

Strong duality in conic linear programming: facial reduction and extended duals

Gábor Pataki
gabor@unc.edu

Department of Statistics and Operations Research
University of North Carolina at Chapel Hill

Abstract

The facial reduction algorithm of Borwein and Wolkowicz and the extended dual of Ramana provide a strong dual for the conic linear program

$$\sup \{ \langle c, x \rangle \mid Ax \leq_K b \} \quad (P)$$

in the absence of any constraint qualification. The facial reduction algorithm solves a sequence of auxiliary optimization problems to obtain such a dual. Ramana's dual is applicable when (P) is a semidefinite program (SDP) and is an explicit SDP itself. Ramana, Tunçel, and Wolkowicz showed that these approaches are closely related; in particular, they proved the correctness of Ramana's dual using certificates from a facial reduction algorithm.

Here we give a clear and self-contained exposition of facial reduction, of extended duals, and generalize Ramana's dual:

- we state a simple facial reduction algorithm and prove its correctness; and
- building on this algorithm we construct a family of extended duals when K is a *nice* cone. This class of cones includes the semidefinite cone and other important cones.

Key words: Conic linear programming; minimal cone; semidefinite programming; facial reduction; extended duals; nice cones

MSC 2010 subject classification: Primary: 90C46, 49N15, 90C22, 90C25; secondary: 52A40, 52A41

1 Introduction

Conic linear programs Conic linear programs generalize ordinary linear programming: they require membership in a closed convex cone in place of the usual nonnegativity constraint. Conic LPs share some of the duality theory of linear optimization:

weak duality always holds in a primal-dual pair, and assuming a suitable constraint qualification (CQ), their objective values agree, and are attained.

When a CQ is lacking and the underlying cone is not polyhedral, pathological phenomena can occur: nonattainment of the optimal values, positive gaps, and infeasibility of the dual even when the primal is bounded. All these pathologies appear in semidefinite programs, second order cone programs, and p -order conic programs, arguably the most important and useful classes of conic LPs ([4, 3, 31, 1, 29, 7, 14, 2]).

Facial reduction and extended duals Here we study two fundamental approaches to duality in conic linear programs that work without assuming any CQ. The first approach is the facial reduction algorithm (FRA) of Borwein and Wolkowicz [6, 5], which constructs a so-called minimal cone of a conic linear system. Using this minimal cone one can always ensure strong duality in a primal-dual pair of conic LPs.

The second approach is Ramana’s extended dual for semidefinite programs [25]. (Ramana named his dual an Extended Lagrange-Slater Dual, or ELSD dual. We use the shorter name for simplicity.) The extended dual is an explicit semidefinite program with a fairly large number (but polynomially many) variables and constraints. It has the following desirable properties: it is feasible if and only if the primal problem is bounded; and when these equivalent statements hold, it has the same value as the primal and attains it. Ramana, Tunçel and Wolkowicz in [26] showed that the two approaches are closely related: in the case of semidefinite programming, the correctness of Ramana’s dual can be proven using certificates from a facial reduction algorithm.

The goal of this paper is to give an exposition of these two approaches, study their connection, and give simple proofs of generalizations of Ramana’s dual. We will use ideas from the paper [26].

We look at two important questions:

- How do we use an *algorithm* to prove the correctness of an *explicit conic linear program*?
- Are there other conic LPs with an extended dual?

Next we present our framework in more detail. A conic linear program can be stated as

$$\begin{aligned} \sup \quad & \langle c, x \rangle \\ \text{s.t.} \quad & Ax \leq_K b, \end{aligned} \tag{P}$$

where $A : X \rightarrow Y$ is a linear map between finite dimensional Euclidean spaces X and Y , and $c \in X, b \in Y$. Also, the set $K \subseteq Y$ is a closed, convex cone, and we write $Ax \leq_K b$ to mean $b - Ax \in K$. We naturally associate a dual program with (P). Letting A^* be the adjoint operator of A , and K^* the dual cone of K , i.e.,

$$K^* = \{ y \mid \langle y, x \rangle \geq 0 \forall x \in K \},$$

the dual problem is

$$\begin{aligned} \inf \quad & \langle b, y \rangle \\ \text{s.t.} \quad & y \geq_{K^*} 0 \\ & A^*y = c. \end{aligned} \tag{D}$$

When (P) is feasible, we say that *strong duality holds between (P) and (D)* if the following conditions are satisfied:

- problem (P) is bounded, if and only if (D) is feasible; and
- when these equivalent conditions hold, the optimal values of (P) and (D) agree and the latter is attained.

We say that (P) is *strictly feasible*, or *satisfies Slater's condition*, if there is an $x \in X$ such that $b - Ax$ is in the relative interior of K . When (P) is strictly feasible, it is well-known that strong duality holds between (P) and (D) .

The facial reduction algorithm (FRA) of Borwein and Wolkowicz constructs a suitable face of K , called the *minimal cone of (P)* , which we here denote by F_{\min} . The minimal cone has two important properties:

- The feasible set of (P) remains the same if we replace its constraint set by

$$Ax \leq_{F_{\min}} b.$$

- The new constraint set satisfies Slater's condition.

Thus, if we also replace K^* by F_{\min}^* in (D) , strong duality holds in the new primal-dual pair. The algorithm in [6, 5] constructs a decreasing chain of faces starting with K and ending with F_{\min} , in each step solving a pair of auxiliary conic linear programs.

Contributions of the paper We first state a simplified FRA and prove its correctness. Building on this algorithm, and assuming that cone K is *nice*, i.e., the set $K^* + F^\perp$ is closed for all F faces of K we show that the dual of the minimal cone has a representation

$$\begin{aligned} F_{\min}^* = \{ & u_{\ell+1} + v_{\ell+1} : \\ & (u_0, v_0) = (0, 0) \\ & (A, b)^*(u_i + v_i) = 0, \quad i = 1, \dots, \ell, \\ & (u_i, v_i) \in K^* \times \tan(u_0 + \dots + u_{i-1}, K^*), \quad i = 1, \dots, \ell + 1 \}, \end{aligned} \tag{1.1}$$

where $\tan(u, K^*)$ denotes the tangent space of the cone K^* at $u \in K^*$ and ℓ is a suitable integer. Plugging this expression for F_{\min}^* in place of K^* in (D) we obtain a dual with the properties of Ramana's dual. We show the correctness of several representations of F_{\min}^* , each leading to a different extended dual.

The cone of positive semidefinite matrices is nice (and also self-dual), so in this case the representation of (1.1) is valid. In this case we can also translate the tangent space constraint into an explicit semidefinite constraint, and recover variants of Ramana's dual.

We attempted to simplify our treatment of the subject as much as possible: as background we use only the fact that strong duality holds in a primal-dual pair of conic LPs, when the primal is strictly feasible and some elementary facts in convex analysis.

