering, University at Buffalo (SUNY), Buffalo, NY 14260

†Department of Decision Sciences, School of Business, The George Washington University, Washington, DC 20052

1 Introduction

In the OR/MS literature, several researchers have developed models and solution methodologies for locating public facilities such as emergency vehicles, libraries, hospitals, and fire stations. The focus usually is on minimizing a single objective function, but there also has been a focus on obtaining efficient solutions satisficing several objectives. The objectives used are usually some function of the response time; typical functions are the average response time and the maximum response time. There are many review articles for location theory and application papers. These include those by Daskin (2008), Revelle and Eiselt (2005), Hale and Moberg (2003), Avella *et al.* (1998), Owen and Daskin (1998), and Brandeau and Chiu (1989). For a good review of recent trends specifically in the context of public facility location modeling, see Serra and Marianov (2004). Despite the impressive list of citations in these papers, there is a significant gap between model development and analysis and usage of these models to make actual locational decisions. There are, of course, some notable exceptions. Dell *et al.* (2009) discuss location of new schools combined with closing of existing schools as a strategy to improve transportation access to school children in Chile. Their work is helping shape the school access debate in the Chilean Education Ministry. Antunes (1999) developed a model for locating waste management facilities in Portugal. The model he developed was a variant of existing location models and provided a good starting point for policy makers in Portugal. Mehrez *et al.* (1996) discussed the location of a hospital in a rural area of Israel. They tried various models though none fit exactly to the specifications of the problem at hand. Finally, they ended up using a screening method to suggest alternatives. A well documented success story is Larson's Hypercube queuing model which is used by several police departments to suggest locations and deployment strategies for vehicles (Larson, 1974). Another successful application of public facility locational analysis is due to Eaton *et al.* (1985) who employed the Maximal Covering Location problem due to Church and ReVelle (1974) to deploy ambulances in Austin, Texas.

In the view of the authors, there are five plausible reasons for the gap between theory and practice pertaining to public facility locational analysis:

1. Each application has its own unique set of constraints, since public facilities are typically community service functions and operate within local political constraints. OR/MS models typically do not adequately capture such constraints. A specific example that illustrates how choices are also increasingly being dictated by social welfare considerations (Bigman and ReVelle 1978) is the co-location of public facilities (Minde 2008).

2. When the public facility to be located is fixed, like a fire station, an ambulance garage,

or a police precinct headquarter, the economics of the decision often dictate the ultimate choice. For example, availability of land, land prices, real-estate taxes, accessibility of plot under consideration from a major roadway, etc. are usually overriding factors over traditional objectives of OR/MS models.

3. Data collection is often difficult, expensive, and time consuming. Many times data is not available in the form that is desired by the OR/MS models. For example, call sheets that are typically kept by ambulance personnel usually contain information on when a call arrived, how long it took the ambulance to arrive at the scene, and how long it took to stabilize the patient. Travel time information has to be extracted from such data, and then has to be extrapolated to obtain an accurate estimate for travel times between arbitrarily chosen pairs of points in the region.

4. Quite often a model's underlying assumptions are simply not valid for the application at hand. For example, the Hypercube model assumes Exponential service times, which may not hold in practice. Another example is the assumption in the Hypercube model of stationary Poisson arrivals, which contrast sharply with the well-known fact that the arrival process for most emergency service systems is cyclic (every 24 hours). One can make amends by trying to patch up a model's weaknesses — like the mean service time calibration procedure in the Hypercube model to take into account the travel time component of service time — but this may still leave a decision maker wary of the model's applicability to their situation.

5. The model may be too simplistic in that it does not capture the large number of conflicting concerns of the decision maker. For example, the decision maker may wish to see trade-offs between frequent relocations and repositionings of vehicles and the response time, as well as evaluate the degree of discomfort and inconvenience that it causes the drivers of the vehicles.

So what choices does an analyst have when they are provided with sketchy data, limited time and resources, and a fair understanding of the location problem that the agency is hopeful of solving? Certainly one way to approach the problem is to look through the large number of available OR/MS models, pick the one that best fits the situation and then use the model to suggest location alternatives to the agency. Two difficulties that could arise in this *scientific* approach are first that no model may fit the situation closely enough, and second that the data requirements of the model might be excessive or it may require a significant amount of time and effort to obtain good estimates for the model's data from the data available. Another approach is to forget about explicit mathematical modeling and simply arrive at a list of reasonable location sites via brainstorming sessions, using the available data, if possible, to get some insight into the relative performance of the feasible sites with respect to performance measures like the p-median objective. The increasing use of GIS (Nedovic-Budic 2000), specifically Spatial Decision Support Systems (SDSS), that "allow decision makers, e.g., county leaders or public representatives, to play a more direct role in determining which data and how data are input into the GIS for analysis" (Minde 2008) is motivating the need for decision processes and models that are more realistic. The focus of this paper is to develop the framework for a prescriptive approach that allows decision makers to incorporate traditional performance measures ensuring that the chosen solution is not too bad with respect to the median and equity objectives.

Before we venture into some technical analysis, we briefly focus on practical public facility location studies to further solidify the motivation for this work. The paper by Price and Turcotte (1986) discusses their work as consultants for locating a blood bank in Quebec City. They adopted the *intuitive* approach, using a simplistic gravity location model to make a comparison between the five sites that they had shortlisted. The reasons that they cited for not using the *scientific* approach were the lack of data availability and time for the analysis. The work of Carson and Batta (1990) is a good example of a failed locational project. They adopted the *scientific* approach, which led to an interesting model for locating a campus ambulance but required considerable effort in terms of data collection. Despite the fact that the model was empirically validated, the results were not implemented because of a change of administration and reluctance on part of the students to wait in the cold for so few calls. The project suffered from its inability to capture "political" constraints and from the "vanishing advocate" phenomenon discussed in Larson and Odoni (1981). In a recent talk (Berman and Krass, 2010) the median problem with unreliable facilities was presented. An interesting result of the analysis was that the facilities tend to colocate (or draw together) when the reliability level drops. At the same time, facilities tend to be dispersed when in less populated areas. The authors examined empirical data of hospital locations in the City of Toronto, Canada, and found that both dispersion and population criteria are present when unreliable public facilities are in question.

In this paper, we attempt to answer the following question in the context of the p-median problem on a network: How bad can a locational choice be provided that the decision makers use a *dispersion*, *population*, and *equity* criteria. Five other papers that the authors are aware of which address similar questions are: (1) Larson and Stevenson (1972), who performed calculations with spatially homogeneous demands which suggest that the mean travel time resulting from totally random distribution of facilities in the region served is only reduced by 25% when the facilities are optimally located; (2) Larson and Odoni (1981), who performed calculations to show that ignoring the size of a city block in computing travel distances induces an average error of no greater than 1/3 the size of the block; (3) Batta and Leifer (1988), who demonstrated that restricting facility location to demand points and barrier vertices gives good results for the p-median problem on a Manhattan plane both with and without travel barriers; (4) Love and Walker (1994), who show that using a combination of norms yields an accurate approximation to actual road distances under a variety of circumstances; and (5) Francis *et al.* (2004), who study the loss in accuracy when demand points are aggregated so as to make the problem more computationally tractable. Their paper provides a summary of the work of several researchers in this area.

In the next section, we introduce some notation, and in Section 3, we develop formulations to study the worst-case performance of location problems that employ the population,dispersion, and equity criteria with respect to the median performance measure. In Section 4, we discuss our computational results and present a sensitivity analysis with respect to a range of values of the parameters used in these three criteria. In Section 5, we present our conclusions.

2 Notation

Consider a network $G = (N, L)$ in which N is the set of nodes and L is the set of links. Let $|N| = n$ and the elements of N be labeled $1, 2, \ldots, n$. Let $l(i, j)$ denote the length of a link $(i, j) \in L$ and $d(x, y)$ denote the length of a shortest length path between two points x and y in G. On any link (i, j) we define a node k breakpoint as a point x_k such that $d(x_k, k)$ through node i and node j are exactly equal. Note that a link may have at most $(n - 2)$ breakpoints. The total arrival rate of calls into the system is λ, with the arrival from node i being λ_i. Thus the relative frequency of calls from node i is $w_i = \lambda_i/\lambda$. There are p facilities to be located on the network. Facility locations can be made at nodes and at any point along a link, and facilities are assumed to be of infinitesimal size. Response units are housed in these facilities.

