Analytical formulas for calculating the extremal ranks and inertias of $A + BXB^*$ when X is a fixed-rank Hermitian matrix

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Abstract. The rank of a matrix and the inertia of a square matrix are two of the most generic concepts in matrix theory for describing the dimension of the row/column vector space and the sign distribution of the eigenvalues of the matrix. Matrix rank and inertia optimization problems are a class of discontinuous optimization problems, in which decision variables are matrices running over certain matrix sets, while the ranks and inertias of the variable matrices are taken as integer-valued objective functions. In this paper, we first establish several groups of explicit formulas for calculating the maximal and minimal ranks and inertias of matrix expression $A+X$ subject to a Hermitian matrix X that satisfies a fixed-rank and semi-definiteness restrictions by using some discrete and matrix decomposition methods. We then derive formulas for calculating the maximal and minimal ranks and inertias of matrix expression $A+BXB^*$ subject to a Hermitian matrix X that satisfies a fixed-rank and semi-definiteness restrictions and use the formulas obtained to characterize behaviors of $A + BXB^*$.

Key words: Hermitian matrix; matrix-valued function; rank; inertia; maximization; minimization; Moore–Penrose inverse; equality; inequality; Löwner partial ordering

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1 Introduction

A matrix-valued function for complex matrices is a map between matrix spaces, which can generally be written as

$$Y = f(X) \quad \text{for} \quad Y \in \mathbb{C}^{m\times n} \quad \text{and} \quad X \in \mathbb{C}^{p\times q},$$

or briefly,

$$f : \mathbb{C}^{m\times n} \to \mathbb{C}^{p\times q},$$

where $\mathbb{C}^{m\times n}$ and $\mathbb{C}^{p\times q}$ are two two complex matrix spaces. As usual, linear matrix-valued functions, as ordinary representatives of all matrix-valued functions, are extensively studied from theoretical and applied points of view.

Throughout this paper,

$\mathbb{C}^{m\times n}$ stands for the set of all $m \times n$ complex matrices;

$\mathbb{C}^{m}_{\mathrm{H}}$ stands for the set of all $m \times m$ complex Hermitian matrices;

A^*, A^T, $r(A)$ and $\mathscr{R}(A)$ stand for the conjugate transpose, transpose, rank and range (column space) of a matrix $A \in \mathbb{C}^{m\times n}$, respectively;

I_m denotes the identity matrix of order m;

$[\,A,\, B\,]$ denotes a row block matrix consisting of A and B;

the Moore–Penrose inverse of $A \in \mathbb{C}^{m\times n}$, denoted by $A^{\dagger}$, is defined to be the unique solution X satisfying the four matrix equations $AXA = A$, $XAX = X$, $(AX)^* = AX$ and $(XA)^* = XA$;

the symbols E_A and F_A stand for $E_A = I_m - AA^{\dagger}$ and $F_A = I_n - A^{\dagger}A$, their ranks are $r(E_A) = m - r(A)$ and $r(F_A) = n - r(A)$;

$i_+(A)$ and $i_-(A)$, called the partial inertia of $A \in \mathbb{C}^{m}_{\mathrm{H}}$, are defined to be the numbers of the positive and negative eigenvalues of A counted with multiplicities, respectively;

$A \succ 0$ ($A \succcurlyeq 0$, $\prec 0$, $\preccurlyeq 0$) means that A is Hermitian positive definite (positive semi-definite, negative definite, negative semi-definite);

two $A, B \in \mathbb{C}^{m}_{\mathrm{H}}$ are said to satisfy the inequality $A \succcurlyeq B$ ($A \succ B$) in the Löwner partial ordering if $A - B$ is positive definite (positive semi-definite).

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Like the simplest linear expression $a + bx$ in elementary mathematics, people can define various linear expressions for matrices and their operations, which we call linear matrix-valued functions (LMFs). It is easy to see that one of the generic linear expressions with symmetric patterns is given by

$$\phi(X) = A + BXB^*, \tag{1.1}$$

where $A \in \mathbb{C}_{\rm H}^{m}$ and $B \in \mathbb{C}^{m\times n}$ are given, and $X \in \mathbb{C}_{\rm H}^{n}$ is a variable matrix. We treat (1.1) as an LMF:

$$\phi : X \in \mathbb{C}_{\rm H}^{n} \to \mathbb{C}_{\rm H}^{m}. \tag{1.2}$$

This basic LMF and its variations were extensively studied in the literature from theoretical and applied points of view, and many results on behaviors of (1.1) were posed and were obtained, for instance,

(i) Expansion formulas for calculating the (global extremal) ranks and inertias of $\phi(X)$ when X running over $\mathbb{C}_{\rm H}^{n}$ [6, 13, 21].

(ii) Nonsingularity, positive definiteness, rank and inertia invariance, etc., of $\phi(X)$ [13, 21].

(iii) Canonical forms of $\phi(X)$ under generalized singular value decompositions and their algebraic properties [6].

(iv) Solutions and least-squares solutions of the matrix equation $\phi(X) = 0$ and their algebraic properties [5, 6, 9, 16, 18, 20].

(v) Solutions of the matrix inequalities $\phi(X) \succ (\succcurlyeq, \prec, \preccurlyeq)\, 0$ and their properties [13, 17].

(vi) Formulas for calculating the extremal ranks and inertias of $\phi(X)$ under $X \in \mathbb{C}_{\rm H}^{n}$, $r(X) \leqslant q$ and/or $\pm X \succcurlyeq 0$ [13, 21].

(vii) Formulas for calculating the extremal ranks and inertias of $\phi(X)$ subject to the Hermitian solution of a consistent matrix equation $CXC^* = D$ [7].

(viii) Formulas for calculating the extremal ranks and inertias of $A + BC^-B^*$, where C^- is a Hermitian generalized inverse of a Hermitian matrix C, [7, 19].