Literature review Borwein and Wolkowicz originally presented their facial reduction algorithm in the two papers [6, 5]. Their algorithm works for a potentially nonlinear conic system of the form $\{x \mid g(x) \in K\}$. Luo, Sturm, and Zhang in [18] studied a so-called *conic expansion method* which finds a sequence of increasing sets starting with K^* and ending with F_{\min}^* : thus their algorithm can be viewed as a dual variant of facial reduction. Sturm in [30] introduced an interesting and novel application of facial reduction: deriving error bounds for semidefinite systems that lack a strictly feasible solution. Luo and Sturm in [17] generalized this approach to mixed semidefinite and second order conic systems. Lewis in [16] used facial reduction to derive duality results without a CQ assumption in partially finite convex programming. Tunçel in [32] constructed an SDP instance with n by n semidefinite matrices that requires $n - 1$ iterations of the facial reduction algorithm to find the minimal cone, which is essentially the theoretical worst case.

Waki and Muramatsu in [33] also described an FRA, rigorously showed its equivalence to the conic expansion approach of Luo et al, and presented computational results on semidefinite programs. A preliminary version of this paper appeared in [20]. Pólik and Terlaky in [24] used the results of [20] to construct strong duals for conic LPs over homogeneous cones. Wang et al in [34] presented a facial reduction algorithm for non-symmetric semidefinite least squares problems. Cheung et al in [8] developed a relaxed version of a facial reduction algorithm, in which one can allow an error in the solution of the auxiliary conic LPs, and applied their method to SDPs.

Nice cones appear in other areas of optimization as well. In [21] we studied the question of when the linear image of a closed convex cone is closed and described necessary and sufficient conditions. These lead to a particularly simple and exact characterization when the dual of the cone in question is nice. We call a conic linear system *well behaved* if for all objective functions the resulting conic linear program has strong duality with its dual and *badly behaved*, otherwise. In related work, [22], we described characterizations of well- and badly behaved conic linear systems. These become particularly simple when the underlying cone is nice, and yield combinatorial type characterizations for semidefinite and second order conic systems.

Chua and Tunçel in [10] showed that if a cone K is nice, then so is its intersection with a linear subspace. Thus, all homogeneous cones are nice, since they arise as the slice of a semidefinite cone with a suitable subspace, as proven independently by Chua in [9] and by Faybusovich in [11]. In [10] the authors also proved that the preimage of a nice cone under a linear map is also nice and in [23] we pointed out that this result implies

that the intersection of nice cones is also nice. In [23] we gave several characterizations of nice cones and proved that they must be facially exposed; facial exposedness with a mild additional condition implies niceness; and conjectured that facially exposed and nice cones are actually the same class of cones. However, this conjecture was proven false by Roshchina [28].

Klep and Schweighofer in [15] derived a strong dual for semidefinite programs that also works without assuming any constraint qualification. Their dual resembles Ramana's dual, but interestingly, it is derived using concepts from algebraic geometry, whereas all other references known to us use convex analysis.

Recently Gouveia et al in [19] studied the following fundamental question: can a convex set be represented as the projection of an affine slice of a suitable closed, convex cone? They gave necessary and sufficient conditions for such a *lift* to exist and showed that some known lifts from the literature are in the lowest dimension possible. The representation of (1.1) is related in spirit, as we also represent the set F_{\min}^* as the projection of a conic linear system in a higher dimensional space.

Organization of the paper and guide to the reader In Section 2 we fix notation, review preliminaries, and present two motivating examples. The reader familiar with convex analysis can skip the first part of this section and go directly to the examples. In Section 3 we present a simple facial reduction algorithm, prove its correctness, and show how F_{\min}^* can be written as the projection of a nonlinear conic system in a higher dimensional space.

Assuming that K is nice, in Section 4 we arrive at the representation in (1.1), i.e., show that F_{\min}^* is the projection of a conic *linear* system, and derive an extended dual for conic LPs over nice cones. Here we obtain our first Ramana-type dual for semidefinite programs which is an explicit SDP itself, but somewhat different from the dual proposed in [25].

In Section 5 we describe variants of the representation in (1.1), of extended duals, and show how we can exactly obtain Ramana's dual. In Section 6 we show that the minimal cone F_{\min} itself also has a representation similar to the representation of F_{\min}^* in (1.1) and discuss some open questions.

The paper is organized to arrive at an explicit Ramana-type dual for SDP as quickly as possible. Thus, if a reader is interested in only the derivation of such a dual, it suffices for him/her to read only Sections 2, 3 and 4.

2 Preliminaries

Matrices and vectors We denote operators by capital letters and matrices (when they are considered as elements of a Euclidean space and not as operators) and vectors by lower case letters. The i th component of vector x is denoted by x_i and the (i, j) th

component of matrix z by z_{ij} . We distinguish vectors and matrices of similar type with lower indices, i.e., writing x_1, x_2, \dots . The j th component of vector x_i is denoted by $x_{i,j}$. This notation is somewhat ambiguous, as x_i may denote a vector, or the i th component of the vector x , but the context will make it clear which one is meant.

Convex sets For a set C we write $\text{cl } C$ for its closure, $\text{lin } C$ for its linear span, and C^\perp for the orthogonal complement of its linear span. For a convex set C we denote its relative interior by $\text{ri } C$. For a one-element set $\{x\}$ we abbreviate $\{x\}^\perp$ by x^\perp . The open line-segment between points x_1 and x_2 is denoted by (x_1, x_2) .

For a convex set C , and an F , a convex subset of C we say that F is a face of C if $x_1, x_2 \in C$ and $(x_1, x_2) \cap F \neq \emptyset$ implies that x_1 and x_2 are both in F . For $x \in C$ there is a unique minimal face of C that contains x , i.e., the face that contains x in its relative interior: we denote this face by $\text{face}(x, C)$. For $x \in C$ we define the set of feasible directions and the tangent space at x in C as

$$\begin{aligned} \text{dir}(x, C) &= \{y \mid x + ty \in C \text{ for some } t > 0\}, \\ \text{tan}(x, C) &= \text{cl } \text{dir}(x, C) \cap -\text{cl } \text{dir}(x, C). \end{aligned}$$

Cones We say that a set K is a cone, if $\lambda x \in K$ holds for all $x \in K$ and $\lambda \geq 0$, and define the dual of cone K as

$$K^* = \{z \mid \langle z, x \rangle \geq 0 \text{ for all } x \in K\}.$$

For an F face of a closed convex cone K and $x \in \text{ri } F$ the complementary, or conjugate face of F is defined alternatively as (the equivalence is straightforward)

$$F^\Delta = K^* \cap F^\perp = K^* \cap x^\perp.$$

The complementary face of a face G of K^* is defined analogously and denoted by G^Δ . A closed convex cone K is facially exposed, i.e., all faces of K arise as the intersection of K with a supporting hyperplane iff for all F faces of K we have $(F^\Delta)^\Delta = F$. For brevity we write $F^{\Delta*}$ for $(F^\Delta)^*$ and $F^{\Delta\perp}$ for $(F^\Delta)^\perp$.

For a closed convex cone K and $x \in K$ we have

$$\text{tan}(x, K) = \text{face}(x, K)^{\Delta\perp}, \tag{2.2}$$

as shown in [22, Lemma 1].

