CALMNESS MODULUS OF LINEAR SEMI-INFINITE PROGRAMS *

M.J. CÁNOVAS†, A.Y. KRUGER‡, M.A. LÓPEZ§, J. PARRA¶, AND M.A. THÉRA‖

Abstract. Our main goal is to compute or estimate the calmness modulus of the argmin mapping of linear semi-infinite optimization problems under canonical perturbations, i.e., perturbations of the objective function together with continuous perturbations of the right-hand-side of the constraint system (with respect to an index ranging in a compact Hausdorff space). Specifically, we provide a lower bound on the calmness modulus for semi-infinite programs with unique optimal solution which turns out to be the exact modulus when the problem is finitely constrained. The relationship between the calmness of the argmin mapping and the same property for the (sub)level set mapping (with respect to the objective function), for semi-infinite programs and without requiring the uniqueness of the nominal solution, is explored too, providing an upper bound on the calmness modulus of the argmin mapping. When confined to finitely constrained problems, we also provide a computable upper bound as it only relies on the nominal data and parameters, not involving elements in a neighborhood. Illustrative examples are provided.

Key words. Isolated calmness, calmness modulus, variational analysis, linear programming, semi-infinite programming.

AMS subject classifications. 90C05, 90C34, 90C31, 49J53, 49J40.

1. Introduction.

We are concerned with the linear optimization problem

$$P(c,b): \quad \begin{array}{ll} \text{minimize} & \langle c, x\rangle \\ \text{subject to} & \langle a_t, x\rangle \leq b_t, \;\; t \in T, \end{array} \tag{1}$$

where $c, x \in \mathbb{R}^p$ (regarded as column-vectors in matrix calculus), $\langle \cdot,\cdot\rangle$ denotes the usual inner product in $\mathbb{R}^p, T$ is a compact Hausdorff space and the functions $t \mapsto a_t \in \mathbb{R}^p$ and $t \mapsto b_t \in \mathbb{R}$ are continuous on T. In the sequel we will use indifferently the notation $\langle x, y\rangle$ or $x'y$ to denote the scalar product between x and y, where in the last formula x' stands for the transpose of x. We assume that $t \mapsto a_t$ is a given function and that perturbations fall on a parameter $(c,b) \in \mathbb{R}^p \times C(T,\mathbb{R})$, with $b \equiv (b_t)_{t\in T}$. The parameter space $\mathbb{R}^p \times C(T,\mathbb{R})$ is endowed with the uniform convergence topology through the norm

$$\|(c,b)\| := \max\left\{\|c\|_*, \|b\|_\infty\right\}, \tag{2}$$

where $\mathbb{R}^p$ is equipped with an arbitrary norm, $\|\cdot\|$, $\|\cdot\|_*$ denotes the associated dual norm, given by $\|u\|_* = \max_{\|x\|\leq 1}\langle u, x\rangle$, and $\|b\|_\infty := \max_{t\in T}|b_t|$. Throughout the

*This research has been partially supported by Grants MTM2011-29064-C03 (02-03) from MINECO, Spain ACOMP/2013/062 from Generalitat Valenciana, Spain, Grant C10E08 from ECOS-SUD, and Grant DP110102011 from the Australian Research Council.

†Operations Research Center, Miguel Hernández University of Elche, 03202 Elche (Alicante), Spain (canovas@umh.es).

‡Centre for Informatics and Applied Optimization, School of Science, Information Technology and Engineering, University of Ballarat, POB 663, Ballarat, Vic, 3350, Australia (a.kruger@ballarat.edu.au).

§Department of Statistics and Operations Research, University of Alicante, Spain and Adjunct professor, University of Ballarat 03071 Alicante, Spain (marco.antonio@ua.es).

¶Operations Research Center, Miguel Hernández University of Elche, 03202 Elche (Alicante), Spain (parra@umh.es).

‖Laboratoire XLIM, UMR-CNRS 6172, University of Limoges, France and Adjunct professor, University of Ballarat (michel.thera@unilim.fr).

paper d_* stands for the distance associated with $\|\cdot\|_*$. Our aim here is to analyze the *solution mapping* (also called *argmin mapping*) of problem (1):

$$\mathcal{S} : (c, b) \mapsto \{x \in \mathbb{R}^p \mid x \text{ solves (1) for } (c, b)\}, \text{ with } (c, b) \in \mathbb{R}^p \times C(T, \mathbb{R}).$$

When c is fixed, we deal with the *partial* solution mapping $\mathcal{S}_c : C(T, \mathbb{R}) \rightrightarrows \mathbb{R}^p$ defined as

$$\mathcal{S}_c(b) := \mathcal{S}(c, b). \tag{3}$$

Definition 1. (calmness). *A mapping S acting from a metric space (Y, d_Y) to a metric space (X, d_X) is said to be* calm *at $(\bar{y}, \bar{x}) \in \text{gph}S$ (the graph of S) if there exist a constant $\kappa \geq 0$ and neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that*

$$d_X(x, S(\bar{y})) \leq \kappa d_Y(y, \bar{y}) \text{ whenever } x \in S(y) \cap U \text{ and } y \in V. \tag{4}$$

Recall that calmness of S turns out to be equivalent to *metric subregularity of* S^{-1} (see [12, Theorem 3H.3 and Exercise 3H.4]) which reads in terms of the existence of a (possibly smaller) neighborhood U of $\bar{x}$ and $\kappa \geq 0$ such that

$$d_X(x, S(\bar{y})) \leq \kappa d_Y\left(\bar{y}, S^{-1}(x)\right), \text{ for all } x \in U, \tag{5}$$

where $S^{-1}(x) := \{y \in Y \mid x \in S(y)\}$. The variant of the previous definition when the point-to-set distance $d_X(x, S(\bar{y}))$ is replaced with $d_X(x, \bar{x})$ in (4) corresponds to the so-called *isolated calmness* property. Isolated calmness of S at $(\bar{y}, \bar{x}) \in \text{gph}S$ implies that $S(\bar{y}) \cap U = \{\bar{x}\}$, so $\bar{x}$ is an isolated point in $S(\bar{y})$, hence the terminology. In fact, the reader can observe that isolated calmness (also called *calmness on selections*, see e.g. [14], or *local upper Lipschitz* property, see e.g. [16]) is nothing else but standard calmness together with this isolatedness condition (i. e., $S(\bar{y}) \cap U = \{\bar{x}\}$ for some neighborhood U of $\bar{x}$). Observe that in this case $S(y) \cap U$ needs not to be a singleton and might be empty for $y \in V \setminus \{\bar{y}\}$. In the case of our convex-valued argmin mapping $\mathcal{S}$, an isolated solution is nothing else but a unique solution. Sometimes along the paper we will require this uniqueness assumption of the nominal problem $P\left(\bar{c}, \bar{b}\right)$. Any linear mapping $A : \mathbb{R}^m \to \mathbb{R}^n$ is isolatedly calm at any point while isolated calmness of the inverse mapping A^{-1} is equivalent to injectivity of A, that is, $A^{-1}(0) = \{0\}$. More generally, from a result by Robinson [21] it follows that a set-valued mapping $S : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ whose graph is the union of finitely many polyhedral convex sets is isolatedly calm at $(\bar{y}, \bar{x})$ if and only if $\bar{x}$ is an isolated point of $S(\bar{y})$. For a function $f : \mathbb{R}^m \to \mathbb{R}^n$ which is smooth in a neighborhood of $\bar{x}$, the inverse f^{-1} is isolatedly calm at $(f(\bar{x}), \bar{x})$ if and only if the derivative mapping $Df(\bar{x})$ is injective. As shown in [10], see also [11], this inverse-function-type result can be extended to mappings of the form $f + F$ where f is a smooth function and F is a set-valued mapping with closed values. In [11] it was shown that, for a problem of minimizing a convex function with linear perturbations over a polyhedral convex set, the property of isolated calmness of the solution set is *equivalent* to the standard second-order sufficient optimality condition. In our semi-infinite framework, the set of feasible solutions is generally *not* polyhedral.

Calmness is known to be a weakened version of the more robust *Aubin property*: We say that S is *Aubin continuous* (or pseudo Lipschitz, or enjoys the Aubin property) at $(\bar{y}, \bar{x}) \in \text{gph}S$ if there exist $\kappa \geq 0$ and neighborhoods V of $\bar{y}$ and U of $\bar{x}$ such that

$$d_X(x, S(y')) \leq \kappa d_Y(y, y') \text{ whenever } y, y' \in V \text{ and } x \in S(y) \cap U. \tag{6}$$

Calmness corresponds to the case when we fix $y' = \bar{y}$ in (6).

The infimum of the values of κ for which (6) holds (for some associated neighborhoods U and V) is called the *Lipschitz modulus* of S at $(\bar{y}, \bar{x})$, and denoted by $\mathrm{lip}S(\bar{y}, \bar{x})$. Concerning calmness, the infimum of the values of κ for which (4) holds (for some associated neighborhoods U and V) is called the *calmness modulus* of S at $(\bar{y}, \bar{x})$, and denoted by $\mathrm{clm}S(\bar{y}, \bar{x})$. As a direct consequence of the definitions we have

$$\mathrm{lip}S(\bar{y}, \bar{x}) = \limsup_{\substack{y, y' \to \bar{y},\ y \neq y' \\ x \to \bar{x},\ x \in S(y)}} \frac{d_X(x, S(y'))}{d_Y(y, y')}, \tag{7}$$

$$\mathrm{clm}S(\bar{y}, \bar{x}) = \limsup_{\substack{y \to \bar{y} \\ x \to \bar{x},\ x \in S(y)}} \frac{d_X(x, S(\bar{y}))}{d_Y(y, \bar{y})}. \tag{8}$$

Accordingly, we always have $\mathrm{clm}S(\bar{y}, \bar{x}) \leq \mathrm{lip}S(\bar{y}, \bar{x})$. It is well-known that $\mathrm{clm}S(\bar{y}, \bar{x})$ coincides with the *modulus of metric subregularity of* S^{-1} at $(\bar{x}, \bar{y})$; i.e., with the infimum of the values of κ for which (5) holds (for associated neighborhoods U). So, we can write

$$\mathrm{clm}S(\bar{y}, \bar{x}) = \limsup_{x \to \bar{x}} \frac{d_X(x, S(\bar{y}))}{d_Y(\bar{y}, S^{-1}(x))}.$$

The main goal of this paper consists in computing or estimating $\mathrm{clm}\mathcal{S}\left(\left(\bar{c}, \bar{b}\right), \bar{x}\right)$, assuming sometimes that $\mathcal{S}\left(\bar{c}, \bar{b}\right) = \{\bar{x}\}$ and the Slater constraint qualification holds at $P\left(\bar{c}, \bar{b}\right)$ (see Section 2). As emphasized in [14], the calmness property plays a key role in optimization theory, specifically in the study of optimality conditions, sensitivity analysis, stability of solutions or existence of error bounds. The calmness modulus in the context of constraint systems under right-hand side perturbations has been widely analyzed in the literature (the reader is addressed to [17] for more details). In particular, [17, Theorem 1] constitutes a starting point for the present paper, as far as it provides a formula for the calmness modulus of constraint systems in terms of the associated supremum function.

Specifically, the structure of the paper is as follows: Section 2 introduces the necessary notation and background. Section 3 provides a lower bound on the calmness modulus of $\mathcal{S}$ for semi-infinite programs with unique optimal solutions. Section 4 shows that this lower bound equals the exact modulus when T is finite. The uniqueness of optimal solution is not required in Section 4 for establishing the upper estimate. Section 5 is concerned with upper bounds on $\mathrm{clm}\mathcal{S}\left(\left(\bar{c}, \bar{b}\right), \bar{x}\right)$. The first part is devoted to explore the relationship between the moduli of $\mathcal{S}$ and the (sub)level set mapping $\mathcal{L}$ introduced in [6] for characterizing the calmness of $\mathcal{S}$ at $\left(\left(\bar{c}, \bar{b}\right), \bar{x}\right)$ (see Section 2). The second part of Section 5 is confined to finitely constrained problems under uniqueness of optimal solution and obtains a new upper bound which may be easily computable as it is formulated exclusively in terms of the nominal data $\bar{c}$, $\bar{b}$, and $\bar{x}$. Section 6 provides examples showing that the latter upper bound may not be attained and also illustrating how the expression for the exact modulus (for finitely constrained problems with unique optimal solutions) given in Section 4 works. Finally in Section 7 we give some concluding remarks.