There is a natural tendency to favor choices that spread or disperse emergency facilities in the geographical region that they service. To capture this *dispersion* criterion mathematically, we assume that the distance between any pair of facilities in a candidate solution is greater than or equal to a threshold λ, or $d(x_i, x_k) \geq \lambda$ for $i, k = 1, \ldots, p, i \neq k$. Dispersion alone is probably not the only criterion that decision makers would use when making public facility location decisions. If the population is not evenly distributed, decision makers are likely to assign more facilities to heavily populated areas. To capture this *population* criterion, we assume that the distance of the closest facility to a population center is less than or equal to a threshold, where the threshold itself is inversely proportional to the population center's weight. Mathematically, if β_s is the distance of the closest facility to a node s, then we impose the constraint $\beta_s \leq A/w_s$, where A is a factor of proportionality. A high value of A implies that the decision makers are not too concerned by choosing locations that don't account for population variations in the geographical region. A low value of A implies much concern for population variations in facility location decisions. A third criterion that is often relevant in the context of public facility location is fair or equitable access to the facilities, where decision makers have to ensure that no demand point is beyond an established threshold distance (B) from its closest facility. This is especially true in contexts that involve emergency response (see, for example, the Oklahoma Fire Station Study 2006). Mathematically, we impose the constraint $\beta_s \leq B$, where β_s is as defined earlier.

3 Analysis

The objective of the p-median model is to locate p facilities so as to minimize the average travel time to a call, under the assumptions that (i) a call is always answered by a response unit at the closest facility, and (ii) such a response is always available for service, i.e., it is not busy servicing another call. Obviously, the model is applicable in the situation when $\lambda \to 0$ or when the number of response units available at each facility are large.

We know from Hakimi (1964) that an optimum solution to the p-median model is to restrict facility locations to nodes. Therefore, the optimum objective function value is

$$f = \min_{x_1, \ldots, x_p \in N} \sum_{s \in N} w_s \; min[d(x_1, s), \ldots, d(x_p, s)]. \tag{1}$$

By definition, the worst-case value is

$$g = \max_{x_1, \ldots, x_p \in G} \sum_{s \in N} w_s \; min[d(x_1, s), \ldots, d(x_p, s)]. \tag{2}$$

Although our objective is essentially to compute g, there are other inherent applications of a solution to (2). For example, the optimization problem on the right side of equation (2) is referred to in the literature as the p-maxian problem (see, e.g., Erkut and Neuman 1989), and this has applications in the location of certain obnoxious facilities. Church and Garfinkel (1978) show that the p-maxian is equivalent to the 1-maxian problem and propose a method to solve it. Minieka (1983) and Ting (1984) provide more efficient methods for solving this problem. All these methods take advantage of the fact that the search for the 1-maxian can be restricted to a finite set of points. For more recent work in this area, see the paper by Cheng and Kang (2010), who study the p-maxian on an interval graph.

The problem of interest can be formulated as:

$$(P) \quad \text{Maximize} \quad \sum_{s \in N} w_s \; \beta_s \tag{3}$$

Subject to the constraints:

$$\begin{aligned}
d(x_j, x_l) \geq \lambda \quad & \text{for} \quad j = 1, \ldots, p-1, \; l = j+1, \ldots, p && (4)\\
Min_j\{d(s, x_j)\} \geq \beta_s \quad & \text{for} \quad j = 1, \ldots, p, \; s \in N && (5)\\
Min_j\{d(s, x_j)\} \leq A/w_s \quad & \text{for} \quad s \in N && (6)\\
x_j \in G \quad & \text{for} \quad j = 1, \ldots, p && (7)\\
\beta_s \leq B \quad & \text{for} \quad s \in N && (8)
\end{aligned}$$

The objective function (3) is to maximize the weighted travel distance from nodes to their closest facility. Constraint (4) ensures that distances between facilities are at least λ. Constraint (5) ensures that the distance between a node and its closest facility is computed correctly. Constraint (6) ensures that the distance between a node and its closest facility is less than or equal to a threshold, where the threshold itself is inversely proportional to the node's weight. Constraint (7) says that a facility must be located on the network itself. Finally, constraint (9) ensures that there is a facility within distance B of each demand node.

The above formulation belongs to a class of problems referred to in the literature as network location problems with distance constraints. One of the earliest reference to this class of problems is Toregas and Revelle (1972). Many researchers have recently studied location problems with distance constraints. For example, Berman and Huang (2008) study the minimum weighted covering problem in such a context.

Network location problems with distance constraints are typically very hard to solve, and efficient procedures are usually possible for only the simplest cases. This is due to the non-convexity of the solution space that the distance constraints introduce. Convexity is retained, however, for a class of problems with upper bounded constraints and when the underlying network structure is a tree. This enables the development of efficient procedures such as in Dearing *et al.* (1976), and Francis *et al.* (1978). Erkut *et al.* (1988) develop a methodology that could be used to convert a continuous tree network location problem with distance constraints to mathematical programming problems. Erkut *et al.* (1992) use this methodology in the context of distance constrained multifacility minimax location problems on tree networks. All these articles deal with the location of 'desirable' facilities, for which the distance between facilities and demand points is, in some sense, minimized. Another class of problems, that is relevant to our problem, is the location of obnoxious facilities. These problems attempt to maximize the distance between facilities and demand points, which is the objective of our problem. These are also referred to as multifacility maxisum problems by Erkut and Neuman (1989). Kuby (1987) and Erkut *et al.* (1988) consider discrete versions of this problem. A recent review article by Farahani *et al.* (2010) covers multiple criteria facility location problems in general, and includes a good discussion of obnoxious facility location.

Problem (P) is hard to solve, especially since the location of facilities is unrestricted. Our approach will be to restrict location to a finite set of sites, and use this to derive a valid upper bound on the optimal value of (P). We will start, however, by analyzing the complexity of (P) for a few simple cases.

3.1 Unrestricted Facility Location

The first case that we consider is the 1-maxian with distance constraints. Constraint set (4) is not relevant in this case. Along a link, the 1-maxian achieves a maximum at either of the end points of the link or at one of the finitely many break points along the link. Since feasibility of a break point with respect to the distance constraints can be easily verified, the optimal solution can be determined by a straightforward procedure, whose order is polynomial in the number of links and vertices of the underlying graph.

The next simple case is when the underlying network structure is a chain. In this case, problem (P) can be reformulated as a linear program. This is possible because all distances can be computed with respect to a common reference point, which could be either end of the chain. The decision variables can be defined as $x_i, i = 1, \ldots, p$, where x_i is the distance of facility i from the reference point. Constraint (4) in (P) can then be written as $|x_i - x_j| \geq \lambda$ for all $i, j = 1, \ldots, p, i \neq j$. Constraint (5) can be written as $|d_s - x_i| \geq \beta_s$ for $i = 1, \ldots, p, s \in N$, where d_s is distance of node s from the reference point. Thus, (P) can be solved in polynomial time on a chain network, as the number of variables and constraints in the resulting linear program are polynomial.

The above argument holds even in the case of a tree network (not necessarily a chain), as long as the facility locations are restricted to lie on a path. The tree network can be represented on a rectangular coordinate system and the distances can be computed correctly. This is illustrated in Figure 1 in which facility location is restricted to be on the the path $1-2-5$. Using node 1 as the origin, the distances between facilities and nodes can be computed correctly. This, however, is not possible if we allow location at any point on the tree. Then the distance between x_i on link $1-3$ and x_j on $2-4$ cannot be computed correctly for the given rectangular coordinate transformation. For any given rectangular coordinate

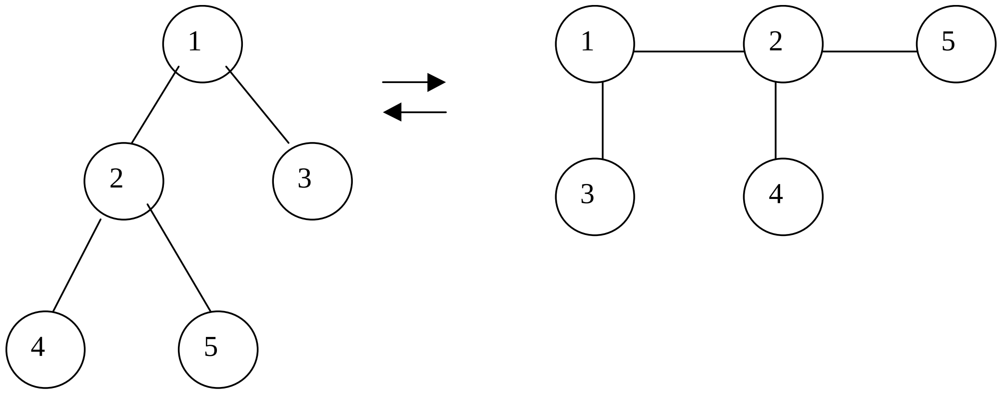


Figure 1: Tree Network Example

transformation, there will exist at least one pair of such links for which the distance cannot be computed correctly.