The semidefinite cone We denote the space of n by n symmetric and the cone of n by n symmetric, positive semidefinite matrices by \mathcal{S}^n , and \mathcal{S}_+^n , respectively. The space \mathcal{S}^n is equipped with the inner product

$$\langle x, z \rangle := \sum_{i,j=1}^n x_{ij} z_{ij},$$

and \mathcal{S}_+^n is self-dual with respect to it. For $y \in \mathcal{S}^n$ we write $y \succeq 0$ to denote that y is positive semidefinite. Using a rotation $v^T(\cdot)v$ by a full-rank matrix v any face of \mathcal{S}_+^n and its conjugate face can be brought to the form

$$F = \left\{ \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \mid x \in \mathcal{S}_+^r \right\}, F^\Delta = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix} \mid y \in \mathcal{S}_+^{n-r} \right\}, \quad (2.3)$$

where r is a nonnegative integer.

For a face of this form and related sets we use the shorthand

$$F = \begin{pmatrix} \oplus & 0 \\ 0 & 0 \end{pmatrix}, F^\Delta = \begin{pmatrix} 0 & 0 \\ 0 & \oplus \end{pmatrix}, F^{\Delta\perp} = \begin{pmatrix} \times & \times \\ \times & 0 \end{pmatrix}, \quad (2.4)$$

when the size of the partition is clear from the context. The \oplus sign denotes a positive semidefinite submatrix and the sign \times stands for a submatrix with arbitrary elements.

For an x positive semidefinite matrix we collect some expressions for $\tan(x, \mathcal{S}_+^n)$ below: these play an important role when constructing explicit duals for semidefinite programs. The second part of Proposition 1 is based on Lemma 1 in [26].

Proposition 1. *The following statements hold.*

(1) *Suppose $x \in \mathcal{S}_+^n$ is of the form*

$$x = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}, \quad (2.5)$$

and $F = \text{face}(x, \mathcal{S}_+^n)$. Then F , F^Δ , and $F^{\Delta\perp}$ are as displayed in equation (2.4), with the upper left block r by r , and

$$\tan(x, \mathcal{S}_+^n) = F^{\Delta\perp} = \begin{pmatrix} \times & \times \\ \times & 0 \end{pmatrix}. \quad (2.6)$$

(2) *For an arbitrary $x \in \mathcal{S}_+^n$ we have*

$$\tan(x, \mathcal{S}_+^n) = \left\{ w + w^T \mid \begin{pmatrix} x & w \\ w^T & \beta I \end{pmatrix} \succeq 0 \text{ for some } \beta \in \mathbb{R} \right\}. \quad (2.7)$$

Proof of (1) This statement is straightforward from the form of x and the expression for the tangent space given in (2.2) with $K = \mathcal{S}_+^n$.

Proof of (2) If x is of the form as in equation (2.5), then our claim follows from part (1).

Suppose now that $x \in \mathcal{S}_+^n$ is arbitrary and let q be a matrix of suitably scaled eigenvectors of x with eigenvectors corresponding to nonzero eigenvalues coming first. Let us write $T(x)$ for the set on the right hand side of equation (2.7). Then one easily checks $\tan(q^T x q, \mathcal{S}_+^n) = q^T \tan(x, \mathcal{S}_+^n) q$ and $T(q^T x q) = q^T T(x) q$, so this case reduces to the previous case. \blacksquare

Conic LPs An ordinary linear program is clearly a special case of (P) . If we choose $X = \mathbb{R}^m$, $Y = \mathcal{S}^n$, and $K = \mathcal{S}_+^n$, then problem (P) becomes a semidefinite program (SDP). Since K is self-dual, the dual problem (D) is also an SDP. The operator A and its adjoint are defined via symmetric matrices a_1, \dots, a_m as

$$Ax = \sum_{i=1}^m x_i a_i \text{ and } A^*y = (\langle a_1, y \rangle, \dots, \langle a_m, y \rangle)^T.$$

We use the operator $\text{Feas}()$ to denote the feasible set of a conic system.

The minimal cone Let us choose $x \in \text{ri Feas}(P)$. We define the minimal cone of (P) as the unique face of K that contains $b - Ax$ in its relative interior and denote this face by F_{\min} .

For an arbitrary $y \in \text{Feas}(P)$ there is $z \in \text{Feas}(P)$ such that $x \in (y, z)$. Hence $b - Ax \in (b - Ay, b - Az)$, so $b - Ay$ and $b - Az$ are in F_{\min} , and (P) is equivalent to

$$Ax \leq_{F_{\min}} b,$$

and this constraint system satisfies Slater's condition.

Nice cones We say that a closed convex cone K is nice if

$$K^* + E^\perp \text{ is closed for all } E \text{ faces of } K.$$

Most cones appearing in the optimization literature, such as polyhedral, semidefinite, p -order, in particular second order cones are nice: see e.g. [6, 5, 21].

Example 1. In the linear inequality system

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.8}$$

all feasible solutions satisfy the last four inequalities at equality, and for, say, $x = (-1, 0, 0)^T$ the first inequality is strict. So the minimal cone of this system is

$$F_{\min} = \mathbb{R}_+^1 \times \{0\}^4.$$

In linear programs strong duality holds even without strict feasibility, so this example illustrates only the concept of the minimal cone.

■

Example 2. In the semidefinite program

$$\begin{aligned} & \sup \quad x_1 \\ & \text{s.t.} \quad x_1 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned} \quad (2.9)$$

a feasible positive semidefinite slack z must have all entries equal to zero, except for z_{11} , and there is a feasible slack with $z_{11} > 0$. So the minimal cone and its dual are

$$F_{\min} = \begin{pmatrix} \oplus & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, F_{\min}^* = \begin{pmatrix} \oplus & \times & \times \\ \times & \times & \times \\ \times & \times & \times \end{pmatrix}. \quad (2.10)$$

The optimal value of (2.9) is clearly zero. Writing y for the dual matrix, the dual program is equivalent to

$$\begin{aligned} & \inf \quad y_{11} \\ & \text{s.t.} \quad \begin{pmatrix} y_{11} & 1/2 & -y_{22}/2 \\ 1/2 & y_{22} & y_{23} \\ -y_{22}/2 & y_{23} & y_{33} \end{pmatrix} \succeq 0. \end{aligned} \quad (2.11)$$

The dual has an unattained 0 minimum: y_{11} can be an arbitrarily small positive number, at the cost of making y_{22} and in turn y_{33} large, however, y_{11} can not be made 0, as y_{12} is $1/2$.

Suppose that in (2.11) we replace the constraint $y \succeq 0$ by $y \in F_{\min}^*$. Then we can set y_{11} to zero, so with this modification the dual attains. ■

We will return to these examples later to illustrate our facial reduction algorithm and extended duals.

We assume throughout the paper that (P) is feasible. It is possible to remove this assumption and modify the facial reduction algorithm of section 3 to either prove the infeasibility of (P) , or to find the minimal cone in finitely many steps; such an FRA was described by Waki and Muramatsu in [33].

3 A simple facial reduction algorithm

We now state a simplified facial reduction algorithm that is applicable when K is an arbitrary closed convex cone. We prove its correctness and illustrate it on Examples 1 and 2.