2. Preliminaries. In this section we introduce some notation and preliminary results. Given $X \subset \mathbb{R}^k$, $k \in \mathbb{N}$, we denote by $\mathrm{int}X$, $\mathrm{co}X$, and $\mathrm{cone}X$ the *interior*, the *convex hull*, and the *conical convex hull* of X, respectively. It is assumed that $\mathrm{cone}X$

always contains the zero-vector 0_k, in particular cone$(\varnothing) = \{0_k\}$. Associated with the parameterized problem (1), we denote by $\mathcal{F}$ the feasible set mapping, which is given by

$$\mathcal{F}(b) := \{x \in \mathbb{R}^p \mid \langle a_t, x\rangle \leq b_t, \ t \in T\}.$$

We also remind that the *set of active indices* and the *active cone* at $x \in \mathcal{F}(b)$ are the sets $T_b(x)$ and $A_b(x)$ defined respectively by

$$T_b(x) := \{t \in T \mid \langle a_t, x\rangle = b_t\} \text{ and}$$

$$A_b(x) := \text{cone}\{a_t \mid t \in T_b(x)\}.$$

Recall that the *Slater constraint qualification* (hereafter called *Slater condition*) holds for the problem $P(c,b)$ if there exists $\widehat{x} \in \mathbb{R}^p$ such that $a_t'\widehat{x} < b_t$ for all $t \in T$. The following result can be traced out from [13]:

Proposition 1. *Let $(\bar{c}, \bar{b}) \in \mathbb{R}^p \times C(T, \mathbb{R})$ and assume that $P(\bar{c}, \bar{b})$ satisfies the Slater condition. Then $\bar{x} \in \mathcal{S}(\bar{c}, \bar{b})$ if and only if the* Karush-Kuhn-Tucker (KKT) *condition holds, i. e.,*

$$\bar{x} \in \mathcal{F}(\bar{b}) \ \text{ and } \ -\bar{c} \in A_{\bar{b}}(\bar{x}). \tag{9}$$

In the next result we appeal to the concept of strong uniqueness of minimizers. We say that $\bar{x} \in \mathbb{R}^p$ is the *strongly unique minimizer* of $P(\bar{c}, \bar{b})$ if for some $\alpha > 0$ we have

$$\langle \bar{c}, x\rangle \geq \langle \bar{c}, \bar{x}\rangle + \alpha \|x - \bar{x}\| \text{ for all } x \in \mathcal{F}(\bar{b}). \tag{10}$$

Theorem 2. [4, Thm. 3] *Consider the parameterized linear optimization problem* (1) *and let $\mathcal{S}$ be the associated solution mapping. Let $((\bar{c}, \bar{b}), \bar{x}) \in$ gph$\mathcal{S}$ be such that $P(\bar{c}, \bar{b})$ satisfies the Slater condition. Then the following are equivalent:*

(i) $-\bar{c} \in int\, A_{\bar{b}}(\bar{x})$;

(ii) $\bar{x}$ *is the strongly unique minimizer of* $P(\bar{c}, \bar{b})$;

(iii) $\mathcal{S}$ *is isolatedly calm at* $((\bar{c}, \bar{b}), \bar{x})$;

(iv) $\mathcal{S}_{\bar{c}}$ (*defined in* (3)) *is isolatedly calm at* $(\bar{b}, \bar{x})$;

If T is finite, we can add the following condition:

(v) $\bar{x}$ *is the unique solution of* $P(\bar{c}, \bar{b})$.

Remark 1. Paper [4] deals with convex optimization problems with canonical perturbations, and provides examples showing that implications

$$\text{(ii)} \Rightarrow \text{(iii)} \Rightarrow \text{(iv)} \Rightarrow \text{(v)}$$

may be strict in the convex case. This paper also extends condition (i) to the convex case and shows the equivalence between (ii) and this extended version of (i).

Concerning the Aubin property, we have the following result (where $|D|$ stands for the cardinality of D):

Theorem 3. [7, Theorem 16] *For the linear semi-infinite program (1), let $((\bar{c}, \bar{b}), \bar{x}) \in$ gph$\mathcal{S}$. Then, the following conditions are equivalent:*

(i) $\mathcal{S}$ is Aubin continuous at $((\bar{c}, \bar{b}), \bar{x})$;

(ii) $\mathcal{S}$ *is strongly Lipschitz stable at* $\left(\left(\bar{c},\bar{b}\right),\bar{x}\right)$ *(i. e., locally single-valued and Lipschitz continuous around* $\left(\bar{c},\bar{b}\right)$*);*

(iii)$\mathcal{S}$ *is locally single-valued and continuous in some neighborhood of* $(\bar{c},\bar{b})$*;*

(iv) $\mathcal{S}$ *is single valued in some neighborhood of* $\left(\bar{c},\bar{b}\right)$*;*

(v) $P\left(\bar{c},\bar{b}\right)$ *satisfies the Slater condition and there is no* $D\subset T_{\bar{b}}\left(\bar{x}\right)$ *with* $|D|<p$ *such that* $-\bar{c}\in\operatorname{cone}\left(\{a_t,\ t\in D\}\right)$*;*

(vi) $P\left(\bar{c},\bar{b}\right)$ *satisfies the Slater condition and for each* $D\subset T_{\bar{b}}\left(\bar{x}\right)$ *with* $|D|=p$ *such that* $-\bar{c}\in\operatorname{cone}\left(\{a_t,\ t\in D\}\right)$*, all possible subsets of* $\{a_t,\ t\in D;-\bar{c}\}$ *with* p *elements are linearly independent;*

(vii) $\left(\bar{c},\bar{b}\right)\in\operatorname{int}\left(\{(c,b)\mid\mathcal{S}\left(c,b\right)\text{ consists of a strongly unique minimizer}\}\right).$

We emphasize condition (v) which will be referred to as the *Nürnberger condition*, since it was stated in [20] (in the equivalent form (vi)) for characterizing the counterpart of condition (vii) when all coefficients (including the a_t's) are subject to perturbations.

In the next theorem we provide a characterization of the calmness of $\mathcal{S}$ at $\left(\left(\bar{c},\bar{b}\right),\bar{x}\right)$, under the Slater condition, without requiring $\mathcal{S}\left(\bar{c},\bar{b}\right)$ to be reduced to the singleton $\{\bar{x}\}$. In this characterization we use the *level set mapping* $\mathcal{L}:\mathbb{R}\times C\left(T,\mathbb{R}\right)\rightrightarrows\mathbb{R}^p$ given by

$$\mathcal{L}\left(\alpha,b\right)=\{x\in\mathbb{R}^p\mid\bar{c}'x\leq\alpha;\ a_t'x\leq b_t,\ t\in T\}\tag{11}$$

and the supremum function $\bar{f}:\mathbb{R}^p\to\mathbb{R}$ defined as

$$\bar{f}\left(x\right):=\sup\left\{\bar{c}'(x-\bar{x});\ a_t'x-\bar{b}_t,\ t\in T\right\}.\tag{12}$$

In the following theorem, we appeal to the zero sublevel set of $\bar{f}$:

$$\left[\bar{f}\leq 0\right]:=\left\{x\in\mathbb{R}^p\mid\bar{f}\left(x\right)\leq 0\right\}=\mathcal{L}\left(\bar{c}'\bar{x},\bar{b}\right)=\mathcal{S}\left(\bar{c},\bar{b}\right).\tag{13}$$

Theorem 4. *Let* $((\bar{c},\bar{b}),\bar{x})\in\operatorname{gph}\mathcal{S}$ *and assume* $P\left(\bar{c},\bar{b}\right)$ *satisfies the Slater condition. Then the following conditions are equivalent:*

(i) $\mathcal{S}$ *is calm at* $((\bar{c},\bar{b}),\bar{x})$*;*

(ii) $\mathcal{S}_{\bar{c}}$ *is calm at* $(\bar{b},\bar{x})$*;*

(iii) $\mathcal{L}$ *is calm at* $((\langle\bar{c},\bar{x}\rangle,\bar{b}),\bar{x})$*;*

(iv) $\bar{f}$ *has a* local error bound *at* $\bar{x}$*; i.e., there exist* $\kappa\geq 0$ *and a neighborhood* U *of* $\bar{x}$ *such that*

$$d\left(x,\left[\bar{f}\leq 0\right]\right)\leq\kappa\left[\bar{f}\left(x\right)\right]_+,\ \text{for all } x\in U\ ;$$

(we recall that given a real a*,* a_+ *stands for* $\max(a,0)$*)*

(v) $\liminf_{x\to\bar{x},\ \bar{f}(x)>0} d_*\left(0_p,\partial\bar{f}\left(x\right)\right)>0.$

Proof. The equivalence $(i)\Leftrightarrow(ii)\Leftrightarrow(iii)$ is exactly [6, Theorem 1], whereas $(iii)\Leftrightarrow(iv)$ is a direct

consequence of the definitions, as far as,

$$\begin{aligned}\left[\bar{f}\left(x\right)\right]_+&=\left[\sup\{\bar{c}'x-\bar{c}'\bar{x};\ a_t'x-\bar{b}_t,\ t\in T\}\right]_+\\&=\sup\{[\bar{c}'x-\bar{c}'\bar{x}]_+;\ \left[a_t'x-\bar{b}_t\right]_+,\ t\in T\}\\&=d\left(\left(\bar{c}'\bar{x},\bar{b}\right),\mathcal{L}^{-1}\left(x\right)\right),\ \text{for all } x\in\mathbb{R}^p,\end{aligned}\tag{14}$$

taking also into account (13) and the equivalence between the calmness of $\mathcal{L}$ and the metric subregularity of $\mathcal{L}^{-1}$ (see (5)).

Finally, $(i) \Leftrightarrow (v)$ can be traced out from [1, Proposition 2.1 and Theorem 5.1] (see also [15]), and constitutes a key tool for establishing $(ii) \Rightarrow (iii)$ in [6, Theorem 1]. □

In order to provide an upper bound on the calmness modulus in Section 5, we use the following notation: for any finite subset of indices $D \subset T, A_D$ denotes the matrix whose rows are $a_t', t \in D$ (given in some prefixed order). Given any $b \in C(T, \mathbb{R})$, we denote $b_D = (b_t)_{t \in D}$. If moreover $x \in \mathcal{S}(\bar{c}, b)$, we denote

$$\mathcal{T}_b(x) = \left\{ D \subset T_b(x) \;\middle|\; \begin{array}{l} |D| = p,\; A_D \text{ is nonsingular,} \\ \text{and } -\bar{c} \in \operatorname{cone}\{a_t,\; t \in D\} \end{array} \right\}. \tag{15}$$

This set was already introduced in [5, p. 520] with the aim of obtaining bounds on the Lipschitz modulus, provided that the Aubin property holds (i. e., under the Nürnberger condition). In that paper, the nonsingularity of A_D was not explicitly required when defining $\mathcal{T}_b(x)$, since it follows as a consequence of the Nürnberger condition (at points $((\bar{c}, b), x) \in \operatorname{gph}\mathcal{S}$ around $((\bar{c}, \bar{b}), \bar{x})$).