Nevertheless, for a specific assignment of facilities to links on the network, the optimal locations of the facilities on the respective links can be found by formulating a linear program. The decision variables can be defined as $x_i, i = 1, \ldots, p$, where x_i is the distance of facility i from either of the two end nodes of the link to which facility i is assigned. The number of such linear programs is, however, exponential in n and p. Therefore, this is not a computationally feasible approach.

We now consider the case of a general network. As in the case of a tree network, the optimal locations of facilities can be determined by solving a linear program for a given assignment of facilities to links. As an example, consider the network shown in Figure 2. Assume that facilities 1 and 2 are to be located on the network, on links $1 - 2$ and $1 - 3$ respectively. Let x_1 and x_2 represent the distances of facilities 1 and 2 from nodes 2 and 3 respectively. Then, the distance between facilities 1 and 2 can be written as

$$d(x_1, x_2) = Min\,\{2 + x_1 + x_2, 7 - x_1 - x_2\}\,.$$

Such a constraint can be incorporated within a linear program and thus the problem is solvable polynomially for a given assignment of facilities to links. As mentioned earlier, such an approach is not computationally feasible, since there are an exponential number of possible assignments of facilities to links.

It may be possible, however, to develop efficient procedures to solve (P) on a general network for certain values of the parameters λ, A and B. For example, Problem (P) for large values of A and B and relatively small values of λ, reduces to the p-maxian (which is equivalent to the 1-maxian), and can therefore be solved polynomially. It is not clear, however, that the problem is polynomially solvable for general values of A, B, and λ. In

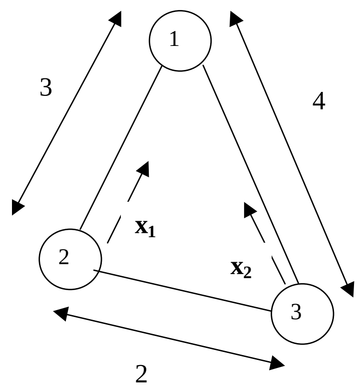


Figure 2: General Network Example

fact there is considerable evidence that leads us to conjecture that problem (P) is NP-complete. For example, for a certain version of the p-maxian problem where the objective is to maximize the sum of the distances between facilities and demand points and the distances between the facilities, Hansen and Moon (1988) have shown that when location is restricted to a discrete set of points the problem is strongly NP-complete; and Tamir (1991) proved NP-completeness for the more general case of unrestricted location.

3.2 Location Restricted to a Finite Set

We now consider a discrete version of problem (P), where location is restricted to a finite set of sites. Although even this problem is hard to solve (as established by the NP-completeness result a little later), we can at least formulate it in a concise manner unlike the unrestricted version. We propose several equivalent formulations for the discrete location problem and examine their features.

Let Z represent the set of sites available for facility location. Furthermore, let d_{ij} denote the length of a shortest distance path between sites i and j, and α_j be a 0-1 decision variable so that $\alpha_j = 1$ if a facility is at site j and 0 otherwise, for $j = 1, \ldots, z$. Let the sites be labeled $1, \ldots, z$. We define the index sets $H_s = \{j : d_{sj} \leq min(A/w_s, B),\ j = 1, \ldots, z\}$, $s = 1, \ldots, n$. Each set H_s contains the indices j of the possible locations which are within acceptable distance (i.e., minimum between A/w_s and B) of node s. With this notation in place, the problem (P) can be reformulated as the following bilinear programming problem:

$$(P_{Z_1}) \quad \text{Maximize} \quad \sum_{s=1}^{n} w_s\ \beta_s \tag{9}$$

Subject to the constraints:

$$\sum_{j=1}^{z} \alpha_j = p \tag{10}$$

$$(1-\alpha_j\ \alpha_l)\ M + \alpha_j\ \alpha_l\ d_{jl} \geq \lambda \quad \text{for} \quad j = 1,\ldots,z-1,\ l = j+1,\ldots,z \tag{11}$$

$$(1-\alpha_j)\ D_s + \alpha_j\ d_{sj} \geq \beta_s \quad \text{for} \quad s = 1,\ldots,n,\ j \in H_s \tag{12}$$

$$\sum_{j \in H_s} \alpha[j] \geq 1, \quad \text{for} \quad s = 1,\ldots,n \tag{13}$$

$$\alpha_j = 0 \ \text{ or } \ 1 \quad \text{for} \quad j = 1,\ldots,z \tag{14}$$

The decision variables in the above formulation are the α_j's which are binary variables (14): $\alpha_j = 1$ indicates that a facility is located at site j and $\alpha_j = 0$ otherwise. Constraint (10) ensures that only p facilities are located among the z sites. Constraints (11), (12), and (13) are analogous to constraints (4), (5), and (6), respectively. Constraints (11) and (12) are big-M constraints. The role of the constraints (12) is to calculate the distance β_s between a node s and the closest open facility. Since a facility must be opened within an acceptable distance (i.e., at most equal to the minimum of $A/[w[s]$ and B) of any node s, we have a number $|H_s|$ (and not $|Z|$) of constraints of form (12) for each node s. Indeed, H_s is the set of sites that are within $min(A/w[s], B)$ from s and constraint (13) ensures that at least one facility will be opened at one of the sites $j \in H_s$. The values of the positive constants M and D_s must be chosen such that they do not exclude any admissible solution. From a computational perspective, it is important to set them equal to the smallest possible value allowing for the above objective to be attained. Assigning a very large value (say 10^5) to M and D_s would produce a valid (integer) model. However, its continuous relaxation would be very loose and would likely generate an excessive number of branching and increase the computational time. It is straightforward to see that λ and $min(A/w_s, B)$ are the smallest values that M and D_s can respectively take to guard us from rejecting any admissible solution.

Theorem 1 *Problem* (P_{Z_1}) *is NP-complete.*

Proof: We show the above when the set Z is restricted to the set of demand nodes N. This can be shown by reducing a known NP-complete problem (in this case the dominating set problem) to the given problem. The decision version of the dominating set problem can be stated as follows: Given a graph $G = (N, L)$, and a positive integer $K \leq |N|$, is there a dominating set of size K for G, i.e, a subset $N' \subseteq N$ with $|N'| = K$ such that for all $i \in N - N'$ there is a $j \in N'$ for which $(i, j) \in L$? Let $G' = (N, L)$ be derived from G such that $l(i, j) = 1$ for $(i, j) \in L$, and $w_s = 1$ for all $s \in N$. Consider the problem P_N over G' with the parameters (A, B and λ) all set to unity, and $p = K$. It is easy to verify that the graph G has a dominating set of size K if and only if the solution to P_N over G' is equal to $n - K$. $\square$

If the set N' is required to be both a dominating and independent set of size K (which is also NP-complete), then letting $A = 1$, $B = 1$ and $\lambda = 2$ in the solution of P_N over G' will also yield the same result.

We introduce the following notations: $M_{jl} \in \mathcal{R}_+, j = 1, \ldots, (z-1), l = j+1, \ldots, z$ is a positive parameter, d_{jl} is the known distance between the potential facility sites j and l. We can now rewrite the nonlinear constraint (11) in a linear form.

Proposition 1 *Let* $M_{jl} = max\left(\lambda, \frac{\lambda + d_{jl}}{2}\right)$. *The linear constraint*

$$M_{jl}(2 - \alpha_j - \alpha_l) + (\alpha_j + \alpha_l - 1)d_{jl} \geq \lambda. \tag{15}$$

is equivalent to

$$(1 - \alpha_j \ \alpha_l) \ \lambda + \alpha_j \ \alpha_l \ d_{jl} \geq \lambda \ . \tag{16}$$

Constraint (16) is obtained by setting M in (11) to its smallest admissible value λ.