Let us recall that F_{\min} denotes the minimal cone of (P) and for brevity define the subspace L as

$$L = \mathcal{N}((A, b)^*). \quad (3.12)$$

Lemma 1. *Suppose that an F face of K satisfies $F_{\min} \subseteq F$. Then the following hold:*

(1) *For all $y \in F^* \cap L$ we have*

$$F_{\min} \subseteq F \cap y^\perp \subseteq F. \quad (3.13)$$

(2) *There exists $y \in F^* \cap L$ such that the second containment in (3.13) is strict, iff $F_{\min} \neq F$. We can find such a y , or prove $F = F_{\min}$ by solving a pair of auxiliary conic linear programs.*

Proof of (1) Suppose that x is feasible for (P) and let $y \in F^* \cap L$. Then $b - Ax \in F_{\min} \subseteq F$, hence $\langle b - Ax, y \rangle = 0$, which implies the first containment; the second is obvious.

Proof of (2) The “only if” part of the statement is obvious. To see the “if” part, let us fix $f \in \text{ri } F$, and consider the primal-dual pair of conic linear programs that we call reducing conic LPs below:

$$\begin{array}{ll} \sup & t \\ (R-P) \quad \text{s.t.} & Ax + ft \leq_F b \end{array} \qquad \begin{array}{ll} \inf & b^*y \\ \text{s.t.} & y \geq_{F^*} 0 \\ & A^*y = 0 \\ & \langle f, y \rangle = 1. \end{array} \quad (R-D)$$

First let us note

$$\begin{aligned} F_{\min} = F &\Leftrightarrow \exists x \text{ s.t. } b - Ax \in \text{ri } F \\ &\Leftrightarrow \exists x \text{ and } t > 0 \text{ s.t. } b - Ax - ft \in F. \end{aligned}$$

Here in the first equivalence the direction \Rightarrow is obvious from how we defined the minimal cone. To see the direction \Leftarrow assume $b - Ax \in \text{ri } F$. Then $\text{ri } F \cap F_{\min} \neq \emptyset$ and F_{\min} is a face of K , so Theorem 18.1 in [27] implies $F \subseteq F_{\min}$, and the reverse containment already given. The second equivalence is obvious.

Therefore, $F_{\min} \neq F$ iff the optimal value of $(R-P)$ is 0. Note that $(R-P)$ is strictly feasible, with some x such that $b - Ax \in F$, and some $t < 0$.

Hence $F_{\min} \neq F$ iff $(R-D)$ has optimal value 0 and attains it, i.e., iff there is $y \in F^* \cap L$ with $\langle f, y \rangle = 1$. Such a y clearly must satisfy $F \cap y^\perp \subsetneq F$. ■

Based on Lemma 1 we now state a simple facial reduction algorithm in Figure 1.

The algorithm of Figure 1 may not terminate in general, as it allows the choice of a y_i in iteration i such that $F_i = F_{i-1}$; it even allows $y_i = 0$ for all i . Based on this general algorithm, however, it will be convenient to construct a representation of F_{\min}^* .

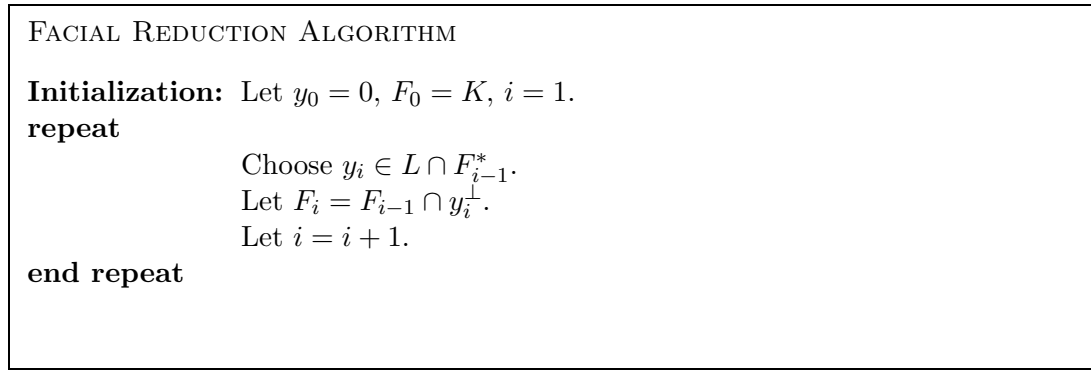


Figure 1: The facial reduction algorithm

We call an iteration of the FRA reducing, if the y_i vector found therein satisfies $F_i \subsetneq F_{i-1}$; we can make sure that an iteration is reducing, or that we have found the minimal cone by solving the pair of conic linear programs $(R-P)$ - $(R-D)$. It is clear that after a sufficient number of reducing iterations the algorithm terminates.

Let us define the quantities

$$\begin{aligned} \ell_K &= \text{the length of the longest chain of faces in } K, \\ \ell &= \min\{\ell_K - 1, \dim L\}. \end{aligned} \tag{3.14}$$

We prove the correctness of our FRA and an upper bound on the number of reducing iterations in Theorem 1:

Theorem 1. *Suppose that the FRA finds y_0, y_1, \dots , and corresponding faces F_0, F_1, \dots . Then*

- (1) $F_{\min} \subseteq F_i$ for $i = 0, 1, \dots$
- (2) *After a sufficiently large number of reducing iterations the algorithm finds $F_{\min} = F_t$ in some iteration t . Furthermore,*

$$F_{\min} = F_i$$

holds for all $i \geq t$.

- (3) *In particular, the number of reducing iterations in the FRA is at most ℓ .*

Proof Let us first note that the face F_i found by the algorithm is of the form

$$F_i = K \cap y_0^\perp \cap \dots \cap y_i^\perp, \quad i = 0, 1, \dots$$

Statement (1) follows from applying repeatedly part (1) of Lemma 1.

Proof of (2) The first part of the claim is straightforward; in particular, the number of reducing iterations cannot exceed $\ell_K - 1$. Suppose $i \geq t$. Since $F_t = F_{\min}$, we have

$$F_{\min} \subseteq F_i = F_t \cap y_{t+1}^\perp \cap \cdots \cap y_i^\perp \subseteq F_{\min}, \quad (3.15)$$

so equality holds throughout in (3.15), which proves $F_i = F_{\min}$.

Proof of (3) Let us denote by k the number of reducing iterations. It remains to show that $k \leq \dim L$ holds, so assume to the contrary $k > \dim L$. Suppose that y_{i_1}, \dots, y_{i_k} are the vectors found in reducing iterations. Since they are all in L , they must be linearly dependent, so there is an index $r \in \{1, \dots, k\}$ such that

$$y_{i_r} \in \text{lin}\{y_{i_1}, \dots, y_{i_{r-1}}\} \subseteq \text{lin}\{y_0, y_1, \dots, y_{i_{r-1}}\}.$$

For brevity let us write $s = i_r$. Then $y_0^\perp \cap \cdots \cap y_{s-1}^\perp \subseteq y_s^\perp$, so

$$F_s = F_{s-1},$$

i.e., the s^{th} step is not reducing, which is a contradiction. ■

Next we illustrate our algorithm on the examples of Section 2.

