Effectiveness-Equity Models for Facility Location Problems on Tree Networks

Miguel A. Lejeune, Srinivas Prasad*

Abstract

We propose models to investigate effectiveness-equity tradeoffs in tree network facility location problems. We use the commonly used median objective as a measure of effectiveness, and the Gini index as a measure of (in)equity, and formulate bicriteria problems involving these objectives. We develop procedures to identify an efficient set of solutions to these problems, analyze the complexity of the proposed procedures, and finally illustrate the procedures with an example.

Key words: Facility Location; Equity; Quasi-Convexity; Multiobjective Programming; Tree Network; Efficiency

1 Introduction

Facility location problems have been studied for a long time in the Operations Research (OR) literature. The motivation to study these problems arises from its numerous applications in both public and private sector contexts. Typical problems involve determining locations on a network so as to optimize some performance measure that is a function of the distance or time from demand points to their nearest facilities. Common functions that have been used in the literature include the average weighted distance (referred to as the median objective), and the maximum distance (referred to as the center objective). Numerous papers have studied these problems incorporating extensions to multiple facilities, multiple criteria, and stochastic versions of the problem. For an extensive review of the literature, see Current *et al.* [7] and Daskin [9].

As much as the median and center type performance measures make sense to consider from an effectiveness or efficiency standpoint, other considerations, such as how equitable (or fair) a solution is, become important especially in situations that involve public facilities or resources. Although one could consider the center objective as a measure that enforces some level of equity, there are several other equity measures that would be considered more appropriate measures of equity. The literature includes papers ranging from a discussion of the desirable properties of an equity measure (Eiselt *et al.* [13], Marsh and Schilling [26]), to those that have focused on developing efficient algorithms to determine equitable locations (Halpern and Maimon [17]). Some of the common equity measures developed include the following: Variance [22], Range [10], Mean or Sum of Absolute Weighted Differences [30], Maximum Weighted Absolute Deviation [20], and the Gini coefficient [23].

In this work, we use equity measures that are compatible with the Principle of Transfers which requires that the transfer of service units from a subgroup to any relatively worse off should result in an improvement in the measure (Allison [1]). Two specific measures that satisfy this condition are the Sum of Absolute Weighted Differences and the Gini coefficient, the latter being one measure that is used often to describe income distributions in Economics. Both of these measures have also been used in the location literature and algorithms have been developed to identify optimal locations based on these measures (see Tamir [35]). Other early work involving equity measures in OR models is that of Erkut [14] and Mandell [25] who demonstrated that the use of the Gini index as an equity measure results in analytically tractable models. More recent studies addressing the role of equity in location are those of Baron *et al.* [2] and Berman *et al.* [4] who studied the problem of

*Department of Decision Sciences, School of Business, The George Washington University, Washington, DC 20052: mlejeune@gwu.edu, prasad@gwu.edu

locating facilities so as to make the loads among service facilities equitable, Drezner *et al.* [11] who employ the Gini coefficient to determine the most equitable location for a single service facility, and Espejo *et al.* [15] who consider facility locations so as to minimize the total *envy* felt by the demand points. Equity measures have also been employed in location problems that involve path shaped facilities (rather than point facilities) - see, for example, Puerto *et al.* [33] and Cáceres *et al.* [6].

The need for the simultaneous consideration of effectiveness and equity measures in the allocation of public resources has been well described in the literature - see for example, the papers by Berman [3] who used an objective function incorporating the p-median and variance measures, Mandell [25] who uses a bicriteria model involving the Gini coefficient as an equity measure, Ogryzcak [31] who uses a multi objective goal programming model, and Maliszewski *et al.* [24] who studied a number of multiple objective models using common dispersion measures to manage critical assets in urban areas. Further motivation for our work is provided in a recent paper by Leclerc and McLay [19] who, in describing issues involved in modeling equity for allocating public resources, call for "research in the operations research domain that systematically analyzes the functions through which equity may be combined with other attributes in an objective function, or that addresses how to incorporate equity into the constraints of a mathematical programming model from a methodological point of view." It should be noted that several of the location models that have been used to address effectiveness equity tradeoffs fall under the class of multiple objective location problems. An early review of multiple objective facility location models is by Current *et al.* [8], followed by two more recent reviews by Nickel *et al.* [29] and Farahania *et al.* [16]. Most of the models studied in this literature typically involve objectives that are either median/center based measures (Hamacher *et al.* [18]), or generalized forms of these measures as in the ordered median problem formulations (Nickel and Puerto [28]). Moreover, bicriteria models that have studied effectiveness equity tradeoffs have either considered discrete location contexts (Mandell [25]) or used equity measures that do not satisy the Principle of Transfers (Halpern and Maimon [17]). Location models have employed other equity measures that do satisfy the Principle of Transfers (Tamir [35]) but not necessarily in the context of examining tradeoffs with efficiency - one exception is the work of Ohsawa *et al.* [32] who studied a bicriteria model for locations in a plane. Our proposed models extend this line of analysis when location is restricted to tree networks.

Our contributions are threefold. First, the proposed models allow one to identify efficient (or Pareto optimal) locations using equity measures that are more appropriate (i.e., have more desirable attributes) than those used in previous studies. Second, we derive some useful properties of the efficient (Pareto) set based on an analysis of the measures, and lastly, we propose algorithmic procedures for solving these models based on the characteristics of the solution set.

The rest of the paper is organized as follows. In Section 2, we introduce the notation and provide a formal definition of the problem. In Section 3, we study the properties of the proposed facility location problems and derive a solution method. The method is illustrated in Section 4, and Section 5 provides concluding remarks.

2 Notation and Problem Definition

Let $G = (N, E)$ be an undirected connected graph with n nodes within node set N and m edges in edge set E. Each edge $e \in E$ has a positive length l_e, and two vertices (nodes) that characterize the edge. Let $A(G)$ denote the continuum set of points in the edges of G, which is also a union of m intervals. An interior point along the edge is identified by its distance from the vertices. The length of the shortest path from node i to a facility at a location x on the graph is denoted by $d(i, x)$, where non-negative w_i is the weight of each node i in the network, such that $\sum_{i=1}^{n} w_i = 1$.

The median function

$$M(x) = \sum_{i=1}^{n} w_i d(i, x) \tag{1}$$

is defined as the sum of the weighted distances to the nearest facility. We consider two particular inequity measures, the first one of which is the summation of absolute weighted distance differences:

$$SAWD(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} |w_i d(i, x) - w_j d(j, x)| . \tag{2}$$

A second measure of inequity is the Gini coefficient

$$G(x) = \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} |w_i d(i, x) - w_j d(j, x)|}{2n \sum_{i=1}^{n} w_i d(i, x)} \tag{3}$$

defined as the ratio of the summation of absolute weighted distance differences to the scaled p-median value. Although, several other variations of the Gini coefficient have been used in the literature, the one that we use here is based on the papers by Tamir [35] and Drezner *et al.* [11]. Clearly, $G(x) = \frac{SAWD(x)}{2n\ M(x)}$.