In this paper, we take the rank and inertia of $A + BXB^*$ as integer-valued objective functions, and study the following constrained optimization problems:

Problem 1.1 For the function in (1.1), solve the following constrained rank and inertia optimization problems

$$\text{maximize } r(A + BXB^*) \quad \text{s.t. } X \in \mathbb{C}_{\rm H}^{n} \text{ and } r(X) = q, \tag{1.3}$$
$$\text{minimize } r(A + BXB^*) \quad \text{s.t. } X \in \mathbb{C}_{\rm H}^{n} \text{ and } r(X) = q, \tag{1.4}$$
$$\text{maximize } i_{\pm}(A + BXB^*) \quad \text{s.t. } X \in \mathbb{C}_{\rm H}^{n} \text{ and } r(X) = q, \tag{1.5}$$
$$\text{minimize } i_{\pm}(A + BXB^*) \quad \text{s.t. } X \in \mathbb{C}_{\rm H}^{n} \text{ and } r(X) = q, \tag{1.6}$$
$$\text{maximize } r(A + BXB^*) \quad \text{s.t. } X \in \mathbb{C}_{\rm H}^{n},\ \pm X \succcurlyeq 0 \text{ and } r(X) = q, \tag{1.7}$$
$$\text{minimize } r(A + BXB^*) \quad \text{s.t. } X \in \mathbb{C}_{\rm H}^{n},\ \pm X \succcurlyeq 0 \text{ and } r(X) = q, \tag{1.8}$$
$$\text{maximize } i_{\pm}(A + BXB^*) \quad \text{s.t. } X \in \mathbb{C}_{\rm H}^{n},\ \pm X \succcurlyeq 0 \text{ and } r(X) = q, \tag{1.9}$$
$$\text{minimize } i_{\pm}(A + BXB^*) \quad \text{s.t. } X \in \mathbb{C}_{\rm H}^{n},\ \pm X \succcurlyeq 0 \text{ and } r(X) = q. \tag{1.10}$$

Problem 1.2 For the function in (1.1), solve the following constrained rank and inertia optimization problems

$$\text{maximize } r(A + BXB^*) \quad \text{s.t. } X \in \mathbb{C}_{\rm H}^{n} \text{ and } p \leqslant r(X) \leqslant q, \tag{1.11}$$
$$\text{minimize } r(A + BXB^*) \quad \text{s.t. } X \in \mathbb{C}_{\rm H}^{n} \text{ and } p \leqslant r(X) \leqslant q \tag{1.12}$$
$$\text{maximize } i_{\pm}(A + BXB^*) \quad \text{s.t. } X \in \mathbb{C}_{\rm H}^{n} \text{ and } p \leqslant r(X) \leqslant q, \tag{1.13}$$
$$\text{minimize } i_{\pm}(A + BXB^*) \quad \text{s.t. } X \in \mathbb{C}_{\rm H}^{n} \text{ and } p \leqslant r(X) \leqslant q, \tag{1.14}$$
$$\text{maximize } r(A + BXB^*) \quad \text{s.t. } X \in \mathbb{C}_{\rm H}^{n},\ \pm X \succcurlyeq 0 \text{ and } p \leqslant r(X) \leqslant q, \tag{1.15}$$
$$\text{minimize } r(A + BXB^*) \quad \text{s.t. } X \in \mathbb{C}_{\rm H}^{n},\ \pm X \succcurlyeq 0 \text{ and } p \leqslant r(X) \leqslant q, \tag{1.16}$$
$$\text{maximize } i_{\pm}(A + BXB^*) \quad \text{s.t. } X \in \mathbb{C}_{\rm H}^{n},\ \pm X \succcurlyeq 0 \text{ and } p \leqslant r(X) \leqslant q, \tag{1.17}$$
$$\text{minimize } i_{\pm}(A + BXB^*) \quad \text{s.t. } X \in \mathbb{C}_{\rm H}^{n},\ \pm X \succcurlyeq 0 \text{ and } p \leqslant r(X) \leqslant q. \tag{1.18}$$

The rank and inertia of a Hermitian matrix are two generic concepts in matrix theory for describing the dimension of the row/column vector space and the sign distribution of the eigenvalues of the matrix, which are well understood and are easy to compute by the well-known elementary or congruent matrix operations and play an essential role in characterizing algebraic properties of matrices. These two integer indices occur only in finite-dimensional algebras and are not replaceable and cannot be approximated by other continuous quantities. Because the rank and inertia of a matrix are always finite nonnegative integers less than or equal to the dimensions of the matrix, it is not hard to give upper and lower bounds for ranks and inertias of matrices, and the global maximal and minimal values of the objective rank and inertia functions always exist, no matter what the decision domain $\mathcal{S}$ is given. Also, due to the integer feature of rank and inertia, inexact or approximate values of maximal and minimal ranks and inertias are less valuable, so that no approximation methods can be used to find exact values of the maximal and minimal possible ranks and inertias of a matrix-valued function. This fact means that solving methods of matrix rank and inertia optimization problems are not consistent with any of the ordinary continuous and discrete problems in optimization theory. It has been known that matrix rank and inertia optimization problems are NP-hard in general due to the discontinuity and combinational nature of rank and inertia of a matrix and the algebraic structure of $\mathcal{S}$, but only a few has analytical solutions.

Generally speaking, matrix rank-optimization problems are a class of discontinuous optimization problems, in which the decision variables are matrices running over certain matrix sets, while the ranks of the variable matrices are taken as integer-valued objective functions. So that analytical formulas for calculating the maximal and minimal ranks of $A - X$ cannot be derived by numerical approximation methods. This fact means that solving methods of matrix rank optimization problems are not consistent with any of the ordinary continuous and discrete problems in optimization theory, so that we cannot apply various common methods of solving continuous optimization problems, such as the well-known differential and Lagrangian methods, to approach these constrained optimization problems. Instead, we can only find the exact maximal and minimal ranks through pure algebraic operations of matrices. It has been known that matrix rank-optimization problems are NP-hard in general due to the discontinuity and combinational nature of rank of a matrix and the algebraic structure of $\mathcal{S}$. Many new researches were conducted on this kind of matrix rank-optimization problems from theory and applied points of view in the past decades; see, e.g., `http://perception.csl.uiuc.edu/matrix-rank/home.html` and the references therein. Because the rank of a matrix can only take finite integers between 0 and the dimensions of the matrix, it is really expected to establish certain analytical formulas for calculating the maximal and minimal ranks for curiosity.