Examples 1 and 2 continued Suppose we run our algorithm on the linear system (2.8). The y_i vectors below, with corresponding faces shown, are a possible output:

$$\begin{aligned} y_0 &= 0, F_0 = \mathbb{R}_+^5, \\ y_1 &= (0, 0, 0, 1, 1)^T, F_1 = \mathbb{R}_+^3 \times \{0\}^2, \\ y_2 &= (0, 1, 1, 0, -1)^T, F_2 = F_{\min} = \mathbb{R}_+^1 \times \{0\}^4. \end{aligned} \quad (3.16)$$

The algorithm may also finish in one step, by finding, say, $y_0 = 0$, and

$$y_1 = (0, 1, 1, 2, 1)^T. \quad (3.17)$$

Of course, in linear systems there is always a reducing certificate that finds the minimal cone in one step, i.e., $F_{\min} = K \cap y_1^\perp$ for some $y_1 \geq 0$; this is straightforward from LP duality.

When we run our algorithm on the instance of (2.9), the y_i matrices below, with corresponding F_i faces, are a possible output:

$$\begin{aligned} y_0 &= 0, F_0 = \mathcal{S}_+^3, \\ y_1 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, F_1 = \begin{pmatrix} \oplus & 0 \\ & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ y_2 &= \begin{pmatrix} 0 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 0 \end{pmatrix}, F_2 = F_{\min} = \begin{pmatrix} \oplus & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (3.18)$$

Indeed it is clear that the y_i are orthogonal to all the constraint matrices in problem (2.9) and that $y_i \in F_{i-1}^*$ for $i = 1, 2$. ■

Let us now consider the conic system

$$\left. \begin{aligned} y_0 &= 0 \\ y_i &\in F_{i-1}^*, \text{ where} \\ F_{i-1} &= K \cap y_0^\perp \cap \dots \cap y_{i-1}^\perp, \quad i = 1, \dots, \ell + 1 \\ y_i &\in L, \quad i = 1, \dots, \ell \end{aligned} \right\}, \quad (EXT)$$

that we call an extended system.

We have the following representation theorem:

Theorem 2. $F_{\min}^* = \{y_{\ell+1} \mid (y_i)_{i=0}^{\ell+1} \text{ is feasible in } (EXT)\}$. ■

Before proving Theorem 2 we make some remarks. First, the two different ranges for the i indices in the constraints of (EXT) are not accidental: the sequence y_0, \dots, y_ℓ is a possible output of our FRA, iff with some $y_{\ell+1}$ it is feasible in (EXT) , and the variable $y_{\ell+1}$ represents the dual of the minimal cone. It also becomes clearer now why we allow nonreducing iterations in our algorithm: in the conic system (EXT) some y_i correspond to reducing iterations, but others do not.

The extended system (EXT) is not linear, due to how the y_i vectors depend on the previous y_j , and in general we also don't know how to describe the duals of faces of K . Hence the representation of Theorem 2 is not yet immediately useful. However, in the next section we state an equivalent conic linear system to represent F_{\min}^* when K is nice, and arrive at the representation of (1.1), and at an extended dual of (P) .

Proof of Theorem 2 Suppose that $(y_i)_{i=0}^{\ell+1}$ is feasible in (EXT) with corresponding faces F_0, \dots, F_ℓ . By part (1) in Theorem 1 we have

$$F_{\min} \subseteq F_\ell, \text{ hence } F_{\min}^* \supseteq F_\ell^*. \quad (3.19)$$

Since $y_{\ell+1} \in F_\ell^*$ in the set on the right hand side, the containment \supseteq follows.

By part (2)-(3) in Theorem 1 equality holds in (3.19) for at least one $(y_i)_{i=0}^{\ell+1}$ that is feasible in (EXT) and this proves the inclusion \subseteq . ■

4 When K is nice: an extended dual of (P)

From now on we make the following assumption:

K is nice.

Let us consider the conic system

$$\left. \begin{aligned} (u_0, v_0) &= (0, 0) \\ (u_i, v_i) &\in K^* \times \tan(u_0 + \cdots + u_{i-1}, K^*), \quad i = 1, \dots, \ell + 1 \\ u_i + v_i &\in L, \quad i = 1, \dots, \ell \end{aligned} \right\}. \quad (EXT_{\text{nice}})$$

This is a conic linear system, since the set

$$\{(u, v) \mid u \in K^*, v \in \tan(u, K^*)\}$$

is a convex cone, although it may not be closed (e.g., if $K^* = \mathbb{R}_+^2$, then $(\epsilon, 1)$ is in this set for all $\epsilon > 0$, but $(0, 1)$ is not).

Theorem 3. $\text{Feas}(EXT) = \{(u_i + v_i)_{i=0}^{\ell+1} : (u_i, v_i)_{i=0}^{\ell+1} \in \text{Feas}(EXT_{\text{nice}})\}$.

Proof of \subseteq Suppose that $(y_i)_{i=0}^{\ell+1}$ is feasible in (EXT) , with faces

$$F_{i-1} = K \cap y_0^\perp \cap \cdots \cap y_{i-1}^\perp, \quad i = 1, \dots, \ell + 1. \quad (4.20)$$

For $i = 1, \dots, \ell + 1$ we have $y_i \in F_{i-1}^*$, and K is nice, so we can write $y_i = u_i + v_i$ for some $u_i \in K^*$ and $v_i \in F_{i-1}^\perp$. Also, let us set $u_0 = v_0 = 0$, then of course $y_0 = u_0 + v_0$.

We show that $(u_i, v_i)_{i=0}^{\ell+1}$ is feasible in (EXT_{nice}) . To do this, it is enough to verify

$$F_{i-1}^\perp = \tan(u_0 + \cdots + u_{i-1}, K^*) \quad (4.21)$$

for $i = 1, \dots, \ell + 1$. Equation (4.21) will follow if we prove

$$F_{i-1} = K \cap (u_0 + \cdots + u_{i-1})^\perp \quad (4.22)$$

for $i = 1, \dots, \ell + 1$; indeed, from (4.22) we directly obtain

$$F_{i-1} = \text{face}(u_0 + \cdots + u_{i-1}, K^*)^\Delta,$$

hence

$$\begin{aligned} F_{i-1}^\perp &= \text{face}(u_0 + \cdots + u_{i-1}, K^*)^{\Delta\perp} \\ &= \tan(u_0 + \cdots + u_{i-1}, K^*), \end{aligned}$$

where the second equality comes from (2.2).

So it remains to prove (4.22). It is clearly true for $i = 1$. Let i be a nonnegative integer at most $\ell + 1$ and assume that (4.22) holds for $1, \dots, i - 1$. We then have

$$\begin{aligned} F_{i-1} &= F_{i-2} \cap y_{i-1}^\perp \\ &= F_{i-2} \cap (u_{i-1} + v_{i-1})^\perp \\ &= F_{i-2} \cap u_{i-1}^\perp \\ &= K \cap (u_0 + \cdots + u_{i-2})^\perp \cap u_{i-1}^\perp \\ &= K \cap (u_0 + \cdots + u_{i-2} + u_{i-1})^\perp. \end{aligned}$$

Here the second equation follows from the definition of (u_{i-1}, v_{i-1}) , the third from $v_{i-1} \in F_{i-2}^\perp$, the fourth from the inductive hypothesis, and the last from all u_j being in K^* .