The problems we are investigating are bi-objective problems which involve the median measure as the effectiveness criterion, and either the $SAWD(x)$ or $G(x)$ measure as the equity criterion. Specifically, we are interested in analyzing the tradeoffs between the two objectives. The two problems of interest, **P1** and **P2**, can thus be formulated as follows:

$$\begin{aligned} \textbf{P1}: \quad & \min_{x \in A(G)} \ M(x) = \sum_{i=1}^{n} w_i d(i, x) \\ & \min_{x \in A(G)} \ SAWD(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} |(w_i d(i, x) - w_j d(j, x))| \\ \text{subject to} \quad & d(i, x) \geqslant 0, \forall i \end{aligned}$$

$$\begin{aligned} \textbf{P2}: \quad & \min_{x \in A(G)} \ M(x) = \sum_{i=1}^{n} w_i d(i, x) \\ & \min_{x \in A(G)} \ G(x) = \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} |(w_i d(i, x) - w_j d(j, x))|}{2n \sum_{i=1}^{n} (w_i d(i, x))} \\ \text{subject to} \quad & d(i, x) \geqslant 0, \forall i \end{aligned}$$

As is typical in multiple objective optimization, the solutions to **P1** and **P2** are defined by the concept of *efficiency*. A solution x' to **P1** (or **P2**) is considered efficient if there is no solution $x \in A(G)$ such that $M(x) \leq M(x')$ and $SAWD(x) \leq SAWD(x')$ with strict inequality holding in at least one (or $M(x) \leq M(x')$ and $G(x) \leq G(x')$, again with strict inequality holding in at least one). One approach to solving multiple criteria optimization problems is to either generate the entire set of efficient solutions to the problem or a representative subset of the efficient set (see, for example, Steuer [34], Ehrgott and Wiecek [12]). Solution procedures vary depending on the nature of the objective functions, and whether the solution space is continuous or discrete

(Nickel *et al.* [29]). We devise procedures to identify such efficient solutions for problems **P1** and **P2** in the next section.

We also discuss a variation of the problems formulated above wherein we consider a convex combination (weighted sum) of the two objectives (problems **P3** and **P4**), where λ_m, λ_g and λ_s are all non-negative weights. Problems **P3** and **P4** can be formulated as follows:

$$\mathbf{P3}: \quad \min_{x \in A(G)} \lambda_m M(x) + \lambda_s SAWD(x)$$
$$\text{subject to} \quad d(i,x) \geq 0, \forall i$$

$$\mathbf{P4}: \quad \min_{x \in A(G)} \lambda_m M(x) + \lambda_g G(x)$$
$$\text{subject to} \quad d(i,x) \geq 0, \forall i$$

We should note here that in **P3**, the objectives being weighted both have the same units, whereas that is not the case in **P4**. As such, the weights in **P4** should be chosen in a way that takes into account the different scales over which the objectives are defined over the network. Also, solving Problems **P1** and **P2** result in a set of efficient solutions that may be further evaluated based on other relevant objectives in a given context. Instead, the solutions to **P3** and **P4** instead would most likely result in unique solutions that capture the tradeoffs between the two objectives that are reflected in the chosen weights in the respective problems.

3 Analysis

In order to characterize the efficient solutions to **P1** and **P2**, we need to first examine the properties of the objective functions.

3.1 Properties of $M(x), SAWD(x)$, and $G(x)$

The general solution approach to optimizing the objective functions on $A(G)$ is based on decomposing the network into edges, identifying the characterizing points on each edge, and solving a restricted problem. Characterizing points are the points where direction changes of the components of the objective functions can occur (see [27] and [21]). We now describe various properties of the objective functions and how the characterizing points can be defined.

For tree networks, it is known [27] that the distance function, d_i, for any node i is linear along any edge of the network. Hence, d_i is minimized at one of the two end vertices (nodes) on a given edge, and it follows that $M(x)$ is also minimized at a node within the network. Also, for a given pair of nodes i and j, the functions $w_i d_i$ and $w_j d_j$ can only intersect once on a given edge. Therefore, the absolute difference between weighted distances, $|(w_i d_i - w_j d_j)|$, can decrease to the value zero only once (if at all) along the edge.

Definition 1 *The set H_e of characterizing points on an edge e are defined by the union of the intersection points and the end vertices that define the edge.*

The absolute weighted differences are clearly convex on each edge, and as such the function $SAWD(x)$ is also convex over each edge, with the minimum occurring at one of the intersection points on the edge where the function changes slope. We summarize these properties in the following propositions.

Proposition 1 *The median function $M(x)$ is linear over any edge of the network, and is minimized at a node of the tree network.*

Proposition 2 *The SAWD function $SAWD(x)$ is convex over any edge of the network, and is minimized at one of the points belonging to the set H_e.*

The above property enables us to restrict the search for the location that minimizes $SAWD(x)$ to the union of H_e over all edges of the network (see Mesa *et al.* [27]).

We now examine the properties of $G(x)$, which is the ratio of $SAWD(x)$ and $M(x)$. Given that over an edge e $SAWD(x)$ is convex and piecewise linear, and $M(x)$ is linear, $G(x)$ can be shown to be quasiconvex over e.

Definition 2 *[5] A function $f : \mathbb{R}^n \to \mathbb{R}$ is quasiconvex if its domain and all its sublevel sets*

$$P_\alpha = \{x \in \mathbf{dom}\ f : f(x) \leq \alpha\}\ \alpha \in \mathbb{R} \tag{4}$$

are convex.

Moreover, $G(x)$ will achieve a minimum at one of the points belonging to the characterizing set H_e. This property enables us to restrict the search for the location that minimizes $G(x)$ on the tree network, again as in the case for $SAWD(x)$, to the union of H_e over all edges of the network. These properties are summarized in the following proposition.

Proposition 3 *The Gini function $G(x)$ is quasiconvex over each edge of the network, and is minimized at one of the points belonging to the set $H(e)$.*

Proof. To show that the Gini function (3) with domain $\mathbf{dom}\ f = \left\{x : \sum_{i=1}^{n} w_i d(i,x) > 0\right\}$ is quasiconvex, we shall prove that its lower sublevel sets P_α (4) are convex. Let $D \in \mathbb{R}_+^{n\times n}$ denote the matrix distance with elements $d_{ij} = w_i d(i,x) - w_j d(j,x), i, j = 1, \ldots, n$. The function $SAWD(x)$ (2) can be rewritten as

$$SAWD(x) = \|D\|_F = \sum_{i=1}^{n}\sum_{j=1}^{n} |d_{ij}|\ .$$

Thus, $SAWD(x)$ is the sum-absolute-value matrix or Frobenius norm $\|D\|_F$ of D, and is a convex function. The lower sublevel set of the Gini function for any $\alpha \in \mathbb{R}$ is given by

$$P_\alpha = \left\{ x : \sum_{i=1}^{n} w_i d(i,x) > 0,\ \frac{\|D\|_F}{2n \cdot \sum_{i=1}^{n} w_i d(i,x)} \leq \alpha \right\} = \left\{ x : \sum_{i=1}^{n} w_i d(i,x) > 0,\ \|D\|_F \leq 2n\alpha \cdot \sum_{i=1}^{n} w_i d(i,x) \right\}\ ,$$

and is convex. Indeed, it is the intersection of a open halfspace with the convex set defined by the nonlinear inequality $\|D\|_F \leq 2n\alpha \cdot \sum_{i=1}^{n} w_i d(i,x)$.
Further, over each sub-edge defined by a pair of adjacent characterizing points, $G(x)$ is the ratio of two linear functions, which is either increasing or decreasing over the entire sub-edge. Thus, $G(x)$ is minimized at one of the end points of each sub-edge, which is essentially the set H_e of characterizing points on the edge. The proposition follows. □

The quasiconvexity concept is an extension of the notion of unimodality for functions with a single real argument. The quasiconvexity of $G(x)$ implies that once $G(x)$ starts increasing on an edge, it will continue to increase along the rest of the edge. We use this property to characterize efficient solutions in the next section.