Given $D \subset T$ with $|D| = p$, we identify the matrix A_D with the 'endomorphism' $\mathbb{R}^p \ni x \mapsto A_D x \in \mathbb{R}^D$, where $\mathbb{R}^p$ is equipped with an arbitrary norm $\|\cdot\|$ and $\mathbb{R}^D$ is endowed with the supremum norm $\|\cdot\|_\infty$. Recall that $(\|\cdot\|_\infty)_* = \|\cdot\|_1$. For our choice of norms, provided that A_D is non-singular, we have

$$\left\|A_D^{-1}\right\| := \max_{\|y\|_\infty \le 1} \left\|A_D^{-1} y\right\| = \max_{y \in \{-1,1\}^p} \left\|A_D^{-1} y\right\| = \left(\min_{\|\lambda\|_1 = 1} \|A_D' \lambda\|_* \right)^{-1}. \tag{16}$$

The second equality comes from the use of $\|\cdot\|_\infty$ in $\mathbb{R}^D$, together with the fact that $\{-1,1\}^p$ is the set of extreme points of the associated closed unit ball and the function to be maximized is convex. The last equality is a straightforward consequence of [3, Corollary 3.2] together with the fact that $\left\|A_D^{-1}\right\|$ coincides with the (metric) regularity modulus of A_D (Lipschitz modulus of A_D^{-1}), at any point of its graph.

Theorem 5. (see [5, Theorems 1 and 2, and Corollary 2]) *Assume, for the linear semi-infinite program (1), that the Nürnberger condition holds at* $((\bar{c}, \bar{b}), \bar{x}) \in \operatorname{gph}\mathcal{S}$. *Then*

$$\operatorname{lip}\mathcal{S}\left((\bar{c}, \bar{b}), \bar{x}\right) \ge \sup_{D \in \mathcal{T}_{\bar{b}}(\bar{x})} \left\|A_D^{-1}\right\|, \tag{17}$$

and equality occurs in (17) provided that either T is finite or $p \le 3$.

In [5], a general sufficient condition –called $(\mathcal{H})$ therein– ensuring equality in (17) was introduced. Condition $(\mathcal{H})$ always holds when either T is finite or $p \le 3$. Examples in [5] show that condition $(\mathcal{H})$ can fail in $\mathbb{R}^4$. The question of whether $(\mathcal{H})$ can be weakened (or even removed) while still ensuring equality in (17) when T is infinite remains an open problem.

In the case when T is finite, it is obvious that, under the assumptions of Theorem 5, $\sup_{D \in \mathcal{T}_{\bar{b}}(\bar{x})} \left\|A_D^{-1}\right\|$ is an upper bound on $\operatorname{clm}\mathcal{S}\left((\bar{c}, \bar{b}), \bar{x}\right)$. We show in Section 5 that, under the Slater condition and assuming that T is finite and $\mathcal{S}(\bar{c}, \bar{b}) = \{\bar{x}\}$, $\sup_{D \in \mathcal{T}_{\bar{b}}(\bar{x})} \left\|A_D^{-1}\right\|$ remains an upper bound of $\operatorname{clm}\mathcal{S}\left((\bar{c}, \bar{b}), \bar{x}\right)$ even when $\mathcal{S}$ is not Aubin continuous at $((\bar{c}, \bar{b}), \bar{x})$. Let us point out that, when T is finite, $\mathcal{S}$ is always calm at any point of its graph as it was proved in [21] (see [6, Section 4] for more details).

3. Lower bound on the calmness modulus. In this section and the next one, we consider, associated with $(b,x)\in \text{gph}\mathcal{S}_{\bar{c}}$, the family of "KKT subsets" of T given by

$$\mathcal{K}_b(x)=\{D\subset T_b(x)\,||D|\leq p \text{ and } -\bar{c}\in \text{cone}\{a_t,\ t\in D\}\},$$

which coincides with $\mathcal{T}_b(x)$ under the Nürnberger condition at $((\bar{c},b),x)$. For any $D\in\mathcal{K}_{\bar{b}}(\bar{x})$, we consider the supremum function, $f_D:\mathbb{R}^p\to\mathbb{R}$ given by

$$\begin{aligned} f_D(x) &:= \sup\left\{\langle a_t,x\rangle-\bar{b}_t,\ t\in T;\ -\langle a_t,x\rangle+\bar{b}_t,\ t\in D\right\}\\ &= \sup\left\{\langle a_t,x\rangle-\bar{b}_t,\ t\in T\setminus D;\ \left|\langle a_t,x\rangle-\bar{b}_t\right|,\ t\in D\right\}. \end{aligned}$$

Roughly speaking, for a given $x\in\mathbb{R}^p$, $f_D(x)$ is the smallest perturbation size on $\bar{b}$, say $\left\|b-\bar{b}\right\|_\infty$, that makes x a KKT point (hence optimal) for problem $P(\bar{c},b)$ with D as an associated "KKT index set". In the semi-infinite framework some technical arrangements are needed to obtain a continuous perturbation. These comments will be formalized when constructing a certain sequence $\{b^n\}_{n\in\mathbb{N}}$ in the following theorem.

For each $D\in\mathcal{K}_{\bar{b}}(\bar{x})$ we may consider the set-valued mapping $\mathcal{L}_D: C(T,\mathbb{R})\times\mathbb{R}^D\rightrightarrows\mathbb{R}^p$ given by

$$\mathcal{L}_D(b,d):=\{x\in\mathbb{R}^p\mid \langle a_t,x\rangle\leq b_t,\ t\in T;\ \langle -a_t,x\rangle\leq d_t,\ t\in D\}.$$

Observe that $\mathcal{L}_D\left(\bar{b},-\bar{b}_D\right)$, where $\bar{b}_D=\left(\bar{b}_t\right)_{t\in D}$, is the referred set of KKT points of problem $P\left(\bar{c},\bar{b}\right)$ associated with D as the KKT index set. Accordingly,

$$\mathcal{L}_D\left(\bar{b},-\bar{b}_D\right)=[f_D=0]\subset\mathcal{S}\left(\bar{c},\bar{b}\right)\ \text{ for all } D\in\mathcal{K}_{\bar{b}}(\bar{x}). \tag{18}$$

Let us also observe that

$$\begin{aligned} \text{clm}\mathcal{L}_D\left(\left(\bar{b},-\bar{b}_D\right),\bar{x}\right) &= \limsup_{x\to\bar{x}}\frac{d\left(x,\mathcal{L}_D\left(\bar{b},-\bar{b}_D\right)\right)}{d\left(\left(\bar{b},-\bar{b}_D\right),\mathcal{L}_D^{-1}(x)\right)}\\ &= \limsup_{x\to\bar{x}}\frac{d\left(x,[f_D\leq 0]\right)}{[f_D(x)]_+}\\ &= \limsup_{\substack{x\to\bar{x}\\ f_D(x)>0}}\frac{1}{d_*\left(0_p,\partial f_D(x)\right)}, \end{aligned} \tag{19}$$

where the last equality comes from [17, Theorem 1] (and the second one appeals to a similar argument as in (14)).

Theorem 6. *Let $\mathcal{S}\left(\bar{c},\bar{b}\right)=\{\bar{x}\}$ and assume that $P\left(\bar{c},\bar{b}\right)$ satisfies the Slater condition. Then*

$$\text{clm}\mathcal{S}\left(\left(\bar{c},\bar{b}\right),\bar{x}\right)\geq\text{clm}\mathcal{S}_{\bar{c}}\left(\bar{b},\bar{x}\right)\geq\sup_{D\in\mathcal{K}_{\bar{b}}(\bar{x})}\ \limsup_{\substack{x\to\bar{x}\\ f_D(x)>0}}\frac{1}{d_*\left(0_p,\partial f_D(x)\right)}.$$

Proof. The first inequality follows straightforwardly from (8). In order to prove the second one, let us consider a fixed $D\in\mathcal{K}_{\bar{b}}(\bar{x})$ and let us show that

$$\text{clm}\mathcal{S}_{\bar{c}}\left(\bar{b},\bar{x}\right)\geq\limsup_{\substack{x\to\bar{x}\\ f_D(x)>0}}\frac{1}{d_*\left(0_p,\partial f_D(x)\right)}. \tag{20}$$

Write

$$\limsup_{\substack{x\to\bar{x}\\ f_D(x)>0}} \frac{1}{d_*\left(0_p,\partial f_D\left(x\right)\right)} = \lim_{n\to+\infty} \frac{1}{d_*\left(0_p,\partial f_D\left(x^n\right)\right)},$$

for a certain sequence $\{x^n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to+\infty} x^n = \bar{x}$ and $f_D\left(x^n\right) > 0$ for all $n \in \mathbb{N}$. Obviously, $x^n \notin \mathcal{S}_{\bar{c}}\left(\bar{b}\right)$ since $\mathcal{S}_{\bar{c}}\left(\bar{b}\right) = \{\bar{x}\}$ and $f_D\left(\bar{x}\right) = 0$. Note that $d_*\left(0_p,\partial f_D\left(x^n\right)\right) > 0$ since $x^n \notin \arg\min_{x\in\mathbb{R}^p} f_D$. From now on, the proof is focused on the construction of a new sequence of parameters $\{b^n\} \subset C\left(T,\mathbb{R}^p\right)$ such that

$$x^n \in \mathcal{S}_{\bar{c}}(b^n) \text{ and } \frac{\|x^n-\bar{x}\|}{\|b^n-\bar{b}\|_\infty} \geq \frac{n}{n+1}\cdot\frac{1}{d_*(0_p,\partial f_D(x^n))}. \tag{21}$$

Firstly, we will provide a lower bound on $\|x^n-\bar{x}\|$. Clearly (21) implies (20). In view of the closedness of $\partial f_D\left(x^n\right)$, we can write $d_*\left(0_p,\partial f_D\left(x^n\right)\right) = \|u^n\|_*$ for a certain

$$\begin{aligned} u^n \in \partial f_D\left(x^n\right) = \operatorname{conv}\big(&\left\{a_t \mid \langle a_t,x^n\rangle - \bar{b}_t = f_D\left(x^n\right),\ t\in T\right\} \\ &\cup\left\{-a_t \mid -\langle a_t,x^n\rangle + \bar{b}_t = f_D\left(x^n\right),\ t\in D\right\}\big), \end{aligned}$$

(where we have appealed to the Ioffe-Tikhomirov theorem –see, for instance, [22, Theorem 2.4.18] or, in the finite dimensional setting, to a consequence of the Danskin Theorem [9]– and to the classical Mazur theorem). Therefore we can write

$$u^n = \sum_{t\in T}\lambda_t^n a_t + \sum_{t\in D}\mu_t^n\left(-a_t\right) \tag{22}$$

where, for all $n\in\mathbb{N}$, $\lambda_t^n \geq 0$ for $t\in T$, $\mu_t^n \geq 0$ for $t \in D$, $\sum_{t\in T}\lambda_t^n + \sum_{t\in D}\mu_t^n = 1$, $\lambda_t^n > 0$ for finitely many elements $t\in T$ verifying $\langle a_t,x^n\rangle - \bar{b}_t = f_D\left(x^n\right)$, and $-\langle a_t,x^n\rangle + \bar{b}_t = f_D\left(x^n\right)$ whenever $\mu_t^n > 0$.

Clearly, appealing to (22), each solution of the system

$$\left\{\langle a_t,x\rangle \leq \bar{b}_t,\ t\in T;\ -\langle a_t,x\rangle \leq -\bar{b}_t,\ t\in D\right\},$$

i. e., each $x \in \mathcal{L}_D\left(\bar{b},-\bar{b}_D\right)$, and $\bar{x}$ in particular, satisfies the inequality

$$\langle u^n,x\rangle \leq \sum_{t\in T}\lambda_t^n\bar{b}_t + \sum_{t\in D}\mu_t^n\left(-\bar{b}_t\right). \tag{23}$$

Accordingly, $\|x^n-\bar{x}\|$ is bounded from below by the distance from x^n to the solution set of (23), which can be computed by means of the well-known Ascoli formula. In other words,

$$\|x^n-\bar{x}\| \geq \frac{\sum_{t\in T}\lambda_t^n\left(\langle a_t,x^n\rangle - \bar{b}_t\right) + \sum_{t\in D}\mu_t^n\left(-\langle a_t,x^n\rangle + \bar{b}_t\right)}{\|u^n\|_*} = \frac{f_D\left(x^n\right)}{\|u^n\|_*}, \tag{24}$$

where the condition stated after formula (22) has been taken into account.