Thus the proof of the containment \subseteq is complete.

Proof of \supseteq Let us choose $(u_i, v_i)_{i=0}^{\ell+1}$ to be feasible in (EXT_{nice}) , define $y_i = u_i + v_i$ for all i , and the faces F_0, \dots, F_ℓ as in (4.20). Repeating the previous argument verbatim, (4.21) holds, so we have

$$y_i \in K^* + F_{i-1}^\perp = F_{i-1}^*, \quad i = 1, \dots, \ell + 1.$$

Therefore $(y_i)_{i=0}^{\ell+1}$ is feasible in (EXT) and this completes the proof. \blacksquare

We now arrive at the representation of F_{\min}^* that we previewed in (1.1), and at an extended dual of (P) :

Corollary 1. *The dual of the minimal cone of (P) has a representation*

$$F_{\min}^* = \{u_{\ell+1} + v_{\ell+1} : (u_i, v_i)_{i=0}^{\ell+1} \text{ is feasible in } (EXT_{\text{nice}})\}, \quad (4.23)$$

and the extended dual

$$\begin{aligned} \inf \quad & \langle b, u_{\ell+1} + v_{\ell+1} \rangle \\ \text{s.t.} \quad & A^*(u_{\ell+1} + v_{\ell+1}) = c \\ & (u_i, v_i)_{i=0}^{\ell+1} \text{ is feasible in } (EXT_{\text{nice}}) \end{aligned} \quad (D_{\text{ext}})$$

has strong duality with (P) .

In particular, if (P) is a semidefinite program with m variables, independent constraint matrices, and $K = \mathcal{S}_+^n$, then the problem

$$\begin{aligned} \inf \quad & \langle b, u_{\ell+1} + v_{\ell+1} \rangle \\ \text{s.t.} \quad & A^*(u_{\ell+1} + v_{\ell+1}) = c \\ & (A, b)^*(u_i + v_i) = 0, \quad i = 1, \dots, \ell \\ & u_i \succeq 0, \quad i = 1, \dots, \ell + 1 \\ (*) \quad & \begin{pmatrix} u_0 + \dots + u_{i-1} & w_i \\ w_i^T & \beta_i I \end{pmatrix} \succeq 0, \quad i = 1, \dots, \ell + 1 \\ & v_i = w_i + w_i^T, \quad i = 1, \dots, \ell + 1 \\ & w_i \in \mathbb{R}^{n \times n}, \quad i = 1, \dots, \ell + 1 \\ & \beta_i \in \mathbb{R}, \quad i = 1, \dots, \ell + 1 \\ & (u_0, v_0) = (0, 0), \end{aligned} \quad (D_{\text{ext,SDP}})$$

where

$$\ell = \min \{ n, n(n+1)/2 - m - 1 \}, \quad (4.24)$$

has strong duality with (P) .

Proof The representation (4.23) follows from combining Theorems 2 and 3. The second statement of the theorem follows, since replacing K^* by F_{\min}^* in (D) yields a strong dual for (P).

Suppose now that (P) is a semidefinite program with $K = \mathcal{S}_+^n$, with m variables, and with independent constraint matrices. The length of the longest chain of faces in \mathcal{S}_+^n is $n + 1$ and the dimension of the subspace $\mathcal{N}((A, b)^*)$ is $n(n + 1)/2 - m - 1$. Hence we can choose ℓ as in (4.24) to obtain a correct extended dual.

Suppose that $v_i \in \mathcal{S}^n$ is a symmetric matrix and $u_0, \dots, u_{i-1} \in \mathcal{S}_+^n$ are symmetric positive semidefinite matrices, where $i \in \{1, \dots, \ell + 1\}$. The representation of the tangent space in \mathcal{S}_+^n in (2.7) implies that $v_i \in \tan(u_0 + \dots + u_{i-1}, K^*)$ holds, iff v_i, u_0, \dots, u_{i-1} with some w_i (possibly nonsymmetric) matrices and β_i scalars satisfies the i th constraint of $(D_{\text{ext,SDP}})$ marked by (*). This proves the correctness of the extended dual $(D_{\text{ext,SDP}})$. ■

For the reducing certificates found for the linear system (2.8) and displayed in (3.16) the reader can easily find the decomposition whose existence we showed in Theorem 3.

Example 2 continued Recall that when we run our FRA on the SDP instance (2.9), matrices y_0, y_1, y_2 shown in equation (3.18) are a possible output.

We illustrate their decomposition as proved in Theorem 3, in particular, as $y_i = u_i + v_i$ with $u_i \in K^*$ and $v_i \in \tan(u_0 + \dots + u_{i-1}, K^*)$ for $i = 1, 2$:

$$\begin{aligned} u_0 &= 0, v_0 = 0, \\ u_1 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, v_1 = 0, \\ u_2 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{4.25}$$

We can check $v_2 \in \tan(u_1, \mathcal{S}_+^3)$ by using the tangent space formula (2.6).

To illustrate the correctness of the extended dual $(D_{\text{ext,SDP}})$, we first note that $n = m = 3$, so by formula (4.24) we can choose $\ell = 2$ to obtain a correct extended dual. Recall that $y \in F_{\min}^*$ is an optimal dual solution if and only if

$$y = \begin{pmatrix} 0 & 1/2 & -y_{22}/2 \\ 1/2 & y_{22} & y_{23} \\ -y_{22}/2 & y_{32} & y_{33} \end{pmatrix} \succeq 0. \tag{4.26}$$

Consider the $(u_i, v_i)_{i=0}^2$ sequence shown in (4.25); we prove that any y optimal matrix satisfies

$$y \in \mathcal{S}_+^3 + \tan(u_0 + u_1 + u_2, \mathcal{S}_+^3). \tag{4.27}$$

Indeed, $\tan(u_0 + u_1 + u_2, \mathcal{S}_+^3)$ is the set of 3 by 3 matrices with the component in the (1, 1) position equal to zero, and the other components arbitrary, and this proves (4.27).

In fact, considering the expression for F_{\min}^* in (2.10), it follows that any $y \in F_{\min}^*$ satisfies (4.27). ■

5 Variants of the representation of F_{\min}^* and of extended duals

So far we proved the correctness of an extended dual of (P) , which is itself an explicit semidefinite program when (P) is. Ramana's original dual is somewhat different from $(D_{\text{ext,SDP}})$ though. Here we describe several variants of extended duals for (P) and show how to derive Ramana's dual.

First let us define a simplified extended system

$$\left. \begin{aligned} (u_0, v_0) &= (0, 0) \\ (u_i, v_i) &\in K^* \times \tan(u_{i-1}, K^*), \quad i = 1, \dots, \ell + 1 \\ u_i + v_i &\in L, \quad i = 1, \dots, \ell. \end{aligned} \right\} \quad (EXT_{\text{nice,simple}})$$

We prove that this system works just as well as (EXT_{nice}) when constructing extended duals.