Proof: The goal is to find the smallest value for M_{jl} that does not exclude any feasible solution for (16). Each pair (α_j, α_l) of binary decision variables in (15) and (16) can take values $\{(1,1); (1,0); (0,1); (0,0)\}$: $(1,1)$ does not impose any restriction on M_{jl}. However, setting (α_j, α_l) to $(0,1)$ or $(1,0)$ in (15) requires $M_{jl} \geq \lambda$. Similarly, $(0,0)$ imposes that $M_{jl} \geq \frac{\lambda + d_{jl}}{2}$. This provides the result that we set out to prove. $\square$

Consider two locations j and l such that $d_{jl} > \lambda$. Setting $M_{jl} = \lambda$ would implicitly force the opening of at least one facility at sites j or at l. Indeed, the option of not opening any facility $(\alpha_j, \alpha_l) = (0,0)$ at j and l results into inequality (15) being rewritten as $2M_{jl} - d_{jl} \geq \lambda$ which is violated if $d_{jl} > \lambda$ and $M_{jl} = \lambda$. Clearly, this would cut a feasible solution (which could be optimal) and alter the feasible set defined by (16).

The reformulation of the bilinear problem (P_{Z_1}) as the mixed-integer programming (MIP) problem (P_{Z_2}) is a direct consequence of Proposition 1:

$$(P_{Z_2}) \quad \text{Maximize} \quad \sum_{s=1}^{n} w_s \ \beta_s$$

Subject to the constraints:

$$(10), (12) - (14)$$

$$M_{jl}(2 - \alpha_j - \alpha_l) + (\alpha_j + \alpha_l - 1)d_{jl} \geq \lambda, \ j = 1, \ldots, (z-1), l = j+1, \ldots, z \tag{17}$$

We shall now derive another MIP formulation (P_{Z_3}) equivalent to (P_{Z_1}) and (P_{Z_2}). We call the tuple set T_2 the set of interdicted joint facility locations and define it as:

$$T_2 = \{(j,l) : d_{jl} < \lambda,\ j = 1, \ldots, (z-1), l = j+1, \ldots, z\}\ .$$

Clearly each tuple $(j,l) \in T_2$ refers to a pair of locations where two facilities are not allowed to operate concurrently.

Proposition 2 can now be derived.

Proposition 2 *The inequalities*

$$M_{jl}(2 - \alpha_j - \alpha_l) + (\alpha_j + \alpha_l - 1)d_{jl} \geq \lambda, j = 1, \ldots, (z-1), l = j+1, \ldots, z$$

can be subsumed by the set of $|T_2|$ *cover inequalities*

$$\alpha_j + \alpha_l \leq 1,\ (j,l) \in T_2\ . \tag{18}$$

It is evident that the set of constraints (18) prevents from opening $(\alpha_j = 1, \alpha_l = 1)$ a facility at both locations j and l if the distance separating them is not at least equal to λ.

It follows from Proposition 2 that the MIP problem (P_{Z_3}) is equivalent to problem (P_{Z_2})

$$(P_{Z_3}) \quad \text{Maximize} \quad \sum_{s=1}^{n} w_s\ \beta_s$$

Subject to the constraints:

$$(10), (12) - (14); (18)$$

We can strengthen the formulation of problem (P_{Z_3}) through the introduction of valid cover inequalities. Let T_k denote the set of k-length tuples $(v_1, \ldots, v_k)$ defined as:

$$T_k = \left\{(v_1, \ldots, v_k) : d_{v_t v_{t'}} < \lambda, t = 1, \ldots, (k-1), t' = t+1, \ldots, k\right\}\ .$$

The set T_k contains the indices of the groups of k facilities among which at most one can be opened. We construct all the sets T_k for $k = 3, \ldots, (z-p-1)$. If the set T_k is non-empty for $k \geq z-p-1$, then the problem is infeasible. Clearly, the larger the value of k, the smaller the cardinality of the set T_k. It is therefore straightforward that:

Proposition 3 *The inequalities*

$$\sum_{j \in T_k} \alpha_j \leq 1,\ k = 3, \ldots, (z-p-1)\ . \tag{19}$$

are valid for (P_{Z_3}).

3.3 An Upper Bound on (P) via (P_Z)

Our focus is to develop a method to obtain a good upper bound for (P). A good upper bound suffices, since our motive is in determining how bad things can get. We attempt to do this by establishing a relationship between the optimal values of (P) and (P_Z). We assume for now that we have chosen a finite set of candidate sites Z; later we will develop a sensible choice for the set Z. An upper bound on (P) can be obtained based on the following theorem:

Theorem 2 *Let* $\delta_Z = max_{x\ \in\ G}[min(d(x,1), \ldots, d(x,z))]$ *be the maximum distance from a point on the network to its closest facility site in the set* Z *and let* $\nu_{\lambda,A,B}(P)$ *denote the optimum objective function value of a problem* P *when a distance standard* λ*, a proportionality standard* A*, and an equity standard* B *are used. Then,* $\nu_{\lambda,A,B}(P) \leq \nu_{\lambda',A',B'}(P_Z^L) + \delta_Z$*, where* $\lambda' = max[0, \lambda - 2\ \delta_Z]$ *and* $A' = A + w_s\delta_Z$ *and* $B' = B + \delta_Z$ *are defined for each node* s*.*

Proof: Let $x_1^*, \ldots, x_p^*$ be the facility locations on the network corresponding to an optimal solution to (P). Associated with $x_1^*, \ldots, x_p^*$, let $v_1^*, \ldots, v_p^*$ be the closest sites (in terms of distance) chosen from the set Z, respectively. The points $x_1^*, \ldots, x_p^*$ certainly satisfy the distance constraints (10) and (12) but the points $v_1^*, \ldots, v_p^*$ may not when the same standards λ, A, B is used. However, the points $v_1^*, \ldots, v_p^*$ will satisfy the constraints when a distance standard of $\lambda' = max[0, \lambda - 2\ \delta_Z]$, a proportionality standard $A' = A + w_s\delta_Z$ and an equity standard $B' = B + \delta_Z$ for each node s is used; the 2 in the expression $max[0, \lambda - 2\ \delta_Z]$ stems from the fact that two facilities may be reassigned to sites in a manner that makes both of them move δ_Z units towards one another. Similarly, the constraint $\beta_s \leq A/w_s$ needs to be modified to $\beta_s \leq A'/w_s$, and $\beta_s \leq B$ needs to be modified to $\beta_s \leq B'$ since a facility that was closest to a node s could have moved δ_Z units away. Let V denote the objective function value at the points $v_1^*, \ldots, v_p^*$. Then $V \geq \nu_{\lambda,A,B}(P) - \delta_Z$, since the maximum reduction in the objective function value is δ_Z, due to the fact that the nodes can each move closer to their nearest facility by at most δ_Z and that $\sum_{s \in N} w_s = 1$ by assumption. Also, $\nu_{\lambda',A',B'}(P_Z^L) \geq V$, since the points $v_1^*, \ldots, v_p^*$ are feasible to P_Z^L when distance standard λ', proportionality standard A', and equity standard B' are used. Combining these two inequalities yields $\nu_{\lambda,A,B}(P) \leq \nu_{\lambda',A',B'}(P_Z^L) + \delta_Z$. The theorem follows. □

Clearly, it is desirable to keep δ_Z small in relation to the value of $\nu_{\lambda,A,B}(P)$. For a chosen value of δ_Z, the sites can be chosen in the following manner: select $max[\lfloor \frac{l_{ij}}{2\delta_Z} \rfloor, 1]$ sites at equally spaced intervals on the interior of a link (i, j), where l_{ij} is the length of a link (i, j), and by having nodes as sites. For example, a link (i, j) of length 20 and a δ_Z value of 3 is handled by choosing $max[\lfloor \frac{20}{6} \rfloor, 1] = 3$ sites at points on link (i, j) that are 5, 10, and 15 units away from node i, respectively. (There is a possibility of having some unnecessary sites if links of length less than 2 δ_Z exist. Such links have one site at the center of the link. It is a simple matter to check if every point on such a link (i, j) is within δ_Z distance of a site on another link — this can be done by checking if the length of a shortest path from the midpoint of link (i, j) to the nearest site on an adjacent link is less than or equal to δ_Z. If such a check is affirmative, the site is removed from consideration.) What remains then is to obtain $\nu_{\lambda',A',B'}(P_{Z_3})$, using which an upper bound can be obtained on $\nu_{\lambda,A,B}(P)$ based on the result from Theorem 2.

