A big bucket time indexed formulation for nonpreemptive single machine scheduling problems

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Abstract

A big bucket time indexed mixed integer linear programming formulation for nonpreemptive single machine scheduling problems is presented in which the length of each period can be as large as the processing time of the shortest job. The model generalises the classical time indexed model to one in which at most two jobs can be processing in each period. The two models are equivalent in the case that the shortest job has unit processing time. For larger minimum processing times the big bucket model can have significantly fewer variables and nonzeros than the time indexed model at the expense of a greater number of constraints. Facet-inducing inequalities for the convex hull of the set of feasible partial schedules, that is, schedules in which not all jobs have to be started, are derived from facet-inducing inequalities for the time indexed model. A computational study using weighted tardiness instances reveals the big bucket model significantly outperforms the time indexed model on instances where the mean processing time of the jobs is large and the range of processing times is small, that is, the processing times are clustered rather than dispersed.

1 Introduction

The single machine scheduling problem is a classical optimisation problem that has been extensively studied in the literature. This prevalent problem can be simply stated and a good solution is often of great practical importance. An instance of the nonpreemptive version of this problem consists of n jobs and a processing time p_j for each job $j \in \{1, \ldots, n\}$. A schedule is a set of starting times, or equivalently completion times, for the jobs on the machine. The problem is to find a schedule such that each job j receives uninterrupted processing for an interval of length p_j and the machine processes at most one job

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at a time. The performance of a schedule is typically evaluated against a standard min-sum or min-max criteria and there may exist additional restrictions on each job j such as a release date r_j, a due date d_j, or a deadline $\bar{d}_j$. Readers not familiar with scheduling are referred to, for example, the survey paper by Lawler, Lenstra, Rinnooy Kan, and Shmoys (1993) or the texts Brucker (1995) and Pinedo (2002) for more information.

In this paper we present a big bucket time indexed (BB) mixed integer linear programming formulation for nonpreemptive single machine scheduling problems. The classical time indexed (TI) integer linear programming formulation assumes the problem data to be integer and discretises a sufficiently large planning horizon into time periods of unit length. Feasible schedules are characterised by every job starting at the beginning of a period, completing at the end of a period, and at most one job being processed in each period. Our BB model is a generalisation of the TI model in that at most two jobs can be processing in each period. That is, the changeover from the completion of the first job to the start of the second job can occur strictly within a time period. Our BB model is an exact formulation and a generalisation of the TI model in the sense that if the problem data are multiples of the length of a single time period then the BB model reduces to the TI model. For the sake of clarity, in the remainder of this paper we will refer to time periods in the BB model as buckets which, as will be seen, consist of multiple consecutive time periods of the classical TI model.

1.1 Literature review

The most common exact methods for solving single machine scheduling problems reported in the literature are based on either dynamic programming, branch-and-bound, or a combination of these. See Abdul-Razaq, Potts, and Van Wassenhove (1990) for a survey of earlier work on the weighted tardiness problem and, for example, Pan (2003), Pan and Shi (2007), Sourd (2009), Tanaka, Fujikuma, and Araki (2009), Tanaka and Fujikuma (2011), and Nessah and Kacem (2012) for more recent work. Among the various branch-and-bound approaches are those that solve mixed integer programming formulations of the problem. Such formulations can be largely characterised by their choice of decision variables. The more common formulations appearing in the literature include natural date variables, TI variables, linear ordering variables, and positional date and assignment variables. See Queyranne and Schulz (1994) for a comprehensive survey, and Keha, Khowala, and Fowler (2009) and Baker and Keller (2010) for recent computational comparisons of these formulations.

The classical TI formulation for nonpreemptive single machine scheduling problems was first investigated by Sousa and Wolsey (1992). They established that while the formulations are notoriously large, they give considerably stronger relaxations than the other mixed integer linear programming formulations. Polyhedral analyses have been performed by Sousa and Wolsey (1992), Crama and Spieksma (1996), van den Akker, van Hoesel, and Savelsbergh (1999), and Waterer, Johnson, Nobili, and Savelsbergh (2002). Computational investigations have been performed by van den Akker, Hurkens, and Savelsbergh (2000), Avella, Boccia, and D'Auria (2005), Pan and Shi (2007), and Bigras, Gamache, and Savard (2008). Recently arc-time indexed formulations have been studied by Sourd (2009), Tanaka et al. (2009), and Pessoa, Uchoa, Poggi de Aragão, and Rodrigues (2010). This formulation is known to be at least as strong as that of the TI formulation.

Dividing the planning horizon of problems involving planning over time into buckets has been used in several areas. The term big bucket is used in lot sizing and it was the use of buckets in this context that in part motivated this work. Pochet and Wolsey (2006, p374) define small-bucket models of production planning and scheduling problems to be problems with small time windows/periods/buckets during which the machine setup status remains constant. The discretisation used in the TI model is consistent with this definition. This is not the case for our formulation. The changeover from one job to the next can occur within a time window/period/bucket and so the setup status of the machine does not remain constant. Models in which this occurs are said to be BB models. The reader is referred to Pochet and Wolsey (2006) for more information on lot sizing and production planning and scheduling problems.

The partitioning of time windows into buckets has been used in the context of the travelling salesman problem with time windows (Dash, Günlük, Lodi, and Tramontani, 2012). In machine scheduling Bigras et al. (2008) use this and other ideas to accelerate the column generation approach of van den Akker et al. (2000). The buckets partition the constraint matrix of the TI formulation into a block diagonal structure, where each block is associated with a subproblem of the column generation approach used to solve the LP relaxation. Not to be confused with the heuristic approach of Hall, Schulz, Shmoys, and Wein (1997) with the same name, the exact interval indexed formulation of Baptiste and Sadykov (2009) partitions the planning horizon into buckets defined by an appropriately chosen superset of the release dates and deadlines of all jobs. The authors prove that for such a partition there exists an optimal schedule in which the jobs assigned to each bucket are scheduled according to weighted shortest processing time. In contrast to our BB model these approaches do not in general use buckets of uniform size. Furthermore, the restriction that at most two jobs can be processing in each bucket means that the length of the buckets in BB model are relatively short, no larger than the processing time of the shortest job.

1.2 Overview of the paper

The remainder of this paper is arranged as follows. In Section 2 the classical TI integer linear programming formulation for nonpreemptive single machine scheduling problems is presented and strong valid inequalities for this formulation from the literature are reviewed. In Section 3 we present the BB TI mixed integer linear programming formulation. This is the main contribution of this paper. We prove the correctness of the BB model, compare the TI and BB models, and discuss the choice of bucket size. We describe how facet-inducing inequalities for the convex hull of the set of feasible partial schedules, that is, schedules in which not all jobs have to be started, are derived from facet-inducing inequalities for the TI model. In Section 4 we present a computational study that compares the TI and BB models on weighted tardiness instances described in the literature. We find that the BB model significantly outperforms the TI model on instances where the the mean processing time of the jobs is large and the range of processing times is small, that is, the processing times are clustered rather than dispersed. Concluding remarks and a discussion of future work are given in Section 5.

2 Time indexed formulation

Let $J = \{1, \dots, n\}$ denote the index set of jobs. We assume that the problem data is integer and that $p_{\min} \geq 1$ where

$$p_{\min} = \min_{j \in J} p_j.$$

Consider a planning horizon consisting of T periods in which period t starts at time $t-1$ and ends at time t. Specifically, period t corresponds to the right half-open real interval $[t-1, t)$. Thus, if $\tau \in [0, T)$ is an instant in time, then by definition time τ is contained in period $t = \lfloor \tau \rfloor + 1$.

We assume that the planning horizon is of sufficient length that it admits a feasible schedule. For example, for standard min-sum scheduling criteria it is sufficient to assume that

$$T \geq \max_{j \in J} r_j + \sum_{j \in J} p_j.$$

For convenience we will use the notation $[t_1, t_2]$ to denote the set $\{t_1, \dots, t_2\} \cap \{1, \dots, T\}$ of periods where the set $[t_1, t_2] = \emptyset$ if period $t_1 > t_2$.

The classical TI formulation of single machine scheduling problems is an integer linear programme with binary variables x_{jt} for all jobs $j \in J$ and periods $t \in [1, T - p_j + 1]$. If the variable $x_{jt} = 1$ then job j starts at the beginning of period t at time $t-1$ and $x_{jt} = 0$ otherwise. For ease of exposition, job release dates and deadlines are omitted in the following formulation. They are easily incorporated into the model by fixing the appropriate variables to zero.

The classical TI formulation is the following:

$$\min \sum_{j \in J} \sum_{t \in [1, T-p_j+1]} c_{jt} x_{jt}, \tag{1}$$

$$\text{s.t.} \sum_{t \in [1, T-p_j+1]} x_{jt} = 1, \qquad j \in J, \tag{2}$$

$$\sum_{j \in J} \sum_{s \in [t-p_j+1, t]} x_{js} \leq 1, \quad t \in [1, T], \tag{3}$$

$$x_{jt} \in \{0, 1\}, \qquad j \in J, \ t \in [1, T - p_j + 1]. \tag{4}$$

All standard min-sum scheduling criteria can be modelled using the objective function (1) as they are linear in the TI variables. Constraints (2) ensure that each job starts exactly once and constraints (3) ensure that at most one job is processed on the machine at a time nonpreemptively. Constraints (4) enforce the integrality restrictions on the variables.

Perhaps the main advantage of the TI model is that it is known to have a strong LP relaxation, that is, the LP relaxation provides a strong lower bound on the value of an optimal solution. Furthermore, the formulation is naturally suited to be included in models of more complex production planning and scheduling problems which are frequently modelled using TI variables.

The main disadvantage of the TI model is that the formulation becomes intractable for practical instances with a large number of jobs and large processing times. The formulation becomes so large that even the LP relaxation cannot be solved in a reasonable time. Even if the integer linear programme could be solved efficiently in polynomial time, the formulation is pseudopolynomial in size and so can provide at best pseudopolynomial algorithms for the corresponding scheduling problem.

The simplest, and seemingly most effective, valid inequalities known for the TI model are the so called clique inequalities

$$\sum_{s\in[l-p_j+1,t]} x_{js} + \sum_{i\in J\setminus\{j\}} \sum_{s\in[t-p_i+1,l]} x_{is} \leq 1 \tag{5}$$

where each inequality is determined by one job j and two time periods l and t such that $l \leq t$. Note that if period $l = t$ these inequalities coincide with the constraints (3). If periods $l = p_j$, $t = T - p_j + 1$, and $l \leq t - p_i$ for all jobs $i \in J \setminus \{j\}$ these inequalities coincide with the constraints (2). These inequalities are polynomial in number and induce facets of the convex hull of the set of feasible solutions to the TI model if they are maximal with respect to set inclusion and

$$T \geq \sum_{j\in J} p_j + 3p_n.$$

For more details on these and other strong valid inequalities see, for example, Sousa and Wolsey (1992), Crama and Spieksma (1996), van den Akker et al. (1999), and Waterer et al. (2002).

3 Big bucket time indexed formulation

As is the case with the TI model we assume that the problem data is integer. However, this assumption is not strictly necessary and is discussed further in Section 3.5.

3.1 Preliminaries

Consider partitioning the planning horizon into B buckets of Δ periods such that $(B-1)\Delta < T \leq B\Delta$. Specifically, bucket b corresponds to the right half-open real interval $[(b-1)\Delta, b\Delta)$ that starts with period $(b-1)\Delta + 1$ and ends with period $b\Delta$. Thus, if $\tau \in [0, T)$ is an instant in time, then by definition time τ is contained in bucket $b = \lfloor \tau/\Delta \rfloor + 1$.

The length of each bucket Δ is chosen to be an integer number of periods no larger than the processing time of the shortest job, that is, $1 \leq \Delta \leq p_{\min}$. Choosing a bucket size $\Delta \leq p_{\min}$ ensures that at most one job can start in each bucket. Since the problem data is integer there exists an optimal solution for all standard min-sum scheduling criteria in which the start times of all jobs are integer valued. Choosing the bucket size Δ to be integer ensures at most Δ possible start times need be considered for a job in each bucket.

A job is said to span m buckets if it starts in bucket b and finishes in bucket $b + m - 1$. We adopt the convention that all jobs span at least two buckets. Consider the case in which a job with processing time Δ starts processing in bucket b at time $(b-1)\Delta + \epsilon$ where $0 < \epsilon \ll \Delta$. This job completes processing in bucket $b+1$ at time $b\Delta + \epsilon$ and thus spans two buckets. For continuity, and to be consist with the definition of a bucket, we require that in the limit as $\epsilon \to 0$ the job continues to span two buckets.

The actual number of buckets $m \geq 2$ that a job spans is dependent on the start time of the job. Let s_j denote the start time of job $j \in J$ in a feasible schedule and let S_j and σ_j be defined by $s_j = (S_j - \sigma_j)\Delta$ where $S_j = \lfloor s_j/\Delta \rfloor + 1$ and $\sigma_j = S_j - s_j/\Delta \in (0,1]$. Thus S_j denotes the bucket that job j starts in and σ_j the fraction of that bucket that was spent processing the job. Let P_j and π_j be defined by $p_j = (P_j - \pi_j)\Delta$ where $P_j = \lfloor p_j/\Delta \rfloor + 1$ and $\pi_j = P_j - p_j/\Delta \in (0,1]$ for all jobs $j \in J$. Note that $P_j \geq 2$ since $p_j \geq p_{\min} \geq \Delta$ and that $1 - \sigma_j = s_j/\Delta - \lfloor s_j/\Delta \rfloor$ and $1 - \pi_j = p_j/\Delta - \lfloor p_j/\Delta \rfloor$.

Proposition 1. *Job $j \in J$ will span $P_j + k_j$ buckets in a feasible schedule if and only if $1 - k_j < \sigma_j + \pi_j \leq 2 - k_j$ where $k_j \in \{0,1\}$.*

Proof. If job $j \in J$ starts processing at time s_j then the job starts processing in bucket $\lfloor s_j/\Delta \rfloor + 1$, completes processing in bucket $\lfloor (s_j + p_j)/\Delta \rfloor + 1$, and spans $m = \lfloor (s_j + p_j)/\Delta \rfloor - \lfloor s_j/\Delta \rfloor + 1$ buckets. Thus

$$\begin{aligned} m &= \left\lfloor \frac{s_j + p_j}{\Delta} \right\rfloor - \left\lfloor \frac{s_j}{\Delta} \right\rfloor + 1 = \left\lfloor \frac{s_j}{\Delta} + \frac{p_j}{\Delta} \right\rfloor - \left\lfloor \frac{s_j}{\Delta} \right\rfloor + 1 \\ &= \left\lfloor (1 - \sigma_j) + \left\lfloor \frac{s_j}{\Delta} \right\rfloor + (1 - \pi_j) + \left\lfloor \frac{p_j}{\Delta} \right\rfloor \right\rfloor - \left\lfloor \frac{s_j}{\Delta} \right\rfloor + 1 \\ &= \left\lfloor \frac{s_j}{\Delta} \right\rfloor + \left\lfloor \frac{p_j}{\Delta} \right\rfloor + \lfloor (1 - \sigma_j) + (1 - \pi_j) \rfloor - \left\lfloor \frac{s_j}{\Delta} \right\rfloor + 1 \\ &= \left\lfloor \frac{p_j}{\Delta} \right\rfloor + 1 + \lfloor 2 - \sigma_j - \pi_j \rfloor \\ &= P_j + \lfloor 2 - \sigma_j - \pi_j \rfloor . \end{aligned}$$

By definition $\sigma_j, \pi_j \geq 0$ and so $0 \leq \lfloor 2 - \sigma_j - \pi_j \rfloor \leq 1$. It then follows that $P_j \leq m \leq P_j + 1$. Furthermore, job j will span $P_j + k_j$ buckets if and only if $1 - k_j < \sigma_j + \pi_j \leq 2 - k_j$ where $k_j \in \{0,1\}$. □

In summary, if a job $j \in J$ starts processing at time s_j in bucket S_j then it will complete processing at time $s_j + p_j$ in bucket $S_j + P_j + k_j - 1$ where $k_j \in \{0,1\}$. The job will span $P_j + k_j$ buckets if and only if $1 - k_j < \sigma_j + \pi_j \leq 2 - k_j$. Finally, if σ_j is the fraction of bucket S_j used in processing the job then the fraction of bucket $S_j + P_j + k_j - 1$ used is $p_j/\Delta - \sigma_j - (P_j + k_j - 2) = 2 - k_j - \pi_j - \sigma_j \geq 0$. Figure 1 depicts the big bucket framework for an instance consisting of five jobs when the bucket size $\Delta = 6$. An illustration of the above notation is given in Figure 2.

3.2 MILP formulation

Our BB formulation of single machine scheduling problems is a mixed integer linear programme with binary variables z_{jbk} and continuous variables u_{jbk} and v_{jbk} for all jobs $j \in J$, indices $k \in K = \{0,1\}$ and buckets $b \in [1, B]$. If $z_{jbk} = 1$ then job j starts in bucket b and spans $P_j + k$ buckets, and $z_{jbk} = 0$ otherwise. The variable u_{jbk} denotes the fraction of bucket b spent processing job j if job j were to start in bucket b and span $P_j + k$ buckets, and $u_{jbk} = 0$ otherwise. Conversely, the variable v_{jbk} denotes the fraction of bucket b spent processing job j if job j were to complete processing in bucket b and spanned $P_j + k$ buckets, and $v_{jbk} = 0$ otherwise. Recall that if job j completes processing at time $(b - 1)\Delta$ then by definition it completes processing in bucket b and

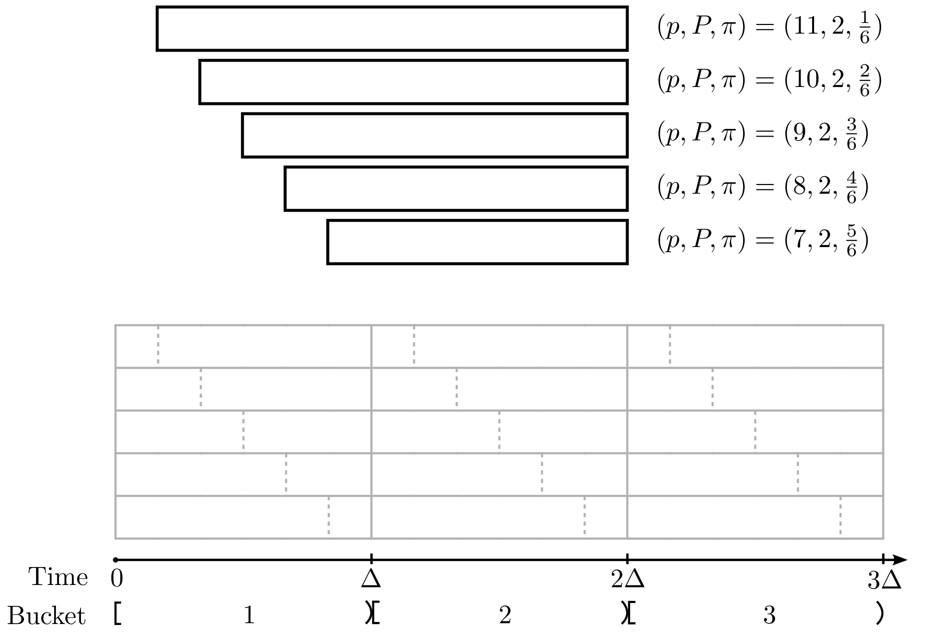


Figure 1: The big bucket framework for an instance consisting of five jobs when the bucket size $\Delta = 6$.

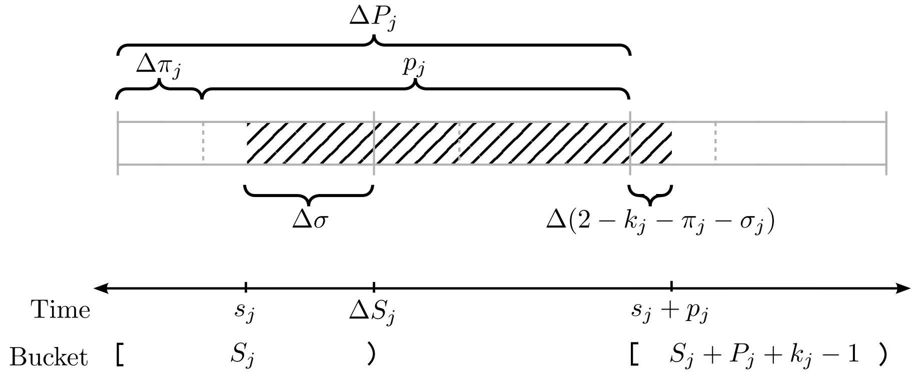


Figure 2: An illustration of the notation used in the big bucket framework.

the fraction of bucket b spent processing job j is zero, that is, the variables $z_{j,b-P_j+1,1} = 1$, $u_{j,b-P_j+1,1} = 1 - \pi_j$, and $v_{jb0} = 0$. For ease of exposition, job release dates and deadlines are omitted in the formulation. We discuss how these restrictions can be incorporated into the model later in the section.

The BB formulation is the following:

$$\text{(6)} \quad \min \sum_{j \in J} \sum_{k \in K} \sum_{b \in [1,B]} c^z_{jb} z_{jbk} + c^u_{jb} u_{jbk} + c^v_{jb} v_{jbk},$$

$$\text{(7)} \quad \text{s.t.} \sum_{k \in K} \sum_{b \in [1,B]} z_{jbk} = 1, \qquad j \in J,$$

$$\text{(8)} \quad \sum_{j \in J} \sum_{k \in K} \sum_{a \in [b-P_j-k+2,b]} z_{jak} \leq 1, \qquad b \in [1,B],$$

$$\text{(9)} \quad \sum_{j \in J} \sum_{k \in K} (u_{jbk} + v_{jbk} + \sum_{a \in [b-P_j-k+2, b-1]} z_{jak}) \leq 1, \quad b \in [1,B],$$

$$\text{(10)} \quad u_{jbk} + v_{j,b+P_j+k-1,k} = (2 - k - \pi_j) z_{jbk}, \qquad j \in J,\ k \in K,\ b \in [1,B],$$

$$\text{(11)} \quad ((1-k)(1-\pi_j) + \frac{1}{\Delta}) z_{jbk} \leq u_{jbk}, \qquad j \in J,\ k \in K,\ b \in [1,B]$$

$$\text{(12)} \quad u_{jbk} \leq (1 - k\pi_j) z_{jbk}, \qquad j \in J,\ k \in K,\ b \in [1,B],$$

$$\text{(13)} \quad (1-k)(1-\pi_j) z_{jbk} \leq v_{j,b+P_j+k-1,k}, \qquad j \in J,\ k \in K,\ b \in [1,B],$$

$$\text{(14)} \quad v_{j,b+P_j+k-1,k} \leq (1 - k\pi_j - \frac{1}{\Delta}) z_{jbk}, \qquad j \in J,\ k \in K,\ b \in [1,B]$$

$$\text{(15)} \quad z_{jbk} \in \{0,1\}, \qquad j \in J,\ k \in K,\ b \in [1,B].$$

With the exception of weighted tardiness, all standard min-sum scheduling criteria can be modelled using the objective function (6) since they are linear in the bucket-indexed variables. We discuss how these criteria can be incorporated into the model later in the section. Constraints (7) ensure that each job starts exactly once. Constraints (8) and (9) ensure that at most one job is processed on the machine at a time. Constraints (8) are the analog of constraints (3) for the TI model. If a job $j \in J$ is being processed at time $b\Delta$ then it must have started in one of the buckets $[b - P_j - k + 2, b]$. However, if the job were to start in bucket $b - P_j - k + 1$ or earlier then the job has completed processing prior to time $b\Delta$. Constraints (9) are a stronger form of the requirement that if two jobs are being processed in a bucket then the fraction of the bucket spent processing these jobs cannot exceed one. Constraints (10)–(14) ensure the correct relationships between the variables and the correct domains for the continuous variables. The constraints (11) and (14) are stronger forms of the simple variable bounds $(1-k)(1-\pi_j) z_{jbk} \leq u_{jbk}$ and $v_{j,b+P_j+k-1,k} \leq (1 - k\pi_j) z_{jbk}$, respectively, that leverage the fact that there are at most Δ possible start times for a job in each bucket. Constraints (15) enforce integrality restrictions on the appropriate variables.

A few observations about the formulation are in order:

1. The variables z_{jb1}, u_{jb1}, and $v_{j,b+P_j,1}$ can be omitted from the formulation for all buckets $b \in [1, B - P_j]$ if $\pi_j = 1$ for job $j \in J$ since such a job can

only ever span P_j buckets.

2. The variables z_{jbk} and u_{jbk} for all jobs $j \in J$, indices $k \in K$, and buckets $b \in [B - P_j - k + 2, B]$, and the variables v_{jbk} for all jobs $j \in J$, indices $k \in K$, and buckets $b \in [1, P_j + k - 1]$ can be omitted from the formulation since jobs can neither start or end in these buckets respectively. While the model remains valid when these variables are included the LP relaxations are noticeably weaker.

3. The variables v_{jbk} are redundant and can be eliminated using the constraints (10). The constraints (13) and (14) are similarly redundant as they are equivalent to the constraints (12) and (11) respectively.

4. If constraints (10)–(14) are multiplied by Δ then all coefficients in these constraints are integer valued since we have assumed that the problem data is integer.

5. The constant $1/\Delta$ in constraints (11) and (14) can be replaced by a constant ϵ where $0 \leq \epsilon \leq 1/\Delta$. If $\epsilon = 1/\Delta$ the constraints are the strongest possible. If $\epsilon < 1/\Delta$ a relaxation to the model is obtained. We note that if the problem data is integer, as ϵ decreases the cardinality of the set of feasible solutions (z, u, v) to the resulting relaxations increases, but the set of optimal feasible solutions remains the same. However, if $\epsilon = 0$ then there are feasible schedules s that do not have a unique representation (z, u, v) in the resulting relaxation and the interpretation of these solutions is inconsistent with various definitions underlying the model. For example, $(z_{jb1}, u_{jb1}) = (1, 0)$ and $(z_{j,b+1,0}, u_{j,b+1,0}) = (1, 1)$ both correspond to job j starting at the beginning of bucket b at time $b\Delta$. Consequently, a job starting at these times can either span P_j or $P_j + 1$ buckets. While the resulting inconsistency could be resolved it is more convenient and much simpler to restrict $\epsilon > 0$ if such relaxations are of interest.

3.3 Correctness of the formulation

In this section we prove the correctness of the BB model. We first show that every feasible schedule of start times corresponds to a feasible solution to the BB model.

Proposition 2. *Every feasible schedule $s = (s_1, s_2, \ldots, s_n)$ of start times corresponds to a feasible solution (z, u, v) to the BB model.*

Proof. Recall that if job $j \in J$ starts at time s_j then it starts in bucket $S_j = \lfloor s_j/\Delta \rfloor + 1$ and $\sigma_j = S_j - s_j/\Delta \in (0, 1]$ is the fraction of that bucket that was spent processing the job. Let index $k_j = 1$ if $\sigma_j + \pi_j \leq 1$ and $k_j = 0$ otherwise. For all jobs $j \in J$, buckets $b \in [1, B]$, and indices $k \in K$, let variable $z_{jbk} = 1$ if $(j, b, k) = (j, S_j, k_j)$ and $z_{jbk} = 0$ otherwise. Similarly, let variable $u_{jbk} = \sigma_j$ if $(j, b, k) = (j, S_j, k_j)$ and $u_{jbk} = 0$ otherwise, and variable $v_{jbk} = 2 - k_j - \sigma_j - \pi_j$ if $(j, b, k) = (j, S_j + P_j + k_j - 1, k_j)$ and $v_{jbk} = 0$ otherwise. Note that $0 \leq u_{jbk} \leq z_{jbk} \in \{0, 1\}$ and $0 \leq v_{j,b+P_j+k-1,k} \leq z_{jbk} \in \{0, 1\}$ for all jobs $j \in J$, buckets $b \in [1, B]$, and indices $k \in K$.