Also, note that for any two integers p and q satisfying $0 \leqslant p \leqslant q \leqslant m$, the following decompositions of the generalized Stiefel manifolds

$$\begin{aligned}
&\{ X \in \mathbb{C}_{\rm H}^{m} \mid p \leqslant r(X) \leqslant q\} \\
&= \{ X \in \mathbb{C}_{\rm H}^{m} \mid r(X) = p\} \cup \{ X \in \mathbb{C}_{\rm H}^{m} \mid r(X) = p+1\} \cup \cdots \cup \{ X \in \mathbb{C}_{\rm H}^{m} \mid r(X) = q\}, \qquad (1.19)\\
&\{ 0 \preccurlyeq X \in \mathbb{C}_{\rm H}^{m} \mid p \leqslant r(X) \leqslant q\} \\
&= \{ 0 \preccurlyeq X \in \mathbb{C}_{\rm H}^{m} \mid r(X) = p\} \cup \{ 0 \preccurlyeq X \in \mathbb{C}_{\rm H}^{m} \mid r(X) = p+1\} \cup \cdots \cup \{ 0 \preccurlyeq X \in \mathbb{C}_{\rm H}^{m} \mid r(X) = q\} \qquad (1.20)
\end{aligned}$$

hold. So that, once Problem 1.1 is solved, solutions of Problem 1.2 can be obtained consequently from the above matrix set decompositions. The constrained optimization problems listed in (1.3)–(1.18) consist of determining the global maximal and minimal ranks and inertias of (1.1), and finding the constrained variable matrix X such that the corresponding (1.1) attains the extremal ranks and inertias. Note that the ranks and inertias of BXB^* and $A + BXB^*$ may vary with respect to the choice of X, which are neither continuous or convex. Also note that the Hermitian matrix X with $r(X) = q$ is not unique, which can be characterized by the canonical decompositions of X under congruence transformation:

$$X = U{\rm diag}\{ I_s, \ -I_{q-s}, \ 0\}U^*, \qquad (1.21)$$

where U is any nonsingular matrix, and $0 \leqslant s \leqslant q$. This canonical decomposition shows that a Hermitian matrix X with rank q is characterized by both a variable integer s with $0 \leqslant s \leqslant q$ and an arbitrary nonsingular matrix U. So that both BXB^* and $A + BXB^*$ depend on the choice of both s and U, and the variation of the rank and inertia of (1.2) with respect to a fixed-rank Hermitian matrix X is quite complicated, while (1.3)–(1.18) can be classified as some special types of constrained integer optimization problems. Int this case, we can only find the global maximal and minimal ranks and inertias through pure algebraic operations of matrices. Although the task of establishing explicit formulas for calculating the extremal ranks and inertias in (1.3)–(1.18) are challenging, fortunately, we now are able to solve (1.3)–(1.18) by using many new known results on ranks and inertias of matrices and some tricky matrix operations.

2 Preliminary results

The following are some known results on ranks and inertias of matrices, which will be used in the latter part of this paper for solving the previous problems.

Lemma 2.1 ([13]) *Let $\mathcal{S}$ be a set consisting of matrices over $\mathbb{C}^{m\times n}$, and let $\mathcal{H}$ be a set consisting of Hermitian matrices over $\mathbb{C}_{\rm H}^{m}$. Then,*

(a) *Under $m = n$, $\mathcal{S}$ has a nonsingular matrix if and only if* $\max_{X\in\mathcal{S}} r(X) = m$.

(b) *Under $m = n$, all $X \in \mathcal{S}$ are nonsingular if and only if* $\min_{X\in\mathcal{S}} r(X) = m$.

(c) $0 \in \mathcal{S}$ *if and only if* $\min_{X\in\mathcal{S}} r(X) = 0$.

(d) $\mathcal{S} = \{0\}$ *if and only if* $\max_{X\in\mathcal{S}} r(X) = 0$.

(e) *$\mathcal{H}$ has a matrix $X \succ 0$ ($X \prec 0$) if and only if* $\max_{X\in\mathcal{H}} i_{+}(X) = m$ ($\max_{X\in\mathcal{H}} i_{-}(X) = m$).

(f) *All $X \in \mathcal{H}$ satisfy $X \succ 0$ ($X \prec 0$) if and only if* $\min_{X\in\mathcal{H}} i_{+}(X) = m$ ($\min_{X\in\mathcal{H}} i_{-}(X) = m$).

(g) *$\mathcal{H}$ has a matrix $X \succcurlyeq 0$ ($X \preccurlyeq 0$) if and only if* $\min_{X\in\mathcal{H}} i_{-}(X) = 0$ ($\min_{X\in\mathcal{H}} i_{+}(X) = 0$).

(h) *All $X \in \mathcal{H}$ satisfies $X \succcurlyeq 0$ ($X \preccurlyeq 0$) if and only if* $\max_{X\in\mathcal{H}} i_{-}(X) = 0$ ($\max_{X\in\mathcal{H}} i_{+}(\,X\,) = 0$).

This lemma indicates that rank and inertia of matrices can be used quantitative tools to characterize some fundamental algebraic properties of a given matrix set. In particular, whether a given matrix-valued function is semi-definite everywhere is ubiquitous in matrix theory and applications. Lemma 2.1(e)–(h) assert that if certain explicit formulas for calculating the global maximal and minimal inertias of Hermitian matrix-valued functions are established, we can use them, as demonstrated in Sections 2–7 below, to derive explicit necessary and sufficient conditions for the matrix-valued functions to be definite or semi-definite. In addition, we are able to use these inertia formulas to establish various matrix inequalities in the Löwner partial ordering, and to solve many matrix optimization problems in the Löwner partial ordering.