Some observations are now made regarding the choice of δ_Z. The smaller δ_Z is, the better the approximation thus resulting in a tighter upper bound. Ideally, δ_Z should be as small as possible, but this results in a harder problem to solve since there will will be a large number of sites. As can be seen from the formulation P_{Z_3}, there are order $\mathcal{O}(z^2)$ number of constraints. Therefore, as z increases, the problem quickly becomes harder to solve. The largest value of δ_Z that can be chosen is given by $\delta_Z = max_{(i,j)} l_{ij}/2$ and results in $Z = N$. When δ_Z is chosen this way, the value of $\nu_{\lambda',A',B'}(P_N)$ which represents an upper bound on $\nu_{\lambda,A,B}(P_N)$, tells us how bad the p-median objective could be if locations are restricted to the set of nodes.

4 Computational Results

We present our results based on three different data sets. The first one is a transportation network in the county of Albany, New York (NY). The second one is a transportation network from the South-East (SE) US that has been used in a disaster preparedness study (Rawls and Turnquist 2010). The last set of results are based on a number of randomly generated networks (RAN) of different sizes and weights on nodes. The problems have been modelled with the AMPL algebraic programming language and solved with the CPLEX 12.4 solver.

Before proceeding with a discussion of the results, some last comments are in order regarding the choice of δ_Z. Recall, that the maximum value of δ_Z, say δ_Z^{max}, is determined by the length of the longest link on the network and is therefore fixed for a given network. Also, this value of δ_Z results in the set of sites, Z, restricted to the set of nodes (N).

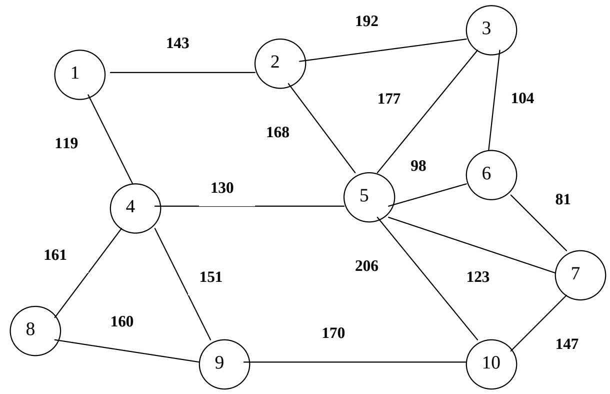


Figure 3: New York Network

Also, the values of the parameters A, λ and B are varied based on base starting values A_p, B_p and λ_p as implied by the p-median solution. Successive values of A, B, and λ are obtained using increments of 10% of A_p, B_p and λ_p in both directions of the base values, representing tighter constraints on one side of the base values and looser constraints on the other. Also, preliminary analysis was conducted for a range of values of δ_Z, the largest value for which the set Z of potential sites essentially reduces to the set N, the set of nodes. Clearly, the tightest bounds on the objective function (when location is unrestricted on the network) are obtained for the smallest value of δ_Z which results in a fairly large number of potential sites over which the problem is solved. Our findings showed that the upper bounds are fairly loose even for the smallest value of δ_Z suggesting that the worst case values when location is unrestricted could be quite bad relative to the p median solution. Moreover, as discussed in the Introduction section, in actual practice it is quite common that location choices are not unrestricted but instead limited to a discrete set of locations which are often the set of demand points (see for example, the study by Carling *et. al.* (2012). As such, in our computational study, we limited our analysis to locations that are restricted to the set of demand nodes.

4.1 New York Data Set

The county of Albany in NY state divides into ten townships with highly variant population densities and relative locations as shown in Figure 3. Only the major roadways have been considered in constructing the network — these are shown in the figure as links connecting nodes and the numbers alongside the links represent the lengths. Consideration of only major road segments is quite typical when studying public facility location problems, e.g., see the recent paper by Erdemir *et al.* (2008) for locating aeromedical bases in the State of New Mexico. The population densities of the nodes are shown in Table 2.

Township	Node(i)	Weight (w_i)
Knox	1	0.0072
Guilderland	2	0.0833
Colonie	3	0.2741
Berne	4	0.0080
New Scotland	5	0.0362
Albany City	6	0.4591
Bethlehem	7	0.0929
Rensselaerville	8	0.0061
Westerlo	9	0.0089
Coeymans	10	0.0266

Table 1: Population Densities on Network

We solve problems with 3, 5, and 7 facilities on this network, and results are presented in Figure 4. The Y-axis on the figure shows the % deviation of the worst-case median value from the optimal p-median value, while the X-axis shows the parameter values. The middle value on the X-axis, as shown in the figure as (ABL6) corresponds to the parameter values as implied by the p-median solution. The rest of the values on the X-axis represent successive 10% changes from this base value.

For tighter values of the parameters (represented in the figure by (ALB1) through (ALB5)), the problem becomes infeasible. For the other values of the parameters, as seen in the graph, the worst case p-median value is very stable and does not change much at all. In fact, for both the 3 and 7 facility case, the deviation from the p-median solution is zero, and about 26% for the 5 facility case. For each computed worst case median solution, we also assessed the deviation with respect to the equity measure from its value as implied by the p-median solution, and that was 0 for the 3 and 7 facility case, and improved by about 15% in the 5 facility case. These results, we suspect, are somewhat sensitive to the particular features of this network which has an extreme variation in the population densities across the nodes - roughly 80% of the entire population of the network is contained in three nodes of the network. As such, the parameter values implied by the p-median solution seem to place rather tight bounds on the corresponding criteria, thus not allowing for too many feasible solutions.

4.2 South-East (SE) US Network

This network was used in a disaster preparedness study by Rawls and Turnquist (2010) and consists of 30 nodes with cities and towns along the gulf from Texas to Florida as shown

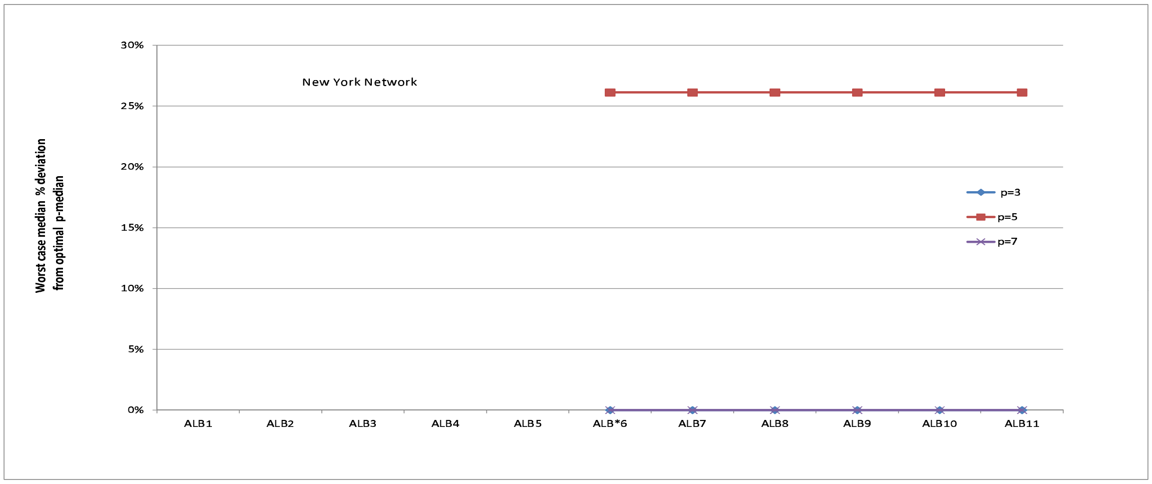


Figure 4: Results for NY Network

in Figure 5. We solve problems with 3,5,7, and 10 facilities on this network, and results are presented in Figure 6. As in the previous figure for the NY network, the Y-axis on this figure again shows the % deviation of the worst-case median value from the optimal p-median value, while the X-axis shows the parameter values.