It is straightforward to show the validity of each of the constraints (7)–(14) with respect to the feasible schedule s. For completeness a proof is given in the Appendix. □

We now show that every feasible solution to the BB model corresponds to a feasible schedule of start times.

Proposition 3. *Every feasible solution (z, u, v) to the BB model corresponds to a feasible schedule s of start times.*

Proof. Define $z_{jb} = z_{jb0} + z_{jb1}$, $u_{jb} = u_{jb0} + u_{jb1}$, and $v_{jb} = v_{jb0} + v_{jb1}$ for all jobs $j \in J$ and buckets $b \in [1, B]$. Note that $z_{jb} \in \{0, 1\}$ by constraints (7) and (15), that $0 \le u_{jb} \le 1$ and $0 \le v_{jb} \le 1$ by constraints (11)–(14), and that $u_{jb} + v_{jb} \le 1$ by constraint (9). Define time

$$s_j = \Delta \sum_{b \in [1,B]} (b z_{jb} - u_{jb})$$

for all jobs $j \in J$. Thus

$$S_j = \sum_{b \in [1,B]} b z_{jb} \text{ and } \sigma_j = \sum_{b \in [1,B]} u_{jb}.$$

Let index $k_j = 1$ if $\sigma_j + \pi_j \le 1$ and $k_j = 0$ otherwise. Note that time $s_j \ge 0$ for all jobs $j \in J$.

To show that s is a feasible schedule it suffices to show that if $S_j \ge S_i$ then time $s_j \ge s_i + p_i$ for all jobs $i, j \in J$. If $S_j \ge S_i$ then $S_j \ge S_i + P_i + k_i - 1$ by constraint (8). Therefore

$$\begin{aligned} s_j - s_i &= \Delta((S_j - u_{jS_j}) - (S_i - u_{iS_i})) \\ &= \Delta((S_j - S_i) - (u_{jS_j} - u_{iS_i})) \\ &= \Delta((S_j - S_i) - (u_{jS_j} - (2 - k_i - \pi_i) + v_{i,S_i+P_i+k_i-1})) \\ &= \Delta((S_j - S_i) + (2 - k_i - \pi_i - u_{jS_j} - v_{i,S_i+P_i+k_i-1})) \end{aligned}$$

by way of constraints (10). We now consider two cases. Firstly, suppose that $S_j = S_i + P_i + k_i - 1$. Since $u_{jb} + v_{ib} \le 1$ it follows that

$$\begin{aligned} s_j - s_i &= \Delta((P_i + k_i - 1) + (2 - k_i - \pi_i - u_{jS_j} - v_{iS_j})) \\ &\ge \Delta((P_i + k_i - 1) + (2 - k_i - \pi_i - 1)) \\ &= \Delta(P_i - \pi_i) \\ &= p_i \end{aligned}$$

and so time $s_j \ge s_i + p_i$. Secondly, suppose that $S_j > S_i + P_i + k_i - 1$. Since $u_{jb} \le 1$ and $v_{jb} \le 1$ it follows that

$$\begin{aligned} s_j - s_i &\ge \Delta((P_i + k_i) + (2 - k_i - \pi_i - u_{jS_j} - v_{i,S_i+P_i+k_i-1})) \\ &\ge \Delta((P_i + k_i) - k_i - \pi_i) \\ &= \Delta(P_i - \pi_i) \\ &= p_i \end{aligned}$$

and so time $s_j \ge s_i + p_i$. $\square$

3.4 Modelling release dates, due dates, deadlines, and min-sum criteria

A job $j \in J$ may be subject to a release date r_j, a due date d_j, or a deadline $\bar{d}_j$ which we assume are specified with respect to the start time of the job. Such restrictions are easily incorporated into the BB model. Let $r_j = (R_j - \rho_j)\Delta$ where $R_j = \lfloor r_j/\Delta \rfloor + 1$ and $\rho_j = R_j - r_j/\Delta \in (0,1]$. Similarly, let $d_j = (D_j - \delta_j)\Delta$ where $D_j = \lfloor d_j/\Delta \rfloor + 1$ and $\delta_j = D_j - d_j/\Delta \in (0,1]$, and $\bar{d}_j = (\bar{D}_j - \delta_j)\Delta$ where $\bar{D}_j = \lfloor \bar{d}_j/\Delta \rfloor + 1$ and $\bar{\delta}_j = \bar{D}_j - \bar{d}_j/\Delta \in (0,1]$. Note that if $0 \leq r_j \leq d_j \leq \bar{d}_j \leq T - p_j + 1$ for a job $j \in J$ then $1 \leq R_j \leq D_j \leq \bar{D}_j \leq B - P_j + 1$.

If $\rho_j \in \{1 - \pi_j, 1\}$ and $\bar{\delta}_j \in \{1 - \pi_j, 1\}$ then no further constraints are necessary. Otherwise, to enforce job release dates or deadlines, the following constraints need to be added to the model:

$$(16) \qquad u_{jR_j0} \leq \rho_j z_{jR_j0}, \quad j \in J\colon 1 - \pi_j < \rho_j < 1,$$

$$(17) \qquad u_{jR_j1} \leq \rho_j z_{jR_j1}, \quad j \in J\colon \rho_j < 1 - \pi_j,$$

$$(18) \qquad \bar{\delta}_j z_{j\bar{D}_j0} \leq u_{j\bar{D}_j0}, \quad j \in J\colon 1 - \pi_j < \bar{\delta}_j < 1,$$

$$(19) \qquad \bar{\delta}_j z_{j\bar{D}_j1} \leq u_{j\bar{D}_j1}, \quad j \in J\colon \bar{\delta}_j < 1 - \pi_j.$$

Note that these constraints are variable upper and lower bounds on the variables u_{jR_jk} and $u_{j\bar{D}_jk}$ respectively since release dates and deadlines are specified relative to the start time of a job. In addition, various variables can be fixed. Specifically, variable $z_{jR_j0} = 0$ if $\rho_j < 1 - \pi_j$ for all jobs $j \in J$, variables $z_{jbk} = u_{jbk} = v_{j,b+P_j+k-1,k} = 0$ for all jobs $j \in J$, indices $k \in K$, and buckets $b \in [1, B - P_j - k + 1] \setminus [R_j, \bar{D}_j]$, and variable $z_{j\bar{D}_j1} = 0$ if $1 - \pi_j < \bar{\delta}_j < 1$ for all jobs $j \in J$.

As previously mentioned, with the exception of weighted tardiness, the standard min-sum criteria are linear in the bucket-indexed variables. For example, to model weighted start time, which is equivalent to modelling weighted completion time since the two criteria differ by a constant, we let $c^z_{jb} = w_j b\Delta$, $c^u_{jb} = -w_j\Delta$, and $c^v_{jb} = 0$ for all jobs $j \in J$ and buckets $b \in [1, B]$. Weighted tardiness, however, is linear in all but possibly the pair of bucket-indexed variables corresponding to the bucket the due date of each job falls in, in which case it is piecewise linear. Thus, to model weighted tardiness we introduce the variables T_{jbk} for all jobs $j \in J$, buckets $b \in [D_j, B]$, and indices $k \in K$, and consider the following objective and constraints:

$$(20) \quad \min \sum_{j \in J} \sum_{k \in K} \sum_{b \in [D_j, B]} w_j \Delta T_{jbk},$$

$$(21) \quad \text{s.t. } (1 - \delta_j) z_{jD_j0} - u_{jD_j0} \leq T_{jD_j0}, \quad j \in J\colon 1 - \pi_j < \delta_j,$$

$$(22) \quad T_{jD_j0} \leq ((1 - \delta_j) - (1 - \pi_j)) z_{jD_j0}, \quad j \in J\colon 1 - \pi_j < \delta_j,$$

$$(23) \quad T_{jD_j1} = \delta_j z_{jD_j1} - u_{jD_j1}, \quad j \in J\colon 1 - \pi_j < \delta_j,$$

$$(24) \quad (1 - \delta_j) z_{jD_j1} - u_{jD_j1} \leq T_{jD_j1}, \quad j \in J\colon \delta_j \leq 1 - \pi_j,$$

$$(25) \quad T_{jD_j1} \leq (1 - \delta_j) z_{jD_j1}, \quad j \in J\colon \delta_j \leq 1 - \pi_j,$$

$$(26) \quad T_{jbk} = (b - D_j + \delta_j) z_{jbk} - u_{jbk}, \quad j \in J,\ k \in K,\ b \in [D_j + 1, B].$$

Note that the variables T_{jD_j0} are only defined if $1 - \pi_j < \delta_j$ for $j \in J$ and the variables T_{jbk} are redundant for all jobs $j \in J$, buckets $b \in [D_j + 1, B]$, and indices $k \in K$, and can be eliminated.

3.5 Comparing the TI and BB models

The TI model requires the problem data to be integer. This requirement is a result of the implicit assumption that jobs must start at the beginning of a time period. The BB model on the other hand can represent feasible schedules in which the jobs have arbitrary rational start times. The model implicitly scales the problem data which, for a sufficiently large choice of bucket size Δ, results in the BB model having fewer variables and nonzeros than the TI model at the expense of a greater number of constraints. This is discussed in more detail in Section 3.6.

A feature of the BB model that is a result of this scaling is that the model does not require the problem data to be integer, only rational. For example, consider an instance of the problem with rational data where $q \geq 2$ is the least common multiple of the denominators when the data is expressed in fractional form, and a corresponding integer instance obtained by multiplying the rational data by q. Suppose that the constant $1/\Delta$ in constraints (11) and (14) has been replaced by the constant ϵ where $0 < \epsilon \leq 1/\Delta$. There is a one-to-one correspondence between the feasible solutions to the instance of the BB model resulting from the integer data with bucket size $\Delta = \Delta_I$ and constant $\epsilon = \epsilon_I = 1/\Delta_I$, and the feasible solutions to the instance resulting from the rational data with bucket size $\Delta = \Delta_R = \Delta_I/q$ and constant $\epsilon = \epsilon_R = \epsilon_I/q = 1/(q\Delta_I)$. Note that if there are at most Δ_I possible start times for a job in each bucket in the instance of the BB model resulting from the integer data then there are at most $q\Delta_I$ possible start times for a job in each bucket in the instance resulting from the rational data when $q\Delta_R = \Delta_I$.

Practically there is little to be gained from considering rational data since validating the choice for the value for the constant ϵ requires knowledge of the least common multiple of the denominators of the data when expressed in fractional form. However, we note that in the case of rational data the requirement that $\Delta \geq 1$ can be relaxed to $\Delta > 0$. If the shortest processing time $p_{\min} < 1$ the BB model implicitly scales up the problem data.

Assuming integer data, there is obviously a correspondence between feasible solutions to the TI and BB models. Every feasible solution to the TI model can be mapped to a feasible solution to the BB model, but the converse is not true: there exist feasible solutions to the BB model that do not have a corresponding solution to the TI model. Recall that a period $t \in [1, T]$ corresponds to the right half-open real interval $[t-1, t)$. Then, by definition of a bucket, period t is contained in bucket $b \in \lfloor (t-1)/\Delta \rfloor + 1$. Let $z_{jb} = z_{jb0} + z_{jb1}$ and $u_{jb} = u_{jb0} + u_{jb1}$ for all jobs $j \in J$ and buckets $b = [1, B - P_j + 1]$. Consider a feasible solution x to the TI model. If $x_{jt} = 1$ then $(z_{jb}, u_{jb}) = (1, (b\Delta - (t-1))/\Delta)$ in the corresponding feasible solution to the BB model where $(z_{jb}, u_{jb}) = (z_{jb1}, u_{jb1})$ if $b\Delta - (t-1) \leq \Delta(1 - \pi_j)$ and $(z_{jb}, u_{jb}) = (z_{jb0}, u_{jb0})$ otherwise. Conversely, consider a feasible solution (z, u) to the BB model. If $u_{jb} > 0$ and Δu_{jb} is integer valued then $x_{jt} = 1$ in the corresponding feasible solution to the TI model where $t = (b - u_{jb})\Delta + 1$. If Δu_{jb} is not integer valued, then there is no corresponding solution to the TI model.

An important feature of the BB model is that the TI model is a special case when all problem data are multiples of Δ and have a greatest common divisor of one, that is, $\Delta = p_{\min} = 1$. Recall that the variables z_{jb1}, u_{jb1}, $v_{j,b+P_j,1}$ for all jobs $j \in J$ and buckets $b \in [1, B - P_j]$ are omitted from the formulation since all processing times are a multiple of Δ. If the problem data is integer and $\Delta = 1$ then the buckets of the BB model are equivalent to the periods of the TI model. Furthermore, constraints (11)–(14) impose the restrictions that variables $u_{jb0} = z_{jb0}$ and $v_{j,b-P_j+1,0} = 0$ for all jobs $j \in J$ and buckets $b \in [1, B-P_j+1]$. Consequently, these variables can be omitted from the model which leaves z_{jb0} as the only variables in the model.

3.6 Choice of bucket size

A parameter that needs to be specified in the BB model is the bucket size Δ where $1 \leq \Delta \leq p_{\min}$. The choice of Δ affects the number of variables, constraints, and nonzeros in the model. As we have seen, if Δ is chosen to be one then the BB model reduces to the TI model. However, if Δ can be chosen to be larger than one then, in general, the number of variables, constraints, and nonzeros in the BB model decreases as the value of Δ increases although this relationship is not linear. As a result, the number of variables and nonzeros in the BB model can be significantly less than those in the TI model. However, the number of constraints in the BB model is never less than those in the TI model and can be several orders of magnitude more.

In the following analysis we assume that the problem data is integer and that the BB model is formulated with just the variables (z, u), that is, the variables v and the corresponding constraints have been eliminated. Note that the number of buckets B, and the parameters P_j and π_j associated with each job $j \in J$, are functions of the buckets size Δ. Let N denote the number of variables in the BB model and M the number of constraints. Let $J_r = \{j \in J : \Delta(1 - \pi_j) = r\}$ and $n_r = |J_r|$ for $r \in \{0, 1, \Delta - 1\}$, and consider the following three cases.

1. Suppose that $\Delta = 1$ and so the BB model reduces to the TI model. Then the set $J_0 = J$ and the variables z_{jb1}, u_{jb1}, and v_{jb1} are not defined for all jobs $j \in J$ and buckets $b \in [1, B-P_j]$. There is one variable (z_{jb0}) for each job $j \in J$ and bucket $b \in [1, B - P_j + 1]$ since variables $u_{jb0} = z_{jb0}$. Thus the number of variables $N = n_0 B = nB$ and the number of constraints $M = n + B$.

2. Suppose that $\Delta = 2$. Then the set $J_1 = J_{(\Delta-1)}$ and there are two variables (z_{jb0} and z_{jb1}) for each job $j \in J_1$, index $k \in K$, and bucket $b \in [1, B - P_j - k + 1]$ since $u_{jbk} = z_{jbk}$. There are also two variables (z_{jb0} and u_{jb0}) for each job $j \in J_0$ and bucket $b \in [1, B - P_j + 1]$. Thus the number of variables $N = 2(n_0 + n_1)B = 2nB$ and the number of constraints $M = n + 2B + 2n_0 B$.

3. Suppose that $\Delta \geq 3$. There are three variables (z_{jb0}, u_{jb0}, and z_{jb1}) for each job $j \in J_1$, index $k \in K$, and bucket $b \in [1, B - P_j - k + 1]$ since variables $u_{jb1} = z_{jb1}$. There are three variables (z_{jb0}, z_{jb1}, and u_{jb1}) for each job $j \in J_{(\Delta-1)}$, index $k \in K$, and bucket $b \in [1, B - P_j - k + 1]$ since variables $u_{jb0} = z_{jb0}$. Again there are two variables (z_{jb0} and u_{jb0}) for each job $j \in J_0$ and bucket $b \in [1, B - P_j + 1]$. Finally, there are four variables for each job $j \in J \setminus \{J_0 \cup J_1 \cup J_{(\Delta-1)}\}$, $k \in K$, and bucket $b \in [1, B - P_j - k + 1]$. Thus the number of variables $N = (4(n - n_0 - n_1 - n_{(\Delta-1)}) + 2n_0 + 3n_1 + 3n_{(\Delta-1)})B = (4n - 2n_0 - n_1 - n_{(\Delta-1)})B$ and

the number of constraints $M = n + 2B + 2(2(n - n_0 - n_1 - n_{(\Delta-1)}) + n_0 + n_1 + n_{(\Delta-1)})B = n + 2B + 2(2n - n_0 - n_1 - n_{(\Delta-1)})B$.

It is reasonable to assume that the overriding goal is to solve the BB model to optimality as quickly as possible. If that is not possible in the prescribed time limit then arguably a provably good feasible solution is sought. Thus a natural question to arise is what is the best choice for the value of Δ to achieve this goal? The effect of different choices for the bucket size Δ are investigated in Section 4.4.

3.7 Valid inequalities

In this section we assume that the problem data is integer and that the BB model is formulated with just the variables (z, u), that is, the variables v and the corresponding constraints have been eliminated.

Let X denote the set of all feasible solutions x to the TI model. Polyhedral studies of the TI model have characterised facet-inducing inequalities for the convex hull $\text{conv}(X^*)$ of the set X^* of feasible partial schedules (Sousa and Wolsey, 1992, Crama and Spieksma, 1996, van den Akker et al., 1999, Waterer et al., 2002). A feasible partial schedule is a feasible schedule in which not all jobs have to be started. The polyhedron $\text{conv}(X)$ is a face of the polyhedron $\text{conv}(X^*)$ and so valid inequalities for the set X^* are also valid for the set X. Conditions under which the facet-inducing inequalities for $\text{conv}(X^*)$ are also facet-inducing for $\text{conv}(X)$ are given.

Let Z denote the set of all feasible solutions (z, u) to the BB model. In this section we will characterise an exponentially sized family of strong valid inequalities for the convex hull $\text{conv}(Z)$ of the set Z and prove that a subset of these inequalities are facet-inducing inequalities for the convex hull $\text{conv}(Z^*)$ of the set Z^* of feasible partial schedules to the BB model. The strong valid inequalities for the polyhedron $\text{conv}(Z)$ are obtained by transforming the facet-inducing inequalities (5) for the polyhedron $\text{conv}(X)$. To do this we must relate the variables x_{jt} of the TI model to the variable pairs (z_{jbk}, u_{jbk}) of the BB model. This requires the introduction of the notion of a half-bucket.

Bucket $b \in [1, B]$ contains the periods $[(b-1)\Delta + 1, b\Delta]$. With respect to each job $j \in J$, bucket b can be partitioned into the two half-buckets $(j, b, 0)$ and $(j, b, 1)$ corresponding to the periods $\mathcal{T}_{jb0} = [(b-1)\Delta + 1, (b - (1 - \pi_j))\Delta]$ and $\mathcal{T}_{jb1} = [(b - (1 - \pi_j))\Delta + 1, b\Delta]$ respectively. That is, the half-bucket $(j, b, k) \in J \times [1, B] \times K$ contains the periods $\mathcal{T}_{jbk} = [(b - (1 - k\pi_j))\Delta + 1, (b - (1 - k)(1 - \pi_j))\Delta]$. Conversely, period $t \in [1, T]$ is contained in the half-bucket (j, b, k) of job $j \in J$ where $b = \lceil t/\Delta \rceil$ and $k = \lceil (t + (1 - \pi_j)\Delta - b\Delta)/((1 - \pi_j)\Delta) \rceil$.

For every feasible solution $x \in X$ there exists a unique feasible solution $(z, u) \in Z$ defined by

(27a)
$$z_{jbk} = \sum_{t \in \mathcal{T}_{jbk}} x_{jt}$$

and

(27b)
$$u_{jbk} = \sum_{t \in \mathcal{T}_{jbk}} \frac{b\Delta - t + 1}{\Delta} x_{jt}.$$

Let the triple (x, z, u) denote a pair of such feasible solutions and let XZ denote the set of all such triples.

Each of the inequalities (5) can be written as

$$\sum_{(j,t)\in\mathcal{S}} x_{jt} \leq 1$$

where the set $\mathcal{S} \subseteq J \times [1, T]$ denotes the index set of variables x_{jt} with nonzero coefficients in the inequality. Van den Akker et al. (1999) showed that each inequality is determined by one job, without loss of generality job 1, and two time periods l and t such that $1 \leq l \leq t \leq T$. Specifically, the sets $\mathcal{S}_j = \{t\colon (j,t) \in \mathcal{S}\} \subseteq [1, T - p_j + 1]$ which denote those time periods associated with the indices of each job $j \in J$ are the sets $\mathcal{S}_1 = [l - p_1 + 1, t]$ and $\mathcal{S}_j = [t - p_j + 1, l]$ for all jobs $j \in J \setminus \{1\}$. In order to differentiate between the inequalities (5) we refer to the specific inequality that is determined by job j and the two time periods t and l as the (j, l, t)-inequality. In this section we will show how to transform the $(1, l, t)$-inequality to a strong valid inequality for the set Z. For ease of exposition we assume that the set $\mathcal{S}_j \neq \emptyset$ for all jobs $j \in J$. If this is not the case then simply reinterpret the set J to be the set of jobs restricted to those for which this is true.

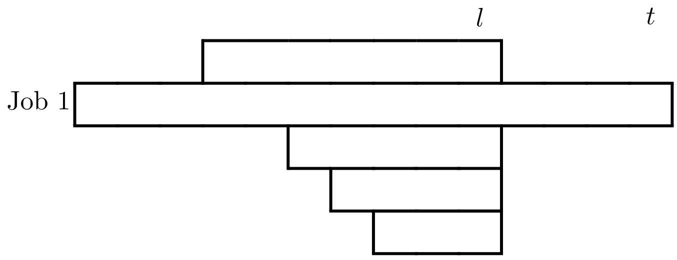
Let the set $\mathcal{J} \subseteq J$ denote those jobs $j \in J$ for which the periods $\mathcal{S}_j$ include either the first or last period of at least one half-bucket (j, b, k). That is, $j \in \mathcal{J}$ if and only if $\mathcal{S}_j \cap \{(b - (1 - k\pi_j))\Delta + 1, (b - (1 - k)(1 - \pi_j))\Delta\} \neq \emptyset$ for at least one half-bucket $(j, b, k) \in J \times [1, B] \times K$. In the remainder of this section we restrict our attention to the set $\mathcal{J}$ of jobs. For each job $j \in \mathcal{J}$ the set $\mathcal{S}_j$ can be partitioned into the three sets $[\mathcal{S}_j^L|\mathcal{S}_j^M|\mathcal{S}_j^R]$ based on the half-buckets it intersects such that a relaxed version of inequality (5) can be written as

$$\sum_{j\in\mathcal{J}} \left(\sum_{t\in\mathcal{S}_j^L} x_{jt} + \sum_{t\in\mathcal{S}_j^M} x_{jt} + \sum_{t\in\mathcal{S}_j^R} x_{jt} \right) \leq 1. \tag{28}$$

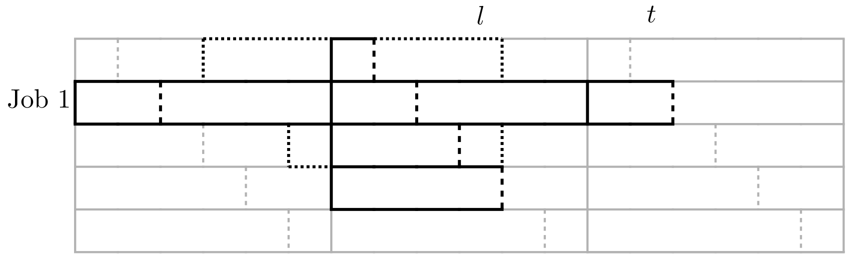
We now define the sets $[\mathcal{S}_j^L|\mathcal{S}_j^M|\mathcal{S}_j^R]$ for each job $j \in \mathcal{J}$. The possibly empty middle interval $\mathcal{S}_j^M$ corresponds to the periods of those half-buckets that are subsets of the interval $\mathcal{S}_j$. That is, if the set $\mathcal{H}_j$ denotes those half-buckets $(j, b, k) \in J \times [1, B] \times K$ whose set $\mathcal{T}_{jbk}$ of periods is a subset of the set $\mathcal{S}_j$, then $\mathcal{H}_j = \{(b, k) \in [1, B] \times K \colon \mathcal{T}_{jbk} \subseteq \mathcal{S}_j\}$ and

$$\mathcal{S}_j^M = \bigcup_{(b,k)\in\mathcal{H}_j} \mathcal{T}_{jbk}$$

for each job $j \in \mathcal{J}$. Note that even if $\mathcal{S}_j^M = \emptyset$ the chronologically ordered set $\mathcal{S}_j \setminus \mathcal{S}_j^M$ of periods naturally separates into a sub-interval $\mathcal{S}_j^L$ of "left" periods and a sub-interval $\mathcal{S}_j^R$ of "right" periods. Furthermore, by definition of the set $\mathcal{S}_j^M$, the possibly empty left interval $\mathcal{S}_j^L$ is a strict subset of the set $\mathcal{T}_{jb_j^L k_j^L}$ of periods for some half-bucket (j, b_j^L, k_j^L) which we denote $\mathcal{T}_j^L$ for ease of exposition. If the set $\mathcal{S}_j^L$ is not empty, then it does not contain the first period $(b_j^L - (1 - k_j^L\pi_j))\Delta + 1$ in the half-bucket (j, b_j^L, k_j^L), but does contain the last period $(b_j^L - (1 - k_j^L)(1 - \pi_j))\Delta$. That is, $\emptyset \subseteq \mathcal{S}_j^L = \mathcal{S}_j \cap \mathcal{T}_j^L$ where $(b_j^L - (1 - k_j^L\pi_j))\Delta + 1 \in \mathcal{T}_j^L \setminus \mathcal{S}_j^L$ and if



(a) Sets $\mathcal{S}_j$ in a TI $(1, l, t)$-inequality



(b) Corresponding sets $[\mathcal{S}_j^L|\mathcal{S}_j^M|\mathcal{S}_j^R]$ in a BB inequality

Figure 3: The sets $\mathcal{S}_j$ for jobs $j \in J$ in a $(1, l, t)$-inequality and the corresponding sets $[\mathcal{S}_j^L|\mathcal{S}_j^M|\mathcal{S}_j^R]$ in a strong valid inequality for the set Z. The sets $\mathcal{S}_j^M$ have a solid outline and the sets $\mathcal{S}_j^G$ have a dotted outline where index $G \in \{L, R\}$.