3.2 Characterizing Efficient Solutions

Given the properties discussed in the previous subsection, we can now characterize the efficient solutions to Problems **P1** and **P2**. Over each edge, we can identify the locations that are dominated with respect to the effectiveness and equity criteria.

Over an edge $e = (i, j)$, let s^*_{ij} be the point where $SAWD(x)$ is minimized, g^*_{ij} be the point where $G(x)$ is minimized, and m^*_{ij} be the point where $M(x)$ is minimized. Based on the properties discussed in the previous section, it is clear that s^*_{ij} and g^*_{ij} can be identified from the characterizing set H_e. Without loss of generality, assume that $M(i) \leq M(j)$. In other words the median objective function value is non-decreasing over the edge $e = (i, j)$ from i to j, and m^*_{ij} is at node i. The edge can now be divided into segments, the segment from node i to s^*_{ij}, and the segment from s^*_{ij} to j. Denote the former as S^1_{ij} and the latter as S^2_{ij}, noting that S^1_{ij} cannot be empty and at least contains node i. We can similarly define the sets G^1_{ij} and G^2_{ij} with respect to the point g^*_{ij}. Based on the properties of $SAWD(x)$ and $G(x)$ discussed in the previous section, we know that $s^*_{ij} \in H_e$ and $g^*_{ij} \in H_e$.
We now formally define the concept of edge efficiency, and describe how edge efficient points can be identified.

Definition 3 *Points on an edge are termed edge efficient if they are not dominated by any other points on the same edge. In other words, a solution $x' \in e$ to problem* **P1** *(or* **P2***) is considered edge-efficient if there is no other point $x \in e$ such that $M(x) \leq M(x')$ and $SAWD(x) \leq SAWD(x')$ with strict inequality holding in at least one (or $M(x) \leq M(x')$ and $G(x) \leq G(x')$, again with strict inequality holding in at least one).*

Lemma 1 *With respect to Problem* **P1***, all points belonging to the segment S^2_{ij} of an edge (i, j), if non-empty, are dominated by the point s^*_{ij}. Similarly, with respect to Problem* **P2***, all points belonging to the segment G^2_{ij} of an edge (i, j), if non-empty, are dominated by the point g^*_{ij}.*

The above follows from the fact that both $M(x)$ and $SAWD(x)$ are non-decreasing over S^2_{ij}, and $M(x)$ and $G(x)$ are non-decreasing over G^2_{ij}.

Lemma 2 *The set of points S^1_{ij} is edge efficient with respect to edge (i, j). Also, the set of points G^1_{ij} is edge efficient with respect to edge (i, j).*

Thus, points in the segment S^1_{ij} (G^1_{ij}) are candidates for the efficient set for Problem **P1**(**P2**), and $S^1_{ij}(G^1_{ij})$ is defined as a segment consisting of efficient sub-edges. The set of efficient solutions to **P1**(**P2**) over the entire tree network, TE(**P1**) (TE(**P2**)), can be defined as a subset of the union of efficient sub-edges, i.e., TE(**P1**) $\subseteq \cup_{ij}\{S^1_{ij}\}$ (TE(**P2**) $\subseteq \cup_{ij}\{G^1_{ij}\}$). Also, based on the definitions of S^1_{ij} and S^2_{ij}, we have the following:

Proposition 4 *If $S^1_{ij}(G^1_{ij})$ contains only node i, which implies that $M(x)$ and $SAWD(x)(G(x))$ are both minimized at node i on edge (i, j), then node i is the only edge efficient point on the edge with respect to* **P1***(***P2***).*

Proposition 5 *If $S^2_{ij}(G^2_{ij})$ is empty, which implies that $M(x)$ and $SAWD(x)(G(x))$ are minimized at nodes i and j respectively, the entire edge is edge efficient with respect to* **P1***(***P2***).*

Further, the relationship between the efficient sub-edges for problems **P1** and **P2** is stated in the following proposition. The proposition follows from the fact that a solution that is efficient with respect to **P1** is also efficient with respect to **P2**.

Proposition 6 *On any given edge (i, j), the efficient sub-edge S^1_{ij} is a subset of the efficient sub-edge G^1_{ij}, i.e., $S^1_{ij} \subseteq G^1_{ij}$.*

Theorem 1 *The objective function of problem* **P3** *is convex over each edge of the network and is minimized at one of the efficient (with respect to* **P1***) characterizing points on the edge.*

Proof. The objective function of **P3** is a linear combination of the linear function $M(x)$ and the convex function $SAWD(x)$ (see Proposition 2), and is therefore also convex (sum of convex functions is convex). Note that $SAWD(x)$ is piecewise linear (since it is the sum of absolute differences of linear functions), and that it changes slope at each characterizing point on the edge. Thus, the objective function of **P3** would also be piece-wise linear, convex, with changing slope at each characterizing point on the edge. This ensures that **P3** is minimized at one of the efficient points in the characterizing set H_e. □

From the above theorem we can also conclude that the optimal solution over the network can be found by searching over H_e over all edges. We now state a formal result regarding the objective function of **P4**.

Theorem 2 *The objective function of problem* **P4** *is quasiconvex over each edge of the network.*

Proof. Let $f(x) = \lambda_m M(x) + \lambda_g G(x)$ with $\mathbf{dom}\ f = \left\{x : \sum_{i=1}^{n} w_i d(i, x) > 0\right\}$ and $\lambda_m, \lambda_g \geq 0$. The quasi-convexity of $f(x)$ follows from the convexity of its lower sublevel sets P^f_α. Using the notations of Proposition 3, we have

$$P^f_\alpha = \left\{x : \sum_{i=1}^{n} w_i d(i, x) > 0,\ \lambda_m \cdot \sum_{i=1}^{n} w_i d(i, x) + \lambda_g \frac{\|D\|_F}{2n \cdot \sum_{i=1}^{n} w_i d(i, x)} \leq \alpha \right\}$$

$$= \left\{x : \sum_{i=1}^{n} w_i d(i, x) > 0,\ \lambda_g \cdot \|D\|_F + 2n\lambda_m \cdot \left(\sum_{i=1}^{n} w_i d(i, x)\right)^2 \leq 2n\alpha \cdot \sum_{i=1}^{n} w_i d(i, x)\right\} . \tag{5}$$

The left-hand side of the inequality

$$\lambda_g \cdot \|D\|_F + 2n\lambda_m \cdot \left(\sum_{i=1}^{n} w_i d(i, x)\right)^2 \leq 2n\alpha \cdot \sum_{i=1}^{n} w_i d(i, x) \tag{6}$$

is a linear combination of the convex functions $\|D\|_F$ and $(M(x))^2 = \left(\sum_{i=1}^{n} w_i d(i, x)\right)^2$ and is hence a convex function itself. Thus, the feasible set defined by (6) is convex. Indeed, (6) requires a convex function to be smaller than or equal to the linear expression $2n\ \alpha \cdot \sum_{i=1}^{n} w_i d(i, x)$. Therefore, any lower sublevel set P^f_α (5) is convex, since it is the intersection of a open halfspace with a convex set. □

It should be noted that, unlike for **P3**, the optimal solution for **P4** can exist within a sub-edge defined by a pair of efficient adjacent characterizing points. This implies that the search for an optimal solution to **P4** cannot be restricted to the set of efficient characterizing points as in the solution for **P3**. However, the quasiconvexity of the objective function enables us to do an efficient search over the efficient sub-edges of each edge to identify the optimal solution to **P4**. For example, if the objective function is increasing along a particular sub-edge, then all the successive sub-edges in the same direction of the increase can be eliminated from the search.