The results indicate that the deviation from the p-median is again quite low (about 20% to 40%) and relatively stable especially for the 10 and 15 facility cases. For the 3 and 5 facility cases, the deviation is slightly higher, but under 80% for the base values of the parameters, and around 80% to 100% for higher values. The deviation with respect to the equity measure from its value as implied by the p-median solution is generally negative or around 0 for parameter values close to the base values. This suggests that the equity values are not as sensitive to variations in parameter values as the median values. This is consistent with the findings in Burkey *et al.* (2010) where they determined that actual locations of hospitals in four states in the US yielded median and equity values very close to the optimal p-median value and equity value implied by the optimal p-median solution, respectively.

4.3 Randomly (RAN) Generated Networks

A number of tests were also conducted using randomly generated problem instances. Three network sizes were generated comprising of 50 nodes, with the number of edges being 125, 175, and 230. For each of these three networks, 4 sets of weights were generated ranging from somewhat uniformly populated nodes ($\mathbf{w_1}$) to nodes that differed considerably in their population densities($\mathbf{w_4}$). For each such problem, 5 random instances were generated and results were averaged over the 5 instances. The experiment thus involved solving a total of 180 problems instances (3 network sizes x 4 sets of weights x 5 problem instances x 3 number

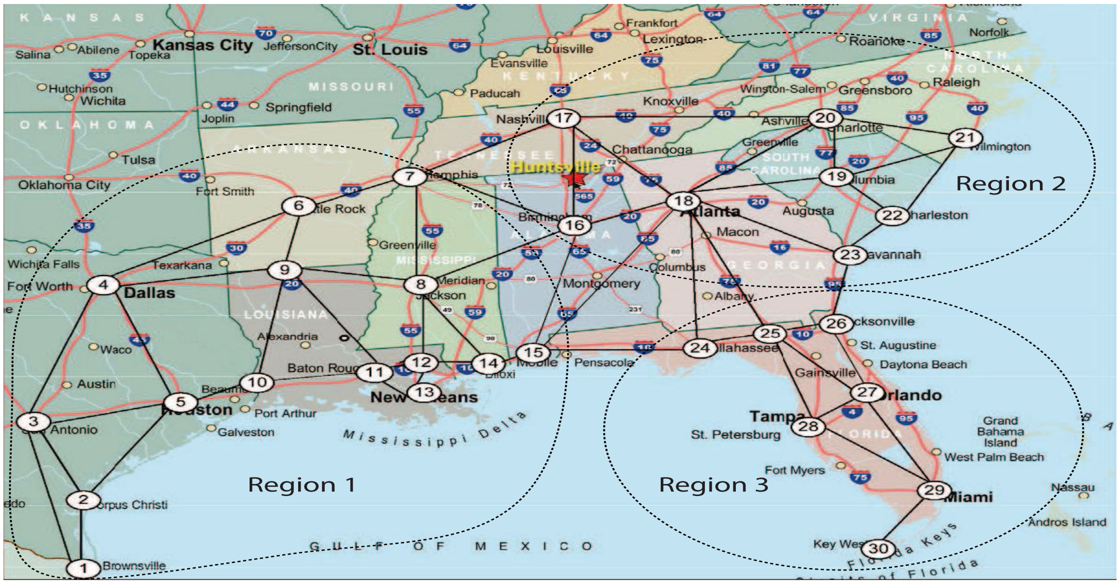


Figure 5: South East Network

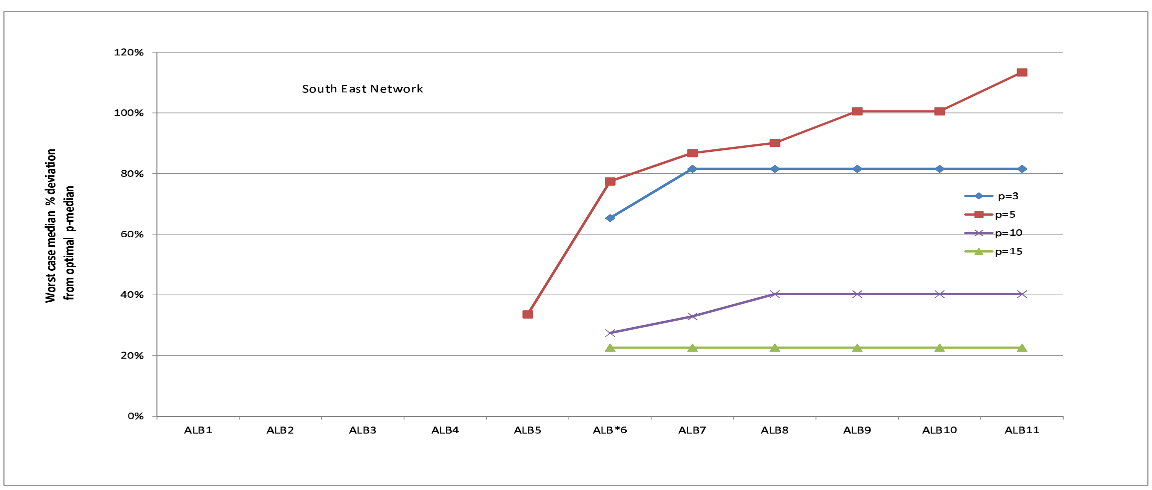


Figure 6: Results for SE Network

of facilities = 180).

The results show that the base parameter values (as implied by the p median solution) when location is restricted to nodes yield solutions that are in the worst case within 100% of the optimal p-median solution, with even lower deviations for the specific cases of $p = 3$ and $p = 5$ (See Table 2). Table 3 shows the deviations of the worst case solutions in terms of the equity criteria relative to equity values implied by the p median solution. It is evident that the equity values are mostly about the same as those implied by the p-median solution or marginally better. These results again confirm the findings in Burkey *et al.* (2010), suggesting that the parameter values as implied by the p-median solution do seem to capture the criteria used by decision makers in practice.

No. Facilities	Network	% deviation			
		$\mathbf{w_1}$	$\mathbf{w_2}$	$\mathbf{w_3}$	$\mathbf{w_4}$
3	r1	23.70	34.90	35.28	24.09
3	r2	16.42	21.48	23.84	35.23
3	r3	23.71	32.44	31.89	15.99
5	r1	40.34	42.12	35.21	49.84
5	r2	35.00	57.27	46.33	69.59
5	r3	38.55	36.52	46.41	42.67
10	r1	51.35	71.08	81.98	90.59
10	r2	73.93	98.14	86.24	87.37
10	r3	70.87	75.84	97.22	89.78

Table 2: Median Deviations for Random Networks

No. Facilities	Network	% deviation			
		$\mathbf{w_1}$	$\mathbf{w_2}$	$\mathbf{w_3}$	$\mathbf{w_4}$
3	r1	-7.44	-2.43	-5.73	-4.22
3	r2	0.00	-1.05	-1.90	-2.90
3	r3	0.00	0.00	-3.68	0.00
5	r1	-2.73	-3.33	-4.98	-6.57
5	r2	-1.05	0.00	-2.11	-1.05
5	r3	0.00	0.00	0.00	-2.50
10	r1	-2.50	0.00	0.00	-5.00
10	r2	-1.25	-1.33	-1.43	-1.25
10	r3	0.00	-1.54	0.00	0.00