$\mathcal{S}_j^L \neq \emptyset$ then $(b_j^L - (1 - k_j^L)(1 - \pi_j))\Delta \in \mathcal{S}_j^L$. Similarly, the possibly empty right interval $\mathcal{S}_j^R$ is a strict subset of the set $\mathcal{T}_{jb_j^R k_j^R}$ of periods for some half-bucket (j, b_j^R, k_j^R) which we denote $\mathcal{T}_j^R$ for ease of exposition. If the set $\mathcal{S}_j^R$ is not empty, then it does contain the first period $(b_j^R - (1 - k_j^R \pi_j))\Delta + 1$ in the half-bucket (j, b_j^R, k_j^R), but does not contain the last period $(b_j^R - (1 - k_j^R)(1 - \pi_j))\Delta$. That is, $\emptyset \subseteq \mathcal{S}_j^R = \mathcal{S}_j \cap \mathcal{T}_j^R$ where $(b_j^R - (1 - k_j^R)(1 - \pi_j))\Delta \in \mathcal{T}_j^R \setminus \mathcal{S}_j^R$ and if $\mathcal{S}_j^R \neq \emptyset$ then $(b_j^R - (1 - k_j^R \pi_j))\Delta + 1 \in \mathcal{S}_j^R$. Figure 3(a) depicts the sets $\mathcal{S}_j$ for each job $j \in J$ in a $(1, l, t)$-inequality and the corresponding sets $[\mathcal{S}_j^L|\mathcal{S}_j^M|\mathcal{S}_j^R]$ in a strong valid inequality for the set Z are shown in Figure 3(b).

For ease of exposition we introduce some further notation. Let the set $\mathcal{J}^G = \{j \in \mathcal{J} \colon \mathcal{S}_j^G \neq \emptyset\}$ for each index $G \in \{L, R\}$. For each job $j \in \mathcal{J}^G$ and index $G \in \{L, R\}$ we denote the pair $(z_{jb_j^G k_j^G}, u_{jb_j^G k_j^G})$ of variables corresponding to the half-bucket (j, b_j^G, k_j^G) by (z_j^G, u_j^G) and define the linear expressions $w_j^G(z_j^G, u_j^G)$ to be

$$w_j^L(z_j^L, u_j^L) := \frac{1}{|\mathcal{S}_j^L|}(((1 - k_j^L)(1 - \pi_j)\Delta + |\mathcal{S}_j^L| + 1)z_j^L - \Delta u_j^L) \tag{29a}$$

and

$$w_j^R(z_j^R, u_j^R) := \frac{1}{|\mathcal{S}_j^R|}(\Delta u_j^R - ((1 - k_j^R \pi_j)\Delta - |\mathcal{S}_j^R|)z_j^R). \tag{29b}$$

Let w_j^G denote the expression $w_j^G(z_j^G, u_j^G)$ and note that these expressions are simply introduced for notational convenience and should not be interpreted as functions or additional variables that have been added to the problem.

Now consider a job $j \in \mathcal{J}^G$ for some index $G = \{L, R\}$. Let

$$y_j^G = \sum_{t \in \mathcal{S}_j^G} x_{jt}$$

and let the set $Y_j^G = \text{proj}_{(y_j^G, z_j^G, u_j^G)} XZ$, that is, the set Y_j^G is the projection of the set XZ on to the (y_j^G, z_j^G, u_j^G)-space. From the constraints (11) and (12), and the relations (27), we see that the set

$$\text{(30a)} \quad Y_j^L = \{(0,0,0)\} \cup \\ \{(1,1,u_j^L) \colon \Delta u_j^L \in [(1-k_j^L)(1-\pi_j)\Delta + 1, (1-k_j^L)(1-\pi_j)\Delta + |\mathcal{S}_j^L|]\} \cup \\ \{(0,1,u_j^L) \colon \Delta u_j^L \in [(1-k_j^L)(1-\pi_j)\Delta + |\mathcal{S}_j^L| + 1, (1-k_j^L\pi_j)\Delta]\}.$$

Similarly, the set

$$\text{(30b)} \quad Y_j^R = \{(0,0,0)\} \cup \\ \{(0,1,u_j^R) \colon \Delta u_j^R \in [(1-k_j^R)(1-\pi_j)\Delta + 1, (1-k_j^R\pi_j)\Delta - |\mathcal{S}_j^R|]\} \cup \\ \{(1,1,u_j^R) \colon \Delta u_j^R \in [(1-k_j^R\pi_j)\Delta - |\mathcal{S}_j^R| + 1, (1-k_j^R\pi_j)\Delta]\}.$$

We now prove two important properties relating to the expressions w_j^G. The first property implies that the expressions w_j^G are the strongest possible.

Lemma 4. *Consider a job $j \in \mathcal{J}^G$ for some index $G \in \{L, R\}$ and let the expression $w_j^G(z_j^G, u_j^G)$ be defined as in equations (29). The inequality $y_j^G \geq w_j^G(z_j^G, u_j^G)$ is a valid inequality for the set Y_j^G and is a facet-inducing inequality for the convex hull $\text{conv}(Y_j^G)$ of the set Y_j^G.*

Proof. Using the explicit description of the sets Y_j^G in (30) it can be verified that the inequality $y_j^G \geq w_j^G(z_j^G, u_j^G)$ is a valid inequality for the set Y_j^G and that the polyhedron $\text{conv}(Y_j^G)$ is full dimensional. To prove that the inequality $y_j^G \geq w_j^G(z_j^G, u_j^G)$ is a facet-inducing inequality of the polyhedron $\text{conv}(Y_j^G)$ it suffices to identify three affinely independent points in the set Y_j^G that satisfy the inequality at equality. Three such points in the set Y_j^L are $(y_j^L, z_j^L, u_j^L) \in \{(0,0,0), (0,1,(1-k_j^L)(1-\pi_j) + (|S_j^L|+1)/\Delta), (1,1,(1-k_j^L)(1-\pi_j)+1/\Delta)\}$. Similarly, three affinely independent points in the set Y_j^R are $(y_j^R, z_j^R, u_j^R) \in \{(0,0,0), (0,1,1-k_j^R\pi_j - |S_j^R|/\Delta), (1,1,1-k_j^R\pi_j)\}$. □

The second property is that if the intervals $\mathcal{S}_j^G$ are maximal with respect to cardinality, then the inequality $w_j^G(z_j^G, u_j^G) \geq 0$ is a valid inequality for the set Y_j^G. Let the set $\mathcal{W}^G = \{j \in \mathcal{J}^G : |\mathcal{T}_j^G \setminus \mathcal{S}_j^G| = 1\}$ for each index $G \in \{L, R\}$. The following claim can be verified using the explicit description of the sets Y_j^G in (30).

Lemma 5. *Consider a job $j \in \mathcal{W}^G$ for some index $G \in \{L, R\}$ and let the expression $w_j^G(z_j^G, u_j^G)$ be defined as in (29). The inequality $w_j^G(z_j^G, u_j^G) \geq 0$ is a valid inequality for the set Y_j^G.* □

Note that unless the set $\mathcal{S}_j^G$ is maximal then it is always the case that $w_j^L(1, u_j^L) < 0$ for all $\Delta u_j^L \in [(1-k_j^L)(1-\pi_j)\Delta + |\mathcal{S}_j^L| + 2, (1-k_j^L\pi_j)\Delta]$ and $w_j^R(1, u_j^R) < 0$ for all $\Delta u_j^R \in [(1-k_j^R)(1-\pi_j)\Delta + 1, (1-k_j^R\pi_j)\Delta - |\mathcal{S}_j^R| - 1]$ for all jobs $j \in \mathcal{J}$.

Having defined the partition $[\mathcal{S}_j^L|\mathcal{S}_j^M|\mathcal{S}_j^R]$ and the expressions w_j^G for each job $j \in \mathcal{J}$ and index $G \in \{L, R\}$ we can now transform the valid inequality (28) for the set X to a strong valid inequality for the set Z. From the relations (27) we have that

$$\sum_{t \in \mathcal{S}_j^M} x_{jt} = \sum_{(b,k) \in \mathcal{H}_j} z_{jbk}.$$

Using this substitution and the result of Lemma 4 we have that the inequality (28) can be relaxed to give the inequality

$$\sum_{j \in \mathcal{J}^L} w_j^L(z_j^L, u_j^L) + \sum_{j \in \mathcal{J}} \sum_{(b,k) \in \mathcal{H}_j} z_{jbk} + \sum_{j \in \mathcal{J}^R} w_j^R(z_j^R, u_j^R) \le 1$$

which is a valid inequality for the set Z. In fact, using the nonnegativity of the variables x, the inequality

$$\sum_{j \in \mathcal{J}^L} a_j^L w_j^L(z_j^L, u_j^L) + \sum_{j \in \mathcal{J}} \sum_{(b,k) \in \mathcal{H}_j} z_{jbk} + \sum_{j \in \mathcal{J}^R} a_j^R w_j^R(z_j^R, u_j^R) \le 1 \tag{31}$$

is also a valid inequality for the set Z for all coefficients $a_j^G \in \{0, 1\}$ where job $j \in \mathcal{J}^G$ and index $G \in \{L, R\}$.

The inequalities (31) form an exponentially sized family $\mathcal{F}_{1lt}$ of valid inequalities for the set Z that is derived from the $(1, l, t)$-inequality which is only one of the facet-inducing inequalities (5) for the polyhedron conv(X). Thus there exists an exponentially sized family $\mathcal{F}_{jl't'}$ of valid inequalities for the set Z derived from each (j, l', t')-inequality (5) for all jobs $j \in J$ and time periods l' and t' such that $1 \le l' \le t' \le T$. An inequality (31) is unique within a family $\mathcal{F}_{jl't'}$, but it is possible that this inequality could be derived from more than one (j, l', t')-inequality in which case the inequality would be a member of more than one family. Furthermore, an inequality (31) in one family can dominate inequalities in another. In fact, the strong inequalities in one family can dominate other inequalities within the same family. The relative strength of the inequalities (31) both within a family, and across families, is explored in the following results.

Strong inequalities in $\mathcal{F}_{1lt}$ are those in which the coefficient $a_j^G = 1$ when the expression $w_j^G(z_j^G, u_j^G) \ge 0$, that is, for each job $j \in \mathcal{W}^G$ and index $G \in \{L, R\}$. This is formalised in the following claim which is a consequence of the sets $\mathcal{S}_j^G$ being mutually exclusive, and therefore the variable pairs (z_j^G, u_j^G) being mutually independent, and Lemma 5.

Proposition 6. *Consider two time periods l and t such that $1 \le l \le t \le T$ and let the coefficient $a_j^G = 1$ for each job $j \in \mathcal{W}^G$ and index $G \in \{L, R\}$, and let coefficient $a_j^G \in \{0, 1\}$ otherwise. The corresponding inequality* (31) *from the family $\mathcal{F}_{1lt}$ of inequalities is not dominated by any other inequality in $\mathcal{F}_{1lt}$.* □

Some of the strong inequalities in $\mathcal{F}_{1lt}$ are among the strongest inequalities from all families $\mathcal{F}_{jl't'}$ where job $j \in J$ and time periods l' and t' are such that $1 \le l' \le t' \le T$. This is formalised in the following claim which is a consequence of Proposition 6 and the fact that the inequalities (5) are facet-inducing for the polyhedron $\mathrm{conv}(X)$.

Proposition 7. *Consider two time periods l and t such that $1 \leq l \leq t \leq T$ and let the coefficient $a_j^G = 1$ for each job $j \in \mathcal{J}^G$ and index $G \in \{L, R\}$. The corresponding inequality* (31) *from the family $\mathcal{F}_{1lt}$ of inequalities is not dominated by any inequality from the family $\mathcal{F}_{jl't'}$ for any job $j \in J$ and any two time periods t' and l' such that $1 \leq l' \leq t' \leq T$.* □

If the period t is not the last period of a half-bucket then the set $\mathcal{S}_1^R \neq \emptyset$. Let period $t' = (b_1^R - (1-k_1^R\pi_1))\Delta$ which is the last period in the previous half-bucket $(1, b_1^R - (1 - k_1^R), 1 - k_1^R)$. If period $l \leq t'$ then there exist strong inequalities in the family $\mathcal{F}_{1lt'}$ that are at least as strong as some of the inequalities in the family $\mathcal{F}_{1lt}$. This is formalised in the following claim and is a consequence of the sets $\mathcal{S}_j = [t - p_j + 1, l] \subseteq [t' - p_j + 1, l]$ for all jobs $j \in \mathcal{J} \setminus \{1\}$.

Lemma 8. *Consider two time periods l and t such that $1 \leq l \leq t \leq T$. Let the coefficients $a_j^G = 0$ for all jobs $j \in \mathcal{J}$ and indices $G \in \{L, R\}$. If period $t \neq b_1^R - (1 - k_1^R)(1 - \pi_1)\Delta$ and period $l \leq t' = (b_1^R - (1 - k_1^R\pi_1))\Delta$ then the corresponding inequality* (31) *from the family $\mathcal{F}_{1lt'}$ is at least as strong as the corresponding inequality* (31) *from the family $\mathcal{F}_{1lt}$.* □

Similarly, if the period $l - p_1 + 1$ is not the first period of a half-bucket then the set $\mathcal{S}_1^L \neq \emptyset$. Let period $l' = (b_1^L + P_1 - 1 + k_1^L(1 - \pi_1))\Delta$ so that period $l' - p_1 + 1$ is the first period in the next half-bucket $(1, b_1^L + k_1^L, 1 - k_1^L)$. If period $l' \leq t$ then there exist strong inequalities in the family $\mathcal{F}_{1l't}$ that are at least as strong as some of the inequalities in the family $\mathcal{F}_{1lt}$. This is formalised in the following claim and is a consequence of the sets $\mathcal{S}_j = [t - p_j + 1, l] \subseteq [t - p_j + 1, l']$ for all jobs $j \in \mathcal{J} \setminus \{1\}$.

Lemma 9. *Consider two time periods l and t such that $1 \leq l \leq t \leq T$ and let the coefficient $a_j^G = 0$ for all jobs $j \in \mathcal{J}^G$ and indices $G \in \{L, R\}$. If period $l \neq (b_1^L + P_1 - 1 - (1 - k_1^L)\pi_1)\Delta$ and period $l' = (b_1^L + P_1 - 1 + k_1^L(1 - \pi_1))\Delta \leq t$ then the corresponding inequality* (31) *from the family $\mathcal{F}_{1l't}$ is at least as strong as the corresponding inequality* (31) *from the family $\mathcal{F}_{1lt}$.* □

In the remainder of this section we show that the family $\mathcal{F}_{1lt}$ of inequalities (31) includes strong valid inequalities for the set Z. We identify a class of valid inequalities in the family $\mathcal{F}_{1lt}$ that is obtained by restricting the choice of the periods l and t and show that each inequality in this class is facet-inducing for the convex hull $\mathrm{conv}(Z^*)$ of the set Z^* of feasible partial schedules.

Let us start by considering the $(1, t, t)$-inequality, that is, period $l = t$. The $(1, t, t)$-inequality is exactly the inequality (3) for the TI model. Similarly, consider the family $\mathcal{F}_{1,b\Delta,b\Delta}$ of inequalities in which period $l = t = b\Delta$ is the last period in bucket b. The periods $t - p_j + 1 = (b - P_j + \pi_j)\Delta + 1$ are the first periods in the half-buckets $(j, b - P_j + 1, 1)$ for all jobs $j \in J$. Thus the sets $\mathcal{J} = J$ and $\mathcal{J}^G = \emptyset$ for all indices $G \in \{L, R\}$, that is, the sets $\mathcal{S}_j^M = S_j$ and $\mathcal{S}_j^G = \emptyset$ for all jobs $j \in J$ and indices $G \in \{L, R\}$. Thus, the family $\mathcal{F}_{1,b\Delta,b\Delta}$ contains a single inequality and this inequality is exactly the inequality (8) for the BB model.

Now let us consider the family $\mathcal{F}_{1,b_l\Delta,b_t\Delta}$ of inequalities in which periods $l = b_l\Delta$ and $t = b_t\Delta$ are the last periods in the buckets b_l and b_t respectively.

The periods $l-p_1+1=(b_l-P_1+\pi_1)\Delta+1$ and $t-p_j+1=(b_t-P_j+\pi_j)\Delta+1$ are the first periods in the half-buckets $(1,b_l-P_1+1,1)$ and $(j,b_t-P_j+1,1)$ for all jobs $j\in J\setminus\{1\}$ respectively. Again we have the set $\mathcal{J}=J$ and $\mathcal{J}^G=\emptyset$ for all indices $G\in\{L,R\}$, that is, the sets $\mathcal{S}_j^M=\mathcal{S}_j$ and $\mathcal{S}_j^G=\emptyset$ for all jobs $j\in J$ and indices $G\in\{L,R\}$. The previously discussed family $\mathcal{F}_{1,b\Delta,b\Delta}$ is the special case in which the buckets $b_l=b_t=b$.

To generalise this further we choose the periods l and t such that the periods $l-p_1+1$ and t are the first and last periods in the half-buckets $(1,b_{l-p_1+1},k_{l-p_1+1})$ and $(1,b_t,k_t)$ respectively. That is, periods $l-p_1+1=(b_{l-p_1+1}-(1-k_{l-p_1+1}\pi_1))\Delta+1$ and $t=(b_t-(1-k_t)(1-\pi_1))\Delta$. By restricting the period $l-p_1+1$ to be the first period of the half-bucket $(1,b_{l-p_1+1},k_{l-p_1+1})$ we can deduce that the period $l=(b_{l-p_1+1}-(1-k_{l-p_1+1}\pi_1)+P_1-\pi_1)\Delta=(b_{l-p_1+1}+P_1-1-(1-k_{l-p_1+1})\pi_1)\Delta$ is contained in the half-bucket $(1,b_l,k_l)$ where bucket $b_l=b_{l-p_1+1}+P_1-1$, and index $k_l=k_{l-p_1+1}$ when $\pi_1\le 1/2$ and $k_l=1$ otherwise. Thus, the period $l=(b_l-(1-k_{l-p_1+1})\pi_1)\Delta$.

Now consider the family $\mathcal{F}_{1,(b_l-(1-k_{l-p_1+1})\pi_1)\Delta,(b_t-(1-k_t)(1-\pi_1))\Delta}$ of inequalities in which $b_l,b_t\in[1,B]$ and $k_{l-p_1+1},k_t\in K$ such that $1\le l\le t\le T$. Again we have the sets $\mathcal{S}_1^M=\mathcal{S}_1$ and $\mathcal{S}_1^G=\emptyset$ for both indices $G\in\{L,R\}$. However, the periods $t-p_j+1=(b_t-(1-k_t)(1-\pi_1)-P_j+\pi_j)\Delta+1=(b_t-P_j-(1-k_t)(1-\pi_1)+\pi_j)\Delta+1$ and $l=(b_l-(1-k_{l-p_1+1})\pi_1)\Delta$ may not be either the first or the last period in a half-bucket for some job $j\in J\setminus\{1\}$. Thus, there may exist a job $j\in\mathcal{J}^G$ for some index $G\in\{L,R\}$, that is, a set $\mathcal{S}_j^G\neq\emptyset$ for some job $j\in\mathcal{J}\setminus\{1\}$ and index $G\in\{L,R\}$. The previously discussed family $\mathcal{F}_{1,b_l\Delta,b_t\Delta}$ is the special case in which indices $k_{l-p_1+1}=k_t=1$.

The choice of the index $k_t\in K$ determines whether job $j\in\mathcal{J}^L$. When the index $k_t=0$ the period $t-p_j+1=(b_t-P_j-(1-\pi_1-\pi_j))\Delta+1$ for all jobs $j\in\mathcal{J}\setminus\{1\}$. If $\pi_1<1-\pi_j$ for some job $j\in\mathcal{J}\setminus\{1\}$ then the period $t-p_j+1$ is in the half-bucket $(j,b_j^L,k_j^L)=(j,b_t-P_j,1)$ and job $j\in\mathcal{J}^L$. If $\pi_1=1/\Delta$ then the set $\mathcal{S}_j^L$ will have maximal cardinality and job $j\in\mathcal{W}^L$. If $\pi_1=1-\pi_j$ for some $j\in\mathcal{J}\setminus\{1\}$ then the period $t-p_j+1$ is the first period in the half-bucket $(j,b_t-P_j+1,0)$. Thus, the half-bucket $(j,b_j^L,k_j^L)=(j,b_t-P_j,1)$ and job $j\in\mathcal{J}\setminus\mathcal{J}^L$. If $\pi_1>1-\pi_j$ for some $j\in\mathcal{J}\setminus\{1\}$ then the period $t-p_j+1$ is in the half-bucket $(j,b_j^L,k_j^L)=(j,b_t-P_j+1,0)$ and job $j\in\mathcal{J}^L$. If $\pi_1=1-\pi_j+1/\Delta$ then the set $\mathcal{S}_j^L$ will have maximal cardinality and the job $j\in\mathcal{W}^L$. Finally, when $k_t=1$ the period $t-p_j+1=(b_t-P_j+\pi_j)\Delta+1$ is the first period in the half-bucket $(j,b_t-P_j+1,1)$ for all jobs $j\in J\setminus\{1\}$. Thus, the half-bucket $(j,b_j^L,k_j^L)=(j,b_t-P_j+1,0)$ and the set $\mathcal{J}^L=\emptyset$, that is, the set $\mathcal{S}_j^L=\emptyset$ for all jobs $j\in\mathcal{J}\setminus\{1\}$.

The choice of the index $k_{l-p_1+1}\in K$ determines whether job $j\in\mathcal{J}^R$. When the index $k_{l-p_1+1}=0$ the period $l=(b_l-\pi_1)\Delta$. If $\pi_1<1-\pi_j$ for some $j\in\mathcal{J}\setminus\{1\}$ then the period l is in the half-bucket $(j,b_j^R,k_j^R)=(j,b_l,1)$ and job $j\in\mathcal{J}^R$. If $\pi_1=1/\Delta$ then the set $\mathcal{S}_j^R$ will have maximal cardinality and job $j\in\mathcal{W}^R$. If $\pi_1=1-\pi_j$ for some $j\in\mathcal{J}\setminus\{1\}$ then the period l is the last period in the half-bucket $(j,b_l,0)$. Thus, the half-bucket $(j,b_j^R,k_j^R)=(j,b_l,1)$ and job $j\in\mathcal{J}\setminus\mathcal{J}^R$. If $\pi_1>1-\pi_j$ for some $j\in\mathcal{J}\setminus\{1\}$ then the period l is in the half-bucket $(j,b_j^R,k_j^R)=(j,b_l,0)$ and job $j\in\mathcal{J}^R$. If $\pi_1=1-\pi_j+1/\Delta$ then the set $\mathcal{S}_j^R$ will have maximal cardinality and job $j\in\mathcal{W}^R$. Finally, when $k_{l-p_1+1}=1$ the period $l=b_l\Delta$ is the last period in the half-bucket $(j,b_l,1)$ for all jobs $j \in J \setminus \{1\}$. Thus, the half-bucket $(j, b_j^R, k_j^R) = (j, b_l + 1, 0)$ and the set $\mathcal{J}^R = \emptyset$, that is, $\mathcal{S}_j^R = \emptyset$ for all jobs $j \in \mathcal{J} \setminus \{1\}$.

The family $\mathcal{F}_{1,(b_l-(1-k_{l-p_1+1})\pi_1)\Delta,(b_t-(1-k_t)(1-\pi_1))\Delta}$ of inequalities (31) with coefficients $a_j^G = 1$ if job $j \in \mathcal{W}^G$, $a_j^G = 0$ if job $j \in \mathcal{J} \setminus \mathcal{J}^G$, and $a_j^G \in \{0,1\}$ otherwise, for all indices $G \in \{L, R\}$, is characterised in the following theorem. The proof is given in the Appendix.

Theorem 10. *Let buckets $b_l, b_t \in [1, B]$ and indices $k_{l-p_1+1}, k_t \in K$ such that time periods $l = (b_l - (1 - k_{l-p_1+1})\pi_1)\Delta$ and $t = (b_t - (1 - k_t)(1 - \pi_1))\Delta$ satisfy $1 \le l \le t \le T$. For each job $j \in \mathcal{J} \setminus \{1\}$ and bucket $b \in \{b_t - P_j + 1, b_l\}$ let the coefficient $a_{jb0} = 1$ if $\pi_1 = 1 - \pi_j + 1/\Delta$ and $a_{jb0} \in \{0,1\}$ otherwise. For each job $j \in \mathcal{J} \setminus \{1\}$ and bucket $b \in \{b_t - P_j, b_l\}$ let the coefficient $a_{jb1} = 1$ if $\pi_1 = 1/\Delta$ and $a_{jb1} \in \{0,1\}$ otherwise. The inequality*

(32)

$$\sum_{\substack{j\in\mathcal{J}\setminus\{1\}:\\ \pi_1<1-\pi_j}} a_{j,b_t-P_j,1}\frac{1-k_t}{1-\pi_1-\pi_j}\left((1-\pi_1-\pi_j+\frac{1}{\Delta})z_{j,b_t-P_j,1}-u_{j,b_t-P_j,1}\right)+$$

$$a_{j,b_t-P_j+1,0}\frac{1-k_t}{1-\pi_1}\sum_{\substack{j\in\mathcal{J}\setminus\{1\}:\\ \pi_1>1-\pi_j}}\left((2-\pi_1-\pi_j+\frac{1}{\Delta})z_{j,b_t-P_j+1,0}-u_{j,b_t-P_j+1,0}\right)+$$

$$\sum_{k\in K}\left(\sum_{\substack{b\in[b_l-P_1+1,\\ b_t-k]}} z_{1bk}+\sum_{\substack{j\in\mathcal{J}\setminus\{1\}:\\ \pi_1\le 1-\pi_j}}\sum_{\substack{b\in[b_t-P_j+1,\\ b_l-k]}} z_{jbk}+\sum_{\substack{j\in\mathcal{J}\setminus\{1\}:\\ \pi_1>1-\pi_j}}\sum_{\substack{b\in[b_t-P_j+2-k,\\ b_l-1]}} z_{jbk}\right)+$$

$$a_{jb_l0}\frac{1-k_{l-p_1+1}}{1-\pi_1}\sum_{\substack{j\in\mathcal{J}\setminus\{1\}:\\ \pi_1>1-\pi_j}}(u_{jb_l0}-\pi_1 z_{jb_l0})+$$

$$\sum_{\substack{j\in\mathcal{J}\setminus\{1\}:\\ \pi_1<1-\pi_j}} a_{jb_l1}\frac{1-k_{l-p_1+1}}{1-\pi_1-\pi_j}(u_{jb_l1}-\pi_1 z_{jb_l1})\le 1$$

is a facet-inducing inequality for the convex hull $\mathrm{conv}(Z^*)$ of the set Z^* of feasible partial schedules to the BB model provided the following conditions hold:

1. *If index $k_{l-p_1+1} = 0$ and fraction $\pi_1 = 1 - 1/\Delta$ then there exists a job $j \in J \setminus \{1\}$ for which either*
   a. *$\pi_j = 1/\Delta$, or*
   b. *coefficient $a_j^R = 1$.*
2. *If index $k_{l-p_1+1} = 0$ and fraction $\pi_1 < 1 - 1/\Delta$ then there exists a job $j \in J \setminus \{1\}$ for which either*
   a. *$\pi_1 \le 1 - \pi_j$ and $\pi_j \ge 2/\Delta$, or*

b. $\pi_1 < 1 - \pi_j$ *and coefficient* $a_j^R = 1$.

3. *If index* $k_t = 0$ *and fraction* $\pi_1 = 1 - 1/\Delta$ *then there exists a job* $j \in J \setminus \{1\}$ *for which either*

   a. $\pi_j = 1/\Delta$, *or*

   b. coefficient $a_j^L = 1$.