Lemma 2.2 *Let $A,\, B \in \mathbb{C}_{\rm H}^{m}$. Then,*

$$r(\,A + B\,) \leqslant r(A) + r(B), \tag{2.1}$$
$$i_{\pm}(\,A + B\,) \leqslant i_{\pm}(A) + i_{\pm}(B), \tag{2.2}$$
$$r(\,A + B\,) \geqslant r(A) - r(B), \tag{2.3}$$
$$i_{\pm}(\,A + B\,) \geqslant i_{\pm}(A) - i_{\mp}(B). \tag{2.4}$$

If $B \succcurlyeq 0$, then

$$r(\,A + B\,) \geqslant i_{+}(\,A + B\,) \geqslant i_{+}(A), \tag{2.5}$$
$$r(\,A - B\,) \geqslant i_{-}(\,A - B\,) \geqslant i_{-}(A). \tag{2.6}$$

Eq. (2.1) is a well-known rank inequality in elementary linear algebra. Eq. (2.2) was given in [2, 11]. Applying (2.1) and (2.2) to $A = (\,A + B\,) + (-B)$ yields (2.3) and (2.4). Eqs. (2.5) and (2.6) follow from (2.4).

Lemma 2.3 ([13]) *Let $A \in \mathbb{C}_{\rm H}^{m}$, $B \in \mathbb{C}_{\rm H}^{n}$, $Q \in \mathbb{C}^{m\times n}$, and assume that $P \in \mathbb{C}^{m\times m}$ is nonsingular. Then,*

$$i_{\pm}(PAP^{*}) = i_{\pm}(A), \tag{2.7}$$
$$i_{\pm}(A^{\dagger}) = i_{\pm}(A), \tag{2.8}$$
$$i_{\pm}(\lambda A) = \begin{cases} i_{\pm}(A) & if\ \lambda > 0 \\ i_{\mp}(A) & if\ \lambda < 0 \end{cases}, \tag{2.9}$$
$$i_{\pm}\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = i_{\pm}(A) + i_{\pm}(B), \tag{2.10}$$
$$i_{+}\begin{bmatrix} 0 & Q \\ Q^{*} & 0 \end{bmatrix} = i_{-}\begin{bmatrix} 0 & Q \\ Q^{*} & 0 \end{bmatrix} = r(Q). \tag{2.11}$$

Lemma 2.4 ([10]) *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$ and $C \in \mathbb{C}^{l\times n}$. Then, the following rank expansion formulas hold*

$$r[\,A,\, B\,] = r(A) + r(E_{A}B) = r(B) + r(E_{B}A), \tag{2.12}$$
$$r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(CF_{A}) = r(C) + r(AF_{C}). \tag{2.13}$$

Lemma 2.5 ([13]) *Let* $A \in \mathbb{C}_{\rm H}^{m}$, $B \in \mathbb{C}^{m\times n}$, $D \in \mathbb{C}_{\rm H}^{n}$, *and let*

$$M_1 = \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} A & B \\ B^* & D \end{bmatrix}.$$

Then, the following expansion formulas hold

$$i_{\pm}(M_1) = r(B) + i_{\pm}(E_BAE_B), \qquad\qquad r(M_1) = 2r(B) + r(E_BAE_B), \tag{2.14}$$

$$i_{\pm}(M_2) = i_{\pm}(A) + i_{\pm}\begin{bmatrix} 0 & E_AB \\ B^*E_A & D - B^*A^{\dagger}B \end{bmatrix}, \quad r(M_2) = r(A) + r\begin{bmatrix} 0 & E_AB \\ B^*E_A & D - B^*A^{\dagger}B \end{bmatrix}. \tag{2.15}$$

In particular,

(a) *If* $A \succcurlyeq 0$, *then*

$$i_{+}(M_1) = r[\,A,\, B\,], \quad i_{-}(M_1) = r(B), \quad r(M_1) = r[\,A,\, B\,] + r(B). \tag{2.16}$$

(b) *If* $A \preccurlyeq 0$, *then*

$$i_{+}(M_1) = r(B), \quad i_{-}(M_1) = r[\,A,\, B\,], \quad r(M_1) = r[\,A,\, B\,] + r(B). \tag{2.17}$$

(c) *If* $\mathscr{R}(B) \subseteq \mathscr{R}(A)$, *then*

$$i_{\pm}(M_2) = i_{\pm}(A) + i_{\pm}(\,D - B^*A^{\dagger}B\,), \quad r(M_2) = r(A) + r(\,D - B^*A^{\dagger}B\,). \tag{2.18}$$

(d) $i_{\pm}(M_2) \geqslant i_{\pm}(A) + i_{\pm}(\,D - B^*A^{\dagger}B\,) \geqslant i_{\pm}(A)$.

(e) $i_{\pm}(M_1) = m \Leftrightarrow i_{\mp}(E_BAE_B) = 0$ *and* $r(E_BAE_B) = r(E_B)$.

(f) $i_{+}(M_2) = i_{+}(A) \Leftrightarrow \mathscr{R}(B) \subseteq \mathscr{R}(A)$ *and* $D - B^*A^{\dagger}B \preccurlyeq 0$.

(g) $i_{-}(M_2) = i_{-}(A) \Leftrightarrow \mathscr{R}(B) \subseteq \mathscr{R}(A)$ *and* $D - B^*A^{\dagger}B \succcurlyeq 0$.

(h) $M_2 \succcurlyeq 0 \Leftrightarrow A \succcurlyeq 0,\ \mathscr{R}(B) \subseteq \mathscr{R}(A)$ *and* $D - B^*A^{\dagger}B \succcurlyeq 0 \Leftrightarrow D \succcurlyeq 0,\ \mathscr{R}(B^*) \subseteq \mathscr{R}(D)$ *and* $A - BD^{\dagger}B^* \succcurlyeq 0$.