Next we proceed with the construction of the aimed sequence $\{b^n\}$ such that (21) holds. To do this, we appeal to Urysohn's Lemma to obtain a certain $\varphi_n \in C\left(T,[0,1]\right)$ such that

$$\varphi_n\left(t\right) = \begin{cases} 0 & \text{if } t\in D, \\ 1 & \text{if } \langle a_t,x^n\rangle - \bar{b}_t \leq -\left(1+\frac{1}{n}\right)f_D\left(x^n\right). \end{cases}$$

Certainly sets D and $\left\{t \in T \mid \langle a_t, x^n\rangle - \bar{b}_t \leq -\left(1+\frac{1}{n}\right) f_D\left(x^n\right)\right\}$ are closed disjoint sets in T, recalling the definition of $f_D\left(x^n\right)$ and the fact that $f_D\left(x^n\right) > 0$.

Let us define, for each $t \in T$,

$$b_t^n = \left(1-\varphi_n\left(t\right)\right)\langle a_t, x^n\rangle + \varphi_n\left(t\right)\left(\bar{b}_t + f_D\left(x^n\right)\right).$$

Then we can easily check that $x^n \in \mathcal{S}_{\bar{c}}\left(b^n\right)$, since $x^n \in \mathcal{F}\left(b^n\right)$ and $D \subset \mathcal{K}_{b^n}\left(x^n\right)$, so that the KKT conditions hold.

Finally, let us observe that $\left|b_t^n - \bar{b}_t\right| \leq f_D\left(x^n\right)$ when $\varphi_n\left(t\right) = 1$ and

$$-\left(1+\frac{1}{n}\right) f_D\left(x^n\right) < a_t' x^n - \ \bar{b}_t \leq f_D\left(x^n\right)$$

when $\varphi_n\left(t\right) < 1$. Accordingly,

$$\left\|b^n - \bar{b}\right\|_\infty \leq \left(1+\frac{1}{n}\right) f_D\left(x^n\right), \tag{25}$$

which, together with (24), entails (21). □

Remark 2. The role of the uniqueness assumption $\mathcal{S}\left(\bar{c}, \bar{b}\right) = \{\bar{x}\}$ consists in ensuring $\mathcal{L}_D\left(\bar{b}, -\bar{b}_D\right) \supset \mathcal{S}\left(\bar{c}, \bar{b}\right)$ for all $D \in \mathcal{K}_{\bar{b}}\left(\bar{x}\right)$, which is the fact underlying (24).

4. Calmness modulus for finitely constrained programs. In this section, we show that, in the case when T is finite, the lower bound on $\mathrm{clm}\mathcal{S}\left(\left(\bar{c}, \bar{b}\right), \bar{x}\right)$ provided in the previous section is also an upper bound, without requiring neither the Slater condition nor the uniqueness of $\bar{x}$ as an optimal solution of $P\left(\bar{c}, \bar{b}\right)$.

Theorem 7. *Assume that T is finite and let $\left((\bar{c}, \bar{b}), \bar{x}\right) \in \mathrm{gph}\mathcal{S}$. Then*

$$\mathrm{clm}\mathcal{S}((\bar{c}, \bar{b}), \bar{x}) = \mathrm{clm}\mathcal{S}_{\bar{c}}(\bar{b}, \bar{x}) \leq \sup_{D \in \mathcal{K}(\bar{x})} \limsup_{\substack{x \to \bar{x} \\ f_D(x) > 0}} \frac{1}{d_*\left(0_p, \partial f_D\left(x\right)\right)}.$$

Proof. Let us write

$$\mathrm{clm}\mathcal{S}((\bar{c}, \bar{b}), \bar{x}) = \lim_{n\to\infty} \frac{d\left(x^n, \mathcal{S}(\bar{c}, \bar{b})\right)}{\left\|(c^n, b^n) - (\bar{c}, \bar{b})\right\|} \tag{26}$$

with $(c^n, b^n) \to (\bar{c}, \bar{b})$, $(c^n, b^n) \neq (\bar{c}, \bar{b})$ and $\mathcal{S}(c^n, b^n) \ni x^n \to \bar{x}$. Since, in ordinary linear programming, optimality is equivalent to the KKT conditions (without requiring the Slater condition), for each $n \in \mathbb{N}$, there exists $D_n \subset T_{b^n}\left(x^n\right)$ with $|D_n| \leq p$ (because of Carathéodory's theorem) such that

$$-c^n \in \mathrm{cone}\left\{a_t,\ t \in D_n\right\}. \tag{27}$$

The finiteness of T entails that the sequence $\left\{D_n\right\}_{n\in\mathbb{N}}$ of subsets of T must contain a constant subsequence. Without loss of generality, we may write

$$D_n = D$$

for all $n \in \mathbb{N}$. Hence, (27) reads as $-c^n \in \mathrm{cone}\left\{a_t,\ t \in D\right\}$ (which is a closed cone) for all $n \in \mathbb{N}$. Accordingly,

$$-\bar{c} \in \mathrm{cone}\left\{a_t,\ t \in D\right\}. \tag{28}$$

Moreover, $D \subset T_{b^n}(x^n)$, and hence, $\langle a_t, x^n\rangle = b_t^n$ for all $t \in D$, clearly implies $D \subset T_{\bar{b}}(\bar{x})$. This, together with (28), entails $D \in \mathcal{K}_{\bar{b}}(\bar{x})$.

In addition, recalling $D_n = D \subset T_{b^n}(x^n)$, (28) yields, for all n,

$$x^n \in \mathcal{S}_{\bar{c}}(b^n).$$

Therefore (26) and the obvious fact that

$$\left\|(c^n, b^n) - (\bar{c}, \bar{b})\right\| \geq \left\|b^n - \bar{b}\right\|_\infty$$

entail

$$\operatorname{clm}\mathcal{S}((\bar{c}, \bar{b}), \bar{x}) \leq \limsup_{n\to\infty} \frac{d\left(x^n, \mathcal{S}(\bar{c}, \bar{b})\right)}{\left\|b^n - \bar{b})\right\|_\infty} \leq \operatorname{clm}\mathcal{S}_{\bar{c}}(\bar{b}, \bar{x}).$$

Since, obviously from the definitions, $\operatorname{clm}\mathcal{S}_{\bar{c}}(\bar{b}, \bar{x}) \leq \operatorname{clm}\mathcal{S}((\bar{c}, \bar{b}), \bar{x})$, the previous upper limit must be an ordinary limit and we can write indeed

$$\operatorname{clm}\mathcal{S}((\bar{c}, \bar{b}), \bar{x}) = \operatorname{clm}\mathcal{S}_{\bar{c}}(\bar{b}, \bar{x}) = \lim_{n\to\infty} \frac{d\left(x^n, \mathcal{S}(\bar{c}, \bar{b})\right)}{\left\|b^n - \bar{b}\right\|_\infty}.$$

Finally, it is easy to see that $D \subset T_{b^n}(x^n)$ translates into $x^n \in \mathcal{L}_D(b^n, -b_D^n)$. To finish the proof, just observe that, from (18), (19), and the obvious fact that $\left\|b^n - \bar{b}\right\|_\infty = \left\|(b^n, -b_D^n) - \left(\bar{b}, -\bar{b}_D\right)\right\|_\infty$, we have

$$\begin{aligned}\lim_{n\to\infty} \frac{d\left(x^n, \mathcal{S}(\bar{c}, \bar{b})\right)}{\left\|b^n - \bar{b}\right\|_\infty} &\leq \limsup_{n\to\infty} \frac{d\left(x^n, \mathcal{L}_D(\bar{b}, -\bar{b}_D)\right)}{\left\|(b^n, -b_D^n) - \left(\bar{b}, -\bar{b}_D\right)\right\|_\infty} \\ &\leq \operatorname{clm}\mathcal{L}_D\left(\left(\bar{b}, -\bar{b}_D\right), \bar{x}\right) = \limsup_{\substack{x\to\bar{x} \\ f_D(x)>0}} \frac{1}{d_*\left(0_p, \partial f_D(x)\right)}.\end{aligned}$$

□

Corollary 8. *If T is finite and $\mathcal{S}(\bar{c}, \bar{b}) = \{\bar{x}\}$, then*

$$\operatorname{clm}\mathcal{S}((\bar{c}, \bar{b}), \bar{x}) = \operatorname{clm}\mathcal{S}_{\bar{c}}(\bar{b}, \bar{x}) = \sup_{D\in\mathcal{K}_{\bar{b}}(\bar{x})} \limsup_{\substack{x\to\bar{x} \\ f_D(x)>0}} \frac{1}{d_*\left(0_p, \partial f_D(x)\right)}.$$

Proof. The result comes straightforwardly from Theorems 6 and 7. The only detail to be taken into account is that the Slater condition in Theorem 6 is only needed at the very beginning of the proof to guarantee the existence of $D \in \mathcal{K}_{\bar{b}}(\bar{x})$. When T is finite, we are dealing with ordinary linear programming problems and, as pointed out at the beginning of the proof of Theorem 7, optimality implies the KKT conditions without requiring the Slater condition. □

5. Upper bounds on the calmness modulus. In this section, we follow various approaches to getting upper estimates of $\operatorname{clm}\mathcal{S}((\bar{c}, \bar{b}), \bar{x})$ for a given $\left((\bar{c}, \bar{b}), \bar{x}\right) \in \operatorname{gph}\mathcal{S}$. In Subsection 5.1, $\operatorname{clm}\mathcal{S}((\bar{c},\ \bar{b}), \bar{x})$ is related to $\operatorname{clm}\mathcal{L}((\langle\bar{c}, \bar{x}\rangle, \bar{b}), \bar{x})$, see Theorem 11 for semi-infinite programs without requiring the uniqueness of $\bar{x}$ as an optimal solution of $P(\bar{c},\ \bar{b})$. On the other hand, Subsection 5.2 is concerned with finitely constrained problems with unique optimal solutions. The main result of this subsection, Theorem 13, reminds of Theorem 5 (which deals with the Lipschitz modulus), although the main perturbation ideas underlying the latter cannot be transferred to the calmness modulus, as we try to illustrate in the next section (and particularly in Example 2).

5.1. A level set approach. In this subsection, we consider a given $\left((\bar{c},\bar{b}),\bar{x}\right) \in \operatorname{gph}\mathcal{S}$ and assume that $P(\bar{c},\bar{b})$ satisfies the Slater condition. Let us recall the level set mapping $\mathcal{L}$ defined in (11). In order to obtain an upper bound on $\operatorname{clm}\mathcal{S}((\bar{c},\bar{b}),\bar{x})$ in terms of $\operatorname{clm}\mathcal{L}((\langle\bar{c},\bar{x}\rangle,\bar{b}),\bar{x})$, first we need a Gauvin-type refinement of the KKT conditions (Proposition 1). This is done in Lemma 9. Before that, we need some additional notation.

For any $(b,x) \in \operatorname{gph}\mathcal{F}$, let

$$C_b(x) := \operatorname{co}\{a_t \mid t \in T_b(x)\}.$$

The compactness of T and $T_{\bar{b}}(x)$ for any $x \in \mathcal{F}\left(\bar{b}\right)$ easily entails the following equivalences.