4. *If index* $k_t = 0$ *and fraction* $\pi_1 < 1 - 1/\Delta$ *then there exists a job* $j \in J \setminus \{1\}$ *for which either*

   a. $\pi_1 \leq 1 - \pi_j$ *and* $\pi_j \geq 2/\Delta$, *or*

   b. $\pi_1 < 1 - \pi_j$ *and coefficient* $a_j^L = 1$.

The inequalities (31) from Proposition 6 that are not captured in the family $\mathcal{F}_{1,(b_l-(1-k_{l-p_1+1})\pi_1)\Delta,(b_t-(1-k_t)(1-\pi_1))\Delta}$ of inequalities (32) described in Theorem 10 are precisely those inequalities for which $\mathcal{S}_1^G \neq \emptyset$ for at least one of the indices $G \in \{L, R\}$.

4 Computational results

In this section we describe a computational study that compares the performance of the BB model with that of the TI model. We compare model size, MILP solve times and optimality gaps, as well as LP relaxation solve times and integrality gaps. Sets of randomly generated weighted tardiness instances are used so that the performance of the models can be established without the additional complexity of release dates or deadlines. The impact of the additional variables and constraints required to model the weighted tardiness criteria in the BB model as described in Section 3.4 are considered negligible. We also investigate the effect of the choice of bucket size on the BB model and the effectiveness of the strong valid inequalities described in Section 3.7.

Five data sets of weighted tardiness instances are generated using a generalisation of the method developed by Potts and Wassenhove (1985) to generate the instances that are part of the widely used OR-Library (Beasley, 1990) benchmark data sets. Each instance in each data set consists of 30 jobs. The weight, processing time, and due date of each job is sampled uniformly from an interval of integers. The job weights in all data sets are sampled from the interval $[1, 10]$. The data sets differ in the intervals from which the processing times are sampled. The intervals $[p_{\min}, p_{\max}]$ considered in each of the five data sets are $[1, 100]$, $[25, 100]$, $[50, 100]$, $[75, 100]$, and $[25, 50]$ respectively. The due dates in all data sets are sampled from the interval $[P(1 - TF - RDD/2), P(1 - TF + RDD/2)]$ where P is the sum of the processing times and the values of the parameters RDD and TF are chosen from the set $\{0.2, 0.4, 0.6, 0.8, 1.0\}$. Any negative due dates are changed to zero since this only changes the value of the objective function in the models by a constant. Five instances are randomly generated for each of the 25 possible (RDD, TF) pairs resulting in 125 instances in each of the data sets ORL30PT1-100, ORL30PT25-100, ORL30PT50-100, ORL30PT75-100, and ORL30PT25-50.

For the computational experiments described in this section the BB model is formulated with just the variables (z, u), that is, the variables v and the corresponding constraints are eliminated. Constraints (10)–(12) are multiplied by the bucket size Δ so that the coefficients in these constraints are integer valued since the instance data is integer.

All computational experiments are performed using CPLEX V12.3 (IBM Corporation, 2011) on a Dell PowerEdge R710 with dual hex core 3.06GHz Intel Xeon X5675 processors and 96GB RAM. CPLEX is restricted to using a single thread with an 1800 second time limit but otherwise default parameter settings are used. To minimise any variability in performance due to hardware effects each experiment with each instance is repeated five times. The result of each of the five experiments is reported as a separate data point. Performance profiles (Dolan and Moreé, 2002) are used to visualise the results.

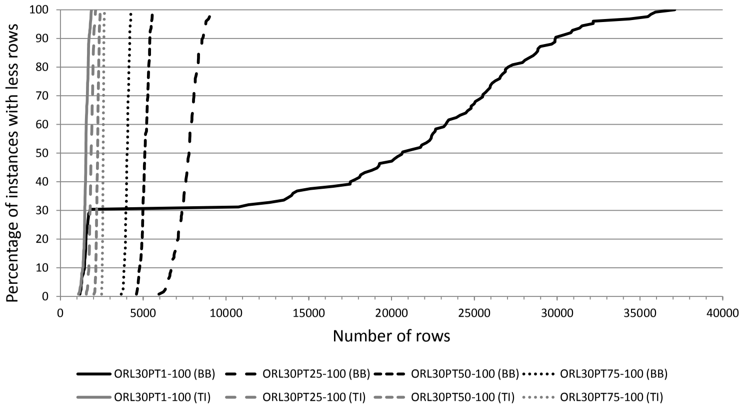
4.1 Model size

The size of a model generally refers to a measure of the number of rows, columns, or nonzeros in the constraint matrix of the formulation. In the TI model the number of rows, columns, and nonzeros are functions of the number of jobs n and the number of periods T in the planning horizon. In the case of weighted tardiness the number of periods T can be taken to be the sum of the processing times. Thus the size of the TI model increases when either of the limits $p_{\min}$ or $p_{\max}$ increase since the processing time p_j of each job $j \in J$ is sampled uniformly over the interval $[p_{\min}, p_{\max}]$ and the expected number of periods $E(T) = nE(p)$ where $E(p) = E(p_j) = (p_{\min} + p_{\max})/2$ is the expected processing time of each job.

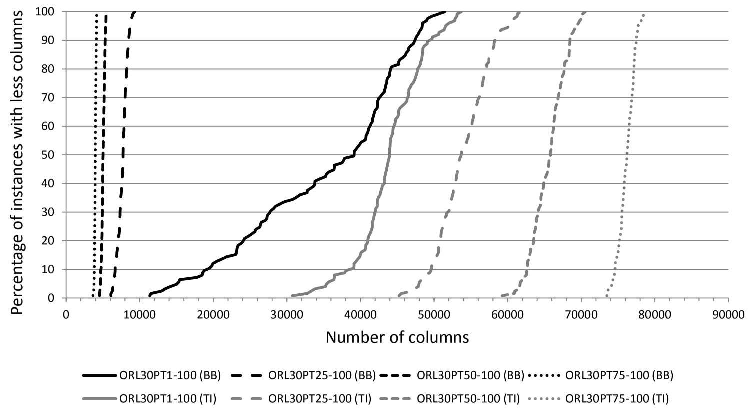
In the BB model, the number of rows, columns, and nonzeros are functions of the number of jobs n and the number of buckets B. The expected number of buckets $E(B) = nE(p)/\Delta \leq nE(p)/p_{\min} = n(1 + p_{\max}/p_{\min})/2$ if the bucket size Δ is chosen to be as large as possible. Thus, the size of the BB model will increase when the limit $p_{\max}$ increases and the limit $p_{\min}$ is held constant. Conversely, the size of the BB model will decrease when the limit $p_{\min}$ increases and the limit $p_{\max}$ is held constant.

The size of the TI and BB models over the first four data sets ORL30PT1-100, ORL30PT25-100, ORL30PT50-100, and ORL30PT75-100 can be seen in Figure 4. The bucket size Δ is chosen to equal the smallest processing time among the jobs in each instance. Consistent with the analysis above we see that as the expected processing time increases, so do the number of rows, columns, and nonzeros in the TI model while they decrease in the BB model. Over all data sets the TI models have fewer rows while the BB models have fewer columns and nonzeros as expected. The density of the models is taken to be the density of the constraint matrix, that is, the number of nonzero entries as a fraction of the total number of entries. The TI model has $O(n+T)$ rows, $O(nT)$ columns, and $O(nT + T^2)$ nonzeros, and so the density is $O(1/n)$. The density also increases as the expected processing time increases. The same behaviour can be observed for the BB model with larger bucket sizes Δ.

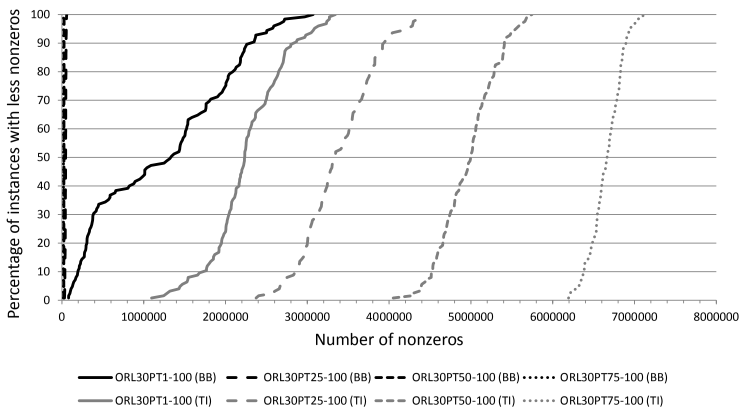
The seemingly strange behaviour of the BB model on data set ORL30PT1-100 in Figures 4(a) and 4(d) is due to the fact that 38 of the 125 instances in this data set have a job with unit processing time. Thus, the bucket size $\Delta = 1$ and the BB model reduces to the TI model. The distribution of the smallest processing time of each instance for these data sets can be seen in Figure 5. The effect of bucket size on the BB model is investigated further in Section 4.4.



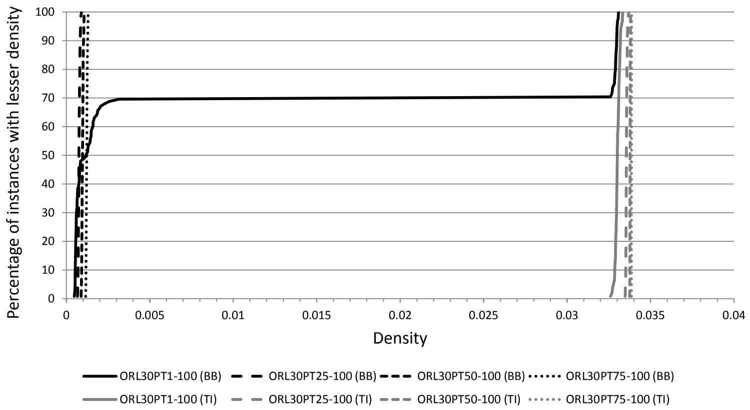
(a) Number of rows



(b) Number of columns

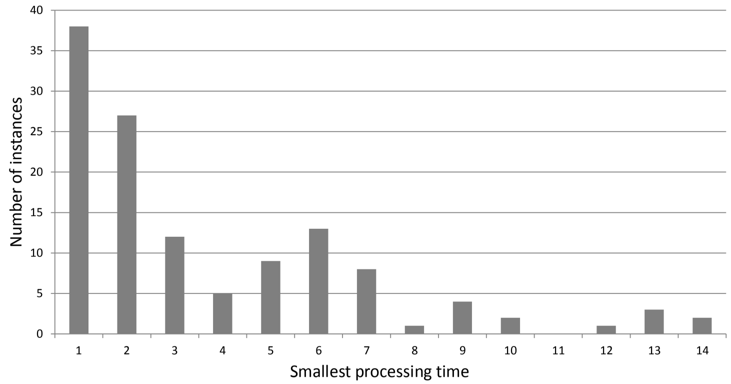


(c) Number of nonzeros

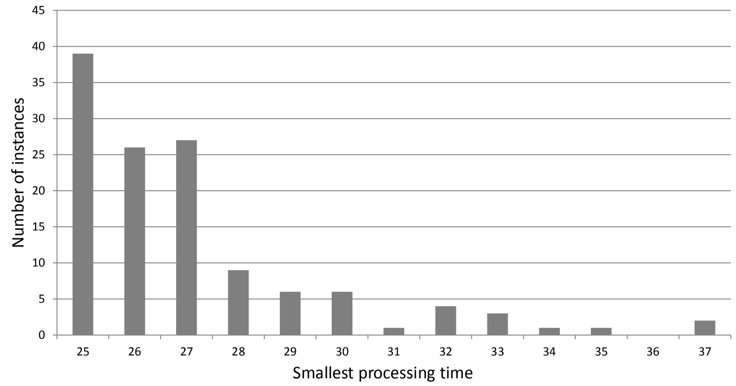


(d) Density

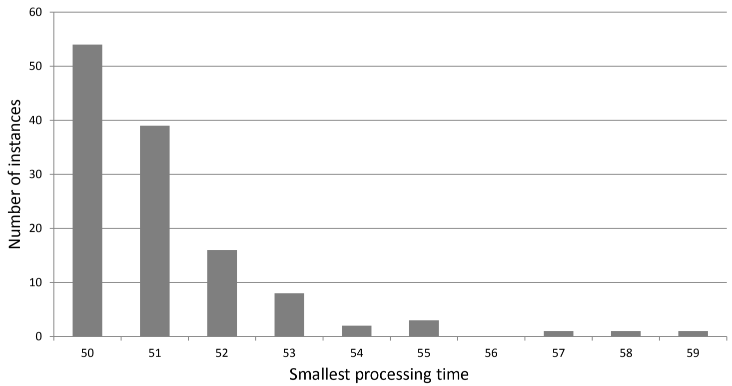
Figure 4: Size of the TI and BB models over the data sets ORL30PT1-100, ORL30PT25-100, ORL30PT50-100, and ORL30PT75-100. The bucket size Δ is chosen to equal the smallest processing time among the jobs in each instance.



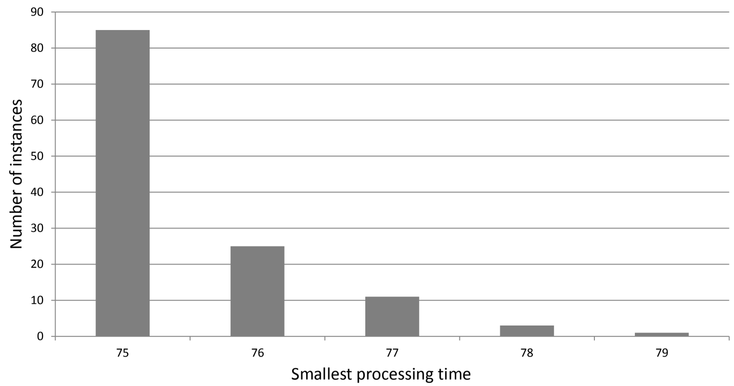
(a) ORL30PT1-100



(b) ORL30PT25-100

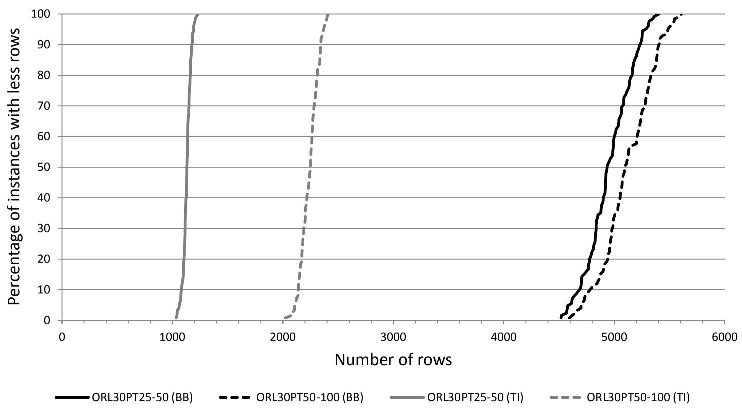


(c) ORL30PT50-100

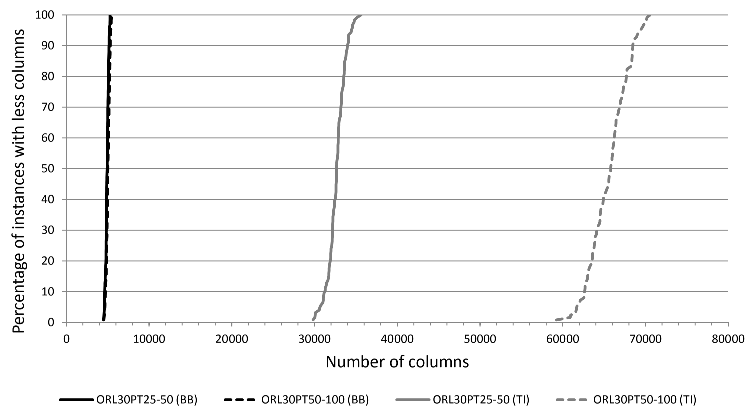


(d) ORL30PT75-100

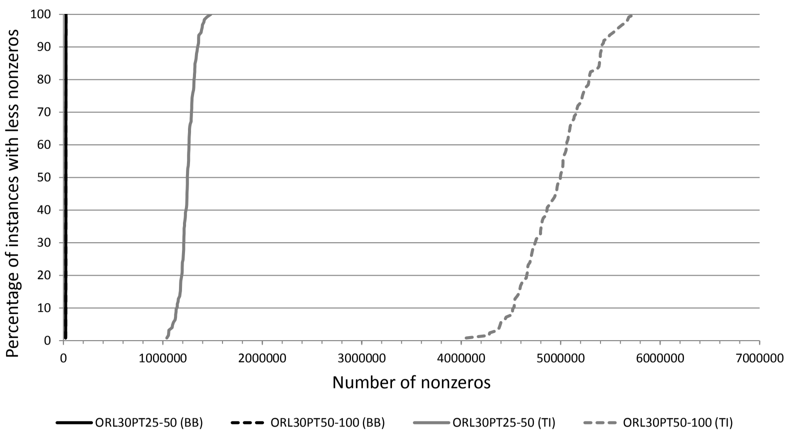
Figure 5: Distribution of the smallest processing time of each instance for the data sets ORL30PT1-100, ORL30PT25-100, ORL30PT50-100, and ORL30PT75-100.



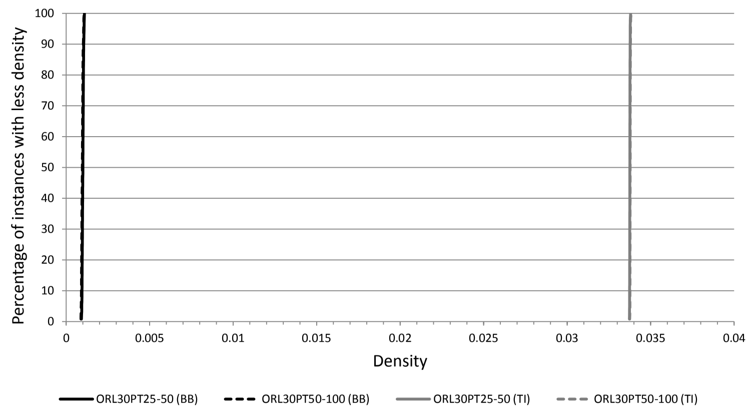
(a) Number of rows



(b) Number of columns



(c) Number of nonzeros



(d) Density

Figure 6: Size of the TI and BB models over the data sets ORL30PT25-50 and ORL30PT50-100. The bucket size Δ is chosen to equal the smallest processing time among the jobs in each instance.

The size of the TI and BB models over the data sets ORL30PT25-50 and ORL30PT50-100 can be seen in Figure 6. The bucket size Δ is again chosen to equal the smallest processing time among the jobs in each instance. The number of rows, columns, and nonzeros in the TI model increase with increasing expected processing time while they are approximately the same for the BB model since the ratio $p_{\max}/p_{\min}$ remains constant.

4.2 MILP solve times and optimality gaps

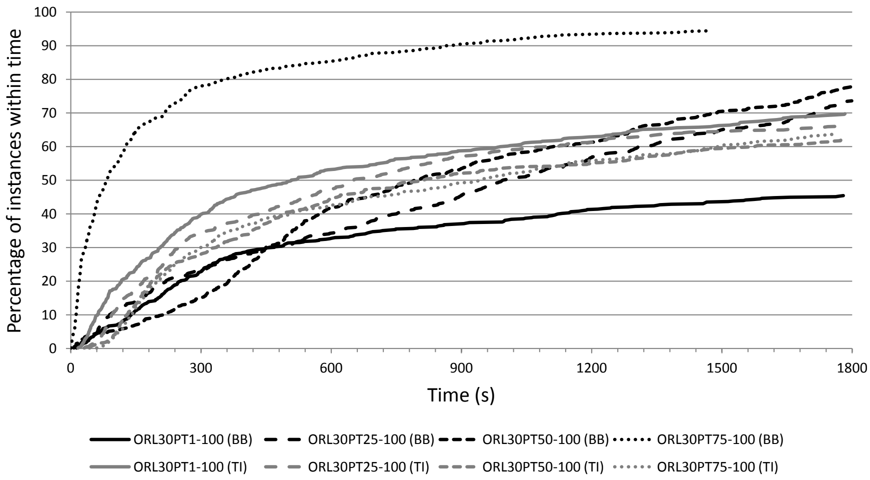
The solve times of the TI and BB models over the first four data sets ORL30PT1-100, ORL30PT25-100, ORL30PT50-100, and ORL30PT75-100 can be seen in Figure 7(a). The trend exhibited by both the TI and BB models is that the solve time increases with increasing model size as might be expected. In the case of the BB model this means that solve times decrease as the smallest processing time increases. Between 62-69% of instances in each of the data sets solve within the 1800 second time limit using the TI model. While only 45% of instances from the data set ORL30PT1-100 solve within the time limit using the BB model, 73%, 77%, and 94% of instances are solved from the data sets ORL30PT25-100, ORL30PT50-100, and ORL30PT75-100 respectively.

The TI model outperforms the BB model over the ORL30PT1-100 data set. Conversely, the BB model outperforms the TI model over the ORL30PT75-100 data set. We see that 77% of instances solve within 300 seconds using the BB model compared to 29% using the TI model. Over the ORL30PT25-100 and ORL30PT50-100 data sets a greater number of instances solve in lesser time using the TI model, but a greater number of instances solve within the time limit using the BB model. Excluding the trivially solved instances, the BB model begins to outperform the TI model on the ORL30PT25-100 data set after approximately 1485 seconds when 64% of instances are solved, and on the ORL30PT50-100 data set after approximately 790 seconds when 50% of instances are solved.

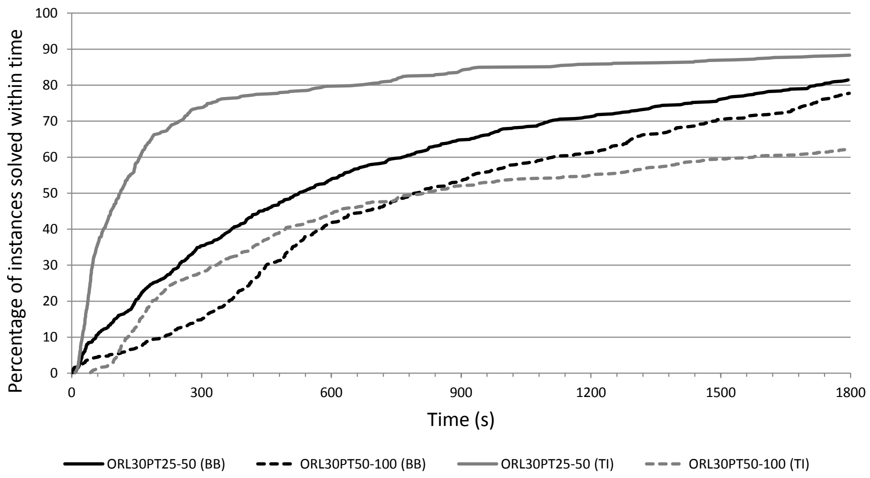
The solve times of the TI and BB models over the data sets ORL30PT25-50 and ORL30PT50-100 can be seen in Figure 7(b). The solve times using the BB model are comparable as expected. The TI model outperforms the BB model over the ORL30PT25-50 data set where 88% of instances solve within the time limit compared to 81% using the BB model.

The optimality gap between the values of the best feasible solution and the best bound over the first four data sets ORL30PT1-100, ORL30PT25-100, ORL30PT50-100, and ORL30PT75-100 can be seen in Figure 8(a). The trend exhibited by both the TI and BB models is that both the gap and the number of instances where a feasible solution is not found increases with increasing model size. In the case of the BB model this means that gaps and the number of instances where a feasible solution is not found decrease as the smallest processing time increases.

With the exception of the ORL30PT1-100 data set, the BB model outperforms the TI model. We see that 91% and 98% of instances of the data sets ORL30PT50-100 and ORL30PT75-100 solve to within 1% of optimality using the BB model compared to 64% and 67% using the TI model. A feasible solution is not found for at most 2% of the instances in each data set using the BB model compared to almost 20% using the TI model. On the ORL30PT1-100 data set 74% of instances solve to within 1% of optimality using the TI model

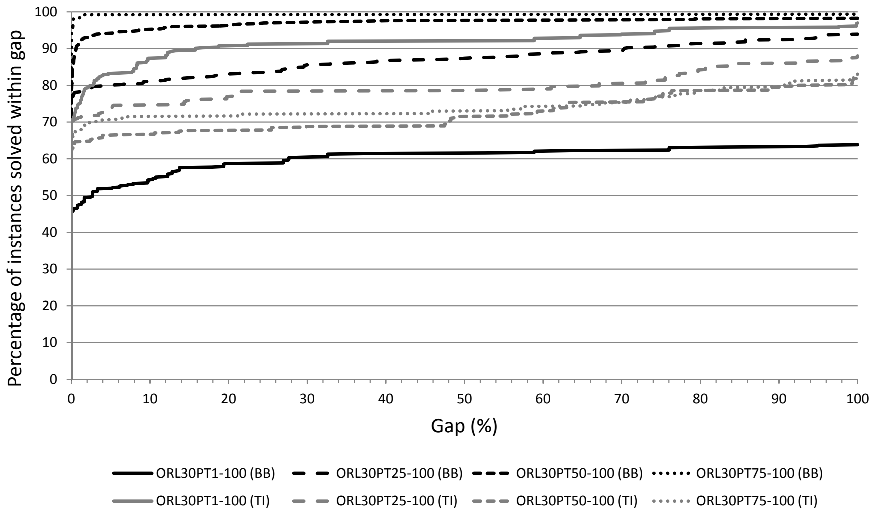


(a) ORL30PT1-100, ORL30PT25-100, ORL30PT50-100, and ORL30PT75-100

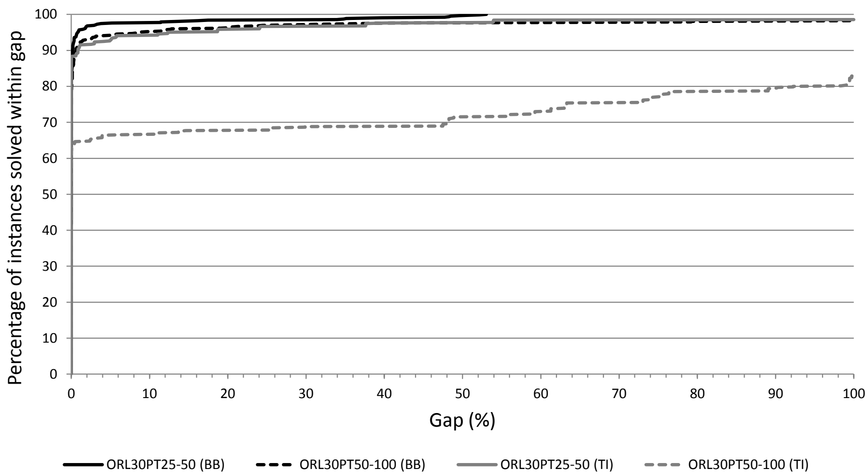


(b) ORL30PT25-50 and ORL30PT50-100

Figure 7: MILP solve times of the TI and BB models.



(a) ORL30PT1-100, ORL30PT25-100, ORL30PT50-100, and ORL30PT75-100



(b) ORL30PT25-50 and ORL30PT50-100

Figure 8: Optimality gaps between the values of the best feasible solution and the best bound of the TI and BB models.

compared to 47% using the BB model. A feasible solution is not found for at most 3% of the instances in each data set using the TI model compared to 36% using the BB model.