(i) $M_2 \succ 0 \Leftrightarrow A \succ 0$ *and* $D - B^*A^{-1}B \succ 0 \Leftrightarrow D \succ 0$ *and* $A - BD^{-1}B^* \succ 0$.

Some useful expansion formulas derived from (2.14) and (2.15) are

$$i_{\pm}(\,D - B^*A^{\dagger}B\,) = i_{\pm}\begin{bmatrix} A^*AA^* & A^*B \\ B^*A^* & D \end{bmatrix} - i_{\pm}(A), \tag{2.19}$$

$$i_{\pm}(\,D - B^*A^{\dagger}B\,) = i_{\pm}\begin{bmatrix} A & B \\ B^* & D \end{bmatrix} - i_{\pm}(A) \ \text{ if } \ \mathscr{R}(B) \subseteq \mathscr{R}(A), \tag{2.20}$$

$$i_{\pm}\begin{bmatrix} A & BF_P \\ F_PB^* & 0 \end{bmatrix} = i_{\pm}\begin{bmatrix} A & B & 0 \\ B^* & 0 & P^* \\ 0 & P & 0 \end{bmatrix} - r(P). \tag{2.21}$$

We shall use them to simplify the inertias of block Hermitian matrices involving Moore–Penrose inverses of matrices.

Lemma 2.6 *Let* $A \in \mathbb{C}_{\rm H}^{m}$ *be given, and* $X \in \mathbb{C}^{m\times n}$ *and* $Y \in \mathbb{C}_{\rm H}^{n}$ *be two variable matrices. Also define*

$$\phi(X,\, Y) = \begin{bmatrix} 0 & X \\ X^* & Y \end{bmatrix}. \tag{2.22}$$

Then,

(a) *The following equalities hold*

$$\max_{X\in\mathbb{C}^{m\times n},\, Y\in\mathbb{C}_{\rm H}^{n}} r[\phi(X,\, Y)] = \min\{\, m+n, \ \ 2n\,\}, \tag{2.23}$$

$$\min_{X\in\mathbb{C}^{m\times n},\, Y\in\mathbb{C}_{\rm H}^{n}} r[\phi(X,\, Y)] = 0, \tag{2.24}$$

$$\max_{X\in\mathbb{C}^{m\times n},\, Y\in\mathbb{C}_{\rm H}^{n}} i_{\pm}[\phi(X,\, Y)] = n, \tag{2.25}$$

$$\min_{X\in\mathbb{C}^{m\times n},\, Y\in\mathbb{C}_{\rm H}^{n}} i_{\pm}[\phi(X,\, Y)] = 0. \tag{2.26}$$

(b) *For any integer q with $0 \leqslant q \leqslant \min\{ m+n, \ \ 2n \}$, there exist $X \in \mathbb{C}^{m\times n}$ and $Y \in \mathbb{C}_{\rm H}^{n}$ such that*

$$r[\phi(X,\,Y)] = q. \tag{2.27}$$

(c) *For any integer q with $0 \leqslant q \leqslant n$, there exist $X_1,\, X_2 \in \mathbb{C}^{m\times n}$ and $Y_1,\, Y_2 \in \mathbb{C}_{\rm H}^{n}$ such that*

$$i_{+}[\phi(X_1,\,Y_1)] = q, \quad i_{-}[\phi(X_2,\,Y_2)] = q \tag{2.28}$$

hold, respectively.

(d) *The following equalities hold*

$$\max_{X\in\mathbb{C}^{m\times n},\, Y\succcurlyeq 0} r[\phi(X,\,Y)] = \min\{ m+n, \ \ 2n \}, \tag{2.29}$$

$$\min_{X\in\mathbb{C}^{m\times n},\, Y\succcurlyeq 0} r[\phi(X,\,Y)] = 0, \tag{2.30}$$

$$\max_{X\in\mathbb{C}^{m\times n},\, Y\succcurlyeq 0} i_{+}[\phi(X,\,Y)] = n, \tag{2.31}$$

$$\min_{X\in\mathbb{C}^{m\times n},\, Y\succcurlyeq 0} i_{+}[\phi(X,\,Y)] = 0, \tag{2.32}$$

$$\max_{X\in\mathbb{C}^{m\times n},\, Y\succcurlyeq 0} i_{-}[\phi(X,\,Y)] = \min\{ m, \ \ n \}, \tag{2.33}$$

$$\min_{X\in\mathbb{C}^{m\times n},\, Y\succcurlyeq 0} i_{-}[\phi(X,\,Y)] = 0. \tag{2.34}$$

(e) *For any integer q with $0 \leqslant q \leqslant \min\{ m+n, \ \ 2n \}$, there exist $X \in \mathbb{C}^{m\times n}$ and $0 \preccurlyeq Y \in \mathbb{C}_{\rm H}^{n}$ such that*

$$r[\phi(X,\,Y)] = q. \tag{2.35}$$

(f) *For any integer q with $0 \leqslant q \leqslant n$, there exist $X \in \mathbb{C}^{m\times n}$ and $0 \preccurlyeq Y \in \mathbb{C}_{\rm H}^{n}$ such that*

$$i_{+}[\phi(X,\,Y)] = q. \tag{2.36}$$

(g) *For any integer q with $0 \leqslant q \leqslant \min\{ m, \ n \}$, there exist $X \in \mathbb{C}^{m\times n}$ and $0 \preccurlyeq Y \in \mathbb{C}_{\rm H}^{n}$ such that*

$$i_{-}[\phi(X,\,Y)] = q. \tag{2.37}$$

Proof. Let $r(X) = \min\{m, \ n\}$ and $Y = I_n$. Then, we find from (2.16) that

$$r[M(X,\,Y)] = r[\,X,\,Y\,] + r(X) = n + r(X) = \min\{ m+n, \ \ 2n \},$$

establishing (2.23). Setting $X = 0$ and $Y = 0$ leads to (2.24). Setting $Y = I_n$, we find from (2.16) that

$$i_{+}[M(X,\,Y)] = r[\,X,\,Y\,] = n;$$

setting $Y = -I_n$, we find from (2.16) that

$$i_{-}[M(X,\,Y)] = r[\,X,\,Y\,] = n,$$

establishing (2.25). Set $X = 0$ and $Y = 0$ leads to $r[M(X,\,Y)] = i_{+}[M(X,\,Y)] = i_{-}[M(X,\,Y)] = 0$, establishing (2.26).