Remark 3. *The following conditions are equivalent:*

(i) $P(\bar{c},\bar{b})$ satisfies the Slater condition;

(ii) There exists $x^0 \in \mathcal{F}\left(\bar{b}\right)$ and some associated $z \in \mathbb{R}^p$, $\|z\| = 1$ satisfying $a_t'z > 0$ for all $t \in T_{\bar{b}}\left(x^0\right)$;

(iii) For each $x \in \mathcal{F}\left(\bar{b}\right)$,

$$\alpha_{\bar{b}}(x) := \sup_{\|z\|=1} \inf_{t\in T_{\bar{b}}(x)} a_t'z > 0. \tag{29}$$

The proof of the equivalence between (i) and (ii) can be found in [13, Theorem 7.2]. Alternatively we can write

$$\alpha_{\bar{b}}(x) = \sup\{\alpha > 0 \mid (\alpha,z) \in \mathcal{G}_{\bar{b}}(x) \text{ for some } z \in \mathbb{S}\},$$

where $\mathbb{S}$ denotes the unit sphere in $\mathbb{R}^p$ and

$$\mathcal{G}_{\bar{b}}(x) := \{(\alpha,z) \in \mathbb{R}\times\mathbb{S} \mid \alpha > 0,\, a_t'z > \alpha,\ \forall t \in T_{\bar{b}}(x)\}.$$

Let us define

$$\bar{\lambda} := \min\nolimits_{(\alpha,z)\in\mathcal{G}_{\bar{b}}(\bar{x})} \frac{-\bar{c}'z}{\alpha}, \tag{30}$$

which obviously satisfies

$$\bar{\lambda} \leq \frac{\|\bar{c}\|_*}{\alpha_{\bar{b}}(\bar{x})}.$$

The following lemma shows that $\bar{\lambda}$ bounds the sum of any KKT multiplier set at $\bar{x} \in \mathcal{S}\left(\bar{c},\bar{b}\right)$.

Lemma 9. *Assume that $P(\bar{c},\bar{b})$ satisfies the Slater condition, and let $\bar{x} \in \mathcal{S}\left(\bar{c},\bar{b}\right)$. Then,*

$$\emptyset \neq \{\lambda \geq 0 \mid -\bar{c} \in \lambda C_{\bar{b}}(\bar{x})\} \subset \left[0,\bar{\lambda}\right].$$

Proof. The KKT conditions (Proposition 1) entail the non-emptiness of the set in question. For any representation

$$-\bar{c} = \lambda \sum_{t\in T_{\bar{b}}(\bar{x})} \mu_t a_t$$

with $\lambda \geq 0$, $\sum_{t\in T_{\bar{b}}(\bar{x})} \mu_t = 1$, μ_t nonnegative for all $t \in T_{\bar{b}}(\bar{x})$ and only finitely many of them being positive, one has, for all $(\alpha, z) \in \mathcal{G}_{\bar{b}}(\bar{x})$,

$$-\bar{c}'z = \lambda \sum_{t\in T_{\bar{b}}(\bar{x})} \mu_t a_t' z \geq \lambda\alpha.$$

Consequently,

$$\lambda \leq \frac{-\bar{c}'z}{\alpha}, \text{ for all } (\alpha, z) \in \mathcal{G}_{\bar{b}}(\bar{x}).$$

□

The next lemma extends the previous one to a neighborhood of $\left(\left(\bar{c}, \bar{b}\right), \bar{x}\right)$ relative to gph$\mathcal{S}$.

LEMMA 10. *Assume that $P(\bar{c}, \bar{b})$ satisfies the Slater condition and let $\bar{x} \in \mathcal{S}\left(\bar{c}, \bar{b}\right)$. Then, for any $\varepsilon > 0$ there exist neighborhoods U of $\bar{x}$ and V of $\left(\bar{c}, \bar{b}\right)$ such that, for any $(c, b) \in V$ and $x \in \mathcal{S}(c, b) \cap U$, it holds*

$$-c \in \lambda C_b(x) \tag{31}$$

for some $\lambda \in [0, \bar{\lambda} + \varepsilon)$.

Proof. First, observe that $P(c, b)$ still satisfies the Slater condition if (c, b) is close enough to $\left(\bar{c}, \bar{b}\right)$; so that the KKT conditions at any $((c, b), x) \in$ gph$\mathcal{S}$ with (c, b) close enough to $\left(\bar{c}, \bar{b}\right)$ yield

$$-c \in \lambda C_b(x) \text{ for some } \lambda \geq 0.$$

Reasoning by contradiction, assume the existence of $\varepsilon > 0$ and a sequence $\{((c^n, b^n), x^n)\}_{n\in\mathbb{N}} \subset$ gph$\mathcal{S}$ converging to $\left(\left(\bar{c}, \bar{b}\right), \bar{x}\right)$ such that

$$-c^n = \lambda^n \sum_{t\in T_{b^n}(x^n)} \mu_t^n a_t,$$

with $\lambda^n \geq 0$, $\sum_{t\in T_{b^n}(x^n)} \mu_t^n = 1$, μ_t^n nonnegative for all $t \in T_{b^n}(x^n)$ and only finitely many of them being positive, in such a way that

$$\lambda^n \geq \bar{\lambda} + \varepsilon \tag{32}$$

for all $n \in \mathbb{N}$.

In this case, appealing to Carathéodory's Theorem and following the lines of [6, Lemma 3] (see also [6, Lemma 1] and references therein), we may assume, for an appropriate subsequence of n's,

$$\lambda^n \to \lambda, \text{ and } \sum_{t\in T_{b^n}(x^n)} \mu_t^n a_t \to \sum_{t\in T_{\bar{b}}(\bar{x})} \mu_t a_t$$

for some $\lambda \geq 0$, μ_t nonnegative for all $t \in T_{\bar{b}}(\bar{x})$ and only finitely many of them being positive, such that $\sum_{t\in T_{\bar{b}}(\bar{x})} \mu_t = 1$. Moreover, (32) yields

$$\lambda \geq \bar{\lambda} + \varepsilon.$$

In this way, we attain a contradiction with Lemma 9. □

The following theorem provides the aimed upper bound on $\operatorname{clm}\mathcal{S}((\bar{c},\bar{b}),\bar{x})$.

THEOREM 11. *Assume that $P(\bar{c},\bar{b})$ satisfies the Slater condition and let $\bar{x} \in \mathcal{S}\left(\bar{c},\bar{b}\right)$. Then,*

$$\operatorname{clm}\mathcal{S}((\bar{c},\bar{b}),\bar{x}) \leq \max\left\{\bar{\lambda},1\right\}\operatorname{clm}\mathcal{L}((\bar{c}'\bar{x},\bar{b}),\bar{x}),$$

where $\bar{\lambda}$ is defined in (30).

Proof. If $\operatorname{clm}\mathcal{L}((\bar{c}'\bar{x},\bar{b}),\bar{x}) = \infty$, then the inequality is trivial. Let $\operatorname{clm}\mathcal{L}((\bar{c}'\bar{x},\bar{b}),\bar{x}) < \gamma < \infty$ and let $\varepsilon > 0$. Then, appealing to [17, Theorem 1], there exists a neighborhood U_1 of $\bar{x}$ such that

$$d\left(x,\mathcal{S}(\bar{c},\bar{b})\right) < \gamma\bar{f}\left(x\right) \tag{33}$$

for all $x \in U_1$ with $\bar{f}\left(x\right) > 0$, where $\bar{f}$ is the supremum function defined in (12). If $(b,x) \in \operatorname{gph}\mathcal{F}$, then

$$\bar{f}\left(x\right) \leq \max\{\bar{c}'(x-\bar{x}),\ \sup_{t\in T}(b_t-\bar{b}_t)\} \leq \max\left\{\bar{c}'(x-\bar{x}),\left\|b-\bar{b}\right\|_\infty\right\}. \tag{34}$$

By Lemma 10, there exist neighborhoods U_2 of $\bar{x}$ and V of $\left(\bar{c},\bar{b}\right)$ such that, for any $(c,b) \in V$ and $x \in \mathcal{S}(c,b) \cap U_2$, (31) holds for some $\lambda \in [0,\bar{\lambda}+\varepsilon)$. Let $(c,b) \in V$ and $x \in \mathcal{S}(c,b) \cap U_1 \cap U_2 \cap B_\varepsilon\left(\bar{x}\right)$ with $\bar{f}\left(x\right) > 0$. Then $-c \in \sum_{t\in T_0}\lambda_t a_t$ for some finite subset $T_0 \subset T_b\left(x\right)$ and some numbers $\lambda_t \geq 0$, $t \in T_0$, satisfying $\sum_{t\in T_0}\lambda_t < \bar{\lambda}+\varepsilon$. Hence,

$$\begin{aligned}-c'(x-\bar{x}) &= \sum_{t\in T_0}\lambda_t a_t'(x-\bar{x}) = \sum_{t\in T_0}\lambda_t(b_t-a_t'\bar{x})\\ &\geq \sum_{t\in T_0}\lambda_t(b_t-\bar{b}_t) \geq -(\bar{\lambda}+\varepsilon)\left\|b-\bar{b}\right\|_\infty,\end{aligned}$$

and consequently

$$\begin{aligned}\bar{c}'(x-\bar{x}) &\leq (\bar{\lambda}+\varepsilon)\left\|b-\bar{b}\right\|_\infty + \left\|c-\bar{c}\right\|_*\left\|x-\bar{x}\right\|\\ &\leq (\bar{\lambda}+2\varepsilon)\left\|(c,b)-(\bar{c},\ \bar{b})\right\|.\end{aligned} \tag{35}$$

Combining (33), (34), and (35), we obtain

$$d\left(x,\mathcal{S}(\bar{c},\bar{b})\right) \leq \gamma\max\{\bar{\lambda}+2\varepsilon,1\}\left\|(c,b)-(\bar{c},\bar{b})\right\|.$$

Hence,

$$\operatorname{clm}\mathcal{S}((\bar{c},\bar{b}),\bar{x}) \leq \gamma\max\{\bar{\lambda}+2\varepsilon,1\}.$$

The aimed equality follows after passing to limits in the right hand side of the last equality as $\varepsilon \downarrow 0$ and $\gamma \downarrow \operatorname{clm}\mathcal{L}((\bar{c}'\bar{x},\bar{b}),\bar{x})$. □

5.2 An upper bound relying on the nominal data

Throughout this subsection we assume that:

(i) T is finite;

(ii) $-\bar{c} \in \operatorname{int} A_{\bar{b}}\left(\bar{x}\right)$.

Condition (ii) implies $\mathcal{S}\left(\bar{c},\bar{b}\right)=\{\bar{x}\}$ and $|T|\geq p$ according to a standard argument in linear programming.

Our goal in this subsection consists of demonstrating that the quantity $\max\limits_{D\in\mathcal{T}_{\bar{b}}(\bar{x})}\left\|A_D^{-1}\right\|$ constitutes an upper bound on $\operatorname{clm}\mathcal{S}((\bar{c},\bar{b}),\bar{x})$. This upper bound has the virtue to be easily computable as it relies exclusively on the nominal data $(\bar{c},\bar{b})$ and the nominal solution $\bar{x}$. In Section 6, we show that this upper bound may not be attained.

Lemma 12. *Under conditions (i) and (ii) above, there exist sequences $b^n\to\bar{b}$ and $\mathcal{S}\left(\bar{c},b^n\right)\ni x^n\to\bar{x}$ with $\mathcal{T}_{b^n}\left(x^n\right)\neq\emptyset$ for all $n\in\mathbb{N}$ such that*

$$\operatorname{clm}\mathcal{S}_{\bar{c}}\left(\bar{b},\bar{x}\right)=\limsup_{n\to\infty}\frac{\|x^n-\bar{x}\|}{\left\|b^n-\bar{b}\right\|_\infty}.$$

Proof. The only point to be proved is that sequence $(x^n)_{n\in\mathbb{N}}$ may be chosen in such a way that $\mathcal{T}_{b^n}\left(x^n\right)\neq\emptyset$ for all n. Let

$$\operatorname{clm}\mathcal{S}_{\bar{c}}\left(\bar{b},\bar{x}\right)=\limsup_{n\to\infty}\frac{\|z^n-\bar{x}\|}{\left\|b^n-\bar{b}\right\|_\infty}$$

with $b^n\to\bar{b}$ and $\mathcal{S}\left(\bar{c},b^n\right)\ni z^n\to\bar{x}$. Assume that, for some $n\in\mathbb{N}$, $\mathcal{T}_{b^n}\left(z^n\right)=\emptyset$, and let us show that z^n may be replaced with $x^n\in\mathcal{S}\left(\bar{c},b^n\right)$ such that $\mathcal{T}_{b^n}\left(x^n\right)\neq\emptyset$ and $\|x^n-\bar{x}\|\geq\|z^n-\bar{x}\|$. This will complete the proof.