Corollary 2. *The dual of the minimal cone of (P) has a representation*

$$F_{\min}^* = \{u_{\ell+1} + v_{\ell+1} : (u_i, v_i)_{i=0}^{\ell+1} \text{ is feasible in } (EXT_{\text{nice,simple}})\}, \quad (5.28)$$

and the extended dual

$$\begin{aligned} \inf \quad & \langle b, u_{\ell+1} + v_{\ell+1} \rangle \\ \text{s.t.} \quad & A^*(u_{\ell+1} + v_{\ell+1}) = c \\ & (u_i, v_i)_{i=0}^{\ell+1} \text{ is feasible in } (EXT_{\text{nice,simple}}), \end{aligned} \quad (D_{\text{ext}})$$

where ℓ is defined in (3.14), has strong duality with (P) .

In particular, if (P) is an SDP as described in Corollary 1, then the problem obtained from $(D_{\text{ext,SDP}})$ by replacing the constraint $(*)$ by

$$(**) \quad \begin{pmatrix} u_{i-1} & w_i \\ w_i^T & \beta_i I \end{pmatrix} \succeq 0, \quad i = 1, \dots, \ell + 1,$$

has strong duality with (P) .

Proof It is enough to prove the representation in equation (5.28); given this, the rest of the proof is analogous to the proof the second and third statements in Corollary 1.

We will use the representation of F_{\min}^* in (4.23). Let us denote by G the set on the right hand side of equation (5.28); we will prove $G = F_{\min}^*$.

To show $G \subseteq F_{\min}^*$ suppose $u_{\ell+1} + v_{\ell+1} \in G$, where $(u_i, v_i)_{i=0}^{\ell+1}$ is feasible in $(EXT_{\text{nice, simple}})$. Then it is also feasible in (EXT_{nice}) , since applying the tangent space formula (2.2) with K^* in place of K implies that

$$\tan(u_{i-1}, K^*) \subseteq \tan(u_0 + \dots + u_{i-1}, K^*)$$

holds for $i = 1, \dots, \ell + 1$.

To prove $G \supseteq F_{\min}^*$ suppose that $u_{\ell+1} + v_{\ell+1} \in F_{\min}^*$, where $(u_i, v_i)_{i=0}^{\ell+1}$ is feasible in (EXT_{nice}) . Again, by (2.2) the sets $\tan(u_0, K^*), \dots, \tan(u_0 + \dots + u_{i-1}, K^*)$ are all contained in $\tan(u_0 + \dots + u_{i-1}, K^*)$ for $i = 1, \dots, \ell$. Hence

$$v_1 + \dots + v_i \in \tan(u_0 + \dots + u_{i-1}, K^*), \quad i = 1, \dots, \ell \quad (5.29)$$

holds, and we also have

$$v_{\ell+1} \in \tan(u_0 + \dots + u_{\ell}, K^*). \quad (5.30)$$

Let us define

$$(u'_i, v'_i) = (u_0 + \dots + u_i, v_0 + \dots + v_i), \quad i = 1, \dots, \ell.$$

By (5.29) and (5.30) it follows that $(u_{\ell+1}, v_{\ell+1})$ with $(u'_i, v'_i)_{i=0}^{\ell}$ is feasible for $(EXT_{\text{nice, simple}})$, so the inclusion follows. \blacksquare

Let us now consider another extended system

$$\left. \begin{aligned} (u_0, v_0) &= (0, 0) \\ (u_i, v_i) &\in K^* \times \tan'(u_{i-1}, K^*), \quad i = 1, \dots, \ell + 1 \\ u_i + v_i &\in L, \quad i = 1, \dots, \ell \end{aligned} \right\}, \quad (EXT'_{\text{nice, simple}})$$

where the set $\tan'(u, K^*)$, satisfies the following two requirements for all $u \in K^*$:

- (1) $\tan'(u, K^*) \subseteq \tan(u, K^*)$.
- (2) For all $v \in \tan(u, K^*)$ there exists $\lambda_v > 0$ such that $v \in \tan'(\lambda_v u, K^*)$.

Corollary 3. *The dual of the minimal cone of (P) has the representation*

$$F_{\min}^* = \{u_{\ell+1} + v_{\ell+1} : (u_i, v_i)_{i=0}^{\ell+1} \text{ is feasible in } (EXT'_{\text{nice, simple}})\}, \quad (5.31)$$

and the extended dual

$$\begin{aligned} \inf \quad & \langle b, u_{\ell+1} + v_{\ell+1} \rangle \\ \text{s.t.} \quad & A^*(u_{\ell+1} + v_{\ell+1}) = c \\ & (u_i, v_i)_{i=0}^{\ell+1} \text{ is feasible in } (EXT_{\text{nice, simple}}), \end{aligned} \quad (D'_{\text{ext, simple}})$$

where ℓ is defined in (3.14), has strong duality with (P) .

In particular, if (P) is an SDP as described in Corollary 1, then the problem obtained from $(D_{\text{ext,SDP}})$ by replacing the constraint $(*)$ by

$$(***) \begin{pmatrix} u_{i-1} & w_i \\ w_i^T & I \end{pmatrix} \succeq 0, \quad i = 1, \dots, \ell + 1,$$

and dropping the β_i variables, has strong duality with (P) .

Proof We use the representation of F_{\min}^* in (5.28). Let us denote by G the set on the right hand side of equation (5.31). We will prove $G = F_{\min}^*$.

It is clear that $G \subseteq F_{\min}^*$, since if $(u_i, v_i)_{i=0}^{\ell+1}$ is feasible in $(EXT'_{\text{nice, simple}})$, then by the first property of the operator \tan' it is also feasible in $(EXT_{\text{nice, simple}})$.

To show the opposite inclusion, suppose $u_{\ell+1} + v_{\ell+1} \in F_{\min}^*$, where $(u_i, v_i)_{i=0}^{\ell+1}$ is feasible in $(EXT_{\text{nice, simple}})$. Let us choose $\lambda_\ell, \lambda_{\ell-1}, \dots, \lambda_1$ positive reals such that

$$\begin{aligned} v_{\ell+1} &\in \tan'(\lambda_\ell u_\ell, K^*), \\ \lambda_\ell v_\ell &\in \tan'(\lambda_{\ell-1} u_{\ell-1}, K^*), \\ &\vdots \\ \lambda_2 v_2 &\in \tan'(\lambda_1 u_1, K^*), \end{aligned} \tag{5.32}$$

and for completeness, set $\lambda_0 = 0$. Then $(u_{\ell+1}, v_{\ell+1})$ with $(\lambda_i u_i, \lambda_i v_i)_{i=0}^\ell$ is feasible in $(EXT'_{\text{nice, simple}})$, and this proves $F_{\min}^* \subseteq G$. ■

We finally remark that in the extended duals for semidefinite programming it is possible to eliminate the v_i variables and use the w_i matrices directly in the constraints; thus one can exactly obtain Ramana's dual. We leave the details to the reader.

6 Conclusion

We gave a self-contained exposition of a facial reduction algorithm and of extended duals: both approaches yield strong duality for a conic linear program, without assuming any constraint qualification. Using ideas from [26] we proved that when K is a nice cone, the set F_{\min}^* has an extended formulation, i.e., it can be written as the projection of the feasible set of a conic linear system in a higher dimensional space. The only nontrivial constraints in this system are of the form $u \succeq_{K^*} 0$, and $v \in \tan(u, K^*)$.