Table 3: Equity Deviations for Random Networks

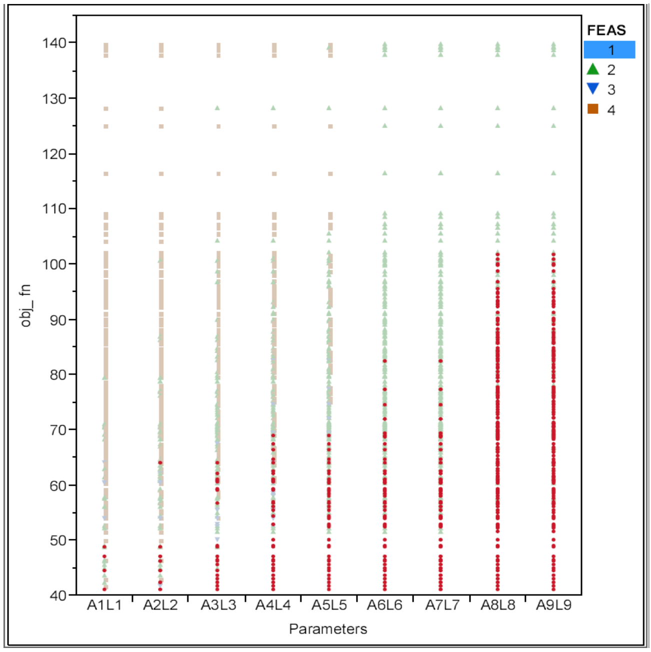


Figure 7: Feasible Solutions

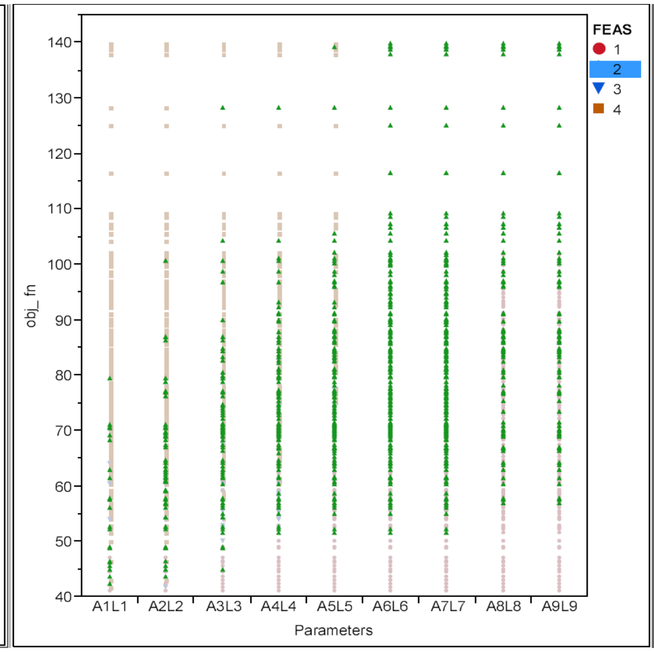
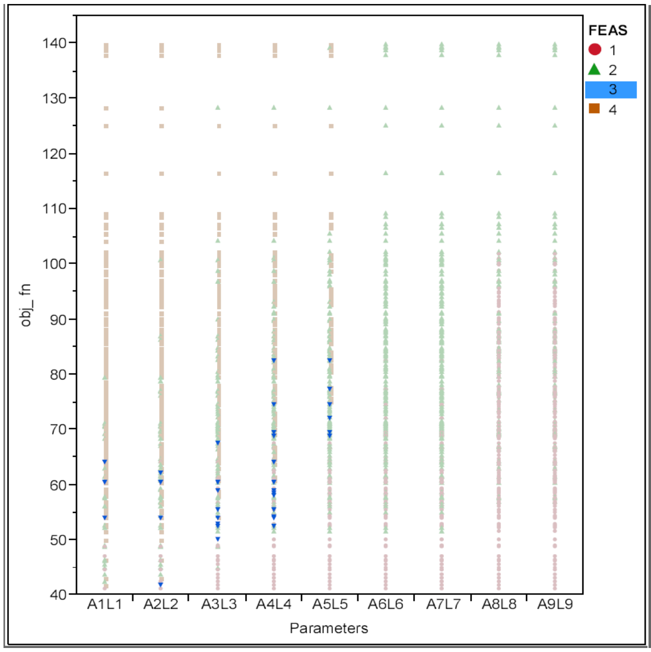


Figure 8: Infeasible wrt Pop. Para. alone

Sensitivity of Solution Set

Finally, we provide some results that illustrate the effectiveness of the parameters in eliminating poor solutions from consideration. In Figures 7 through 10, we illustrate for a particular instance of the NY data set, the effect of varying the population and dispersion parameter values on the set of feasible solutions. The parameter values are plotted on the X-axis, starting with A_p and λ_p as A1L1, and the successive values A2L2 through A9L9 representing 10% successive relaxations on the starting values which were established based on the p median solution. Figure 7 shows that a number of the poor solutions are indeed eliminated at tighter values of A and λ, and the parameters are indeed quite effective in eliminating the worst solutions for a range of values. Figures 8 through 10 show solutions that are infeasible with respect to the population parameter alone, the dispersion parameter alone, and both, respectively. We can see from these graphs, that the dispersion parameter is not as effective as the population parameter in eliminating poor solutions - as shown by Figures 9 and 10, almost all solutions eliminated by the dispersion parameter are also eliminated by the population parameter. This suggests that with a proper choice of the population parameter, solutions are naturally well dispersed and automatically satisfy the dispersion parameter.



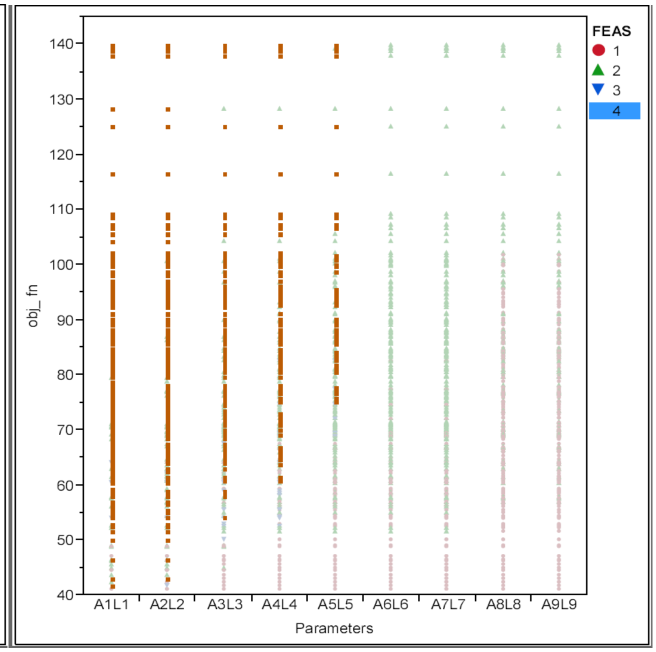


Figure 9: Infeasible wrt Disp. Para,. alone

Figure 10: Infeasible wrt Both

5 Conclusions

In conclusion, our analysis suggests that when when location is restricted to nodes, parameters based on the population, dispersion, and equity criteria seem effective in eliminating most of the poor solutions. Prescriptions derived in this manner based on the p-median solution may serve as a useful prescriptive aid for decision makers to help generate/assess a number of good quality solutions that they can choose from. In the context of larger problem instances, if the p median itself is hard to solve, a question arises as to how *good* values for the parameters can be generated? Preliminary testing using the NY and SE networks with parameter values implied by solutions that are within a specified small% (25%) of the p-median solution suggests that the worst case median value is often close to the optimal p-median value, and within 100% in almost all cases. This suggests the use of a heuristic to get a good solution based on which the parameter values can be derived. For example, one option is to use values based on a solution found using a heuristic such as the 1-opt interchange (Teitz and Bart (1968), Rosing and Revelle (1997), Rosing and Revelle (2002)). If the objective is to actually generate a set of solutions to present to the DMs for consideration, we could employ a modification of the Teitz and Bart heuristic to generate, say, the k best solutions starting with the best heursitic solution that would satisfy the parameters. The modification would be to consider the swap in the 1-opt interchange on the basis of two conditions, 1) that the new solution satisfies the parameters, and 2) the new solution results in the smallest increase with respect to the p-median value. It would be interesting to assess the average and worst solution among the generated solutions relative to the p-median.

References

Antunes, A.P. Location Analysis Helps Manage Solid Waste in Central Portugal. *Interfaces* 29(4), 1999, 32-43.

Avella, P., Benati, S., Martinez, L.C., Dalby, K. Some Personal Views of the Current State and the Future of Locational Analysis. *European Journal of Operational Research* 104(2), 1998, 269-287.

Batta, R., Leifer, L.A. On the Accuracy of Demand Point Solutions to the Planar, Manhattan Metric, p-Median Problem, With and Without Barriers to Travel. *Computers and Operations Research* 15(3), 1988, 253-262.

Batta, R., Dolan, J.M., Krishnamurthy, N.N. The Maximal Expected Covering Location Problem: Revisited. *Transportation Science* 23(4), 1989, 277-287.