3.3 Complexity of Solution Procedures

The procedures we develop to identify the efficient solutions on the network are based on the characterizing points defined on each edge of the network. In other words, the efficient sub-edges on a given edge are completely defined by a subset or perhaps all of the characterizing points on that edge. As such, the complexity of the search procedures is a direct function of the size of the characterizing set of solutions on the network. Some previous work that is relevant in this regard are the complexity results derived for procedures to identify the optimal solutions with respect to a variety of equality measures (see Lopez de los Mozos *et al.* [21] for an extensive discussion). Procedures have been developed to identify the optimal solution to both the $SAWD(x)$ and Gini measures on tree networks whose complexity is shown to be $O(n^2 \log^2 n)$ - this is an improvement over the algorithm by Maimon [23] which is of the $O(n^3 \log n)$ for the optimal Gini location. The search procedures take advantage of the properties of the objective functions and do not require enumeration of all the characterizing points on a given edge to identify the optimal location. However, since our objective is to identify efficient solutions, our search procedure entails a little more work in terms of identifying all the characterizing points on a given edge, and not just the optimal one with respect to the equality measure. Note that there are upto $O(n^2)$ possible characterizing points on a given edge, for which the computation and sorting of the objective function values could take $O(n^2 \log n)$ steps. Doing this across all edges of the tree network results in the complexity being $O(n^3 \log n)$, leading us to the following result.

Proposition 7 *The computational complexity of the procedure to identify all the edge efficient sub-edges on the tree network is* $O(n^3 \log n)$.

3.3.1 Efficient Solutions over the Entire Tree Network

The search for efficient solutions over the entire tree network can be restricted to the set of edge efficient sub-edges on each edge. Let $EESE(\mathbf{P1})$ represent the set of Edge Efficient Sub-Edges for **P1**, and similarly $EESE(\mathbf{P2})$ for **P2**. Identifying the efficient solutions over the entire tree network can be challenging since an edge efficient sub-edge can end up being either completely or partially dominated by some other points on a different edge. In other words, the efficient set over the network can be comprised of a number of partial edge efficient sub-edges belonging to different edges. As such, we attempt to first identify the set of edge efficient sub-edges that cannot be efficient over the entire network from the set of all edge efficient sub-edges. Such sub-edges are defined by adjacent characterizing points that are both dominated by the same point on some other edge. For example, if x' and x'' correspond to the two adjacent points, with corresponding objective function values $(M(x'), SAWD(x'))$ and $(M(x''), SAWD(x''))$. A point x dominates both x' and x'', as long as $M(x) \leq M(x'), SAWD(x) \leq SAWD(x')$, and $M(x) \leq M(x''), SAWD(x) \leq SAWD(x'')$. These conditions imply that $M(x) \leq \min(M(x'), M(x'')) \leq M(y), SAWD(x) \leq \min(SAWD(x'), SAWD(x'')) \leq SAWD(y)$, where y corresponds to any point on the sub-edge joining x' and x''. The second set of inequalities in these conditions result from the fact that the sub-edge is efficient. Thus, x dominates the entire sub-edge, and we are able to state formally the following proposition.

Proposition 8 *If two adjacent characterizing points that define an edge efficient sub-edge are dominated by a point x on the network, then the entire sub-edge is dominated and can be removed from further consideration.*

Sub-edges that can be eliminated in this fashion are defined as being Tree-Dominated. These edges can be removed from further consideration, and the search be restricted to the remaining sub-edges in $EESE(\mathbf{P1})$ and $EESE(\mathbf{P2})$, namely $EESE^*(\mathbf{P1})$ and $EESE^*(\mathbf{P2})$. In other words, the efficient set of solutions over the tree network, $TE(\mathbf{P1})$ and $TE(\mathbf{P2})$ (Tree Efficient solution sets) are subsets of $EESE^*(\mathbf{P1})$ and $EESE^*(\mathbf{P2})$ respectively, i.e., $TE(\mathbf{P1}) \subseteq EESE^*(\mathbf{P1})$, and $TE(\mathbf{P2}) \subseteq EESE^*(\mathbf{P2})$. We propose the use of the characterizing points for the sub-edge dominance testing procedure. The complexity of such a testing procedure is $O(n^6)$ since there are $O(n^3)$ sub-edges which are tested for dominance by any of the $O(n^3)$ characterizing points.

One approach to identify an approximation of the tree efficient solutions is to discretize the sets $EESE^*(\mathbf{P1})$ and $EESE^*(\mathbf{P2})$, and identify the efficient solutions among the discrete sets. An even faster, but perhaps looser, approximation can be found using the efficient characterizing points as the discrete set of points. These approximations are described next.

3.3.2 A Discrete Approximation of the Tree Efficient Set

In many location problems, it is desirable to restrict location to a set of finite sites such as the vertices (population centers) or some such other discrete set of locations. As such, it would be relevant to consider the efficient set of solutions to our problems within such a restricted set. We consider two such discrete sets of locations among which we identify efficient solutions: the set of vertices and the set of characterizing points.

Location restricted to Vertices

When location is restricted to vertices, the efficient set can be at one extreme a singleton (when the objective function values are optimized at the same vertex), or the other extreme when the entire set of vertices is efficient. Once the objective function values are computed at each of the vertices, the efficient set can be identified by eliminating those vertices that are dominated by other vertices. This procedure can be done in $O(n^2)$ steps, since checking for dominated vertices can take at most $n(n-1)/2$ comparisons, resulting in the following proposition.

Proposition 9 *When locations are restricted to the set of vertices (nodes), the efficient set of solutions can be computed in $O(n^2)$ steps.*

Location restricted to Characterizing Points

When location is restricted to the characterizing points on the network, the efficient set of solutions on the entire network can be found by eliminating those that are dominated by other solutions in the same set. Since there are $O(n^3)$ characterizing points on the network, the efficient set can be computed in $O(n^6)$ steps.

Proposition 10 *When locations are restricted to the set of characterizing points, the efficient set of solutions can be computed in $O(n^6)$ steps.*

3.4 Procedure for Finding Efficient Sets Using Characterizing Points

Based on the properties of the functions described in the previous sub section, we now outline the procedure for finding the efficient sets for Problems **P1** and **P2**. The procedure is based on first identifying the efficient sub-edges on each edge, and subsequently determining which of these are possible candidates for efficient solutions over the entire tree network. The procedure described below is illustrated on an example in the next section.

Step 1: Determine Characterizing Points
Compute the set H_e of characterizing points for each edge and sort the characterizing points with respect to $M(x)$, $SAWD(x)$, and $G(x)$.
As explained in the previous section, the number of characterizing points on a given edge is $O(n^2)$, and the complexity of this step is $O(n^3 \log n)$.

Step 2: Determine Efficient Sub-Edges
For each edge, compute the points where $M(x), SAWD(x)$, and $G(x)$ are minimized, and define the segments S^1_{ij} and G^1_{ij}. In each of these segments, identify efficient sub-edges defined by a pair of adjacent efficient characterizing points.
It is possible that on a given edge, there are no edge efficient sub-edges as is the case when, say, $M(x)$ and $SAWD(x)$ are minimized at the same node. Or the entire edge may be efficient if the two objectives are minimized at the opposite ends of the edge in consideration. We illustrate the number of sub-edges that are eliminated in the example used in the next section.