The optimality gaps of the TI and BB models over the data sets ORL30PT25-50 and ORL30PT50-100 can be seen in Figure 8(b). The gaps using the BB model are fairly comparable. The BB model outperforms the TI model over the ORL30PT25-50 data set and finds a feasible solution to every instance compared to 98% of instances using the TI model.

4.3 LP relaxation solve times and integrality gaps

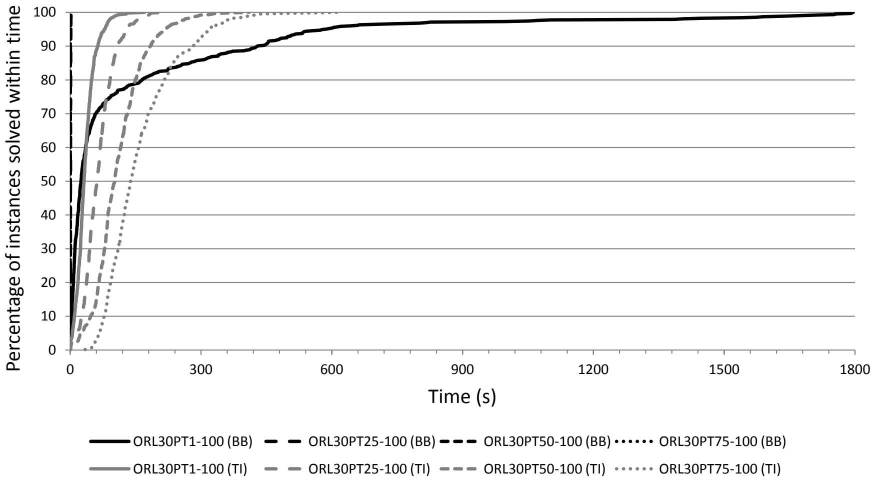
As the size of the TI model increases it is increasingly difficult to solve the LP relaxation in reasonable time let alone solve the MILP. The increasing LP relaxation solve times for the TI model can be seen in Figure 9(a). Both the TI and BB models solve the LP relaxation of every instance within the time limit. With the exception of the ORL30PT1-100 data set, the solve times of the LP relaxation of the BB model are orders of magnitude better than those of the TI model. The LP relaxations of the TI model solve faster than those of the BB model on those instances of the ORL30PT1-100 data set where the smallest processing time is small but greater than one. We investigate this phenomenon further in Section 4.4. For the moment it suffices to note that if we omit those instances that lead to small bucket sizes then the relative performance of the BB model dramatically improves. Figure 9(b) shows the solve times of the LP relaxations of the TI and BB models on the 48 instances of the ORL30PT1-100 data set for which the bucket size $\Delta \geq 4$.

The integrality gap between the values of the optimal solution and the LP relaxation over the first four data sets ORL30PT1-100, ORL30PT25-100, ORL30PT50-100, and ORL30PT75-100 can be seen in Figure 10(a). The trend exhibited by both the TI and BB models is that the gap decreases with increasing model size. In the case of the BB model this means that the gap increases as the smallest processing time increases. We see that 90% of instances over all data sets have a gap of at most 5% but there are also instances with gaps in excess of 30% for both models. The TI and BB models are fairly comparable on the ORL30PT1-100 data set, however, the TI model exhibits smaller integrality gaps over all suggesting that the strength of the LP relaxation of the TI model is superior to that of the BB model.

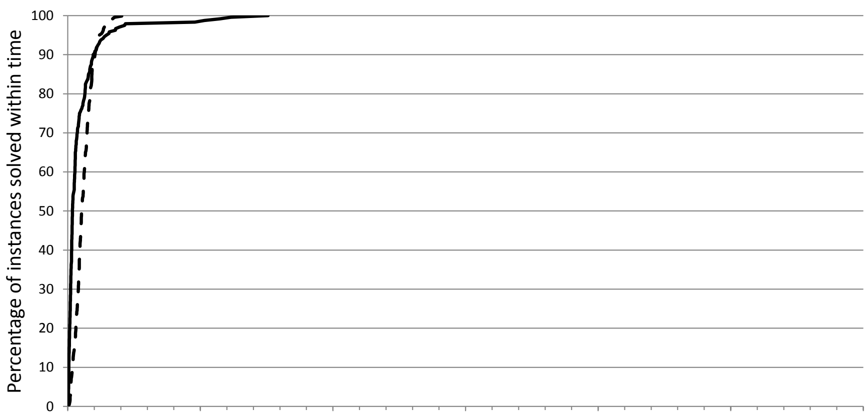
The integrality gaps of the TI and BB models over the data sets ORL30PT25-50 and ORL30PT50-100 can be seen in Figure 10(b). The gaps using the BB model are fairly comparable. The TI model outperforms the BB model over the ORL30PT25-50 data set.

4.4 Choice of bucket size

In the preceding sections the bucket size Δ of the BB model has been equal to the smallest processing time among the jobs in each instance. This is the largest possible choice for the bucket size. In this section we investigate the effect that different choices of bucket size have when solving the instances of the ORL30PT50-100 data set. The set of bucket sizes considered are $\Delta \in \{1, 2, \ldots, 5\} \cup \{10, 15, \ldots, 50\}$.

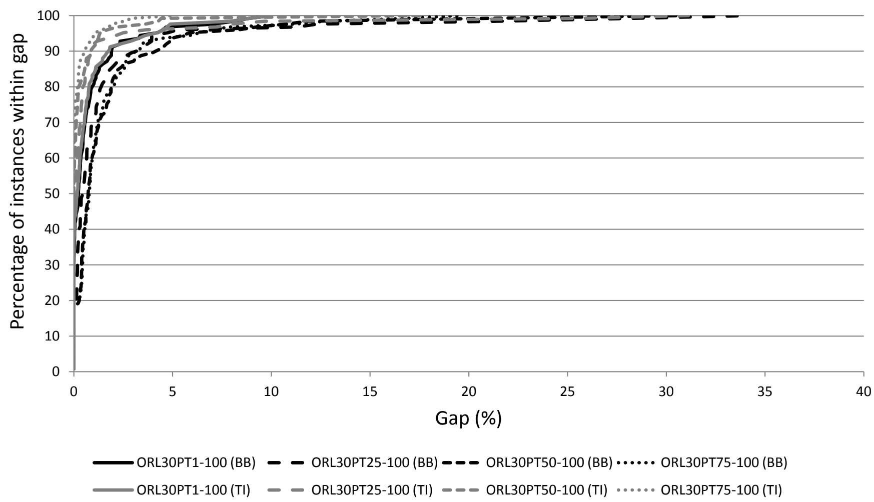


(a) All bucket sizes Δ over all data sets

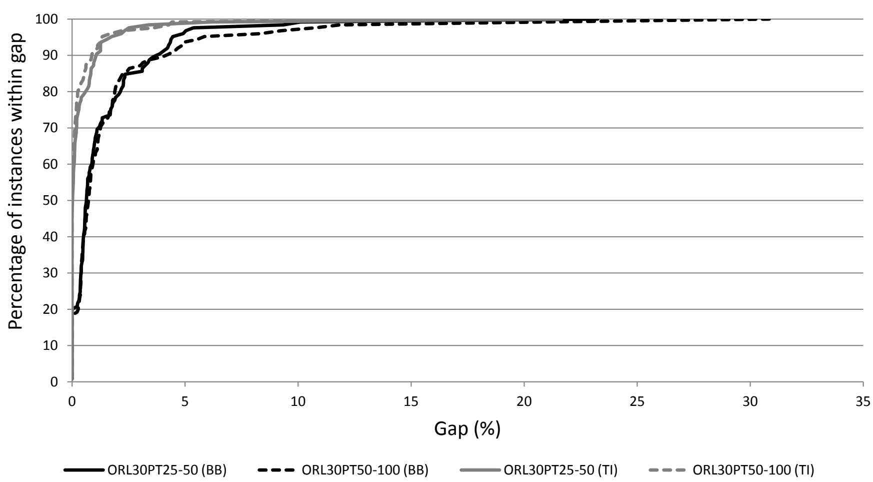


(b) Bucket size $\Delta \geq 4$ on the ORL30PT1-100 data set

Figure 9: LP relaxation solve times of the TI and BB models over the first four data sets ORL30PT1-100, ORL30PT25-100, ORL30PT50-100, and ORL30PT75-100.



(a) ORL30PT1-100, ORL30PT25-100, ORL30PT50-100, and ORL30PT75-100



(b) ORL30PT25-50 and ORL30PT50-100

Figure 10: Integrality gaps between the values of the optimal solution and the LP relaxation of the TI and BB models.

Recall that if the bucket size $\Delta = 1$ then we have the special case in which the BB model is equivalent to the TI model. There is a binary z variable for each job and each bucket. For bucket sizes $\Delta > 1$ the BB model will likely also have continuous u variables and each of these variables incurs two additional variable bound constraints. If the bucket size $\Delta = 2$ then we have the special case in which the u variables are only present for jobs with even processing times. For bucket sizes $\Delta \geq 3$ then we are in the most general case and roughly speaking the number of rows $M = n + 2B + 2N$ where the number of columns $N > B > n$ is a function of the number of buckets B and the remainder of the processing times when divided by the bucket size Δ. See Section 3.6 for a detailed discussion.

The size of the BB model for the different choices of bucket size on the data set ORL30PT50-100 can be seen in Figure 11. Clearly evident are the special cases corresponding to bucket sizes $\Delta \in \{1, 2\}$ as is the trend in the general case for bucket sizes $\Delta \geq 3$, that is, the model size decreases seemingly monotonically and asymptotically as the bucket size increases.

The MILP solve times of the BB model for the different choices of bucket size on the data set ORL30PT50-100 can be seen in Figure 12. Interestingly the best choice for bucket size, with respect to both the number of instances solved within the time limit, namely 82%, and solving more instances in lesser time, appears to be $\Delta = 25$. Buckets sizes $\Delta \in \{40, 45, 50\}$ are the next best choices with respect to the number of instances solved within the time limit, solving 79%, 81%, and 78% respectively. However, buckets sizes $\Delta \in \{20, 30\}$ outperform these choices with respect to solving more instances in lesser time while still solving 75% and 78% of instances within the time limit respectively.

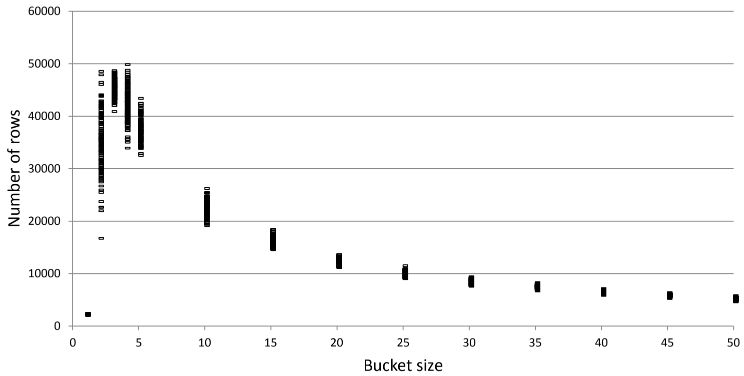
The optimality gaps of the BB model for the different choices of bucket size on the data set ORL30PT50-100 can be seen in Figure 13. The trend in the general case for bucket sizes $\Delta \geq 3$ is that as the bucket size increases both the gap and the number of instances where a feasible solution is not found decreases. The bucket sizes $\Delta \in \{40, 45, 50\}$ outperform bucket sizes $\Delta \in \{20, 25, 30\}$ with respect to having more instances with smaller gaps but bucket sizes $\Delta \in \{25, 40, 45, 50\}$ all have fewer than 1% of instances where a feasible solution is not found.

The LP relaxation solve times of the BB model for the different choices of bucket size on the data set ORL30PT50-100 can be seen in Figure 14. Again the special cases corresponding to bucket sizes $\Delta \in \{1, 2\}$ are clearly evident as is the trend in the general case for bucket sizes $\Delta \geq 3$, that is, the solve times decrease with increasing bucket size. For the larger bucket sizes the solve times are negligible.

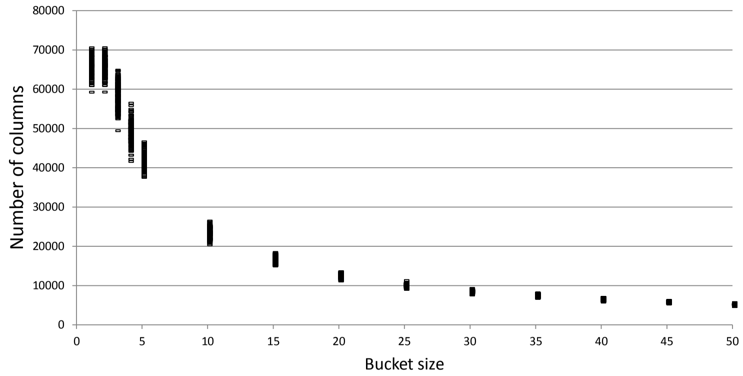
Integrality gaps of the BB model for the different choices of bucket size on the data set ORL30PT50-100 can be seen in Figure 15. The trend evident in the general case for bucket sizes $\Delta \geq 3$ is that the gap increases with increasing bucket size, that is, the strength of the LP relaxation decreases.

4.5 Strong valid inequalities

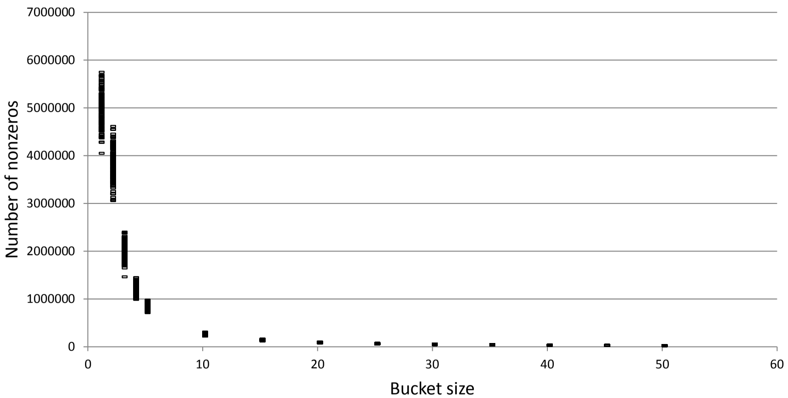
In this section we investigate the effectiveness of using a class of valid inequalities (31) from the family $\mathcal{F}_{1,(b_l-(1-k_{l-p_1+1})\pi_1)\Delta,(b_t-(1-k_t)(1-\pi_1))\Delta}$ as cuts on the data set ORL30PT25-100. The class of inequalities considered are those with coefficients $a_j^G = 0$ for all jobs $j \in \mathcal{J}^G$ and indices $G \in \{L, R\}$, for



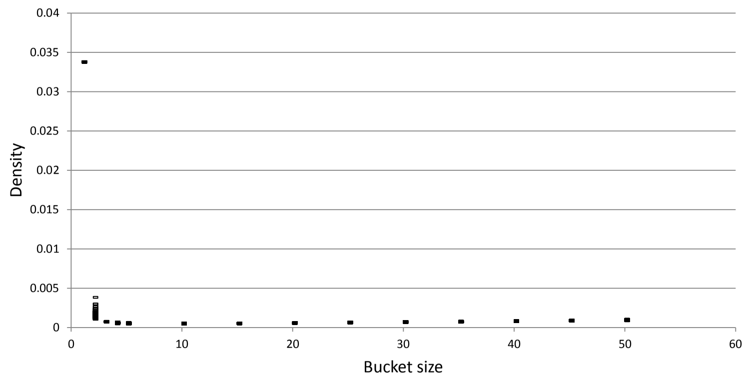
(a) Number of rows



(b) Number of columns

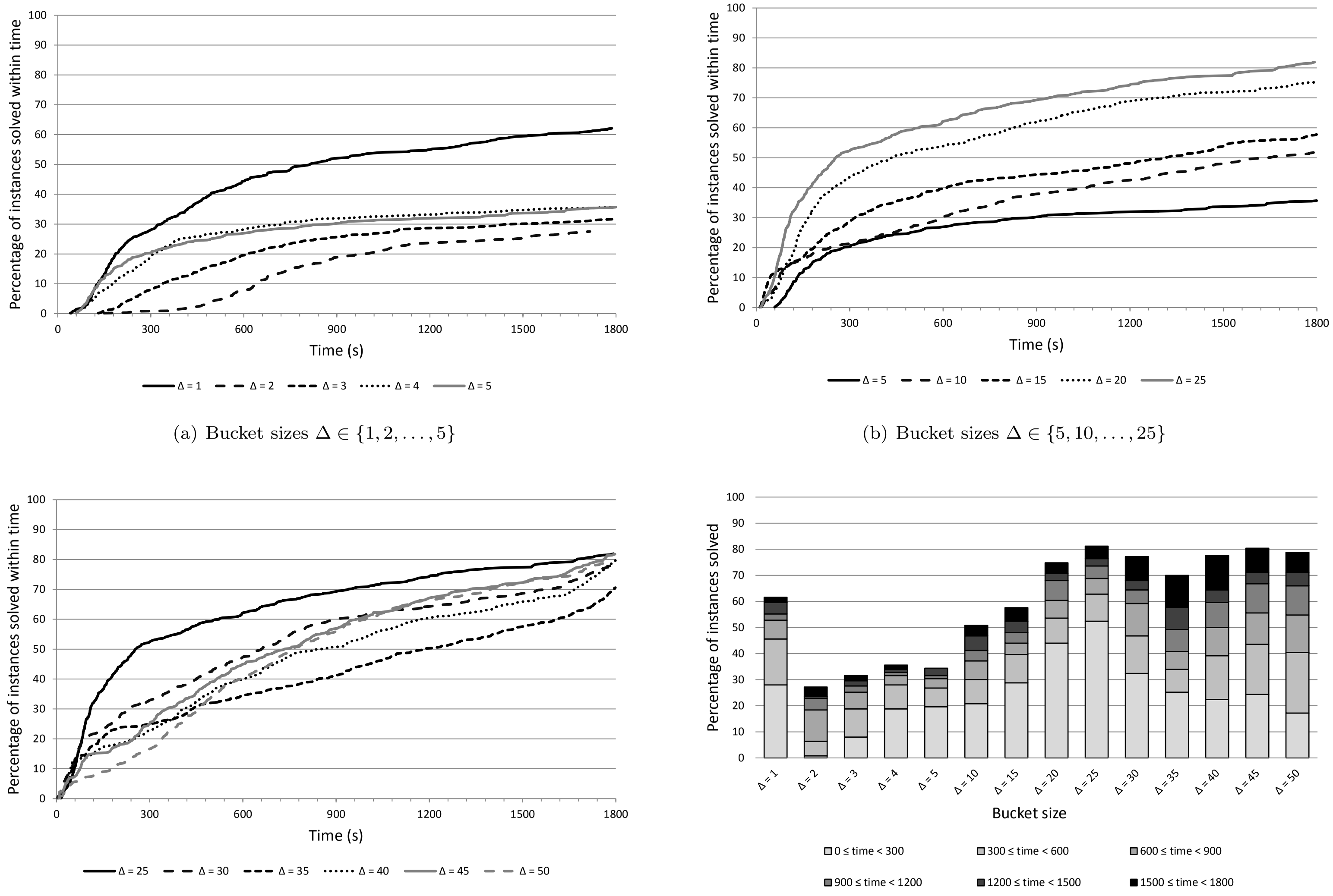


(c) Number of nonzeros



(d) Density

Figure 11: Size of the BB model on the data set ORL30PT50-100 for different choices of bucket size.



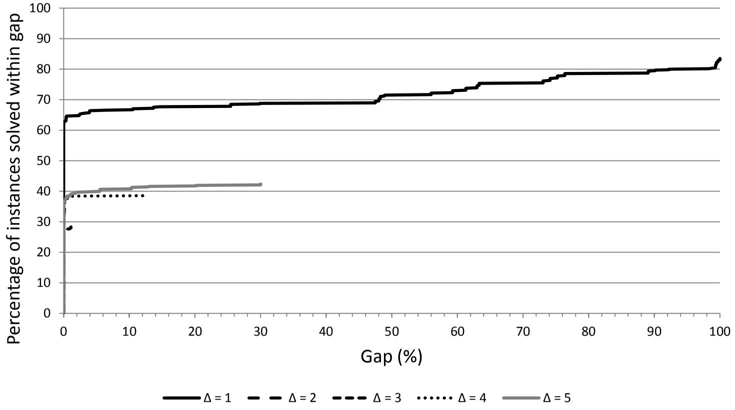
(a) Bucket sizes $\Delta \in \{1, 2, \ldots, 5\}$

(b) Bucket sizes $\Delta \in \{5, 10, \ldots, 25\}$

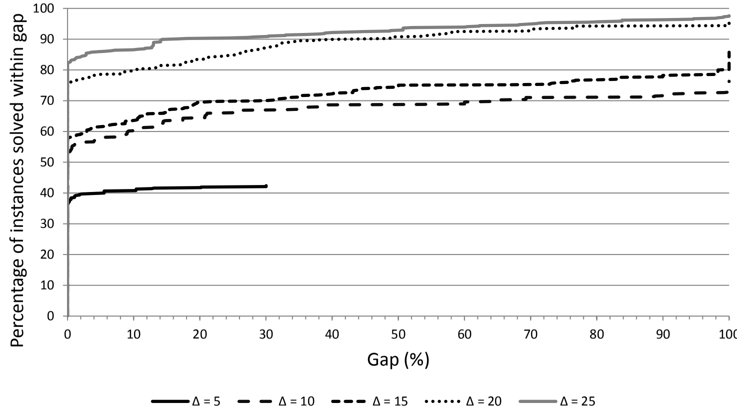
(c) Bucket sizes $\Delta \in \{25, 30, \ldots, 50\}$

(d) All bucket sizes

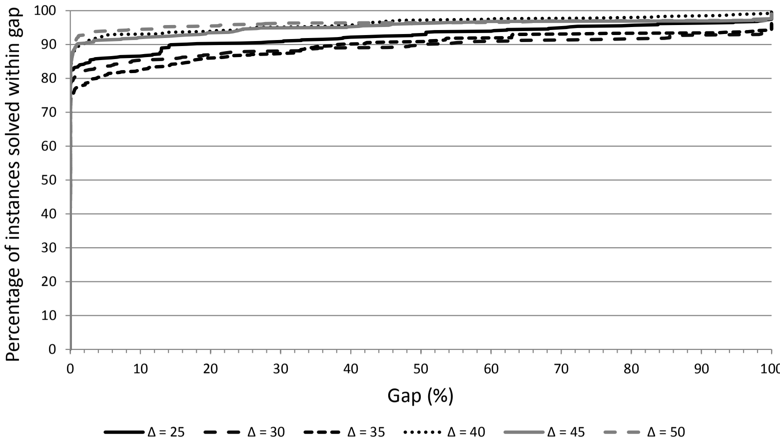
Figure 12: MILP solve times of the BB model on the data set ORL30PT50-100 for different choices of bucket size.



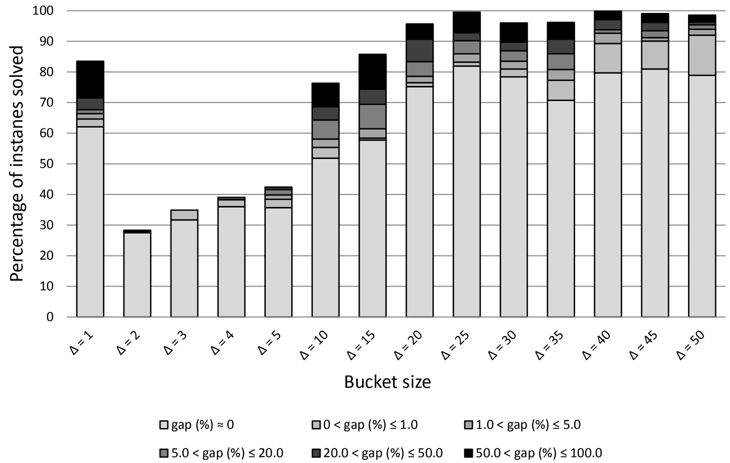
(a) Bucket sizes $\Delta \in \{1, 2, \ldots, 5\}$



(b) Bucket sizes $\Delta \in \{5, 10, \ldots, 25\}$

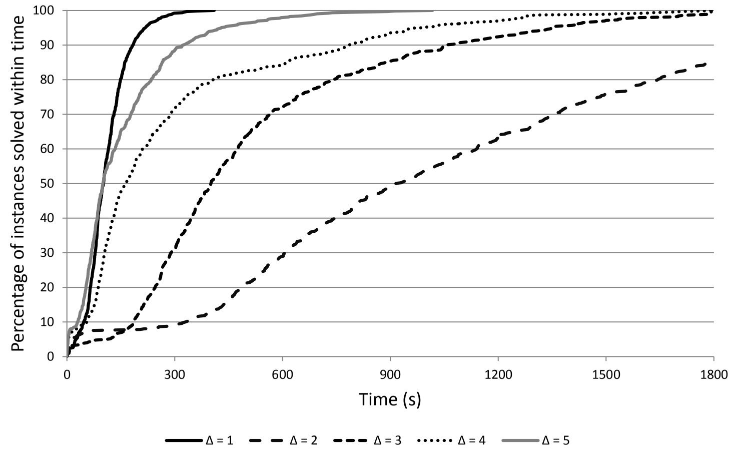


(c) Bucket sizes $\Delta \in \{25, 30, \ldots, 50\}$

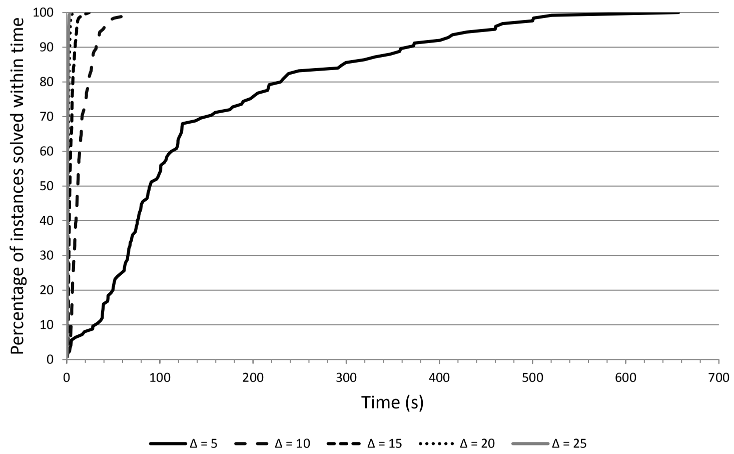


(d) All bucket sizes

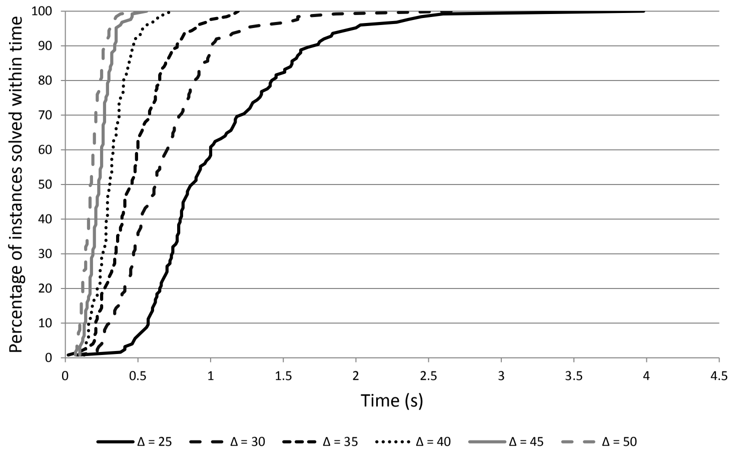
Figure 13: Optimality gaps of the BB model on the data set ORL30PT50-100 for different choices of bucket size.