Berman, O., Huang, R. The Mimimum Weighted Covering Location Problem with Distance Constraints. *Computers and Operations Research* 35(2), 2008, 356-372.

Berman, O., Krass, D. Median Problems with Undesirable Facilities on a Line. Presentation at *INFORMS Austin Conference*, 2010.

Berman, O., Larson, R.C., Chiu, S.S. Optimal Server Location on a Network Operating as an M/G/1 Queue. *Operations Research* 33, 1985, 746-771.

Berman, O., Larson, R.C., Parkan, C. The Stochastic Queue p-Median Problem. *Transportation Science* 21, 1987, 207-216.

Bigman,D., ReVelle, C. The Theory of Welfare Considerations in Public Facility Location Problems. *Geographical Analysis* 10(3),1978, 229-240.

Brandeau, M.L., Chiu, S.S. An Overview of Representative Problems in Location Research. *Management Science* 31(6), 1989, 645-674.

Burkey, M.L., Bhadury, J., Eiselt, H.A. A Location-Based Comparison of Health Care Services in Four U.S. States with Efficiency and Equity. *Socio-Economic Planning Sciences* 46(2), 2012, 157-163.

Carling, K., Han,M., Hakansson,J. Does Euclidean Distance Work Well when the p-median Model is Applied in Rural Areas? *Annals of Operations Research*, 2012, 201:8397.

Carson, Y.M., Batta, R. Locating an Ambulance on the Amherst Campus of the State University of New York at Buffalo. *Interfaces* 20(5), 1990, 43-49.

Cheng,Y., Kang, L. The p-Maxian Problem on Interval Graphs. *Discrete Applied Mathematics* 158(18), 2010, 1986-1993.

Church, R.L., Garfinkel, R.S. Locating an Obnoxious Facility on a Network. *Transportation Science* 12, 1978, 107-118.

Church, R.L., ReVelle, C. The Maximal Covering Location Problem. *Papers in Regional Science Association* 32, 1974, 101-118.

Daskin, M.S. What you Should Know About Location Modeling. *Naval Research Logistics* 55, 2008, 283-294.

Dearing, P.M., Francis, R.L., Lowe, T.J. Convex Location problems on Tree Networks. *Operations Research* 24, 1976, 628-642.

Dell, R., Araya, F., Doneso, P., Marianov, V., Martinez, F., Weintraub, A., Zuniga, C. Resizing, Opening, and Closing of Schools in Chile. Presentation at *INFORMS San Diego Conference*, 2009.

Eaton, D.J., Daskin, M.S., Simmons, D.S., Bulloch, B., Jansma, G. Determining Emergency Medical Service Vehicle Deployment in Austin, Texas. *Interfaces* 15(1), 1985, 96-108.

Erdemir, E.T., Batta, R., Spielman, S., Rogerson, P., Blatt, A., Flanigan, M. Evaluating the Performance of Aeromedical Base Locations of New Mexico by Considering Nodal and Path Demand. *Accident Analysis and Prevention* 40, 2008, 1105-1114.

Erkut, E., Baptie, T., Hohenbalken, B.V. The Discrete p-Maxian Location Problem, *Computers & Operations Research* 17, 1990, 51-61.

Erkut, E., Francis, R.L., Lowe, T.J. A Multimedian Problem with Interdistance Constraints. *Env. Plan. B* 15, 1988, 181-190.

Erkut, E., Francis, R.L., Tamir, A. Distance-Constrained Multifacility Minimax Location Problems on Tree Networks. *Networks* 22, 1992, 37-54.

Erkut, E., Neuman, S. Analytical Models for Locating Undesirable Facilities. *European Journal of Operational Research* 40, 1989, 275-291.

Farahani, R.Z., Seifi, M.S., Asgari, N. Multiple Criteria Facility Location Problems: A Survey. *Applied Mathematical Modelling* 34(7), 2010, 1689-1709.

Francis, R.L., Lowe, T.J., Ratliff, H.D. Distance Constraints for Tree Network Multifacility Location Problems. *Operations Research* 26, 1978, 570-596.

Francis, R.L., Lowe,T.J., Tamir, A. Demand Point Aggregation for Location Models," in *Facility Location: Application and Theory*, Z. Drezner and H. Hamacher (eds.), 2004, Second Edition, Springer Verlag.

Francis, R.L., McGinnis, L.F., White, J.A. *Facility Layout and Location: An Analytical Approach*, Second Edition, Prentice Hall, 1992.

Hakimi, S.L. Optimum Distribution of Switching Centers and the Absolute Centers and Medians of a Graph. *Operations Research* 12, 1964, 450-459.

Hale, T.S., Maberg, C.R. Location Science Research: A Review. *Annals of Operations Research* 123, 2003, 21-35.

Hansen, P., Moon, D. Dispersing Facilities on a Network. *RRR #52-88*, RUTCOR, Rutgers University, (1988).

Kuby, M.J. Programming Models for Facility Dispersion: the p-Dispersion and Maxisum Dispersion Problems. *Geogr. Anal.* 19, 1987, 315-329.

Larson, R.C. A Hypercube Queuing Model for Facility Location and Redistricting in Urban Emergency Services. *Computers and Operations Research* 1, 1974, 67-95.

Larson, R.C., Odoni, A.R. *Urban Operations Research*, Prentice-Hall, Englewood Cliffs, N.J., 1981.

Larson, R.C., Stevenson, K.A. On Insensitivities in Urban Redistricting and Facility Location. *Operations Research* 20, 1972, 595-612.

Love, R.F., Walker, J. An Empirical Comparison of Block and Round Norms for Modeling Actual Distances. *Location Science* 2, 1994, 21-43.

Minieka, E. Anticenters and Antimedians of a Network. *Networks* 13, 1983, 359-364.

Minde, J. *Building a Framework for a Spatial Decision Support System for Co-locating Public Facilities*. Master's Thesis, Fall 2008, George Mason University, VA.

Moon, I.D., Chaudhry, S.S. An Analysis of Network Location Problems with Distance Constraints. *Management Science* 30, 1984, 290-307.

Nedovic-Budic, Z. Geographic Information Science Implications for Urban and Regional Planning. *Journal of the Urban and Regional Information Systems Association* 12(2), 2000, 81-93.

Oklahoma City Fire Department Fire Station Location Study: http://www.okc.gov/fire/fire_report.pdf, 2006.

Owen, S., Daskin, M.S. Strategic Facility Location: A Review. *European Journal of Operational Research* 111, 1998, 423-447.

Price, W.L., Turcotte, M. Relocation of the Red Cross Blood Donor Clinic and Transfusion Center in Quebec City. *Interfaces* 16, 1986, 17-26.

Rawls, C.G. and Turnquist, M.A. Pre-Positioning of Emergency Supplies for Disaster Response,*Transportation Research Part B: Methodological*, 44 (4), 2010, 521-534.

ReVelle, C.S., Eiselt, H.A. Location Analysis: A Synthesis and Survey. *European Journal of Operational Research* 165, 2005, 1-19.

Rosing, K.E., Hodgson, M.J. Heuristic Concentration for the p-median: An Example Demonstrating How and Why It Works. *Computers & Operations Research* 29(10), 2002, 1317-1330.

Rosing, K.E., ReVelle, C.S. Heuristic concentration: Two stage solution Construction. *European Journal of Operational Research* 97(1), 1997, 75-86.

Serra, D., Marianov, V. New Trends in Public Facility Location Modeling. 2004. *Economics and Business Working Paper* No. 755. Available at http://ssrn.com/abstract=563843

Sinuany-Stern, Z., Arad-Geva, T., Binyamin, S. On the Implementation of Quantitative Facility Location Models: The Case of a Hospital in a Rural Region. *Journal of the Operational Research Society* 47, 1998, 612-625.

Tamir, A. Obnoxious Facility Location on Graphs. *SIAM J. Disc. Math.* 4, 1991, 550-567.

Teitz, M.B., Bart P. Heuristic Methods for Estimating the Generalized Vertex Median of a Weighted Graph. *Operations Research* 16, 1968, 955-961.

Ting, S.S. A Linear Time Algorithm for Maxisum Facility Location on Tree Networks. *Transportation Science* 18, 1984, 76-84.

Toregas, C., ReVelle, C. Optimal Location under Time or Distance Constraints. *Papers of the Regional Science Association* 28, 1972, 133-144.