Step 3: Determine Efficient Sub-Edges Over the Tree
A candidate set of tree efficient points (TE(**P1**) and TE(**P2**)) is identified from the set of efficient sub-edges on each edge of the network. An edge efficient sub-edge is eliminated as being inefficient with respect to the tree network if both the characterizing points defining the sub-edge are dominated by the same point on a different edge. The efficient sub-edges that remain after this step are candidate sets for the efficient set over the entire tree network.
Again, as explained in the previous section, we propose the use of the characterizing points for the sub-edge dominance testing procedure. The complexity of such a testing procedure is $O(n^6)$ since there are $O(n^3)$ sub-edges which are tested for dominance by any of the $O(n^3)$ characterizing points.

Step 4: Discrete Approximation of Efficient Set Using the Characterizing Points
Identify the characterizing points belonging to the segments S^1_{ij} and G^1_{ij}, and call these sets S^h_{ij} and G^h_{ij} respectively. A discrete approximation of the efficient sets for **P1** and **P2** over the tree network is identified by searching over the sets, $\cup_{ij}\{S^h_{ij}\}$ and $\cup_{ij}\{G^h_{ij}\}$, respectively.
As described in the previous section, when locations are restricted to the set of characterizing points, the efficient set of solutions can be computed in $O(n^6)$ steps.

It is clear based on the procedure described in steps 1 through 3, that any edge efficient solution that is discarded is one that is dominated by another edge efficient solution on a different edge, and as such, the resulting solution set is guaranteed to contain the tree efficient set of solutions. We can thus state the following proposition regarding the above procedure.

Proposition 11 *The procedure described in steps 1 through 3 results in a solution set that will contain a possibly reduced set of edge efficient solutions that is guaranteed to contain the tree efficient set of solutions.*

4 An Example

We illustrate the procedure of identifying efficient points using the sample tree network shown in Figure 1. The weights on the nodes and edge lengths are as shown in Figure 1.

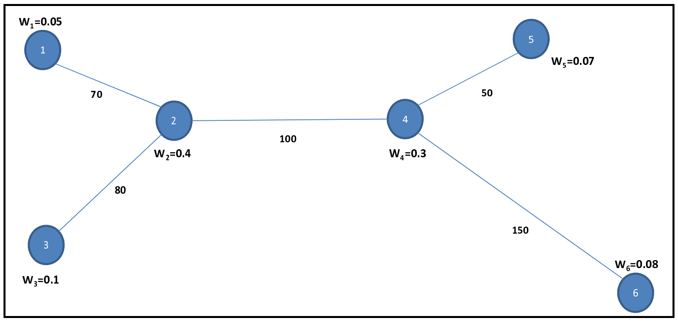

Figure 1: Sample Network

Table 1 shows the results obtained from the procedure discussed in the previous section to identify the efficient sub-edges of the tree network. For each edge (i, j), the characterizing points on the edge are identified by labels shown in column 3 of the table, where the distance from each point to node i is shown in column 2 of the table. Columns 4, 5, and 6 contain the $M(x)$, $SAWD(x)$, and $G(x)$ values respectively, for the location corresponding to a given characterizing point. Columns 7 and 8 identify if the corresponding characterizing point is edge efficient (1) or not (0) with respect to Problems **P1** and **P2** respectively. In other words, the efficient sub-edges on a given edge can be determined by the union of the sub-edges defined by the adjacent efficient characterizing points. These efficient sub-edges are illustrated for edge $(1, 2)$ in Figure 2 for Problems **P1** and **P2**. Notice that only the sub-edge between points labeled 5 and 6 is edge efficient in **P1**, indicated by the set S^1_{12} in the Figure 2. This is due to the fact that $M(x)$ is minimized at node 2 (label 6), and $SAWD(x)$ is minimized at the point labeled 5. With respect to problem **P2**, we can see that the edge-efficient set G^1_{12} is the union of sub-edges identified by the labels 3 through 6. Again, in this case, we can see that $G(x)$ is minimized on edge $(1, 2)$ at the point indicated by label 3, and $M(x)$ is minimized at node 2 (label 6). As such, the set G^1_{12} is edge efficient and contains the points between labels 3 and 6. We can also see that, as stated in Proposition 10, the efficient sub-edge S^1_{ij} is a subset of the efficient sub-edge G^1_{ij}. The efficient sub-edges on the other edges can be identified in a similar fashion. Table 2 shows the number of sub-edges eliminated on each edge based on Steps 1 and 2 of the procedure.

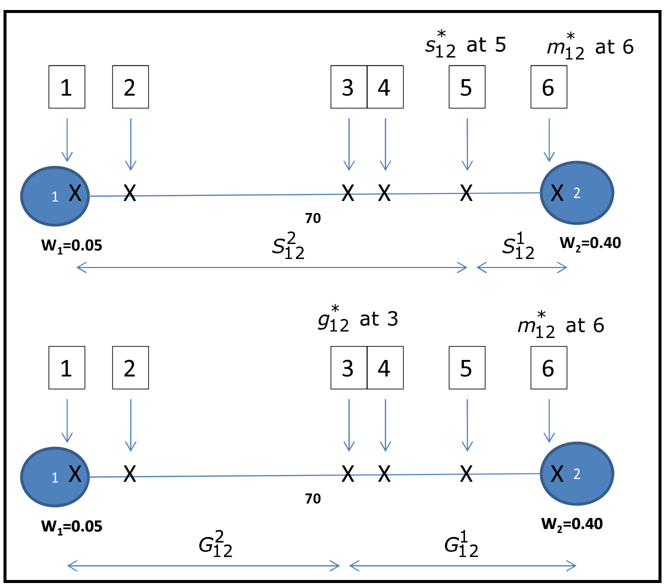

Figure 2: Edge Efficient Sets on Edge $(1, 2)$ for Problems **P1** and **P2**

Edge (i, j)	Dist. from i	Label	$M(x)$	$SAWD(x)$	$G(x)$	**P1** Edge Eff.	**P2** Edge Eff
(1,2)	0.00	1	135.00	304.20	2.25	0	0
(1,2)	7.50	2	128.25	284.25	2.22	0	0
(1,2)	38.18	3	100.64	222.27	2.21	0	1
(1,2)	43.33	4	96.00	215.27	2.24	0	1
(1,2)	62.22	5	79.00	200.91	2.54	1	1
(1,2)	70.00	6	72.00	202.00	2.81	1	1
(2,3)	0.00	1	72.00	202.00	2.81	1	1
(2,3)	10.00	2	80.00	199.60	2.50	1	1
(2,3)	16.00	3	84.80	202.36	2.39	0	1
(2,3)	30.00	4	96.00	222.80	2.32	0	1
(2,3)	31.82	5	97.45	226.00	2.32	0	1
(2,3)	62.50	6	122.00	300.25	2.46	0	0
(2,3)	80.00	7	136.00	353.80	2.60	0	0
(2,4)	0.00	1	72.00	202.00	2.81	1	1
(2,4)	10.00	2	73.00	161.40	2.21	1	1
(2,4)	14.71	3	73.47	145.59	1.98	1	1
(2,4)	22.34	4	74.23	122.53	1.65	1	1
(2,4)	26.67	5	74.67	113.53	1.52	1	1
(2,4)	41.67	6	76.17	91.33	1.20	1	1
(2,4)	42.86	7	76.29	90.71	1.19	1	1
(2,4)	45.45	8	76.55	93.00	1.21	0	0
(2,4)	55.00	9	77.50	105.60	1.36	0	0
(2,4)	58.33	10	77.83	112.67	1.45	0	0
(2,4)	66.67	11	78.67	132.33	1.68	0	0
(2,4)	75.71	12	79.57	156.94	1.97	0	0
(2,4)	84.78	13	80.48	187.96	2.34	0	0
(2,4)	100.00	14	82.00	247.00	3.01	0	0
(4,5)	0.00	1	82.00	247.00	3.01	1	1
(4,5)	9.46	2	90.14	256.84	2.85	0	1
(4,5)	34.00	3	111.24	300.52	2.70	0	1
(4,5)	50.00	4	125.00	337.00	2.69	0	1
(4,6)	0.00	1	82.00	247.00	3.01	1	1
(4,6)	15.22	2	94.78	254.00	2.68	0	1
(4,6)	26.92	3	104.62	264.77	2.53	0	1
(4,6)	31.58	4	108.53	270.26	2.49	0	1
(4,6)	34.00	5	110.56	274.96	2.48	0	1
(4,6)	56.67	6	129.60	330.27	2.55	0	0
(4,6)	90.00	7	157.60	421.60	2.68	0	0
(4,6)	150.00	8	208.00	610.00	2.93	0	0