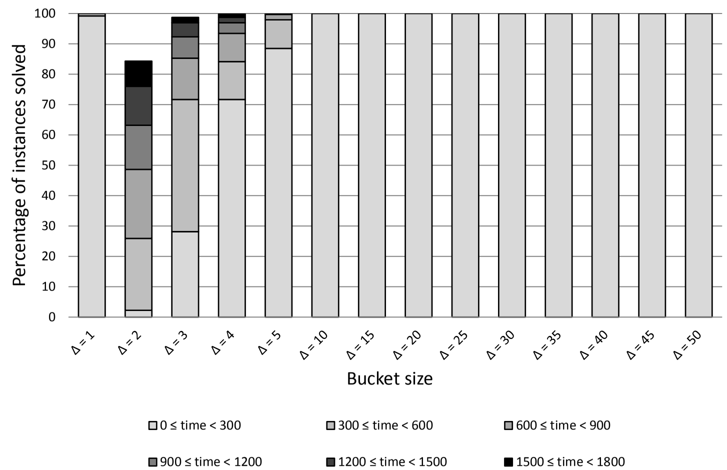
(a) Bucket sizes $\Delta \in \{1, 2, \ldots, 5\}$



(b) Bucket sizes $\Delta \in \{5, 10, \ldots, 25\}$

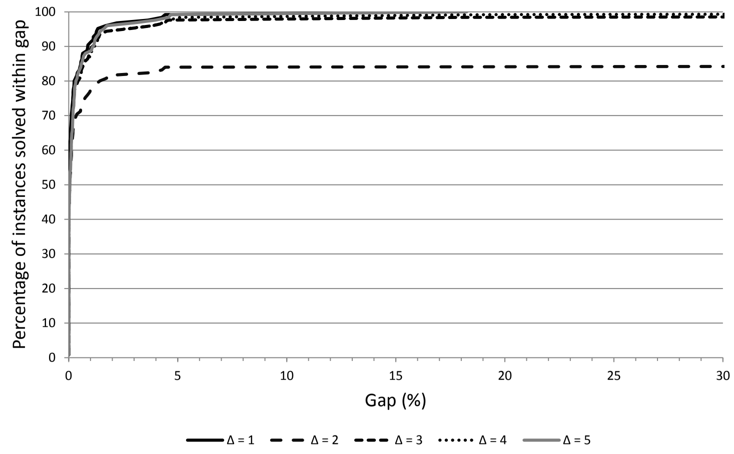


(c) Bucket sizes $\Delta \in \{25, 30, \ldots, 50\}$

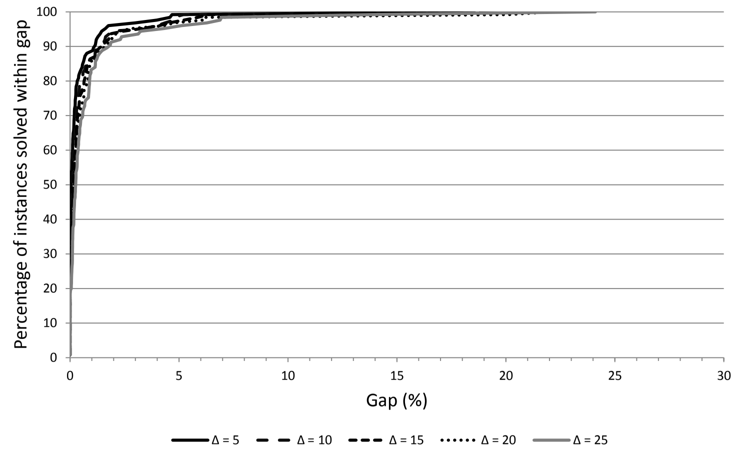


(d) All bucket sizes

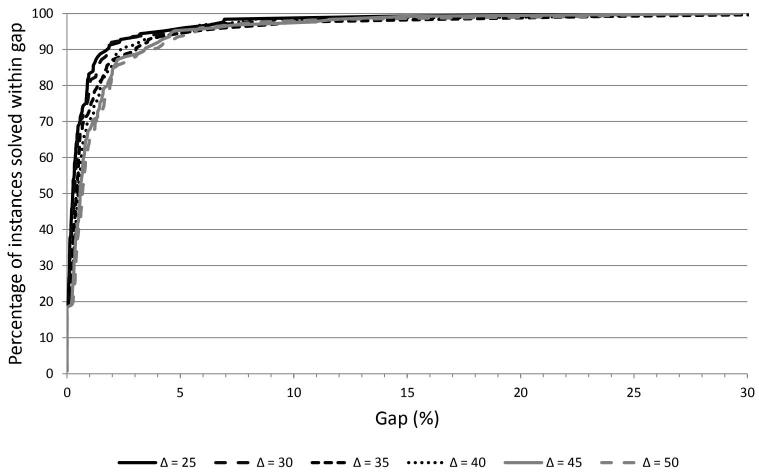
Figure 14: LP relaxation solve times of the BB model on the data set ORL30PT50-100 for different choices of bucket size.



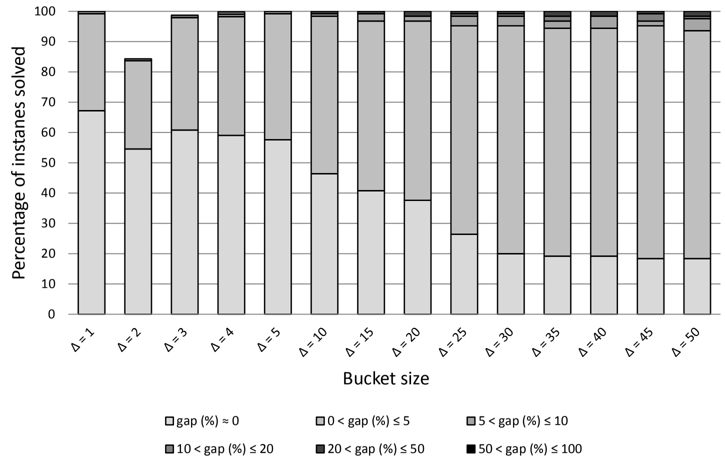
(a) Bucket sizes $\Delta \in \{1, 2, \ldots, 5\}$



(b) Bucket sizes $\Delta \in \{5, 10, \ldots, 25\}$



(c) Bucket sizes $\Delta \in \{25, 30, \ldots, 50\}$



(d) All bucket sizes

Figure 15: Integrality gaps of the BB model on the data set ORL30PT50-100 for different choices of bucket size.

all buckets $b_l, b_t \in [1, B]$ and indices $k_{l-p_1+1}, k_t \in K$ such that time periods $l = (b_l - (1-k_{l-p_1+1})\pi_1)\Delta$ and $t = (b_t - (1-k_t)(1-\pi_1))\Delta$ satisfy $1 \leq l \leq t \leq T$. The inequalities in this class are not as strong as the facet-inducing inequalities (32) described in Theorem 10 but are easier to implement. The inequalities are added as user cuts in CPLEX.

The solve times of the instances with only CPLEX cuts, with only our cuts, and with both CPLEX and our cuts, can be seen in Figure 16(a). Over all there does not appear to be any benefit from including our cuts in the model. However, a closer analysis reveals that when our cuts are not present in the cut pool, CPLEX added its own cuts to only 31 of the 125 instances. When our cuts are present in the cut pool, the number of instances in which any cut, either CPLEX's or ours, is added reduced to 27. In each of these 27 instances at least one of our cuts is added. In the remaining 98 instances a cut pool is being maintained even though no cuts are added which increases the solve times.

Restricting our attention to the 27 instances in which one of our cuts is added we find that in 19 of these instances the solve time decreases, in 7 it increases, and there is one instance that is not solved in the time limit. The solve times of these 27 instances with only CPLEX cuts, with only our cuts, and with both CPLEX and our cuts, can be seen in Figure 16(b).

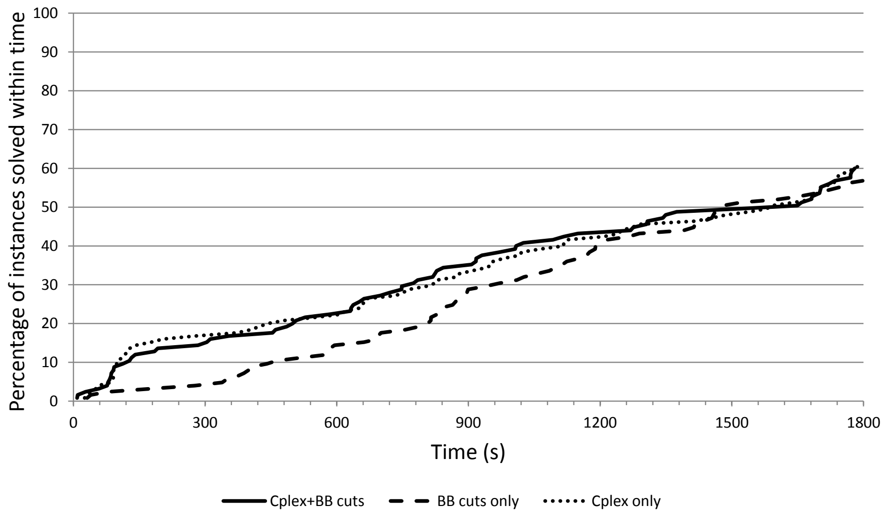
The effectiveness of this class of valid inequalities on strengthening the LP relaxation of the BB model can be seen in Figure 17. The inequalities are added to the model as constraints. There is very little improvement as the integrality gap of 90% of instances improved by at most 0.125% and less than 2% of instances saw an improvement of 1% or more.

5 Conclusions and future work

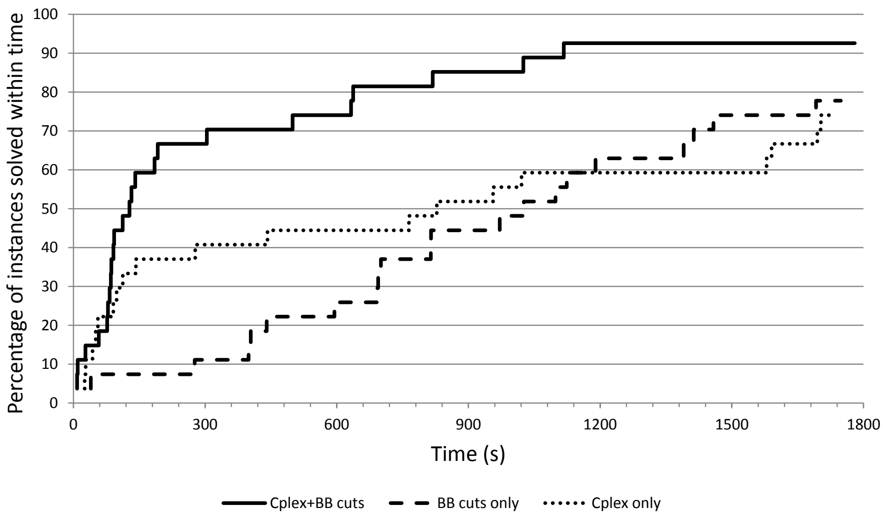
In this paper we present a big bucket time indexed mixed integer linear programming formulation for nonpreemptive single machine scheduling problems. The BB model is a generalisation of the classical TI model to one in which at most two jobs can be processing in each period. The size of a bucket in the model can be no larger than the length of the smallest processing time among all the jobs. If the problem data is integer and the bucket size is chosen to be one period then the BB model reduces to the TI model. However, if the bucket size is chosen to be larger than one then the BB model can have significantly fewer variables and nonzeros than the TI model at the expense of a greater number of constraints.

We show that an exponentially sized family of valid inequalities can be derived from a single facet-inducing inequality of the TI model and give conditions under which inequalities in this family will be strong. We prove that a class of valid inequalities in these families is facet-inducing for the convex hull of the set of feasible partial schedules, that is, schedules in which not all jobs have to be started. It remains to be shown under what conditions these inequalities are facet-inducing for the convex hull of the set of complete feasible schedules, and further work is needed to determine the effectiveness of these inequalities computationally.

Our computational study on data sets of weighted tardiness instances reveals that although the BB model appears to have a weaker LP relaxation, it significantly outperforms the TI model in terms of both solve times and optimality



(a) All 125 instances



(b) Restricted set of 27 instances

Figure 16: Solve times on the data set ORL30PT25-100 with only CPLEX cuts, with only our cuts, and with both CPLEX and our cuts.

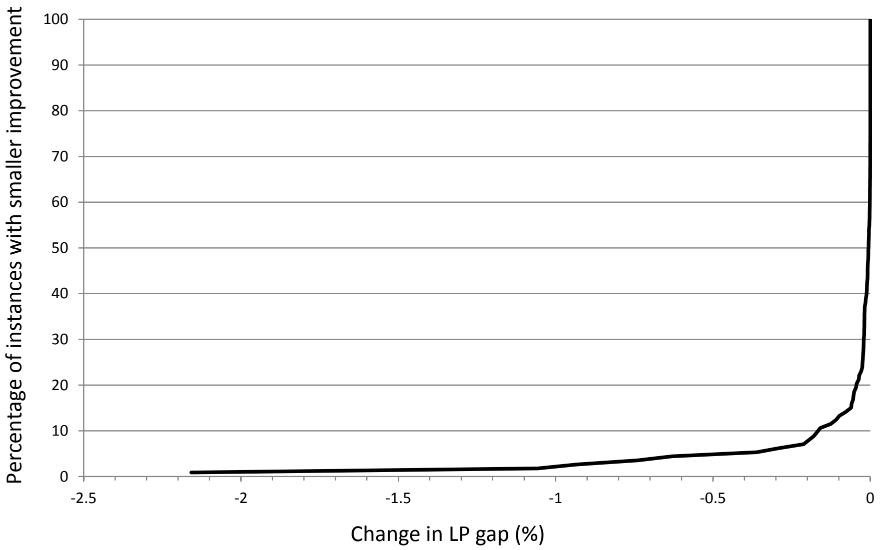


Figure 17: Change in integrality gap on the data set ORL30PT25-100 when our cuts are added to the model as constraints.

gaps on instances where the mean processing time of the jobs is large and the range of processing times is small, that is, the processing times are clustered rather than dispersed. Our investigation into the best choice of bucket size was inconclusive besides revealing that there is a trade-off between the size of the BB model and the strength of the LP relaxation. Further computational studies are needed to compare the performance of the BB model to that of the TI model on data sets for min-sum criteria other than weighted tardiness.

This work was motivated by the use of buckets in lot sizing. Paralleling the work done there, the big bucket framework introduced in this paper could be extended to model the case in which at most three jobs can be processing in each period. Such an extension is possible as there is no ambiguity as to the order in which jobs must be scheduled in each bucket when the jobs processing at the beginning and the end of the bucket are known.

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Appendix

Proof of Proposition 2

Recall that if job $j \in J$ starts at time s_j then it starts in bucket $S_j = \lfloor s_j/\Delta \rfloor + 1$ and $\sigma_j = S_j - s_j/\Delta \in (0, 1]$ is the fraction of that bucket that was spent processing the job. Let index $k_j = 1$ if $\sigma_j + \pi_j \leq 1$ and $k_j = 0$ otherwise. For all jobs $j \in J$, buckets $b \in [1, B]$, and indices $k \in K$, let variable $z_{jbk} = 1$ if $(j, b, k) = (j, S_j, k_j)$ and $z_{jbk} = 0$ otherwise. Similarly, let variable $u_{jbk} = \sigma_j$ if $(j, b, k) = (j, S_j, k_j)$ and $u_{jbk} = 0$ otherwise, and variable $v_{jbk} = 2 - k_j - \sigma_j - \pi_j$ if $(j, b, k) = (j, S_j + P_j + k_j - 1, k_j)$ and $v_{jbk} = 0$ otherwise. Note that $0 \leq u_{jbk} \leq z_{jbk} \in \{0, 1\}$ and $0 \leq v_{j,b+P_j+k-1,k} \leq z_{jbk} \in \{0, 1\}$ for all jobs $j \in J$, buckets $b \in [1, B]$, and indices $k \in K$.

To show the validity of each of the constraints (7)–(14) with respect to the feasible schedule s we consider each constraint in turn.

1. Consider constraints (7). It follows from the definition of the z variables and the fact that s is a feasible schedule that these constraints are satisfied.

2. Consider constraints (8). To see that these constraints are satisfied it suffices to show that if time $s_j > s_i$ then $S_j - S_i \geq P_i + k_i - 1$ for all jobs $i, j \in J$ as then at most one variable z_{jak} is nonzero for all jobs $j \in J$, buckets $b \in [1, B]$, indices $k \in K$, and periods $a \in [b - P_j - k + 2, b]$. If time $s_j > s_i$ then $s_j \geq s_i + p_i$ and $\lfloor s_j/\Delta \rfloor \geq \lfloor (s_i + p_i)/\Delta \rfloor$ as $\Delta > 0$. Therefore

$$\begin{aligned} S_j = \left\lfloor \frac{s_j}{\Delta} \right\rfloor + 1 &\geq \left\lfloor \frac{s_i + p_i}{\Delta} \right\rfloor + 1 \\ &= \left\lfloor (1 - \sigma_i) + \left\lfloor \frac{s_i}{\Delta} \right\rfloor + (1 - \pi_i) + \left\lfloor \frac{p_i}{\Delta} \right\rfloor \right\rfloor + 1 \\ &= \left\lfloor \frac{s_i}{\Delta} \right\rfloor + \left\lfloor \frac{p_i}{\Delta} \right\rfloor + \lfloor (1 - \sigma_i) + (1 - \pi_i) \rfloor + 1 \\ &= S_i + P_i + \lfloor 2 - \sigma_i - \pi_i \rfloor - 1 \\ &= S_i + P_i + k_i - 1 \end{aligned}$$

and so $S_j - S_i \geq P_i + k_i - 1$.

3. Consider constraints (9). To see that these constraints are satisfied we consider two cases. Firstly, if $S_j - S_i > P_i + k_i - 1$ when time $s_j > s_i$ for all jobs $i, j \in J$ then these constraints are satisfied since at most one of the variables u_{jbk}, v_{jbk}, and z_{jak} is nonzero for all jobs $j \in J$, buckets $b \in [1, B]$, indices $k \in K$, and periods $a \in [b - P_j - k + 2, b - 1]$. Secondly, if $S_j - S_i = P_i + k_i - 1$ when time $s_j > s_i$ for some job $i, j \in J$ then it remains to be shown that variables $u_{jS_jk_j} + v_{i,S_i+P_i+k_i-1,k_i} \leq 1$. Now

$$\begin{aligned} u_{jS_jk_j} + v_{i,S_i+P_i+k_i-1,k_i} &= \sigma_j + 2 - k_i - \sigma_i - \pi_i \\ &= S_j - \frac{s_j}{\Delta} + 2 - k_i - S_i + \frac{s_i}{\Delta} - \pi_i \\ &= S_i + P_i + k_i - 1 - \frac{s_j}{\Delta} + 2 - k_i - S_i + \frac{s_i}{\Delta} - \pi_i \\ &= \frac{s_i}{\Delta} + \frac{p_i}{\Delta} - \frac{s_j}{\Delta} + 1 \\ &\leq 1 \end{aligned}$$

since time $s_j \geq s_i + p_i$ and $\Delta > 0$.

4. Consider constraints (10). It follows from the definition of the z variables that these constraints are satisfied.

5. Consider constraints (11). By definition variable $u_{jbk_j} \geq 0$ for all jobs $j \in J$, buckets $b \in [1, B]$, and indices $k \in K$. Thus it remains to be shown that variable $u_{jS_j0} \geq 1 - \pi_j$ when index $k_j = 0$. Now $\sigma_j + \pi_j > 1$ when $k_j = 0$ and so $u_{jS_j0} = \sigma_j > 1 - \pi_j$.

6. Consider constraints (12). By definition variable $u_{jbk_j} \leq z_{jbk_j}$ for all jobs $j \in J$, buckets $b \in [1, B]$, and indices $k \in K$. Thus it remains to be shown that variable $u_{jS_j1} \leq 1 - \pi_j$ when index $k_j = 1$. Now $\sigma_j + \pi_j \leq 1$ when $k_j = 1$ and so $u_{jS_j1} = \sigma_j \leq 1 - \pi_j$.

7. Consider constraints (13) and (14). Since constraints (10), (11) and (12) are valid it follows that these constraints are also satisfied since they are equivalent to (12) and (11) respectively.

□

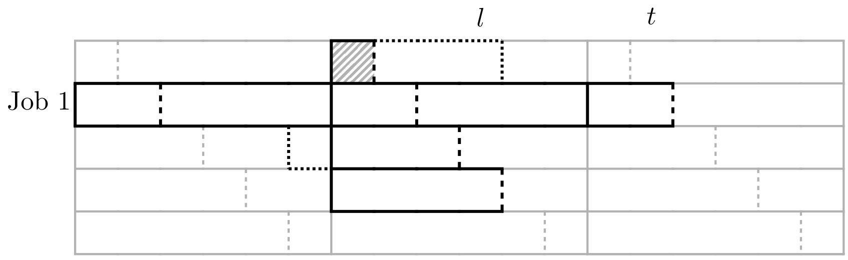
Proof of Theorem 10

Let the inequality $\mu^z z + \mu^u u \leq 1$ denote the inequality (32). To prove the claim we show that there exist sufficient feasible partial schedules in the set Z^* satisfying the inequality $\mu^z z + \mu^u u \leq 1$ at equality that we are able to uniquely identify the coefficients of a generic hyperplane $\lambda^z z + \lambda^u u = 1$ and show that the coefficients $(\lambda^z, \lambda^u) = (\mu^z, \mu^u)$. A consequence of this is that we will have identified $\dim(Z^*)$ affinely independent feasible partial schedules in the set Z^* thereby proving that $\mu^z z + \mu^u u \leq 1$ is a facet-inducing inequality for the polyhedron $\text{conv}(Z^*)$.

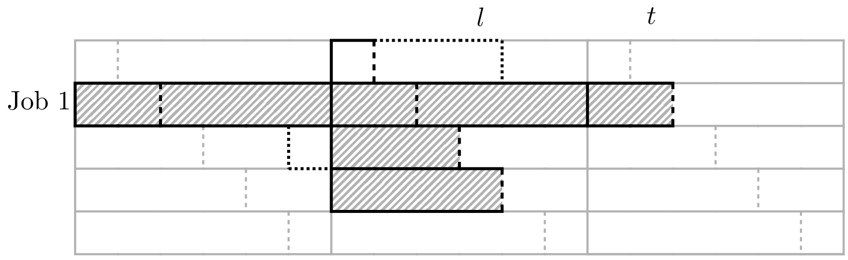
The coefficients of the generic hyperplane $\lambda^z z + \lambda^u u = 1$ can be determined by considering each half-bucket $(j, b, k) \in J \times [1, B] \times K$ in turn. If the half-bucket (j, b, k) contains only one period, that is, $|\mathcal{T}_{jbk}| = 1$, then only one variable z_{jbk}, with coefficient λ^z_{jbk}, is associated with the half-bucket. If the half-bucket (j, b, k) contains more than one period, that is, $|\mathcal{T}_{jbk}| \geq 2$, then the variable pair (z_{jbk}, u_{jbk}), with coefficients $(\lambda^z_{jbk}, \lambda^u_{jbk})$, are associated with this half-bucket. For each half-bucket we claim that one or more partial schedules are feasible, that they satisfy the inequality $\mu^z z + \mu^u u \leq 1$ at equality, and that $(\lambda^z_{jbk}, \lambda^u_{jbk}) = (\mu^z_{jbk}, \mu^u_{jbk})$. We omit the proofs of these claims as they are easily verified. Feasibility of a partial schedule can be verified by observing that no two jobs are being processed at the same time. That these partial feasible schedules satisfy the inequality $\mu^z z + \mu^u u \leq 1$ at equality can be verified by substitution. Finally, verifying that $(\lambda^z_{jbk}, \lambda^u_{jbk}) = (\mu^z_{jbk}, \mu^u_{jbk})$ may require solving a simple system of linear equations the details of which we omit. More details of these claims including proofs are given in Clement (2013).

In what follows e_{jbk} denotes column (j, b, k) of an identity matrix that is indexed by the half-buckets $(j, b, k) \in J \times [1, B] \times K$. For ease of exposition let e^G_j denote $e_{jb^G_j k^G_j}$ and (μ^{zG}_j, μ^{uG}_j) denote $(\mu^z_{jb^G_j k^G_j}, \mu^u_{jb^G_j k^G_j})$ for all jobs $j \in J$ and indices $G \in \{L, R\}$. The proof proceeds by considering various cases that are distinguished by the location of the half-bucket under consideration. Cases 2b and 6a are similar, as are Cases 3b and 6b, and Cases 5, 7, and 9.

1. Consider a half-bucket $(j, b, k) \in \mathcal{H}_j$ for some job $j \in \mathcal{J}$.



(a) Case 1a example



(b) Case 1b example

Figure 18: Case 1 examples. The shaded half-buckets indicate the possible half-buckets $(j, b, k) \in \mathcal{H}_j$ for some job $j \in \mathcal{J}$.

a. If the half-bucket (j, b, k) contains only one period, that is, $|\mathcal{T}_{jbk}| = 1$, then the variable z_{jbk} has coefficient $\mu^z_{jbk} = 1$. The feasible partial schedule $(z, u) = (e_{jbk}, 0)$ satisfies the inequality at equality. See, for example, Figure 18(a).

b. If the half-bucket (j, b, k) contains more than one period, that is, $|\mathcal{T}_{jbk}| \geq 2$, then the variable pair (z_{jbk}, u_{jbk}) has coefficients $(\mu^z_{jbk}, \mu^u_{jbk}) = (1, 0)$. The feasible partial schedules $(z, u) \in \{(e_{jbk}, (1 - k\pi_j)e_{jbk}), (e_{jbk}, (1 - k\pi_j - 1/\Delta)e_{jbk})\}$ satisfy the inequality at equality. See, for example, Figure 18(b).