For any integer $0 \leqslant q \leqslant n$, setting $X = 0$ and $r(Y) = q$ leads to

$$r[\phi(X,\,Y)] = r(Y) = q;$$

for any integer $n < q \leqslant \min\{ m+n, \ \ 2n \}$, setting $r(X) = q - n$ and $Y = I_n$ and applying (2.16) lead to

$$r[M(X,\,Y)] = r[\,X,\,Y\,] + r(X) = n + (\,q - n\,) = q,$$

establishing (2.27).

For any integer $0 \leqslant q \leqslant n$, setting $X = 0$ and $Y_1 = {\rm diag}\{\,I_q,\,0\,\}$ and $Y_2 = {\rm diag}\{\,-I_q,\,0\,\}$ and applying (2.16) leads to

$$i_{+}[\phi(X,\,Y_1)] = r(Y_1) = q, \quad i_{-}[\phi(X,\,Y_2)] = r(Y_2) = q,$$

establishing (2.28).

Let $r(X) = \min\{m, \ n\}$ and $Y = I_n \succcurlyeq 0$. Then, we find from (2.16) that

$$r[M(X,\,Y)] = r[\,X,\,Y\,] + r(X) = n + r(X) = \min\{ m+n, \ \ 2n \},$$

establishing (2.29). Setting $X = 0$ and $Y = 0$ leads to (2.30). For any X and $Y = I_n \succcurlyeq 0$, we find from (2.16) that

$$i_+[M(X, Y)] = r[\, X, Y\,] = n,$$

establishing (2.31). Setting $X = 0$ and any $Y = 0$ leads to (2.32). Setting $r(X) = \min\{m,\ n\}$ and $Y \succcurlyeq 0$, we find from (2.16) that

$$i_-[M(X, Y)] = r(X) = \min\{m,\ n\},$$

establishing (2.33). Setting $X = 0$ and $Y = 0$ leads to (2.34).

For any integer $0 \leqslant q \leqslant n$, setting $X = 0$ and $Y = \mathrm{diag}\{\, I_q,\, 0\,\} \succcurlyeq 0$ leads to

$$r[\phi(X, Y)] = r(Y) = q;$$

for any integer $n < q \leqslant \min\{\, m + n,\ \ 2n\,\}$, setting $r(X) = q - n$ and $Y = I_n \succcurlyeq 0$ and applying (2.16) lead to

$$r[M(X, Y)] = r[\, X, Y\,] + r(X) = n + (\, q - n\,) = q,$$

establishing (2.35).

For any integer $0 \leqslant q \leqslant n$, setting $X = 0$ and $Y = \mathrm{diag}\{\, I_q,\, 0\,\} \succcurlyeq 0$ leads to

$$i_+[\phi(X, Y)] = r(Y) = q;$$

establishing (2.36).

For any integer $0 \leqslant q \leqslant \min\{\, m,\ n\,\}$, setting $r(X) = q$ and $Y = 0$ and applying (2.16) lead to

$$i_-[M(X, Y)] = r(X) = q,$$

establishing (2.37). □

Lemma 2.7 *Let $A \in \mathbb{C}_{\mathrm{H}}^{m}$ be given, $X \in \mathbb{C}^{m\times n}$ and $Y \in \mathbb{C}_{\mathrm{H}}^{n}$ be two variable matrices, and define*

$$\phi(X, Y) = \begin{bmatrix} A & X \\ X^* & Y \end{bmatrix}. \tag{2.38}$$

Then,

(a) *The following equalities hold*

$$\max_{X\in\mathbb{C}^{m\times n},\, Y\in\mathbb{C}_{\mathrm{H}}^{n}} r[\phi(X, Y)] = \min\{\, m + n,\ \ r(A) + 2n\,\}, \tag{2.39}$$

$$\min_{X\in\mathbb{C}^{m\times n},\, Y\in\mathbb{C}_{\mathrm{H}}^{n}} r[\phi(X, Y)] = r(A), \tag{2.40}$$

$$\max_{X\in\mathbb{C}^{m\times n},\, Y\in\mathbb{C}_{\mathrm{H}}^{n}} i_\pm[\phi(X, Y)] = i_\pm(A) + n, \tag{2.41}$$

$$\min_{X\in\mathbb{C}^{m\times n},\, Y\in\mathbb{C}_{\mathrm{H}}^{n}} i_\pm[\phi(X, Y)] = i_\pm(A). \tag{2.42}$$

(b) *For any integer q with $r(A) \leqslant q \leqslant \min\{\, m + n,\ \ r(A) + 2n\,\}$, there exist $X \in \mathbb{C}^{m\times n}$ and $Y \in \mathbb{C}_{\mathrm{H}}^{n}$ such that $r[\phi(X, Y)] = q$.*

(c) *For any integer q with $i_+(A) \leqslant q \leqslant i_+(A)+n$, there exist $X \in \mathbb{C}^{m\times n}$ and $Y \in \mathbb{C}_{\mathrm{H}}^{n}$ such that $i_+[\phi(X, Y)] = q$.*

(d) *For any integer q with $i_-(A) \leqslant q \leqslant i_-(A)+n$, there exist $X \in \mathbb{C}^{m\times n}$ and $Y \in \mathbb{C}_{\mathrm{H}}^{n}$ such that $i_-[\phi(X, Y)] = q$.*