Table 1: Characterizing Points

Edge (i, j)	Num. of Sub Edges	Num Sub-Edges Elim.**P1**	Num Sub-Edges Elim.**P2**
(1,2)	5	4	2
(2,3)	6	5	2
(2,4)	13	7	7
(4,5)	3	3	0
(4,6)	7	7	3

Table 2: Number of Sub-Edges Eliminated

The efficient set over the entire network with respect to the objective functions can now be identified based on the efficient sub-edges which are defined by the characterizing points on each edge. The characterizing points are shown for problems **P1** and **P2** in Figures 3 and 4, and the corresponding edge efficient sub-edges are shown in Figures 5 and 6. A discrete approximation of the efficient edges can be constructed using the efficient characterizing points, and based on this it can be seen that the edge efficient characterizing points of edge $(2, 4)$ are the only tree efficient characterizing points. Also, as can be seen from the figures, most of the edge efficient sub-edges are tree dominated by the edge efficient characterizing points on edge $(2, 4)$, with only the edge efficient sub-edges of edge $(2, 4)$, and the edge efficient sub-edges between points labeled 1 and 2 on edge $(2, 3)$, and points labeled 5 and 6 on edge $(1, 2)$ remaining as the candidate efficient solutions over the entire network ($EESE^*_S$ and $EESE^*_G$) for both problems **P1** and **P2**. In observing the plots of these efficient sub-edges in Figures 5 and 6, we can see that only sub-edges on edge $(2, 4)$ are efficient over the tree network and define the sets TE(**P1**) and TE(**P2**).