According to the KKT conditions (recall that T is finite), and taking into account the current assumption that $\mathcal{T}_{b^n}\left(z^n\right)=\emptyset$, we can write

$$-\bar{c}=\sum_{i=1}^{k}\lambda_i^n a_{t_i^n}, \tag{36}$$

with $k<p$, $\lambda_i^n\geq 0$ for $1\leq i\leq k$, $D_n:=\{t_1^n,...,t_k^n\}\subset T_{b^n}\left(z^n\right)$ and $\left\{a_{t_1^n},...,a_{t_k^n}\right\}$ linearly independent (by Carathéodory's Theorem). Without loss of generality, assume that D_n is maximal, in the sense that there does not exist $\widetilde{D}_n\supsetneq D_n$ such that $\widetilde{D}_n\subset T_{b^n}\left(z^n\right),-\bar{c}\in\operatorname{cone}\left\{a_t,\ t\in\widetilde{D}_n\right\}$ and $\left\{a_t,\ t\in\widetilde{D}_n\right\}$ is linearly independent. Pick $0_p\neq u^n\in\left\{a_{t_1^n},...,a_{t_k^n}\right\}^\perp$ (hence $u^n\perp\bar{c}$). It is a basic fact that either $\|z^n+\alpha u^n-\bar{x}\|\geq\|z^n-\bar{x}\|$ for all $\alpha>0$ or $\|z^n-\alpha u^n-\bar{x}\|\geq\|z^n-\bar{x}\|$ for all $\alpha>0$ (see, [5, Lemma 5]). Without loss of generality, assume that the first case holds. Note that $D_n\subset T_{b^n}\left(z^n+\alpha u^n\right)$ and $z^n+\alpha u^n\in\mathcal{S}\left(\bar{c},b^n\right)$ whenever $z^n+\alpha u^n$ is feasible for b^n. On the other hand, the boundedness of $\{\bar{x}\}=\mathcal{S}\left(\bar{c},\bar{b}\right)$ entails the boundedness of $\mathcal{S}\left(\bar{c},b^n\right)$ for n large enough (by the upper semicontinuity of $\mathcal{S}$ at $\left(\bar{c},\bar{b}\right)$; see, e.g., [2, Theorem 4.3.3]). Hence, $\alpha^n:=\sup\left\{\alpha\geq 0\mid z^n+\alpha u^n\in\mathcal{F}\left(b^n\right)\right\}$ is finite and, then, there exists a new index, say t_{k+1}^n, such that $t_{k+1}^n\in T_{b^n}\left(z^n+\alpha^n u^n\right)$ and that $\left\{a_{t_1^n},...,a_{t_k^n},a_{t_{k+1}^n}\right\}$ is linearly independent. Specifically, the maximality of D_n implies that $\alpha^n\neq 0$ and the definition of α^n entails the existence of $t_{k+1}^n\in T_{b^n}\left(z^n+\alpha^n u^n\right)$ such that $\left\langle a_{t_{k+1}^n},u^n\right\rangle>0$ yielding the linear independence of $\left\{a_{t_1^n},...,a_{t_k^n},a_{t_{k+1}^n}\right\}$. Obviously we can extend (36) by defining $\lambda_{k+1}^n=0$. If $k+1=p$, take $x^n=z^n+\alpha^n u^n$, otherwise, we repeat the process until finding the aimed point x^n and an associated set of active indices $\left\{t_1^n,...,t_p^n\right\}$ belonging to $\mathcal{T}_{b^n}\left(x^n\right)$.
□

Theorem 13 (Upper bound on the calmness modulus).

$$\operatorname{clm}\mathcal{S}\left(\left(\bar{c},\bar{b}\right),\bar{x}\right)\leq\max_{D\in\mathcal{T}_{\bar{b}}(\bar{x})}\left\|A_D^{-1}\right\|. \tag{37}$$

Proof. According to the previous lemma together with the fact that $\mathrm{clm}\mathcal{S}((\bar{c},\bar{b}),\bar{x}) = \mathrm{clm}\mathcal{S}_{\bar{c}}(\bar{b},\bar{x})$, established in Theorem 7 we write

$$\mathrm{clm}\mathcal{S}\left(\left(\bar{c},\bar{b}\right),\bar{x}\right) = \lim_{n\to+\infty}\frac{\|x^n-\bar{x}\|}{\left\|b^n-\bar{b}\right\|_\infty}$$

with $b^n \to \bar{b}$, $\mathcal{S}(\bar{c},b^n) \ni x^n \to \bar{x}$, and $\mathcal{T}_{b^n}(x^n) \neq \emptyset$ for all $n \in \mathbb{N}$. Choose $D_n \in \mathcal{T}_{b^n}(x^n)$ for each n, and write $x^n = A_{D_n}^{-1}b_{D_n}^n$. From the finiteness of T, the sequence $(D_n)_{n\in\mathbb{N}}$ of subsets of T has a constant subsequence. Thus, we may assume (without renumbering) that $D_n = D_0$ for all $n \in \mathbb{N}$, and therefore $x^n = A_{D_0}^{-1}b_{D_0}^n$ for $n \in \mathbb{N}$. Since $D_0 \subset T_{b^n}(x^n)$, $n = 1, 2, ...$, one easily checks that $D_0 \subset T_{\bar{b}}(\bar{x})$, and then we can write $\bar{x} = A_{D_0}^{-1}\bar{b}_{D_0}$. Moreover, the fact that, for any n, $D_0 = D_n \in \mathcal{T}_{b^n}(x^n)$ trivially implies $D_0 \in \mathcal{T}_{\bar{b}}(\bar{x})$. Therefore,

$$\begin{aligned}\mathrm{clm}\mathcal{S}\left(\left(\bar{c},\bar{b}\right),\bar{x}\right) &= \lim_{n\to+\infty}\frac{\|x^n-\bar{x}\|}{\left\|b^n-\bar{b}\right\|_\infty} \leq \limsup_{n\to+\infty}\frac{\left\|A_{D_0}^{-1}\left(b_{D_0}^n-\bar{b}_{D_0}\right)\right\|}{\left\|b_{D_0}^n-\bar{b}_{D_0}\right\|_\infty}\\ &\leq \left\|A_{D_0}^{-1}\right\| \leq \max_{D\in\mathcal{T}_{\bar{b}}(\bar{x})}\left\|A_D^{-1}\right\|.\end{aligned}$$

□

6. Illustrative examples. Example 1 (Upper bound in Theorem 13 attained). Consider the parameterized linear optimization problem, in $\mathbb{R}^2$ endowed with the Euclidean norm:

$$\begin{array}{ll} \textit{minimize} & x_1 + \frac{1}{3}x_2 \\ \textit{subject to} & -x_1 \leq b_1, \\ & -x_1 - \frac{1}{2}x_2 \leq b_2, \\ & -x_1 - x_2 \leq b_3, \end{array}$$

with the nominal parameter $\bar{b} = 0_3$ and $\bar{c} = (1, 1/3)'$. In this case, $\bar{x} = 0_2$ is the unique optimal solution. According to Theorem 3 (v), it is easy to see that $\mathcal{S}$ is Aubin continuous (and therefore isolatedly calm) at $\left(\bar{c},\bar{b}\right)$ for $\bar{x}$. Moreover,

$$\mathcal{T}_{\bar{b}}(\bar{x}) = \{\{1,2\},\{1,3\}\}, \quad \left\|A_{\{1,2\}}^{-1}\right\| = \sqrt{17}, \quad \left\|A_{\{1,3\}}^{-1}\right\| = \sqrt{5}.$$

(According to the last expression in (16), $\left\|A_D^{-1}\right\|$ is the inverse of the $\|\cdot\|_*$-distance from the origin to the boundary of the parallelogram whose vertices are $\{\pm a_t \mid t \in D\}$.) If we consider the sequence $\{(b^n, x^n)\}_{n\in\mathbb{N}}$ in gph $\mathcal{S}_{\bar{c}}$ given by $b^n = (1/n, -1/n, 0)'$ and $x^n = (-1/n, 4/n)$, we have

$$\lim_{n\to+\infty}\frac{\|x^n-\bar{x}\|}{\left\|b^n-\bar{b}\right\|_\infty} = \sqrt{17}. \tag{38}$$

This expression, together with Theorem 13, entails $\mathrm{clm}\mathcal{S}\left(\left(\bar{c},\bar{b}\right),\bar{x}\right) = \sqrt{17}$.

Observe that in this example the expression for $\mathrm{clm}\mathcal{S}\left(\left(\bar{c},\bar{b}\right),\bar{x}\right)$ provided in Corollary 8 can be written as

$$\mathrm{clm}\mathcal{S}\left(\left(\bar{c},\bar{b}\right),\bar{x}\right) = \lim_{n\to+\infty}\frac{1}{d_*\left(0_2,\partial f_D\left(x^n\right)\right)} = \sqrt{17},$$

where x^n is the same as above and $D = \{1,2\}$. The reader can easily check that $\partial f_D(x^n) = \text{co}\{a_1, -a_2\}$ and $d_*(0_2, \partial f_D(x^n)) = 1/\sqrt{17}$, appealing to (16).

Remark 4. *If we remove the last inequality in the previous example, then alternative b^n and x^n is given by $(-1/n, 1/n)$ and $(1/n, -4/n)$. Observe that the latter point does not fulfill the referred last inequality (for index $t = 3$). To make this point feasible, we would need to perturb $\bar{b}_3$ until $-3/n$, and the limit in (38) is $\sqrt{17}/3$. This fact is used in the following example to show that the upper bound on the calmness modulus may not be attained.*

Example 2 (Upper bound in Theorem 13 not attained).