This formulation leads to an extended, strong dual of (P) ; and when $K = K^*$ is the semidefinite cone, by writing the tangent space constraint as a semidefinite constraint, we obtain an extended strong dual, which is an SDP itself.

One may wonder, whether there is an extended formulation for F_{\min} itself. Suppose that K is an arbitrary closed convex cone. When a fixed $\bar{s} \in \text{ri } F_{\min}$ is given, then

obviously

$$F_{\min} = \{s \mid 0 \leq_K s \leq_K \alpha \bar{s} \text{ for some } \alpha \geq 0\}.$$

The minimal cone can also be represented without such an \bar{s} , since

$$F_{\min} = \{s \mid 0 \leq_K s \leq_K \alpha b - Ax \text{ for some } x, \text{ and } \alpha \geq 0\}. \quad (6.33)$$

This representation was obtained by Freund [12], based on the article by himself, Roundy and Todd [13].

It is also natural to ask, whether there are other nice cones, for which the set

$$\{(u, v) \mid u \in K^*, v \in \tan(u, K^*)\}$$

has a formulation in terms of K^* ; e.g., is this true for the second order cone? Conic linear programs over such cones would also have Ramana-type (ie. expressed only in terms of K^*) extended duals.

Acknowledgement Thanks are due to an anonymous referee for his/her helpful comments on an earlier version of the paper.

References

- [1] Farid Alizadeh and Donald Goldfarb. Second-order cone programming. *Math. Program. Ser. B*, 95:3–51, 2003.
- [2] Erling D. Andersen, Cees Roos, and Tamás Terlaky. Notes on duality in second order and p -order cone optimization. *Optimization*, 51(4):627–643, 2002.
- [3] Aharon Ben-Tal and Arkadii Nemirovskii. *Lectures on modern convex optimization*. MPS/SIAM Series on Optimization. SIAM, Philadelphia, PA, 2001.
- [4] Frédéric J. Bonnans and Alexander Shapiro. *Perturbation analysis of optimization problems*. Springer Series in Operations Research. Springer-Verlag, 2000.
- [5] Jonathan M. Borwein and Henry Wolkowicz. Facial reduction for a cone-convex programming problem. *J. Aust. Math. Soc.*, 30:369–380, 1981.
- [6] Jonathan M. Borwein and Henry Wolkowicz. Regularizing the abstract convex program. *J. Math. Anal. App.*, 83:495–530, 1981.
- [7] Stephen Boyd and Lieven Vandenbergh. *Convex Optimization*. Cambridge University Press, 2004.
- [8] Vris Cheung, Henry Wolkowicz, and Simon Schurr. Preprocessing and regularization for degenerate semidefinite programs. Technical report, University of Waterloo, 2012.
- [9] Check-Beng Chua. Relating homogeneous cones and positive definite cones via T-algebras. *SIAM J. Optim.*, 14:500–506, 2003.

- [10] Check-Beng Chua and Levent Tunçel. Invariance and efficiency of convex representations. *Math. Program. B*, 111:113–140, 2008.
- [11] Leonid Faybusovich. On Nesterov’s approach to semi-definite programming. *Acta Appl. Math.*, 74:195–215, 2002.
- [12] Robert M. Freund. Talk at the University of Waterloo. 1994.
- [13] Robert M. Freund, Robin Roundy, and Michael J. Todd. Identifying the set of always-active constraints in a system of linear inequalities by a single linear program. Technical Report Sloan W.P. No. 1674-85, Sloan School of Business, MIT, 1985.
- [14] Osman Güler. *Foundations of Optimization*. Graduate Texts in Mathematics. Springer, 2010.
- [15] Igor Klep and Markus Schweighofer. An exact duality theory for semidefinite programming based on sums of squares. Technical report, Universität Konstanz, 2012, <http://arxiv.org/abs/1207.1691>.
- [16] Adrian S. Lewis. Facial reduction in partially finite convex programming. *Math. Program. B*, 65:123–138, 1994.
- [17] Zhi-Quan Luo and Jos Sturm. Error analysis. In Romesh Saigal, Lieven Vandenbergh, and Henry Wolkowicz, editors, *Handbook of semidefinite programming*. Kluwer Academic Publishers, 2000.
- [18] Zhi-Quan Luo, Jos Sturm, and Shuzhong Zhang. Duality results for conic convex programming. Technical Report Report 9719/A, Erasmus University Rotterdam, Econometric Institute, The Netherlands, 1997.
- [19] João Gouveia, Pablo Parrilo, and Rekha Thomas. Lifts of convex sets and cone factorizations. *Math. Oper. Res.*, to appear.
- [20] Gábor Pataki. A simple derivation of a facial reduction algorithm and extended dual systems. Technical report, Columbia University, 2000.
- [21] Gábor Pataki. On the closedness of the linear image of a closed convex cone. *Math. Oper. Res.*, 32(2):395–412, 2007.
- [22] Gábor Pataki. Bad semidefinite programs: they all look the same. Technical Report available from <http://arxiv.org/abs/1112.1436>, University of North Carolina at Chapel Hill, 2010.
- [23] Gábor Pataki. On the connection of facially exposed and nice cones. *J. Math. Anal. App.*, 400:211–221, 2013.
- [24] Imre Pólik and Tamás Terlaky. Exact duality for optimization over symmetric cones. Technical report, Lehigh University, Bethlehem, PA, USA, 2009.

- [25] Motakuri V. Ramana. An exact duality theory for semidefinite programming and its complexity implications. *Math. Program. Ser. B*, 77:129–162, 1997.
- [26] Motakuri V. Ramana, Levent Tunçel, and Henry Wolkowicz. Strong duality for semidefinite programming. *SIAM J. Opt.*, 7(3):641–662, 1997.
- [27] Tyrrel R. Rockafellar. *Convex Analysis*. Princeton University Press, Princeton, NJ, USA, 1970.
- [28] Vera Roshchina. Facially exposed cones are not nice in general. Technical report, University of Ballarat, <http://arxiv.org/abs/1301.1000>.
- [29] Romesh Saigal, Lieven Vandenbergh, and Henry Wolkowicz, editors. *Handbook of semidefinite programming*. Kluwer Academic Publishers, 2000.
- [30] Jos Sturm. Error bounds for linear matrix inequalities. *SIAM J. Optim.*, 10:1228–1248, 2000.
- [31] Michael J. Todd. Semidefinite optimization. *Acta Numer.*, 10:515–560, 2001.
- [32] Levent Tunçel. *Polyhedral and Semidefinite Programming Methods in Combinatorial Optimization*. Fields Institute Monographs, 2011.
- [33] Hayato Waki and Masakazu Muramatsu. Facial reduction algorithms for conic optimization problems. Technical Report Report CS-09-01, Department of Computer Science, The University of Electro-Communications, 1-5-1 Chofugaoka, Chofu-shi, Tokyo, 182-8585 Japan, 2009.
- [34] Yingnan Wang, Naihua Xiu, and Ziyang Luo. A regularized strong duality for nonsymmetric semidefinite least squares problem. *Optim. Lett.*, (5):665–682, 2011.