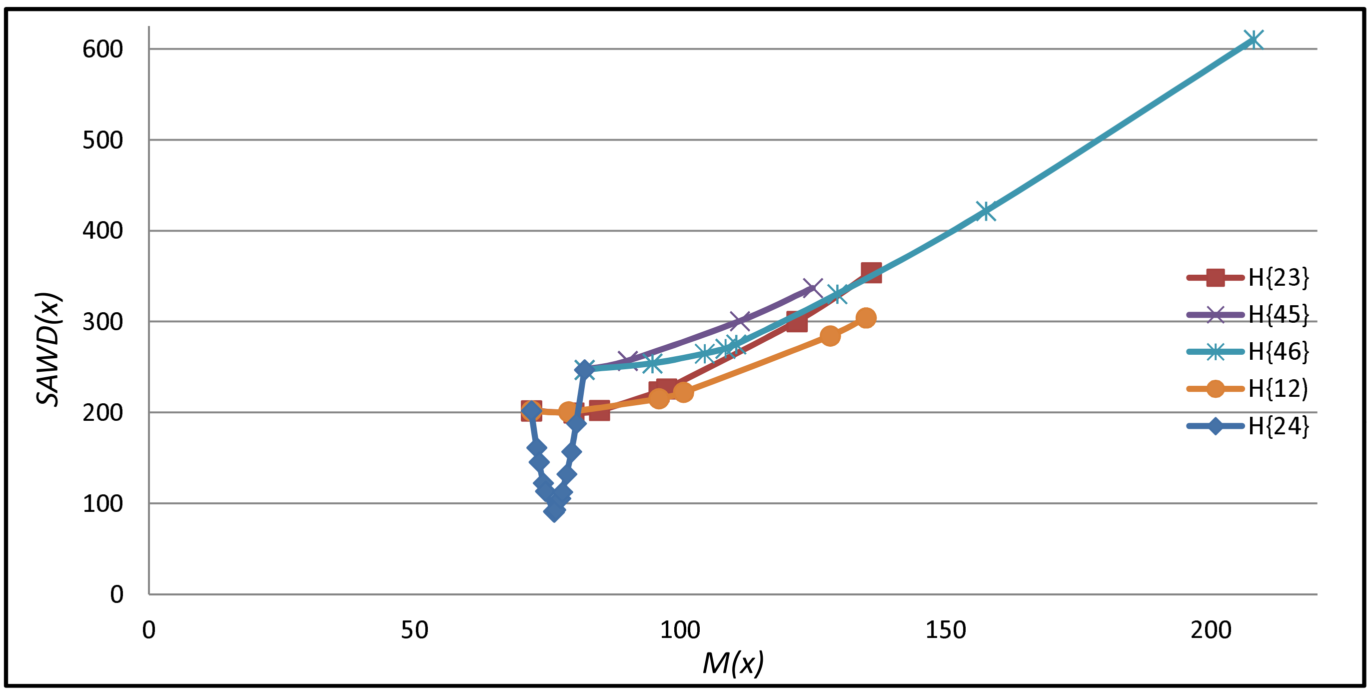

Figure 3: Characterizing Points for Problem **P1**

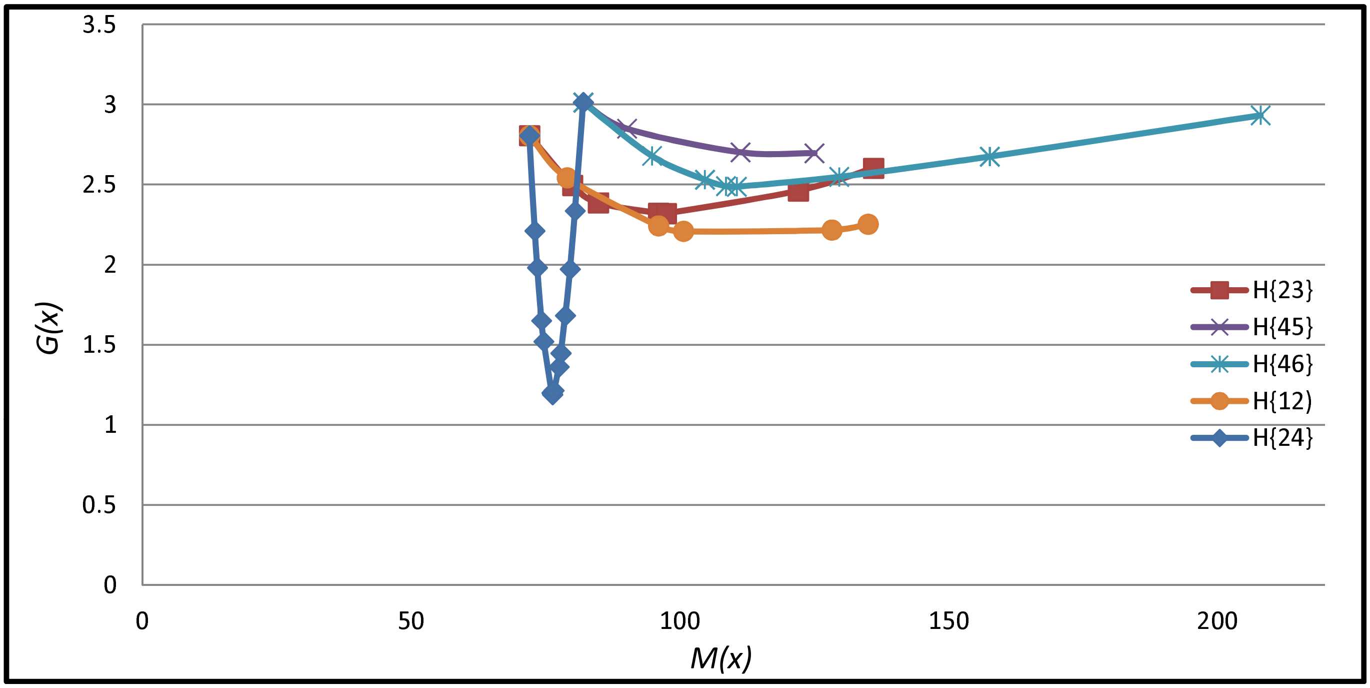

Figure 4: Characterizing Points for Problem **P2**

With respect to both Problems **P3** and **P4**, we can see that the optimal solution, for some non-negative weights used in the respective objective functions, would belong to the set of efficient sub-edges of edge $(2, 4)$. In the case of **P3**, the optimal solution on the tree network would be one of the efficient characterizing points on edge $(2, 4)$. However, the optimal solution for problem **P4** could be within one of the efficient sub-edges of edge $(2, 4)$. For example, using $\lambda_m = 0.3$ and $\lambda_s = \lambda_g = 0.5$ results in an optimal solution for **P3** at the point labeled 7 on edge $(2, 4)$, and an optimal solution for **P4** at a point on the sub-edge between points labeled 1 and 2 on edge $(2, 4)$ at about a distance of 2 units from node 2.