(e) *The following equalities hold*

$$\max_{X\in\mathbb{C}^{m\times n},\, Y\succcurlyeq 0} r[\phi(X, Y)] = \min\{\, m + n,\ \ r(A) + 2n\,\}, \tag{2.43}$$

$$\min_{X\in\mathbb{C}^{m\times n},\, Y\succcurlyeq 0} r[\phi(X, Y)] = r(A), \tag{2.44}$$

$$\max_{X\in\mathbb{C}^{m\times n},\, Y\succcurlyeq 0} i_+[\phi(X, Y)] = i_+(A) + n, \tag{2.45}$$

$$\min_{X\in\mathbb{C}^{m\times n},\, Y\succcurlyeq 0} i_+[\phi(X, Y)] = i_+(A), \tag{2.46}$$

$$\max_{X\in\mathbb{C}^{m\times n},\, Y\succcurlyeq 0} i_-[\phi(X, Y)] = \min\{\, m,\ \ i_-(A) + n\,\}, \tag{2.47}$$

$$\min_{X\in\mathbb{C}^{m\times n},\, Y\succcurlyeq 0} i_-[\phi(X, Y)] = i_-(A). \tag{2.48}$$

(f) *For any integer* q *with* $r(A) \leqslant q \leqslant \min\{\, m+n, \quad r(A)+2n \,\}$, *there exist* $X \in \mathbb{C}^{m\times n}$ *and* $0 \preccurlyeq Y \in \mathbb{C}_{\rm H}^{n}$ *such that* $r[\phi(X, Y)] = q$.

(g) *For any integer* q *with* $i_{+}(A) \leqslant q \leqslant i_{+}(A)+n$, *there exist* $X \in \mathbb{C}^{m\times n}$ *and* $0 \preccurlyeq Y \in \mathbb{C}_{\rm H}^{n}$ *such that* $i_{+}[\phi(X, Y)] = q$.

(h) *For any integer* q *with* $i_{-}(A) \leqslant q \leqslant \min\{\, m, \quad i_{-}(A)+n \,\}$ *there exist* $X \in \mathbb{C}^{m\times n}$ *and* $0 \preccurlyeq Y \in \mathbb{C}_{\rm H}^{n}$ *such that* $i_{-}[\phi(X, Y)] = q$.

(i) *Under* $A \succcurlyeq 0$, *the following equalities hold*

$$\max_{X\in\mathbb{C}^{m\times n},\, Y\in\mathbb{C}_{\rm H}^{n}} r[\phi(X, Y)] = \min\{\, m+n, \quad r(A)+2n \,\}, \tag{2.49}$$

$$\min_{X\in\mathbb{C}^{m\times n},\, Y\in\mathbb{C}_{\rm H}^{n}} r[\phi(X, Y)] = r(A), \tag{2.50}$$

$$\max_{X\in\mathbb{C}^{m\times n},\, Y\in\mathbb{C}_{\rm H}^{n}} i_{+}[\phi(X, Y)] = r(A)+n, \tag{2.51}$$

$$\min_{X\in\mathbb{C}^{m\times n},\, Y\in\mathbb{C}_{\rm H}^{n}} i_{+}[\phi(X, Y)] = r(A), \tag{2.52}$$

$$\max_{X\in\mathbb{C}^{m\times n},\, Y\in\mathbb{C}_{\rm H}^{n}} i_{-}[\phi(X, Y)] = n, \tag{2.53}$$

$$\min_{X\in\mathbb{C}^{m\times n},\, Y\in\mathbb{C}_{\rm H}^{n}} i_{-}[\phi(X, Y)] = 0, \tag{2.54}$$

$$\max_{X\in\mathbb{C}^{m\times n},\, Y\succcurlyeq 0} r[\phi(X, Y)] = \min\{\, m+n, \quad r(A)+2n \,\}, \tag{2.55}$$

$$\min_{X\in\mathbb{C}^{m\times n},\, Y\succcurlyeq 0} r[\phi(X, Y)] = r(A), \tag{2.56}$$

$$\max_{X\in\mathbb{C}^{m\times n},\, Y\succcurlyeq 0} i_{+}[\phi(X, Y)] = r(A)+n, \tag{2.57}$$

$$\min_{X\in\mathbb{C}^{m\times n},\, Y\succcurlyeq 0} i_{+}[\phi(X, Y)] = r(A), \tag{2.58}$$

$$\max_{X\in\mathbb{C}^{m\times n},\, Y\succcurlyeq 0} i_{-}[\phi(X, Y)] = \min\{\, m, \quad n \,\}, \tag{2.59}$$

$$\min_{X\in\mathbb{C}^{m\times n},\, Y\succcurlyeq 0} i_{-}[\phi(X, Y)] = 0. \tag{2.60}$$

(j) *Under* $A \succcurlyeq 0$, *for any integer* q *with* $r(A) \leqslant q \leqslant \min\{\, m+n, \quad r(A)+2n \,\}$, *there exist* $X \in \mathbb{C}^{m\times n}$ *and* $Y \in \mathbb{C}_{\rm H}^{n}$ *such that* $r[\phi(X, Y)] = q$.

(k) *Under* $A \succcurlyeq 0$, *for any integer* q *with* $r(A) \leqslant q \leqslant r(A)+n$, *there exist* $X \in \mathbb{C}^{m\times n}$ *and* $Y \in \mathbb{C}_{\rm H}^{n}$ *such that* $i_{+}[\phi(X, Y)] = q$.

(l) *Under* $A \succcurlyeq 0$, *for any integer* q *with* $0 \leqslant q \leqslant n$, *there exist* $X \in \mathbb{C}^{m\times n}$ *and* $Y \in \mathbb{C}_{\rm H}^{n}$ *such that* $i_{-}[\phi(X, Y)] = q$.

(m) *Under* $A \succcurlyeq 0$, *for any integer* q *with* $r(A) \leqslant q \leqslant \min\{\, m+n, \quad r(A)+2n \,\}$, *there exist* $X \in \mathbb{C}^{m\times n}$ *and* $Y \succcurlyeq 0$ *such that* $r[\phi(X, Y)] = q$.