2. Consider the half-bucket $(1, b^L_1, k^L_1)$ for which the set $\mathcal{S}^L_1 = \emptyset$. Recall that the choice of index $k_{l-p_1+1} \in K$ determines whether job $j \in \mathcal{J}^R$, that is, whether the set $\mathcal{S}^R_j \neq \emptyset$ for each job $j \in J \setminus \{1\}$.

   a. If the index $k_{l-p_1+1} = 0$ then the period $l = (b_l - \pi_1)\Delta$ and the half-bucket $(1, b^L_1, k^L_1) = (1, b_{l-p_1+1} - 1, 1)$. This case requires either condition 1 or 2 to hold.

      i. If the half-bucket $(1, b^L_1, k^L_1)$ contains only one period, that is, $|\mathcal{T}^L_1| = 1$, then $\pi_1 = 1 - 1/\Delta$ and the variable z^L_1 has coefficient $\mu^{zL}_1 = 0$. If condition 1 holds then there exists a job $j \in J \setminus \{1\}$ for which the feasible partial schedule $(z, u) = (e_{1b^L_1 1}, 0) + (e_{jb_l 0}, e_{jb_l 0})$ satisfies the inequality at equality. See, for example, Figure 19(a).

      ii. If the half-bucket $(1, b^L_1, k^L_1)$ contains more than one period, that is, $|\mathcal{T}^L_1| \geq 2$, then $\pi_1 < 1 - 1/\Delta$ and the variable pair (z^L_1, u^L_1) has coefficients $(\mu^{zL}_1, \mu^{uL}_1) = (0, 0)$. If condition 2a holds then there exists a job $j \in J \setminus \{1\}$ for which the feasible partial schedules

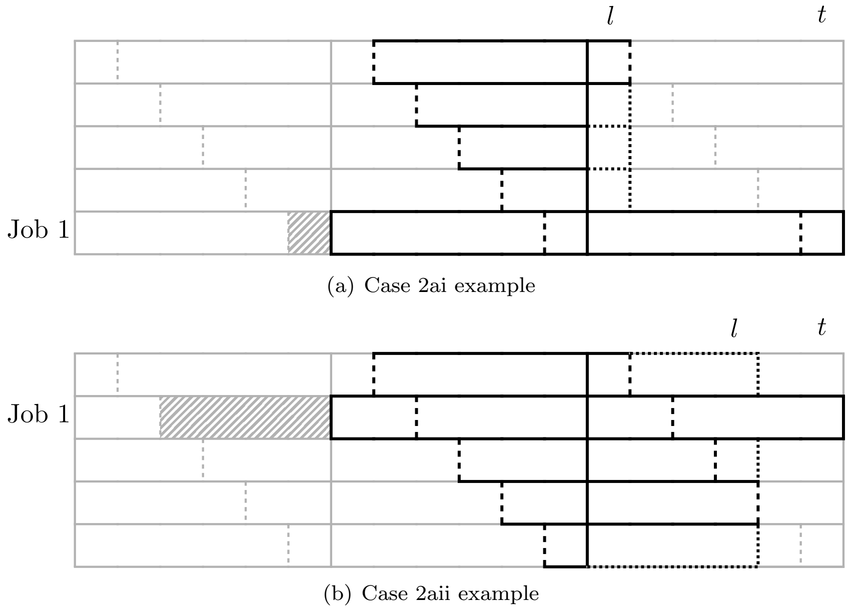


Figure 19: Case 2a examples. The shaded half-bucket indicates the half-bucket $(1, b_1^L, k_1^L)$.

$$(z,u) = \{(e_{1b_1^L 1}, (1-\pi_1)e_{1b_1^L 1}) + (e_{jb_l 0}, (1-1/\Delta)e_{jb_l 0}),$$
$$(e_{1b_1^L 1}, (1-\pi_1-1/\Delta)e_{1b_1^L 1}) + (e_{jb_l 0}, (1-1/\Delta)e_{jb_l 0})\}$$

satisfy the inequality at equality. If condition 2b holds then there exists a job $j \in J \setminus \{1\}$ for which the feasible partial schedules

$$(z,u) = \{(e_{1b_1^L 1}, (1-\pi_1)e_{1b_1^L 1}) + (e_{jb_l 1}, (1-\pi_j)e_{jb_l 1}),$$
$$(e_{1b_1^L 1}, (1-\pi_1-1/\Delta)e_{1b_1^L 1}) + (e_{jb_l 1}, (1-\pi_j)e_{jb_l 1})\}$$

satisfy the inequality at equality. See, for example, Figure 19(b).

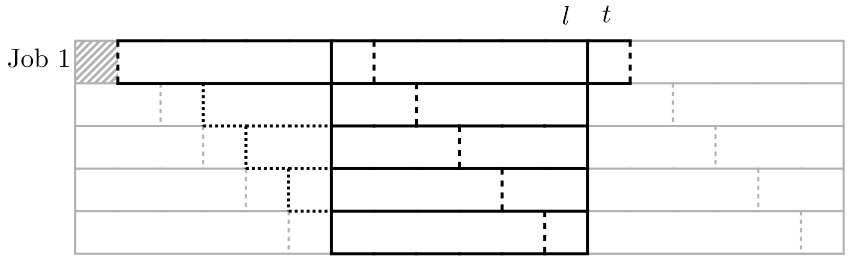
b. If the index $k_{l-p_1+1} = 1$ then the period $l = b_l\Delta$ and the half-bucket $(1, b_1^L, k_1^L) = (1, b_{l-p_1+1}, 0)$.

   i. If the half-bucket $(1, b_1^L, k_1^L)$ contains only one period, that is, $|\mathcal{T}_1^L| = 1$, then the variable z_1^L has coefficient $\mu_1^{zL} = 0$ and the feasible partial schedule $(z,u) = (e_1^L, 0) + (e_{jb_l 1}, 1/\Delta e_{jb_l 1})$ satisfies the inequality at equality. See, for example, Figure 20(a). This case is similar to Case 6ai.

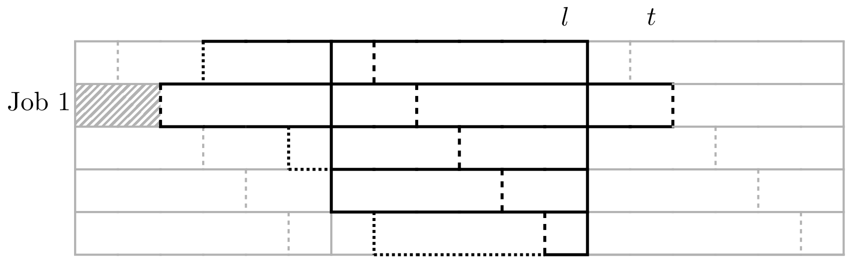
   ii. If the half-bucket $(1, b_1^L, k_1^L)$ contains more than one period, that is, $|\mathcal{T}_1^L| \geq 2$, then the variable pair (z_1^L, u_1^L) has coefficients $(\mu_1^{zL}, \mu_1^{uL}) = (0,0)$ and the feasible partial schedules

$$(z,u) = \{(e_1^L, ((1-k_1^L)(1-\pi_1) - 1/\Delta)e_1^L) + (e_{jb_l 1}, 1/\Delta e_{jb_l 1}),$$
$$(e_1^L, ((1-k_1^L)(1-\pi_1) - 2/\Delta)e_1^L) + (e_{jb_l 1}, 1/\Delta e_{jb_l 1})\}$$

satisfy the inequality at equality. See, for example, Figure 20(b). This case is similar to Case 6aii.



(a) Case 2bi example



(b) Case 2bii example

Figure 20: Case 2b examples. The shaded half-bucket indicates the half-bucket $(1, b_1^L, k_1^L)$.

3. Consider the half-bucket $(1, b_1^R, k_1^R)$ for which the set $\mathcal{S}_1^R = \emptyset$. Recall that the choice of index $k_t \in K$ determines whether job $j \in \mathcal{J}^L$, that is, whether the set $\mathcal{S}_j^L \neq \emptyset$ for each job $j \in J \setminus \{1\}$.

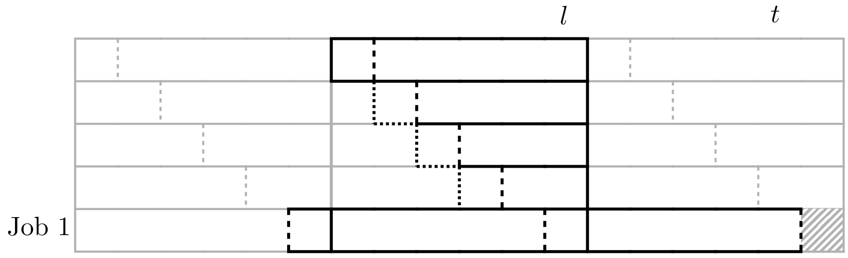
   a. If the index $k_t = 0$ then the period $t = (b_t - (1 - \pi_1))\Delta$ and the half-bucket $(1, b_1^R, k_1^R) = (1, b_t, 1)$. This case requires either condition 3 or 4 to hold.

      i. If the half-bucket $(1, b_1^R, k_1^R)$ contains only one period, that is, $|\mathcal{T}_1^R| = 1$, then $\pi_1 = 1 - 1/\Delta$ and the variable z_1^R has coefficient $\mu_1^{zR} = 0$. If condition 3a holds then there exists a job $j \in J \setminus \{1\}$ for which the feasible partial schedule $(z, u) = (e_{1b_1^R1}, 0) + (e_{j,b_t-P_j+1,0}, e_{j,b_t-P_j+1,0})$ satisfies the inequality at equality. If condition 3b holds then there exists a job $j \in J \setminus \{1\}$ for which the feasible partial schedule $(z, u) = (e_{1b_1^R1}, 0) + (e_{j,b_t-P_j+1,0}, (1 - \pi_j + 1/\Delta)e_{j,b_t-P_j+1,0})$ satisfies the inequality at equality. See, for example, Figure 21(a).

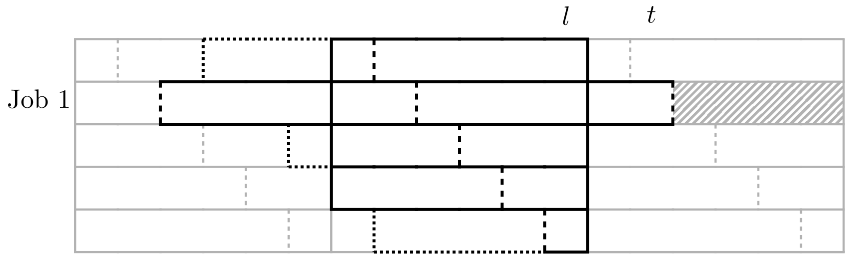
      ii. If the half-bucket $(1, b_1^R, k_1^R)$ contains more than one period, that is, $|\mathcal{T}_1^R| \geq 2$, then $\pi_1 < 1 - 1/\Delta$ and the variable pair (z_1^R, u_1^R) has coefficients $(\mu_1^{zR}, \mu_1^{uR}) = (0, 0)$. If condition 4a holds then there exists a job $j \in J \setminus \{1\}$ for which the feasible partial schedules

$$
\begin{aligned}
(z, u) = \{&(e_{1b_1^R1}, 1/\Delta e_{1b_1^R1}) + (e_{j,b_t-P_j+1,0}, e_{j,b_t-P_j+1,0}),\\
&(e_{1b_1^R1}, 2/\Delta e_{1b_1^R1}) + (e_{j,b_t-P_j+1,0}, e_{j,b_t-P_j+1,0})\}
\end{aligned}
$$

      satisfy the inequality at equality. If condition 4b holds then there exists a job $j \in J \setminus \{1\}$ for which the feasible partial schedules



(a) Case 3ai example



(b) Case 3aii example

Figure 21: Case 3a examples. The shaded half-bucket indicates the half-bucket $(1, b_1^R, k_1^R)$.

$$(z, u) = \{(e_{1b_1^R 1}, 1/\Delta e_{1b_1^R 1}) + (e_{j,b_t-P_j+1,1}, 1/\Delta e_{j,b_t-P_j+1,1}),$$
$$(e_{1b_1^R 1}, 2/\Delta e_{1b_1^R 1}) + (e_{j,b_t-P_j+1,1}, 1/\Delta e_{j,b_t-P_j+1,1})\}$$

satisfy the inequality at equality. See, for example, Figure 21(b).

b. If the index $k_t = 1$ then the period $t = b_t\Delta$ and the half-bucket $(1, b_1^R, k_1^R) = (1, b_t + 1, 0)$.

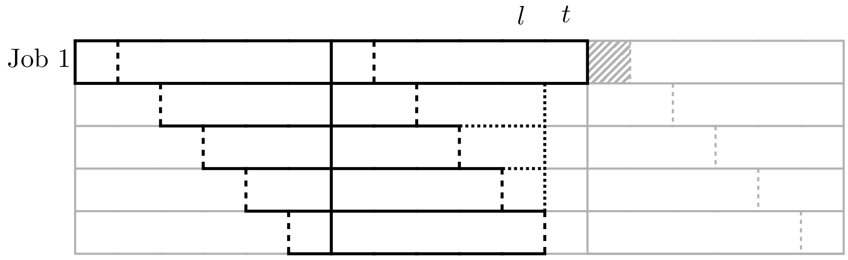
i. If the half-bucket $(1, b_1^R, k_1^R)$ contains only one period, that is, $|\mathcal{T}_1^R| = 1$, then the variable z_1^R has coefficient $\mu_1^{zR} = 0$ and there exists a job $j \in J \setminus \{1\}$ for which the feasible partial schedule $(z, u) = (e_1^R, 0) + (e_{j,b_t-P_j+1,1}, (1 - \pi_j)e_{j,b_t-P_j+1,1})$ satisfies the inequality at equality. See, for example, Figure 22(a). This case is similar to Case 6bi.

ii. If the half-bucket $(1, b_1^R, k_1^R)$ contains more than one period, that is, $|\mathcal{T}_1^R| \geq 2$, then the variable pair (z_1^R, u_1^R) has coefficients $(\mu_1^{zR}, \mu_1^{uR}) = (0, 0)$ and there exists a job $j \in J \setminus \{1\}$ for which the feasible partial schedules

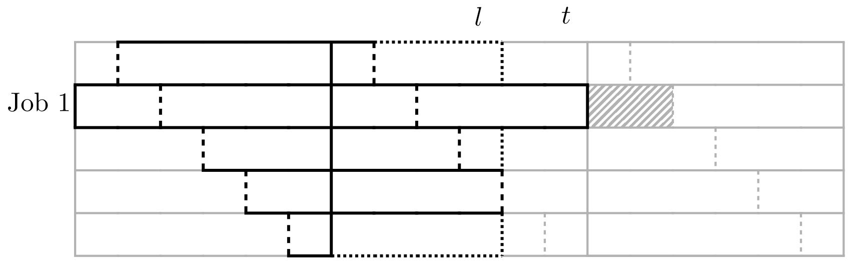
$$(z, u) = \{(e_1^R, (1 - k_1^R\pi_1)e_1^R) + (e_{j,b_t-P_j+1,1}, (1 - \pi_j)e_{j,b_t-P_j+1,1}),$$
$$(e_1^R, (1 - k_1^R\pi_1 - 1/\Delta)e_1^R) + (e_{j,b_t-P_j+1,1}, (1 - \pi_j)e_{j,b_t-P_j+1,1})\}$$

satisfy the inequality at equality. See, for example, Figure 22(b). This case is similar to Case 6bii.

4. Consider the half-bucket (j, b_j^G, k_j^G) for some job $j \in \mathcal{J}^G$ and index $G \in \{L, R\}$. Note that job $1 \notin \mathcal{J}^G$. Since the set $\mathcal{S}_j^G \neq \emptyset$ and the half-bucket $(j, b_j^G, k_j^G) \notin \mathcal{H}_j$ it must be the case that the half-bucket contains more than one period, that is, $|\mathcal{T}_j^G| \geq 2$.



(a) Case 3bi example



(b) Case 3bii example

Figure 22: Case 3b examples. The shaded half-bucket indicates the half-bucket $(1, b_1^R, k_1^R)$.

a. Suppose that $G = L$. Recall that if job $j \in \mathcal{J}^L$, that is, the set $\mathcal{S}_j^L \neq \emptyset$, then the index $k_t = 0$ and the period $t - p_j + 1 = (b_t - P_j - (1 - \pi_1) + \pi_j)\Delta + 1$.

   i. If $a_j^L = 0$ then the variable pair (z_j^L, u_j^L) has coefficients $(\mu_j^{zL}, \mu_j^{uL}) = (0, 0)$ and the feasible partial schedules

$$(z, u) \in \{(e_{1b_t0}, (1 - \pi_1 + 1/\Delta)e_{1b_t0}) + (e_j^L, (1 - k_j^L \pi_j)e_j^L),$$
$$(e_{1b_t0}, (1 - \pi_1 + 1/\Delta)e_{1b_t0}) + (e_j^L, (1 - k_j^L \pi_j - 1/\Delta)e_j^L)\}$$

   satisfy the inequality at equality. See, for example, Figure 23(a).

   ii. If $a_j^L = 1$ then the variable pair (z_j^L, u_j^L) has coefficients

$$(\mu_j^{zL}, \mu_j^{uL}) = \left(\frac{2 - k_j^L - \pi_j - \pi_1 + 1/\Delta}{1 - \pi_1 - k_j^L \pi_j}, \frac{-1}{1 - \pi_1 - k_j^L \pi_j} \right)$$

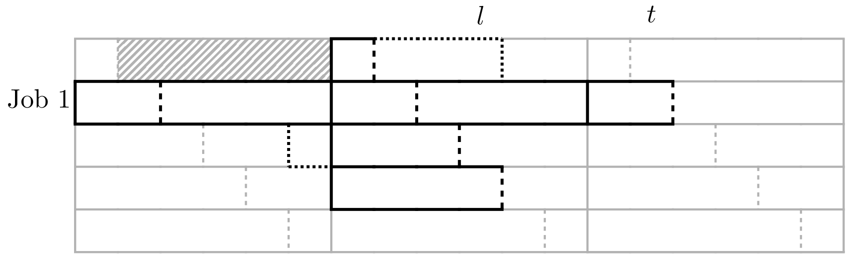
   and the feasible partial schedules

$$(z, u) \in \{(e_j^L, ((1 - k_j^L)(1 - \pi_j) + 1/\Delta)e_j^L),$$
$$(e_{1b_t0}, (1 - \pi_1 + 1/\Delta)e_{1b_t0}) + (e_j^L, (2 - k_j^L - \pi_j - \pi_1 + 1/\Delta)e_j^L)\}$$

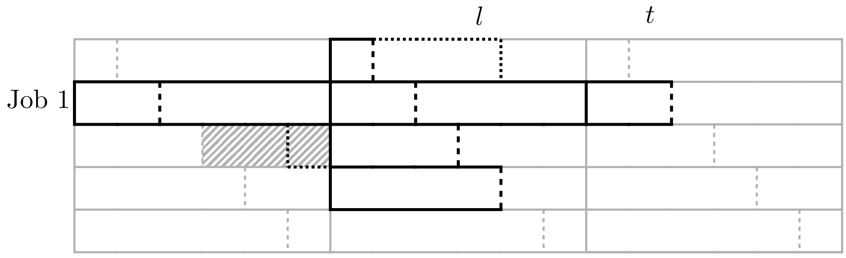
   satisfy the inequality at equality. See, for example, Figure 23(b).

b. Suppose that $G = R$. Recall that if job $j \in \mathcal{J}^R$, that is, the set $\mathcal{S}_j^R \neq \emptyset$, then the index $k_{l-p_1+1} = 0$ and the period $l = (b_l - \pi_1)\Delta$.

   i. If $a_j^R = 0$ then the variable pair (z_j^R, u_j^R) has coefficients $(\mu_j^{zR}, \mu_j^{uR}) = (0, 0)$ and the feasible partial schedules



(a) Case 4ai example



(b) Case 4aii example

Figure 23: Case 4a examples. The shaded half-buckets indicate the possible half-buckets (j, b_j^L, k_j^L) for jobs $j \in \mathcal{J}^L$.

$$(z,u) \in \{(e_{1,b_l-P_1+1,0}, e_{1,b_l-P_1+1,0}) + (e_j^R, ((1-k_j^R)(1-\pi_j)+1/\Delta)e_j^R),$$
$$(e_{1,b_l-P_1+1,0}, e_{1,b_l-P_1+1,0}) + (e_j^R, ((1-k_j^R)(1-\pi_j)+2/\Delta)e_j^R)\}$$

satisfy the inequality at equality. See, for example, Figure 24(a).

ii. If $a_j^R = 1$ then the variable pair (z_j^R, u_j^R) has coefficients

$$(\mu_j^{zR}, \mu_j^{uR}) = \left(\frac{-\pi_1}{1-\pi_1-k_j^R\pi_j}, \frac{1}{1-\pi_1-k_j^R\pi_j} \right)$$

and the feasible partial schedules

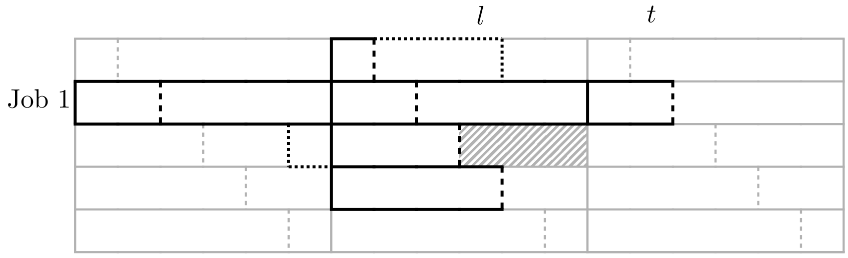
$$(z,u) \in \{(e_j^R, (1-k_j^R\pi_j)e_j^R),$$
$$(e_{1,b_l-P_1+1,0}, e_{1,b_l-P_1+1,0}) + (e_j^R, \pi_j e_j^R)\}$$

satisfy the inequality at equality. See, for example, Figure 24(b).

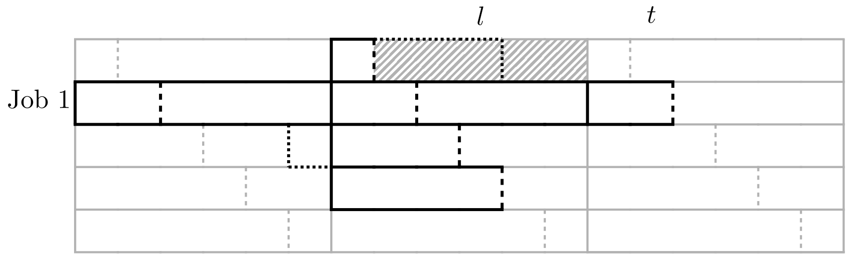
5. Consider the half-bucket (j, b_j^G, k_j^G) for some job $j \in \mathcal{J} \setminus (\{1\} \cup \mathcal{J}^G)$ and index $G \in \{L, R\}$. Note that the set $\mathcal{S}_j^G = \emptyset$.

   a. Suppose that $G = L$.

      i. If the half-bucket (j, b_j^L, k_j^L) contains only one period, that is, $|\mathcal{T}_{jb_j^L k_j^L}| = 1$, then the variable z_j^L has coefficient $\mu_j^{zL} = 0$ and the feasible partial schedule $(z_j^L, u_j^L) = (e_{1b_tk_t}, ((1-k_t)(1-\pi_1)+1/\Delta)e_{1b_tk_t}) + (e_j^L, 0)$ satisfies the inequality at equality. See, for example, Figure 25(a). This case is similar to Cases 7ai and 9ai.



(a) Case 4bi example



(b) Case 4bii example

Figure 24: Case 4b examples. The shaded half-buckets indicate the possible half-buckets (j, b_j^R, k_j^R) for jobs $j \in \mathcal{J}^R$.

ii. If the half-bucket (j, b_j^L, k_j^L) contains more than one period, that is, $|\mathcal{T}_{jb_j^L k_j^L}| \geq 2$, then the variable pair (z_j^L, u_j^L) has coefficients $(\mu_j^{zL}, \mu_j^{uL}) = (0, 0)$ and the feasible partial schedules

$$(z_j^L, u_j^L) = \{(e_{1b_tk_t}, ((1-k_t)(1-\pi_1)+1/\Delta)e_{1b_tk_t})+(e_j^L, ((1-k_j^L)(1-\pi_j)+1/\Delta)e_j^L), (e_{1b_tk_t}, ((1-k_t)(1-\pi_1)+1/\Delta)e_{1b_tk_t})+(e_j^L, ((1-k_j^L)(1-\pi_j)+2/\Delta)e_j^L)\}$$

satisfy the inequality at equality. See, for example, Figure 25(b). This case is similar to Cases 7aii and 9aii.

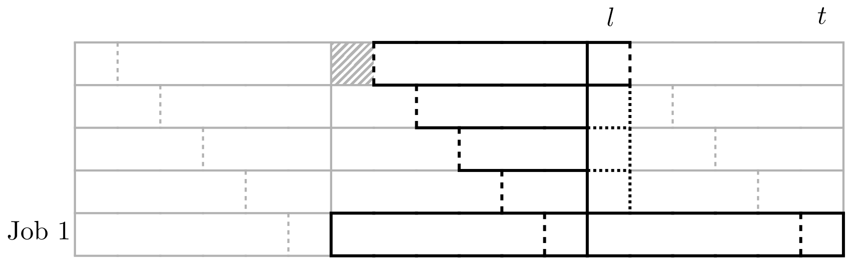
b. Suppose that $G = R$.

i. If the half-bucket (j, b_j^R, k_j^R) contains only one period, that is, $|\mathcal{T}_{jb_j^R k_j^R}| = 1$, then variable z_j^R has coefficient $\mu_j^{zR} = 0$ and the feasible partial schedule $(z_j^R, u_j^R) = (e_{1,b_{l-p_1+1},k_{l-p_1+1}}, (1-k_{l-p_1+1}\pi_1)e_{1,b_{l-p_1+1},k_{l-p_1+1}})+(e_j^R, 0)$ satisfies the inequality at equality. See, for example, Figure 26(a). This case is similar to Cases 7bi and 9bi.

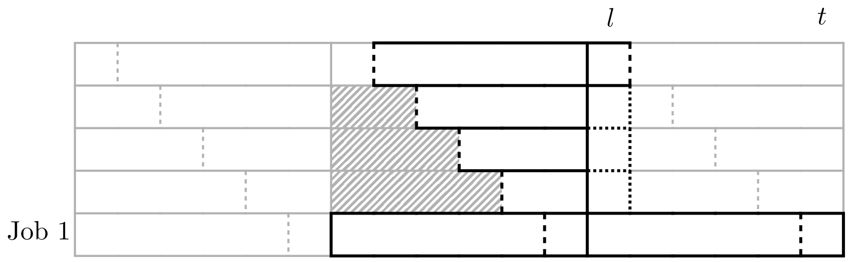
ii. If the half-bucket (j, b_j^R, k_j^R) contains more than one period, that is, $|\mathcal{T}_{jb_j^R k_j^R}| \geq 2$, then the variable pair (z_j^R, u_j^R) has coefficients $(\mu_j^{zR}, \mu_j^{uR}) = (0, 0)$ and the feasible partial schedules

$$(z_j^R, u_j^R) = \{(e_{1,b_{l-p_1+1},k_{l-p_1+1}}, (1-k_{l-p_1+1}\pi_1)e_{1,b_{l-p_1+1},k_{l-p_1+1}})+(e_j^R, (1-k_j^R\pi_j)e_j^R), (e_{1,b_{l-p_1+1},k_{l-p_1+1}}, (1-k_{l-p_1+1}\pi_1)e_{1,b_{l-p_1+1},k_{l-p_1+1}})+(e_j^R, (1-k_j^R\pi_j-1/\Delta)e_j^R)\}$$

satisfy the inequality at equality. See, for example, Figure 26(b). This case is similar to Cases 7bii and 9bii.