Consider the linear optimization problem, in $\mathbb{R}^2$ endowed with the Euclidean norm:

$$\begin{array}{lll} \textit{minimize} & x_1 + \frac{1}{3}x_2 & \\ \textit{subject to} & -x_1 \leq b_1, & (t=1) \\ & -x_1 - \frac{1}{2}x_2 \leq b_2, & (t=2) \\ & -x_1 - x_2 \leq b_3, & (t=3) \\ & -x_1 + x_2 \leq b_4, & (t=4) \end{array}$$

with the nominal data $\bar{b} = 0_4$, $\bar{c} = (1, 1/3)'$ and $\bar{x} = 0_2$. One has

$$\mathcal{T}_{\bar{b}}(\bar{x}) = \{\{1,2\}, \{1,3\}, \{2,4\}, \{3,4\}\}$$
$$\left\| A_{\{1,2\}}^{-1} \right\| = \sqrt{17}, \quad \left\| A_{\{1,3\}}^{-1} \right\| = \sqrt{5}, \quad \left\| A_{\{2,4\}}^{-1} \right\| = \sqrt{17}/3, \quad \left\| A_{\{3,4\}}^{-1} \right\| = 1.$$

As in Example 1, we have Aubin continuity of $\mathcal{S}$ at these nominal data. But now, following the lines of the previous example and remark, both x^n and $-x^n$ are infeasible for $b^n = (1/n, -1/n, 0, 0)'$, and we would need at least either $b_4^n = 5/n$ or $b_3^n = \frac{3}{n}$ for the feasibility of x^n or $-x^n$, respectively. Indeed, it is easy to see that for any $b \in \mathbb{R}^4$ with $\|b\|_\infty \leq \varepsilon$ (for any given $\varepsilon > 0$), one has $(\varepsilon, 0)' \in \mathcal{F}(b)$ and $\mathcal{F}(b) \subset \mathcal{F}(\varepsilon e)$, with $e = (1,1,1,1)'$; and consequently $\mathcal{S}(\bar{c}, b) \subset \left\{ x \in \mathcal{F}(\varepsilon e) \mid x_1 + \frac{1}{3}x_2 \leq \varepsilon, \right\}$, where the last inequality is nothing else but $\langle \bar{c}, x \rangle \leq \left\langle \bar{c}, (\varepsilon, 0)' \right\rangle$. We can go a bit further by saying that $x_1 \leq \varepsilon$ for all $x \in \mathcal{S}(\bar{c}, b)$, since otherwise neither $t = 1$ nor $t = 4$ might be active indices at x ($t = 4$ cannot be active as $x_1 > \varepsilon$ and $x_1 + \frac{1}{3}x_2 \leq \varepsilon$ imply $x_2 < 0$), and therefore x would not be optimal according to the KKT conditions. Thus,

$$\mathcal{S}(\bar{c}, b) \subset \left\{ x \in \mathcal{F}(\varepsilon e) \mid x_1 + \frac{1}{3}x_2 \leq \varepsilon, \; x_1 \leq \varepsilon \right\},$$

and consequently (recalling that we are working with the Euclidean norm) $\|x - \bar{x}\| \leq \left\| \bar{x} - (\varepsilon, -2\varepsilon)' \right\| = \sqrt{5}\varepsilon$ whenever $\|b\|_\infty \leq \varepsilon$ and $x \in \mathcal{S}(\bar{c}, b)$. Moreover $(\varepsilon, -2\varepsilon)' \in \mathcal{S}(\bar{c}, b^\varepsilon)$ with $b^\varepsilon := (-\varepsilon, \varepsilon, \varepsilon, \varepsilon)'$ and $\mathcal{T}_{b^\varepsilon}\left((\varepsilon, -2\varepsilon)'\right) = \{1,3\}$. Gathering all of this, we have

$$\text{clm}\mathcal{S}\left(\left(\bar{c}, \bar{b}\right), \bar{x}\right) = \limsup_{\varepsilon \to 0^+} \frac{\sqrt{5}\varepsilon}{\varepsilon} = \sqrt{5}. \tag{39}$$

Concerning the expression of Corollary 8, in this case we can write, once that we know (39)

$$\text{clm}\mathcal{S}\left(\left(\bar{c}, \bar{b}\right), \bar{x}\right) = \lim_{\varepsilon \to 0^+} \frac{1}{d_*(0_2, \partial f_D(x^\varepsilon))} = \sqrt{5},$$

where $x^{\varepsilon} := (\varepsilon, -2\varepsilon)'$ and $D = \{1,3\}$. The reader can easily check that $\partial f_D(x^{\varepsilon}) = \mathrm{co}\{-a_1, a_3\}$ and $d_*(0_2, \partial f_D(x^{\varepsilon})) = 1/\sqrt{5}$ appealing once more to (16).

The next remark emphasizes the difficulties which may arise when trying to compute the calmness modulus in comparison with the Lipschitz modulus.

Remark 5. In the previous example, we may write, taking Theorem 5 into account,

$$\mathrm{lip}\mathcal{S}\left(\left(\bar{c}, \bar{b}\right), \bar{x}\right) = \sqrt{17} = \lim_{n\to\infty} \frac{\|x^n - \widetilde{x}^n\|}{\left\|b^n - \widetilde{b}^n\right\|_\infty}$$

with $x^n = (-1/n, 4/n)'$, $\widetilde{x}^n = \bar{x} = 0_2$, $b^n = (1/n, -1/n, 0, 5/n)'$, $\widetilde{b}^n = (0,0,0,5/n)'$. We finish this section we a semi-infinite example which tries to show some geometrical ideas that make us conjecture that perturbations of $\bar{c}$ are negligible when computing the calmness modulus of $\mathcal{S}$ at $\left(\bar{c}, \bar{b}\right)$; i. e., $\mathrm{clm}\mathcal{S}\left(\left(\bar{c}, \bar{b}\right), \bar{x}\right) = \mathrm{clm}\mathcal{S}_{\bar{c}}\left(\bar{b}, \bar{x}\right)$. In this example one has that T is infinite, $P(\bar{c}, \bar{b})$ satisfies the Slater condition, $\bar{x} \in \mathcal{S}(\bar{c}, \bar{b})$, and $-\bar{c} \in \mathrm{int}\, A_{\bar{b}}(\bar{x})$ (which already implies $\mathcal{S}\left(\bar{c}, \bar{b}\right) = \{\bar{x}\}$ according to Theorem 2).

Example 3. Consider the linear optimization problem, in $\mathbb{R}^2$ endowed with the Euclidean norm:

$$\begin{array}{lll} \textit{minimize} & x_1 & \\ \textit{subject to} & (\cos t)\, x_1 + (\sin t)\, x_2 \leq b_t, & t \in [-\pi, \pi] \\ & -x_1 - x_2 \leq b_4, & (t=4) \\ & -x_1 + x_2 \leq b_5. & (t=5) \end{array}$$

According to Theorem 2, $\mathcal{S}$ is isolatedly calm at $\left(\bar{c}, \bar{b}\right)$ for $\bar{x}$, with $\bar{c} = (1,0)'$, $\bar{b}_t = 1$ for all $t \in T := [-\pi, \pi] \cup \{4,5\}$ and $\bar{x} = (-1,0)'$. Here $T_{\bar{b}}(\bar{x}) = \{-\pi, \pi, 4, 5\}$ and

$$\mathcal{T}_{\bar{b}}(\bar{x}) = \{\{-\pi, 4\}, \{-\pi, 5\}, \{\pi, 4\}, \{\pi, 5\}, \{4, 5\}\}.$$

Note that $\{-\pi, \pi\} \notin \mathcal{T}_{\bar{b}}(\bar{x})$ since $a_{-\pi}$ and a_π are not linearly independent. We can easily see that $\left\|A_D^{-1}\right\| = \sqrt{5}$ for all $D \in \mathcal{T}_{\bar{b}}(\bar{x}) \setminus \{\{4,5\}\}$ and $\left\|A_{\{4,5\}}^{-1}\right\| = 1$. We will show that in this example

$$\mathrm{clm}\mathcal{S}\left(\left(\bar{c}, \bar{b}\right), \bar{x}\right) = \mathrm{clm}\mathcal{S}_{\bar{c}}\left(\bar{b}, \bar{x}\right) = \sqrt{5}.$$

Let $0 < \varepsilon < 3 - \sqrt{8}$ be given (the upper bound $3 - \sqrt{8}$ will guarantee the existence of some forthcoming elements). We can check that, for all $(c, b) \in \mathbb{R} \times C(T, \mathbb{R})$ with $\left\|(c, b) - \left(\bar{c}, \bar{b}\right)\right\| \leq \varepsilon$, we have

$$\mathcal{S}(c, b) \subset \left\{ x \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} -x_1 - x_2 \leq 1 + \varepsilon, \ \ -x_1 + x_2 \leq 1 + \varepsilon, \\ x_1 \leq -\sqrt{(1-\varepsilon)^2 - x_2^2} \end{array} \right\}. \tag{40}$$

The first two inequalities need to hold for feasibility reasons. The last inequality obeys the KKT conditions, in order to ensure the existence of active indices at $x \in \mathcal{S}(c, b)$ to enable $-c \in A_b(x)$.

Routine calculus shows that the furthest points from the origin in the right-hand side set of (40) are

$$\frac{1}{2}\left(-1 - \varepsilon - \sqrt{1 - 6\varepsilon + \varepsilon^2}, \pm\left(1 + \varepsilon - \sqrt{1 - 6\varepsilon + \varepsilon^2}\right)\right) \approx (-1 + \varepsilon, \pm 2\varepsilon).$$

This yields $\mathrm{clm}\mathcal{S}\left(\left(\bar{c},\bar{b}\right),\bar{x}\right)\leq\sqrt{5}$. Next we provide a sequence $\mathrm{gph}\mathcal{S}_{\bar{c}}\ni\left(b^n,x^n\right)\to\left(\bar{b},\bar{x}\right)$ such that

$$\lim_{n\to\infty}\frac{\|x^n-\bar{x}\|}{\left\|b^n-\bar{b}\right\|_\infty}=\sqrt{5}. \tag{41}$$

Setting $\varepsilon=1/n$ with $n\in\mathbb{N}$ in the above expressions, let

$$\widetilde{b}_t^n:=\begin{cases}1-\dfrac{1}{n} & \text{if } t\in[-\pi,\pi],\\ 1+\dfrac{1}{n} & \text{if } t\in\{4,5\},\end{cases}$$

$$x^n=(x_1^n,x_2^n):=\frac{1}{2}\left(-1-\frac{1}{n}-\sqrt{1-\frac{6}{n}+\frac{1}{n^2}},-1-\frac{1}{n}+\sqrt{1-\frac{6}{n}+\frac{1}{n^2}}\right).$$

Then $T_{\widetilde{b}^n}\left(x^n\right)=\left\{-\arccos\frac{x_1^n}{1-\frac{1}{n}},\ 4\right\}$. The problem here is that making x^n optimal for $\widetilde{b}^n$ yields a large (in relative terms) perturbation of $\bar{c}$, namely, $c^n=-x^n$, that would enlarge the ratio as $\lim_{n\to\infty}\frac{\|x^n-\bar{x}\|}{\left\|(c^n,\widetilde{b}^n)-\left(\bar{c},\bar{b}\right)\right\|}=1$. Thus, it would be convenient to perform a small perturbation of $\widetilde{b}^n$ for getting a b^n such that $x^n\in\mathcal{S}_{\bar{c}}\left(b^n\right)$ and (41) holds. The reader can check that such an element is given by

$$b_t^n:=\begin{cases}1-\dfrac{1}{n} & \text{if } |t|\leq\alpha_n,\\ \left(1-\dfrac{1}{n}\right)\cos\left(|t|-\alpha_n\right) & \text{if } |t|>\alpha_n,\\ 1+\dfrac{1}{n} & \text{if } t\in\{4,5\},\end{cases}$$

where $\alpha_n:=\arccos\frac{x_1^n}{1-\frac{1}{n}}$.

7. Concluding remarks. The paper provides lower and upper estimates for the calmness modulus of $\mathcal{S}$ under different assumptions. The main contributions of this work appeal to three different constants:

$$\begin{aligned}C1&:=\sup_{D\in\mathcal{K}_{\bar{b}}(\bar{x})}\limsup_{\substack{x\to\bar{x}\\ f_D(x)>0}}\frac{1}{d_*\left(0_p,\partial f_D\left(x\right)\right)}\\ &=\sup_{D\in\mathcal{K}_{\bar{b}}(\bar{x})}\mathrm{clm}\mathcal{L}_D((\bar{b},-\bar{b}_D),\bar{x}),\\ C2&:=\max\left\{\bar{\lambda},1\right\}\mathrm{clm}\mathcal{L}((\bar{c}'\bar{x},\bar{b}),\bar{x}),\end{aligned}$$

and

$$C3:=\max_{D\in\mathcal{T}_{\bar{b}}(\bar{x})}\left\|A_D^{-1}\right\|.$$

(Recall that set $\mathcal{K}_{\bar{b}}\left(\bar{x}\right)$, functions f_D, and mappings $\mathcal{L}_D$, for $D\in\mathcal{K}_{\bar{b}}\left(\bar{x}\right)$, are defined at the beginning of Section 3; $\bar{\lambda}$ is defined in (30), and $\mathcal{T}_{\bar{b}}\left(\bar{x}\right)$ is in (15)). The main results of the present paper concerning the semi-infinite case, with T being a compact Hausdorff space, are:

- If $P\left(\bar{c},\bar{b}\right)$ satisfies the Slater condition, then:

$$\text{clm}\mathcal{S}_{\bar{c}}\left(\bar{b},\bar{x}\right) \leq \text{clm}\mathcal{S}\left(\left(\bar{c},\bar{b}\right),\bar{x}\right) \leq C2.$$

- Moreover, if $\bar{x}$ is the unique solution, then

$$C1 \leq \text{clm}\mathcal{S}_{\bar{c}}\left(\bar{b},\bar{x}\right).$$

Observe that $C1$ and $C2$ translate the question of estimating the calmness modulus of our optimal set mapping $\mathcal{S}$ into the –more exploited in the literature– question of computing the calmness moduli of appropriate feasible set mappings. Specifically, one can find in the literature different approaches and expressions applicable to the feasible set mapping of semi-infinite linear inequality systems. In addition to the already mentioned [16, Theorem 1], the reader is addressed to the basic constraint qualification approach by Zheng and Ng [23]. An operative matricial expression for the calmness modulus of feasible set mappings when confined to finite linear inequality systems can be found in [8].