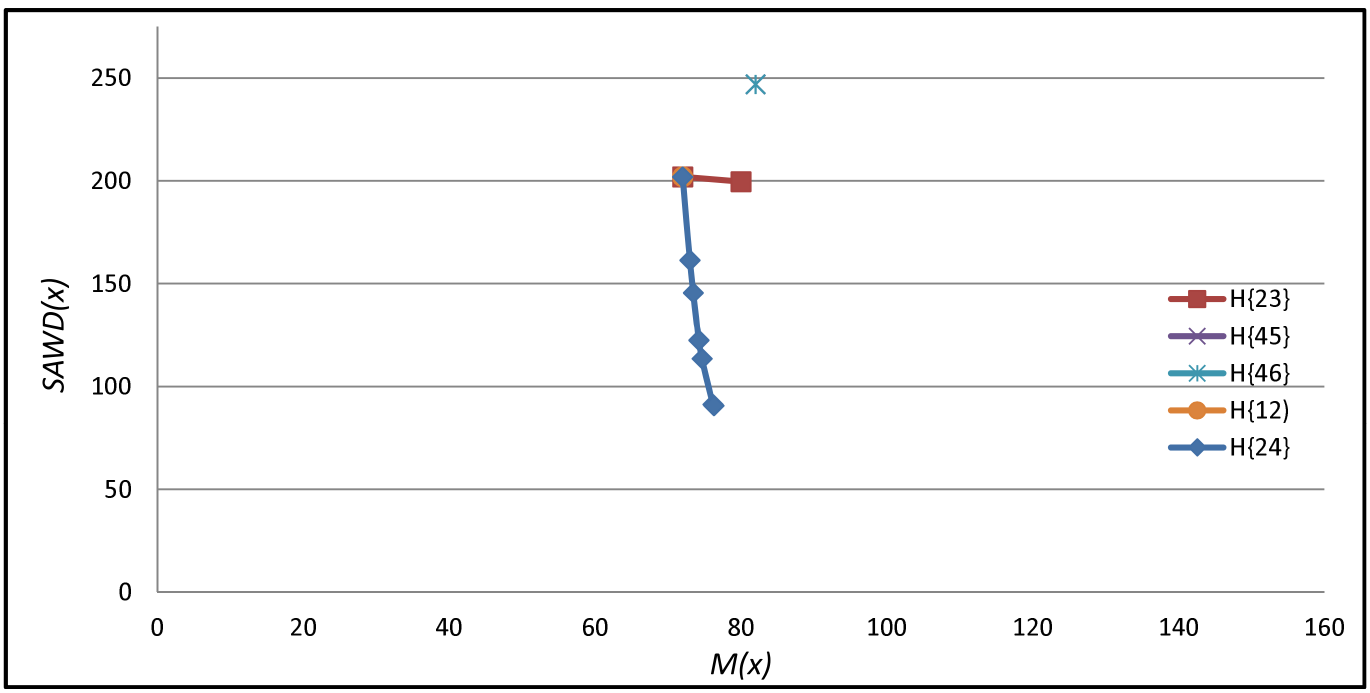

Figure 5: Efficient Sub-Edges for Problem **P1**

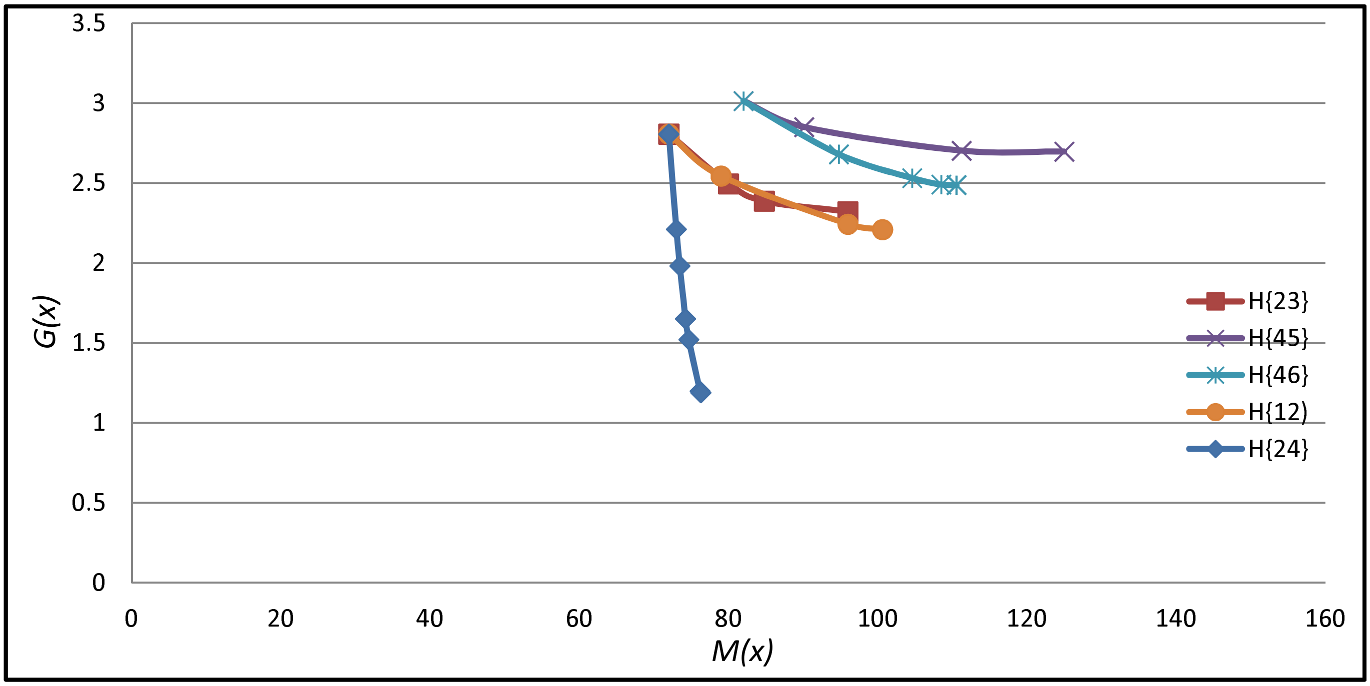

Figure 6: Efficient Sub-Edges for Problem **P2**

5 Conclusions

In this paper, we consider facility location models incorporating effectiveness and equity measures, and propose ways to characterize the efficient solutions to such problems. Specifically, we develop procedures to identify the efficient sets for a one facility location problem on a tree network incorporating the median and a couple of equity measures based on absolute differences, and illustrate the procedures with an example. Extensions to the models proposed in this work would include models for general network contexts, and also problems involving multiple facilities. Also, it would be worthwhile to compare the proposed models with effectivness-equity models that employ other measures of equity (such as the range and variance).

Acknowledgment: M. Lejeune is grateful for the support provided by the Army Research Office under Grant # W911NF-09-1-0497.

References

[1] Allison P. 1978. Measures of Inequality. *American Sociological Review* 43 (6): 865-880.

[2] Baron O., Berman O., Krass D., Wang Q. 2007. The Equitable Location Problem on the Plane. *European Journal of Operational Research* 90: 183-578.

[3] Berman O. 1990. Mean-Variance Location Problems *Transportation Science* 24 (4): 287-293.

[4] Berman O., Drezner Z., Tamir A., Wesolowsky G.O. 2009. Optimal Location with Equitable Loads. *Annals of Operations Research* 167: 307-326.

[5] Boyd S., Vandenberghe L. 2004. *Convex Optimization*. Cambridge University Press.

[6] Cáceres T., Cruz Lopez-de-los-Mozos M., Mesa J.A. 2004. The Path-Variance Problem on Tree Networks. *Discrete Applied Mathematics* 145: 72-79.

[7] Current J., Daskin M., Schilling D. 2002. Discrete Network Location Problems. In: *Facility Location: Applications and Theory*. Eds: Drezner Z., Hamacher H. Berlin: Springer, 81-118.

[8] Current J., Min H., Schilling D. 1990. Multiobjective Analysis of Facility Location Decisions. *European Journal of Operational Research* 49 (3): 295-307.

[9] Daskin M.S. 1995. *Network and Discrete Location: Models, Algorithms, and Applications*. Wiley, Interscience, New York, NY.

[10] Drezner Z. 1986. Location of Regional Facilities. *Naval Research Logistics Quarterly* 33: 523-529.

[11] Drezner T., Drezner Z., Guyse J. 2009. Equitable Service by a Facility: Minimizing the Gini Coefficient. *Computers & Operations Research* 36: 3240-3246.

[12] Ehrgott M., Wiecek M.M. 2005. Multiobjective Programming. In: *Multiple Criteria Decision Analysis: State of the Art Surveys*. Eds: Figueira J., Greco S., Ehrgott M. Springer Verlag, 667-722.

[13] Eiselt H.A., Laporte G. 1995. Objectives in Location Problems. *In: Facility Location: A Survey of Applications and Methods*. Ed.: Drezner Z. Springer, Berlin, 151-180.

[14] Erkut E. 1993. Inequality Measures for Location Problems. *Location Science* 1 (3): 199-217.

[15] Espejo I., Marn A., Puerto J., Rodrguez-Cha A. 2009. A Comparison of Formulations and Solution Methods for the Minimum-Envy Location Problem. *Computers and Operations Research* 36: 1966-1981.

[16] Farahania R.Z., SteadieSeifib M., Asgaria N. 2010. Multiple Criteria Facility Location Problems: A Survey. *Applied Mathematical Modelling*, 34 (7): 1689-1709.

[17] Halpern J., Maimon O. 1983. Accord and Conflict Among Several Objectives in Locational Decisions on Tree Networks. *In: Locational Analysis of Public Facilities*. Eds: Thisse J.F., Zoller H.G. Series in Mathematica and Managerial Economics, North Holland, 301-314.

[18] Hamacher H.W., Labbé M., Nickel S. 1999. Multicriteria Network Location Problems with Sum Objectives. *Networks* 33(2): 79-92.

[19] Leclerc P.D., McLay L.A. 2012. Modeling Equity for Allocating Public Resources. In: *Community-Based Operations Research*. Ed: Johnson M.P. International Series in Operations Research & Management Science, Volume 167, Part 1, 97-118.

[20] Lopez-de-los-Mozos M.C., Mesa J.A. 2001. The Maximum Absolute Deviation Measure in Location Problems on Networks. *European Journal of Operational Research* 135 (7): 184-194.

[21] Lopez de los Mozos M.C., Mesa J.A., Puerto J. 2008. A Generalized Model of Equality Measures in Network Location Problems. *Computers* & *Operations Research* 35: 651-660.

[22] Maimon O. 1986. The Variance Equity Measure in Locational Decision Theory. *Annals of Operations Research* 6: 147-160.

[23] Maimon O. 1988. An Algorithm for the Lorenz Measure in Locational Decisions on Trees. *Journal of Algorithms* 9: 583-596.

[24] Maliszewski P.J., Kuby M.J., M.W. Horner. 2012. A Comparison of Multi-Objective Spatial Dispersion Models for Managing Critical Assets in Urban Areas. To appear in *Computers, Environment and Urban Systems*.

[25] Mandell M.B. 1991. Modelling Effectiveness-Equity Trade-Offs in Public Service Delivery Systems. *Discrete Applied Mathematics* 37 (4): 467-482.

[26] Marsh M.T., Schilling D.A. 1994. Equity Measurement in Facility Location Analysis: A Review and Framework. *European Journal of Operational Research* 74 (1): 1-17.

[27] Mesa J.A., Puerto J., Tamir A. 2003. Improved Algorithms for Several Network Location Problems with Equality Measures. *Discrete Applied Mathematics* 130: 437-448.

[28] Nickel S., Puerto J. 1999. A Unified Approach to Network Location Problems. *Networks*, 34:283290.

[29] Nickel S., Puerto J., Rodrguez-Cha A.M. 2005. MCDM Location Problems. *In: Multiple Criteria Decision Analysis: State of the Art Surveys*. Eds: Figueira J., Greco S., Ehrogott M. International Series in Operations Research & Management Science Volume 78, 761-787.

[30] Ogryczak W. 2000. Inequality Measures and Equitable Approaches to Location Problems. *European Journal of Operational Research* 122 (2): 374-39.

[31] Ogryczak W. 2009. Inequality Measures and Equitable Locations. *Annals of Operations Research* 167: 61-86.

[32] Ohsawa,Y., Ozaki,N., and Plastria, F. 2008. Equity-Efficiency Bicriteria Location with Squared Euclidean Distances *Operations Research* 56 (1), 79-87.

[33] Puerto J., Ricca F., Scozzari A. 2009. Extensive Facility Location Problems on Networks with Equity Measures. *Discrete Applied Mathematics* 157 (5): 1069-1085.

[34] Steuer R.E. 1986. *Multiple Criteria Optimization: Theory, Computation and Application*. John Wiley & Sons. New York, NY.

[35] Tamir A. 1992. On the Complexity of Some Classes of Location Problems. *Transportation Science* 26 (4): 352-354.