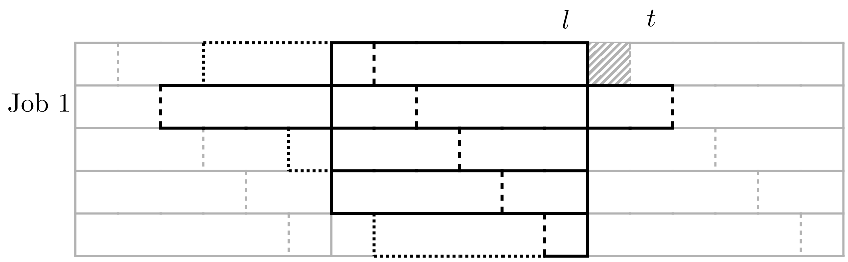


(a) Case 5ai example

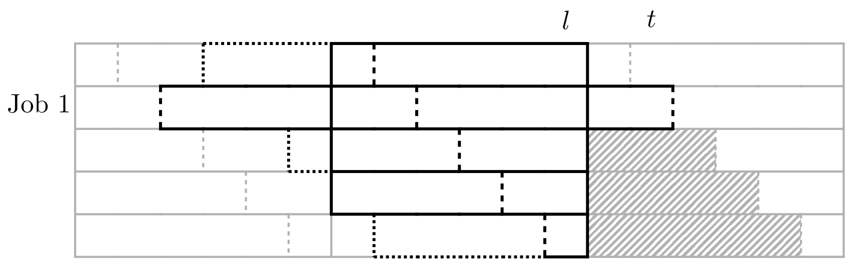


(b) Case 5aii example

Figure 25: Case 5a examples. The shaded half-buckets indicate the possible half-buckets $(j, b_j^L, k_j^L) \in (\mathcal{J} \setminus (\{1\} \cup \mathcal{J}^L)) \times [1, B] \times K$.



(a) Case 5bi example



(b) Case 5bii example

Figure 26: Case 5b examples. The shaded half-buckets indicate the possible half-buckets $(j, b_j^R, k_j^R) \in (\mathcal{J} \setminus (\{1\} \cup \mathcal{J}^R)) \times [1, B] \times K$.

6. Consider a half-bucket $(1, b, k) \in \{1\} \times [1, B] \times K$ such that $(1, b, k) \notin \mathcal{H}_1$ and $(1, b, k) \neq (1, b_j^G, k_j^G)$ for either index $G \in \{L, R\}$.

   a. Suppose that the half-bucket $(1, b, k)$ precedes half-bucket $(1, b_1^L, k_1^L)$, that is, $(b - (1 - k)(1 - \pi_1))\Delta < (b_1^L - (1 - k_1^L \pi_1))\Delta + 1$.

      i. If the half-bucket $(1, b, k)$ contains only one period, that is, $|\mathcal{T}_{1bk}| = 1$, then the variable z_{1bk} has coefficient $\mu_{1bk}^z = 0$ and the feasible partial schedule $(z, u) = (e_{1bk}, 0) + (e_{jb_l1}, 1/\Delta e_{jb_l1})$ satisfies the inequality at equality. See, for example, Figure 27(a). This case is similar to Case 2bi.

      ii. If the half-bucket $(1, b, k)$ contains more than one period, that is, $|\mathcal{T}_{1bk}| \geq 2$, then the variable pair (z_{1bk}, u_{1bk}) has coefficients $(\mu_{1bk}^z, \mu_{1bk}^u) = (0, 0)$ and the feasible partial schedules

$$(z, u) = \{(e_{1bk}, ((1 - k_1^L)(1 - \pi_1) - 1/\Delta)e_{1bk}) + (e_{jb_l1}, 1/\Delta e_{jb_l1}),$$
$$(e_{1bk}, ((1 - k_1^L)(1 - \pi_1) - 2/\Delta)e_{1bk}) + (e_{jb_l1}, 1/\Delta e_{jb_l1})\}$$

      satisfy the inequality at equality. See, for example, Figure 27(b). This case is similar to Case 2bii.

   b. Suppose that the half-bucket $(1, b, k)$ follows half-bucket $(1, b_1^R, k_1^R)$, that is, $(b_1^R - (1 - k_1^R)(1 - \pi_1))\Delta < (b - (1 - k\pi_1))\Delta + 1$.

      i. If the half-bucket $(1, b, k)$ contains only one period, that is, $|\mathcal{T}_{1bk}| = 1$, then the variable z_{1bk} has coefficient $\mu_{11bk}^z = 0$ and there exists a job $j \in J \setminus \{1\}$ for which the feasible partial schedule $(z, u) = (e_{1bk}, 0) + (e_{j,b_t - P_j + 1,1}, (1 - \pi_j)e_{j,b_t - P_j + 1,1})$ satisfies the inequality at equality. See, for example, Figure 28(a). This case is similar to Case 3bi.

      ii. If the half-bucket $(1, b, k)$ contains more than one period, that is, $|\mathcal{T}_{1bk}| \geq 2$, then the variable pair (z_{1bk}, u_{1bk}) has coefficients $(\mu_{1bk}^z, \mu_{1bk}^u) = (0, 0)$ and there exists a job $j \in J \setminus \{1\}$ for which the feasible partial schedules

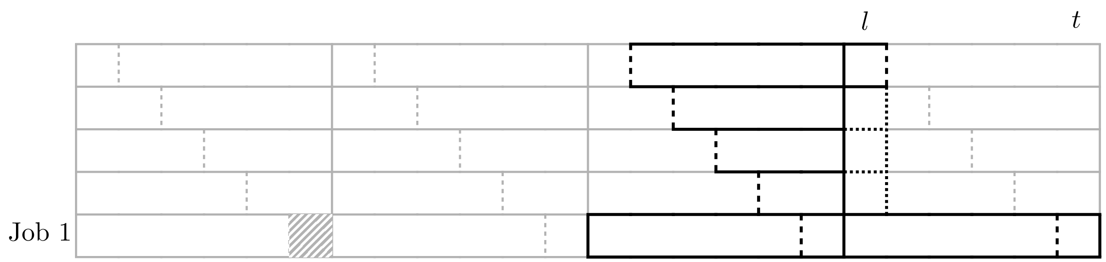
$$(z, u) = \{(e_{1bk}, (1 - k_1^R \pi_1)e_{1bk}) + (e_{j,b_t - P_j + 1,1}, (1 - \pi_j)e_{j,b_t - P_j + 1,1}),$$
$$(e_{1bk}, (1 - k_1^R \pi_1 - 1/\Delta)e_{1bk}) + (e_{j,b_t - P_j + 1,1}, (1 - \pi_j)e_{j,b_t - P_j + 1,1})\}$$

      satisfy the inequality at equality. See, for example, Figure 28(b). This case is similar to Case 3bii.

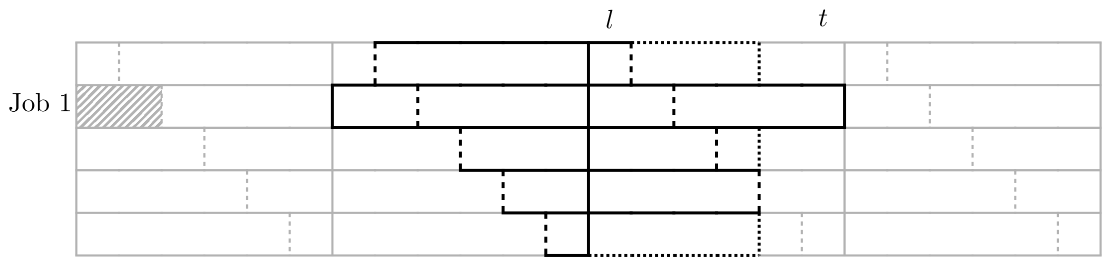
7. Consider a half-bucket $(j, b, k) \in (\mathcal{J} \setminus \{1\}) \times [1, B] \times K$ such that $(j, b, k) \notin \mathcal{H}_j$ and $(j, b, k) \neq (j, b_j^G, k_j^G)$ for either index $G \in \{L, R\}$.

   a. Suppose that the half-bucket (j, b, k) precedes half-bucket (j, b_j^L, k_j^L), that is, $(b - (1 - k)(1 - \pi_j))\Delta < (b_j^L - (1 - k_j^L \pi_j))\Delta + 1$.

      i. If the half-bucket (j, b, k) contains only one period, that is, $|\mathcal{T}_{jbk}| = 1$, then the variable z_{jbk} has coefficient $\mu_{jbk}^z = 0$ and the feasible partial schedule $(z, u) = (e_{1b_tk_t}, ((1 - k_t)(1 - \pi_1) + 1/\Delta)e_{1b_tk_t}) + (e_{jbk}, 0)$ satisfies the inequality at equality. See, for example, Figure 29(a). This case is similar to Case 5ai.

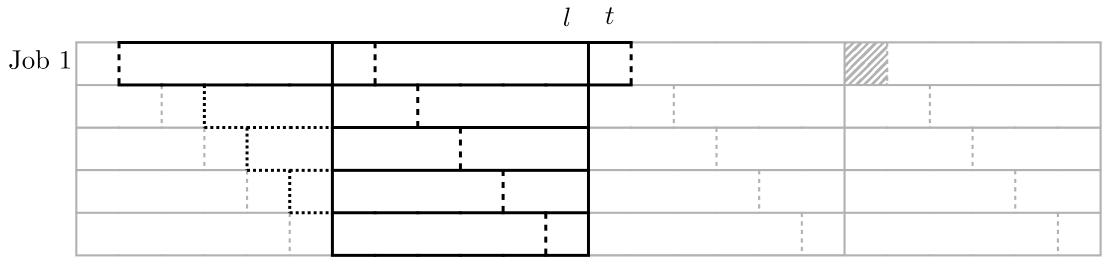


(a) Case 6ai example

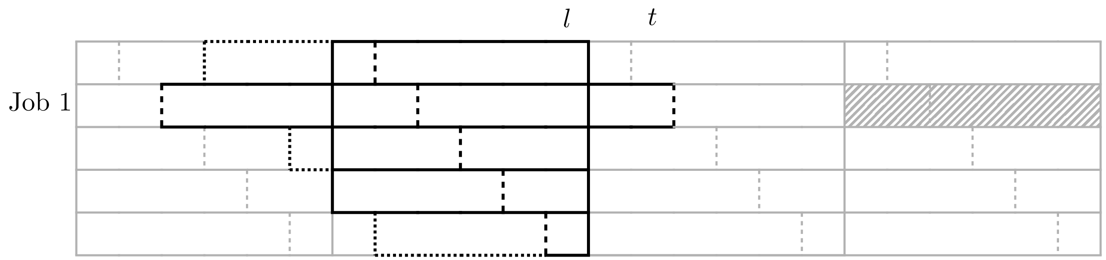


(b) Case 6aii example

Figure 27: Case 6a examples. The shaded half-buckets indicate the possible half-buckets $(1, b, k) \in \{1\} \times [1, B] \times K$ such that $(1, b, k) \notin \mathcal{H}_1$ and $(1, b, k) \neq (1, b_j^L, k_j^L)$.

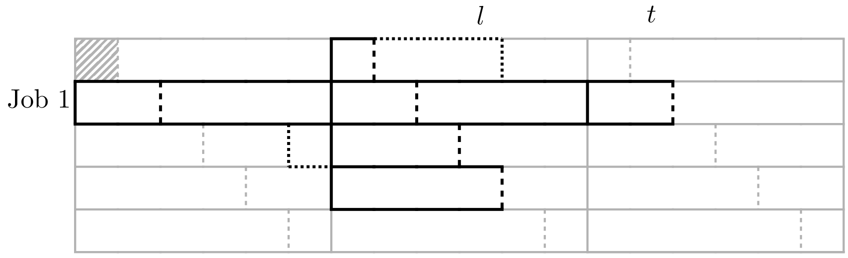


(a) Case 6bi example

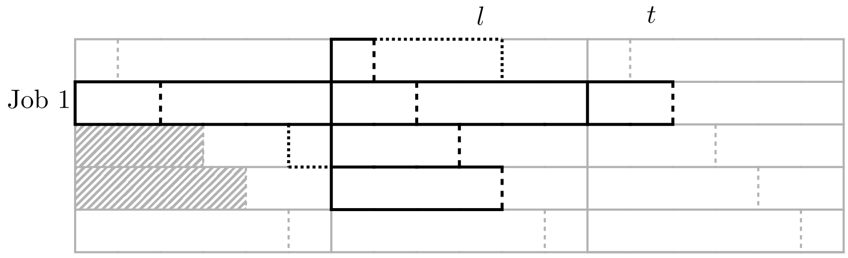


(b) Case 6bii example

Figure 28: Case 6b examples. The shaded half-buckets indicate the possible half-buckets $(1, b, k) \in \{1\} \times [1, B] \times K$ such that $(1, b, k) \notin \mathcal{H}_1$ and $(1, b, k) \neq (1, b_j^R, k_j^R)$.



(a) Case 7ai example



(b) Case 7aii example

Figure 29: Case 7a examples. The shaded half-buckets indicate the possible half-buckets $(j, b, k) \in (\mathcal{J} \setminus \{1\}) \times [1, B] \times K$ such that $(j, b, k) \notin \mathcal{H}_j$ and $(j, b, k) \neq (j, b_j^L, k_j^L)$.

ii. If the half-bucket (j, b, k) contains more than one period, that is, $|\mathcal{T}_{jbk}| \geq 2$, then the variable pair (z_{jbk}, u_{jbk}) has coefficients $(\mu^z_{jbk}, \mu^u_{jbk}) = (0, 0)$ and the feasible partial schedules

$$(z, u) = \{(e_{1b_tk_t}, ((1-k_t)(1-\pi_1)+1/\Delta)e_{1b_tk_t})+(e_{jbk}, ((1-k)(1-\pi_j)+1/\Delta)e_{jbk}),$$
$$(e_{1b_tk_t}, ((1-k_t)(1-\pi_1)+1/\Delta)e_{1b_tk_t})+(e_{jbk}, ((1-k)(1-\pi_j)+2/\Delta)e_{jbk})\}$$

satisfy the inequality at equality. See, for example, Figure 29(b). This case is similar to Case 5aii.

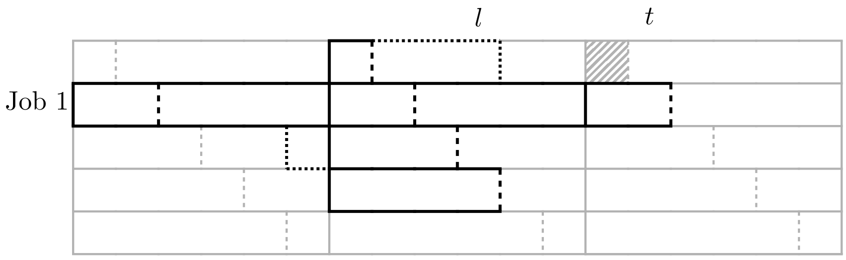
b. Suppose that the half-bucket (j, b, k) follows half-bucket (j, b_j^R, k_j^R), that is, $(b_j^R - (1 - k_j^R)(1 - \pi_j))\Delta < (b - (1 - k\pi_j))\Delta + 1$.

i. If the half-bucket (j, b, k) contains only one period, that is, $|\mathcal{T}_{jbk}| = 1$, then variable z_{jbk} has coefficient $\mu^z_{jbk} = 0$ and the feasible partial schedule $(z, u) = (e_{1,b_{l-p_1+1},k_{l-p_1+1}}, (1-k_{l-p_1+1}\pi_1)e_{1,b_{l-p_1+1},k_{l-p_1+1}})+ (e_{jbk}, 0)$ satisfies the inequality at equality. See, for example, Figure 30(a). This case is similar to Case 5bi.

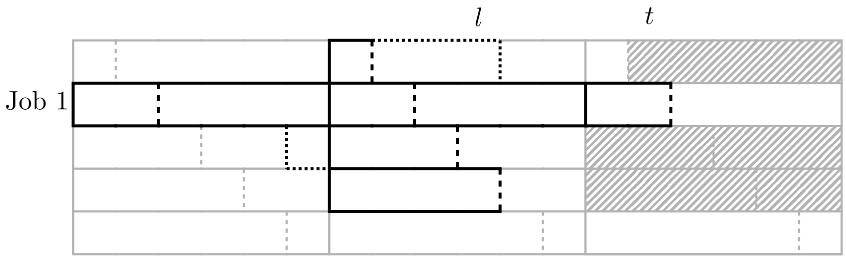
ii. If the half-bucket (j, b, k) contains more than one period, that is, $|\mathcal{T}_{jbk}| \geq 2$, then the variable pair (z_{jbk}, u_{jbk}) has coefficients $(\mu^z_{jbk}, \mu^u_{jbk}) = (0, 0)$ and the feasible partial schedules

$$(z, u) = \{(e_{1,b_{l-p_1+1},k_{l-p_1+1}}, (1-k_{l-p_1+1}\pi_1)e_{1,b_{l-p_1+1},k_{l-p_1+1}})+(e_{jbk}, (1-k\pi_j)e_{jbk}),$$
$$(e_{1,b_{l-p_1+1},k_{l-p_1+1}}, (1-k_{l-p_1+1}\pi_1)e_{1,b_{l-p_1+1},k_{l-p_1+1}})+(e_{jbk}, (1-k\pi_j-1/\Delta)e_{jbk})\}$$

satisfy the inequality at equality. See, for example, Figure 30(b). This case is similar to Case 5bii.



(a) Case 7bi example



(b) Case 7bii example

Figure 30: Case 7b examples. The shaded half-buckets indicate possible half-buckets $(j, b, k) \in (\mathcal{J} \setminus \{1\}) \times [1, B] \times K$ such that $(j, b, k) \notin \mathcal{H}_j$ and $(j, b, k) \neq (j, b_j^R, k_j^R)$.

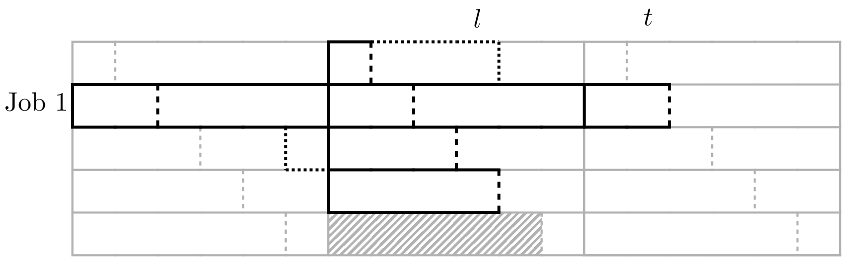


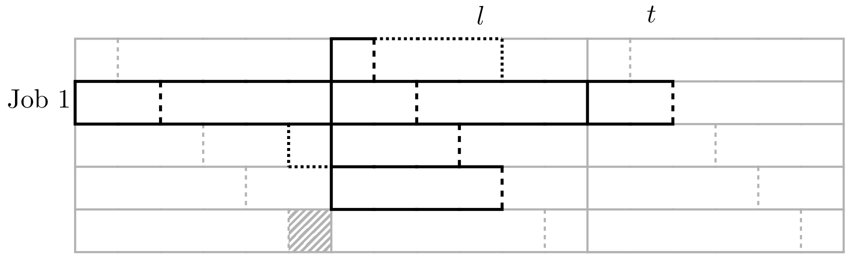
Figure 31: Case 8 example. Shaded half-buckets indicate the possible half-buckets $(j, b, k) \in (J \setminus \mathcal{J}) \times [1, B] \times K$ such that $\mathcal{T}_{jbk} \cap \mathcal{S}_j \neq \emptyset$.

8. Consider a half-bucket $(j, b, k) \in (J \setminus \mathcal{J}) \times [1, B] \times K$ such that $\mathcal{T}_{jbk} \cap \mathcal{S}_j \neq \emptyset$. Note that for $J \setminus \mathcal{J} \neq \emptyset$ it must be the case that the indices $k_{l-p_1+1} = k_t = 0$. Consequently, the bucket $b_t - P_j + 1 - k = b_l$ and the half-bucket (j, b, k) contains at least three periods, that is, $|\mathcal{T}_{jbk}| \geq 3$. The variable pair (z_{jbk}, u_{jbk}) has coefficients $(\mu^z_{jbk}, \mu^u_{jbk}) = (0, 0)$ and the feasible partial schedules

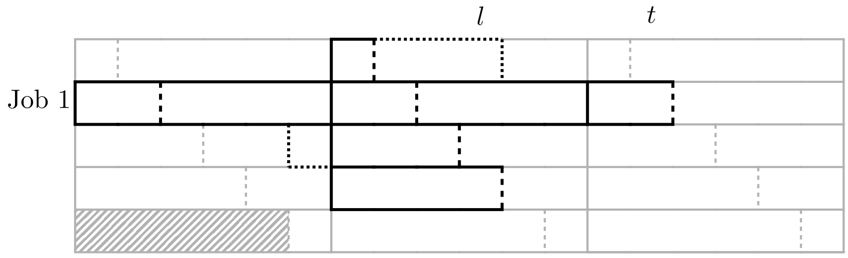
$$(z, u) = \{(e_{1,b_l-P_1+1,0}, e_{1,b_l-P_1+1,0}) + (e_{jbk}, ((1-k)(1-\pi_j) + 1/\Delta)e_{jbk}),$$
$$(e_{1b_t0}, (1 - \pi_1 + 1/\Delta)e_{1b_t0}) + (e_{jbk}, (1 - k\pi_j)e_{jbk})\}$$

satisfy the inequality at equality. See, for example, Figure 31.

9. Consider a half-bucket $(j, b, k) \in (J \setminus \mathcal{J}) \times [1, B] \times K$ such that $\mathcal{T}_{jbk} \cap \mathcal{S}_j = \emptyset$. Note that for $J \setminus \mathcal{J} \neq \emptyset$ it must be the case that the indices $k_{l-p_1+1} = k_t = 0$.



(a) Case 9ai example



(b) Case 9aii example

Figure 32: Case 9a examples. The shaded half-buckets indicate the possible half-buckets $(j, b, k) \in (J \setminus \mathcal{J}) \times [1, B] \times K$ that precede the half-bucket (j, b', k') from Case 8.

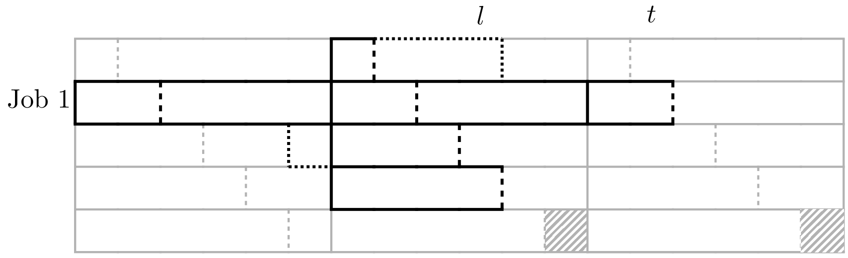
Suppose that the half-bucket $(j, b', k') \in (J \setminus \mathcal{J}) \times [1, B] \times K$ is the unique half-bucket from Case 8 where $\mathcal{T}_{jb'k'} \cap \mathcal{S}_j \neq \emptyset$.

a. Suppose that the half-bucket (j, b, k) precedes half-bucket (j, b', k'), that is, $(b - (1 - k)(1 - \pi_j))\Delta < (b' - (1 - k'\pi_j))\Delta + 1$.
   i. If the half-bucket (j, b, k) contains only one period, that is, $|\mathcal{T}_{jbk}| = 1$, then the variable z_{jbk} has coefficient $\mu^z_{jbk} = 0$ and the feasible partial schedule $(z, u) = (e_{1b_t k_t}, ((1 - k_t)(1 - \pi_1) + 1/\Delta)e_{1b_t k_t}) + (e_{jbk}, 0)$ satisfies the inequality at equality. See, for example, Figure 32(a). This case is similar to Cases 5ai and 7ai.
   ii. If the half-bucket (j, b, k) contains more than one period, that is, $|\mathcal{T}_{jbk}| \geq 2$, then the variable pair (z_{jbk}, u_{jbk}) has coefficients $(\mu^z_{jbk}, \mu^u_{jbk}) = (0, 0)$ and the feasible partial schedules

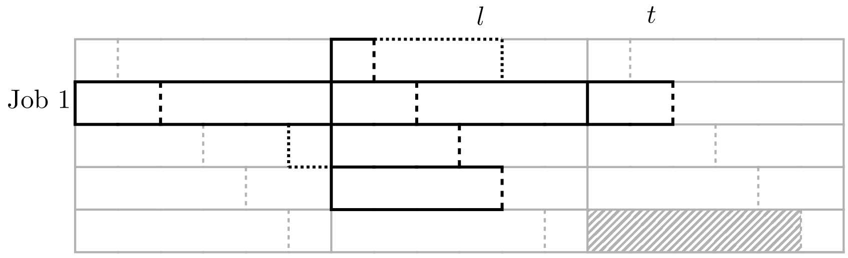
$$(z, u) = \{(e_{1b_t k_t}, ((1-k_t)(1-\pi_1)+1/\Delta)e_{1b_t k_t})+(e_{jbk}, ((1-k)(1-\pi_j)+1/\Delta)e_{jbk}),$$
$$(e_{1b_t k_t}, ((1-k_t)(1-\pi_1)+1/\Delta)e_{1b_t k_t})+(e_{jbk}, ((1-k)(1-\pi_j)+2/\Delta)e_{jbk})\}$$

   satisfy the inequality at equality. See, for example, Figure 32(b). This case is similar to Cases 5aii and 7aii.

b. Suppose that the half-bucket (j, b, k) follows half-bucket (j, b', k'), that is, $(b' - (1 - k')(1 - \pi_j))\Delta < (b - (1 - k\pi_j))\Delta + 1$.
   i. If the half-bucket (j, b, k) contains only one period, that is, $|\mathcal{T}_{jbk}| = 1$, then variable z_{jbk} has coefficient $\mu^z_{jbk} = 0$ and the feasible partial schedule $(z, u) = (e_{1,b_{l-p_1+1},k_{l-p_1+1}}, (1-k_{l-p_1+1}\pi_1)e_{1,b_{l-p_1+1},k_{l-p_1+1}})+$ $(e_{jbk}, 0)$ satisfies the inequality at equality. See, for example, Figure 33(a). This case is similar to Cases 5bi and 7bi.



(a) Case 9bi example



(b) Case 9bii example

Figure 33: Case 9b examples. The shaded half-buckets indicate the possible half-buckets $(j, b, k) \in (J \setminus \mathcal{J}) \times [1, B] \times K$ that follow the half-bucket (j, b', k') from Case 8.

ii. If the half-bucket (j, b, k) contains more than one period, that is, $|\mathcal{T}_{jbk}| \geq 2$, then the variable pair (z_{jbk}, u_{jbk}) has coefficients $(\mu^z_{jbk}, \mu^u_{jbk}) = (0, 0)$ and the feasible partial schedules

$$(z, u) = \{(e_{1,b_{l-p_1+1},k_{l-p_1+1}}, (1-k_{l-p_1+1}\pi_1)e_{1,b_{l-p_1+1},k_{l-p_1+1}}) + (e_{jbk}, (1-k\pi_j)e_{jbk}),$$
$$(e_{1,b_{l-p_1+1},k_{l-p_1+1}}, (1-k_{l-p_1+1}\pi_1)e_{1,b_{l-p_1+1},k_{l-p_1+1}}) + (e_{jbk}, (1-k\pi_j - 1/\Delta)e_{jbk})\}$$

satisfy the inequality at equality. See, for example, Figure 33(b). This case is similar to Cases 5bii and 7bii.

□

For ease of exposition we had assumed that the sets $\mathcal{S}_j \neq \emptyset$ for all jobs $j \in J$. If this is not true then an additional case needs to be considered in the above proof. Specifically, suppose that $\mathcal{T}_{jbk} \cap \mathcal{S}_j = \emptyset$ for all half-buckets $(j, b, k) \in (J \setminus \mathcal{J}) \times [1, B] \times K$. The proof of this case is similar to Cases 5, 7, and 9.