Remark 6. *Both lower and upper bounds on* $\text{clm}\mathcal{S}\left(\left(\bar{c},\bar{b}\right),\bar{x}\right)$ *constitute information about the quantitative stability of the nominal problem. An upper bound provides a measure of stability of the optimal solution set, while any positive lower bound constitutes a measure of its instability. Specifically, in the case when* $0 < C1 \leq \text{clm}\mathcal{S}\left(\left(\bar{c},\bar{b}\right),\bar{x}\right)$, *given any constant* C *such that* $0 < C < C1$, *and any* $\varepsilon > 0$, *as a direct consequence of the definitions, there exist a point* x *and a parameter* (c,b) *verifying that*

$$d\left(x,\mathcal{S}\left(\bar{c},\bar{b}\right)\right) > C\ d\left((c,b),\left(\bar{c},\bar{b}\right)\right),\ \ \|x-\bar{x}\| < \varepsilon \text{ and } \left\|(c,b)-\left(\bar{c},\bar{b}\right)\right\| < \varepsilon\ .$$

Remark 7. *We point out that if we know some upper bound for* $\text{clm}\mathcal{S}\left(\left(\bar{c},\bar{b}\right),\bar{x}\right)$, *the definition of* $C1$ *sometimes provides an operative strategy for computing the aimed calmness modulus, provided that* $C1$ *is a lower bound. The strategy consists of finding* $D \in \mathcal{K}_{\bar{b}}\left(\bar{x}\right)$ *and a sequence of points* $\{x^r\}$ *approaching* $\bar{x}$, *with* $f_D\left(x^r\right) > 0$ *for all* r *and checking if* $\lim_{r\to\infty}\frac{1}{d_*(0_p,\partial f_D(x^r))}$ *coincides with the known upper bound. This, in fact, the argument followed in Example 1 for deriving the exact value of* $\text{clm}\mathcal{S}\left(\left(\bar{c},\bar{b}\right),\bar{x}\right)$ *In this example,* T *is finite and the known upper bound is* $C3$ *(see the next paragraphs for details about the finite case).*

When confined to the special case of ordinary linear programming problems, we have

- If T is finite, then

$$\text{clm}\mathcal{S}_{\bar{c}}\left(\bar{b},\bar{x}\right) = \text{clm}\mathcal{S}\left(\left(\bar{c},\bar{b}\right),\bar{x}\right) \leq C1.$$

- Moreover, if $\mathcal{S}\left(\bar{c},\bar{b}\right) = \{\bar{x}\}$, then

$$\text{clm}\mathcal{S}\left(\left(\bar{c},\bar{b}\right),\bar{x}\right) = C1 \leq C3.$$

Remark 8. *One can find in the literature many contributions to the stability theory and sensitivity analysis in ordinary (finite) linear programming. As a direct antecedent to the previous results, confined to the finite case, we underline the classical works of W. Li ([18], [19]). In fact, the appearence of constant* $C3$ *reminds of the Lipschitz constant for* $\mathcal{S}_{\bar{c}}$ *introduced in [18]. More in detail, in relation to* $\mathcal{S}_{\bar{c}}$, *[18]* *analyzes the (global) Lipschitz property which (adapted to the current notation) is satisfied by this mapping if there exists* $\gamma > 0$ *such that*

$$d_H\left(\mathcal{S}_{\bar{c}}(b), \mathcal{S}_{\bar{c}}(b')\right) \leq \gamma d(b, b'), \text{ for all } b \text{ and } b', \tag{42}$$

where d_H *represents the Hausdorff distance. In contrast with calmness property, this is not a local property in the sense that parameters are allowed to run over the whole space; moreover, the distances in the left hand side are between the whole optimal sets. In any case, it is obvious that any Lispchitz constant* γ *verifying (42) is, in particular, a calmness constant, and then an upper bound on* $\mathrm{clm}\mathcal{S}_{\bar{c}}\left(\bar{b}, \bar{x}\right)$. *For comparative purposes with our Theorem 13, we recall here the expression of* γ *in [18] under the assumption* $\mathcal{S}\left(\bar{c}, \bar{b}\right) = \{\bar{x}\}$ *(yielding that the rank of the coefficient vectors in the left hand side of the constraints is* p) :

$$\gamma = \max_{\substack{D \subset T,\ |D| = p \\ A_D \text{ is non-singular}}} \left\| A_D^{-1} \right\|.$$

One can easily find examples in which

$$\mathrm{C3} < \gamma.$$

(See for instance [5, Example 4] where ($C3$ *provides the exact Lispchitz modulus associated to the Aubin property.) In fact, it is proven in [18] that* γ *is a Lispchitz constant for* $\mathcal{S}_{\bar{c}}\left(\bar{b}, \bar{x}\right)$, *for all* $\bar{c}$ *(observe that* γ *does not depend on* $\bar{c}$*) and [18, Theorem. 3.5] establishes the sharpness of* γ *for some* $\bar{c}$.

Finally, we underline an specific fruitful argument in the finite case (for analyzing $\mathrm{clm}\mathcal{S}\left(\left(\bar{c}, \bar{b}\right), \bar{x}\right)$) which is no longer applicable when T in infinite. The argument reads as follows: if T is finite, when we consider a sequence of points $\{x^r\}$ converging to $\bar{x}$ and a sequence of parameters $\{(c^r, b^r)\}$ converging to $\left(\bar{c}, \bar{b}\right)$ such that $x^r \in \mathcal{S}(c^r, b^r)$ (i.e., x^r is a KKT point of $P(c^r, b^r)$, entailing

$$-c^r \in cone\{a_t : t \in D_r\}, \text{ with } D_r \subset T_{b^r}(x^r))$$

we always can extract a constant subsequence of $\{D_r\}$; moreover, for r large enough

$$T_{b^r}(x^r) \subset T_{\bar{b}}(\bar{x}).$$

Example 3 provides a situation in which T is infinite and the previous argument is not applicable, and illustrates the difficulty of the generalization of some results from finite to semi-infinite programs. In fact, the problem of providing an exact formula for $\mathrm{clm}\mathcal{S}\left(\left(\bar{c}, \bar{b}\right), \bar{x}\right)$, perhaps under appropriate assumptions, still remains open and constitutes one of the further lines of research.

Acknowledgement. The authors would like to thank the reviewers and the Associate Editor for the criticism and suggestions. The second and the last author would like to thank the universities of Alicante and Miguel Hernández de Elche for hopsitality and support.

REFERENCES

[1] D. Azé and J.-N. Corvellec, *Characterizations of error bounds for lower semicontinuous functions on metric spaces*, ESAIM Control Optim. Calc. Var., 10 (2004), pp. 409–425.

[2] B. Bank, J. Guddat, D. Klatte, B. Kummer, and K. Tammer, *Nonlinear Parametric Optimization*, Birkhäuser Verlag, Basel, 1983.

[3] M. J. Cánovas, A. L. Dontchev, M. A. López, and J. Parra, *Metric regularity of semi-infinite constraint systems*, Math. Program., 104 (2005), pp. 329–346.

[4] ———, *Isolated calmness of solution mappings in convex semi-infinite optimization*, J. Math. Anal. Appl., 350 (2009), pp. 829–837.

[5] M. J. Cánovas, F. J. Gómez-Senent, and J. Parra, *On the Lipschitz modulus of the argmin mapping in linear semi-infinite optimization*, Set-Valued Anal., 16 (2008), pp. 511–538.

[6] M. J. Cánovas, A. Hantoute, J. Parra, and F. J. Toledo, *Calmness of the argmin mapping in linear semi-infinite optimization*, submitted.

[7] M. J. Cánovas, D. Klatte, M. A. López, and J. Parra, *Metric regularity in convex semi-infinite optimization under canonical perturbations*, SIAM J. Optim., 18 (2007), pp. 717–732.

[8] M. J. Cánovas, M. A. López, J. Parra, and F. J. Toledo, *Calmness of the feasible set mapping for linear inequality systems*, (2013).

[9] J. M. Danskin, *The theory of* $\max - \min$*, with applications*, SIAM J. Appl. Math., 14 (1966), pp. 641–664.

[10] A. L. Dontchev, *Characterizations of Lipschitz stability in optimization*, in Recent Developments in Well-posed Variational Problems, vol. 331 of Math. Appl., 1995, pp. 95–115.

[11] A. L. Dontchev and R. T. Rockafellar, *Ample parameterization of variational inclusions*, SIAM J. Optim., 12 (2001), pp. 170–187.

[12] ———, *Implicit Functions and Solution Mappings, A View from Variational Analysis*, Springer-Verlag, Berlin, 2009.

[13] M. A. Goberna and M. A. López, *Linear Semi-infinite Optimization*, vol. 2 of Wiley Series in Mathematical Methods in Practice, John Wiley & Sons Ltd., Chichester, 1998.

[14] R. Henrion, A. Jourani, and J. Outrata, *On the calmness of a class of multifunctions*, SIAM J. Optim., 13 (2002), pp. 603–618.

[15] A. Jourani, *Hoffman's error bound, local controllability, and sensitivity analysis*, SIAM J. Control Optim., 38 (2000), pp. 947–970.

[16] D. Klatte and B. Kummer, *Nonsmooth Equations in Optimization. Regularity, Calculus, Methods and Applications*, vol. 60 of Nonconvex Optimization and its Applications, Kluwer Academic Publishers, Dordrecht, 2002.

[17] A. Y. Kruger, H. V. Ngai, and M. Théra, *Stability of error bounds for convex constraint systems in Banach spaces*, SIAM J. Optim., 20 (2010), pp. 3280–3296.

[18] W. Li, *The sharp lipschitz constants for feasible and optimal solutions of a perturbed linear program*, Linear Algebra Appl, 187 (1993), pp. 15–40.

[19] ———, *Sharp lipschitz constants for basic optimal solutions and basic feasible solutions of linear programs*, SIAM J. Optim., 14 (1994), pp. 140–153.

[20] G. Nürnberger, *Unicity in semi-infinite optimization*, in Parametric optimization and approximation (Oberwolfach, 1983), vol. 72 of Internat. Schriftenreihe Numer. Math., Birkhäuser, Basel, 1985, pp. 231–247.

[21] S. M. Robinson, *Some continuity properties of polyhedral multifunctions*, Math. Programming Stud., (1981), pp. 206–214. Mathematical Programming at Oberwolfach (Proc. Conf., Math. Forschungsinstitut, Oberwolfach, 1979).

[22] C. Zălinescu, *Convex Analysis in General Vector Spaces*, World Scientific Publishing Co. Inc., River Edge, NJ, 2002.

[23] X. Zheng and K. Ng, *Metric regularity and constraint qualifications for convex inequalities on banach spaces*, SIAM J. Optim., 14 (2003), pp. 757–772.