(n) *Under* $A \succcurlyeq 0$, *for any integer* q *with* $r(A) \leqslant q \leqslant r(A)+n$, *there exist* $X \in \mathbb{C}^{m\times n}$ *and* $Y \succcurlyeq 0$ *such that* $i_{+}[\phi(X, Y)] = q$.

(o) *Under* $A \succcurlyeq 0$, *for any integer* q *with* $0 \leqslant q \leqslant \min\{\, m, \quad n \,\}$, *there exist* $X \in \mathbb{C}^{m\times n}$ *and* $Y \succcurlyeq 0$ *such that* $i_{-}[\phi(X, Y)] = q$.

(p) *Under* $A \succcurlyeq 0$, *the following equalities hold*

$$\max_{\phi(X,\, Y)\succcurlyeq 0} r[\phi(X, Y)] = r(A)+n, \tag{2.61}$$

$$\min_{\phi(X,\, Y)\succcurlyeq 0} r[\phi(X, Y)] = r(A). \tag{2.62}$$

(q) *Under* $A \succcurlyeq 0$, *for any integer* q *with* $r(A) \leqslant q \leqslant r(A)+n$, *there exist* $X \in \mathbb{C}^{m\times n}$ *and* $Y \succcurlyeq 0$ *such that* $\phi(X, Y) \succcurlyeq 0$ *and* $r[\phi(X, Y)] = q$.

Proof. Without lost generality, we assume that A is given by

$$A = \mathrm{diag}\{\, I_s, \, -I_t, \, 0 \,\}. \tag{2.63}$$

Correspondingly, $M(X, Y)$ can be written as

$$M(X,Y) = \begin{bmatrix} I_s & 0 & 0 & \widehat{X}_1 \\ 0 & -I_t & 0 & \widehat{X}_2 \\ 0 & 0 & 0 & \widehat{X}_3 \\ \widehat{X}_1^* & \widehat{X}_2^* & \widehat{X}_3^* & Y \end{bmatrix}. \tag{2.64}$$

Applying (2.15) to (2.64) leads to

$$r[M(X,Y)] = s + t + r\begin{bmatrix} 0 & \widehat{X}_3 \\ \widehat{X}_3^* & Y - \widehat{X}_1^*\widehat{X}_1 + \widehat{X}_2^*\widehat{X}_2 \end{bmatrix}, \tag{2.65}$$

$$i_+[M(X,Y)] = s + i_+\begin{bmatrix} 0 & \widehat{X}_3 \\ \widehat{X}_3^* & Y - \widehat{X}_1^*\widehat{X}_1 + \widehat{X}_2^*\widehat{X}_2 \end{bmatrix}, \tag{2.66}$$

$$i_-[M(X,Y)] = t + i_-\begin{bmatrix} 0 & \widehat{X}_3 \\ \widehat{X}_3^* & Y - \widehat{X}_1^*\widehat{X}_1 + \widehat{X}_2^*\widehat{X}_2 \end{bmatrix}. \tag{2.67}$$

Applying Lemma 2.5(a) to (2.65)–(2.67) leads to

$$\max_{\widehat{X}\in\mathbb{C}^{m\times n},\, Y\in\mathbb{C}_{\rm H}^n} r\begin{bmatrix} 0 & \widehat{X}_3 \\ \widehat{X}_3^* & Y - \widehat{X}_1^*\widehat{X}_1 + \widehat{X}_2^*\widehat{X}_2 \end{bmatrix} = \min\{\, m - r(A) + n, \ \ 2n \,\}, \tag{2.68}$$

$$\min_{\widehat{X}\in\mathbb{C}^{m\times n},\, Y\in\mathbb{C}_{\rm H}^n} r\begin{bmatrix} 0 & \widehat{X}_3 \\ \widehat{X}_3^* & Y - \widehat{X}_1^*\widehat{X}_1 + \widehat{X}_2^*\widehat{X}_2 \end{bmatrix} = 0, \tag{2.69}$$

$$\max_{\widehat{X}\in\mathbb{C}^{m\times n},\, Y\in\mathbb{C}_{\rm H}^n} i_\pm\begin{bmatrix} 0 & \widehat{X}_3 \\ \widehat{X}_3^* & Y - \widehat{X}_1^*\widehat{X}_1 + \widehat{X}_2^*\widehat{X}_2 \end{bmatrix} = n, \tag{2.70}$$

$$\min_{\widehat{X}\in\mathbb{C}^{m\times n},\, Y\in\mathbb{C}_{\rm H}^n} i_\pm\begin{bmatrix} 0 & \widehat{X}_3 \\ \widehat{X}_3^* & Y - \widehat{X}_1^*\widehat{X}_1 + \widehat{X}_2^*\widehat{X}_2 \end{bmatrix} = 0. \tag{2.71}$$

Substituting (2.68)–(2.71) into (2.65)–(2.67) yields (2.39)–(2.42).

Applying Lemma 2.5(b) and (c) to (2.39)–(2.42) leads to (b)–(d).

Applying Lemma 2.5(e) to (2.65)–(2.67) leads to

$$\max_{\widehat{X}\in\mathbb{C}^{m\times n},\, Y\succcurlyeq 0} r\begin{bmatrix} 0 & \widehat{X}_3 \\ \widehat{X}_3^* & Y - \widehat{X}_1^*\widehat{X}_1 + \widehat{X}_2^*\widehat{X}_2 \end{bmatrix} = \min\{\, m - r(A) + n, \ \ 2n \,\}, \tag{2.72}$$

$$\min_{\widehat{X}\in\mathbb{C}^{m\times n},\, Y\succcurlyeq 0} r\begin{bmatrix} 0 & \widehat{X}_3 \\ \widehat{X}_3^* & Y - \widehat{X}_1^*\widehat{X}_1 + \widehat{X}_2^*\widehat{X}_2 \end{bmatrix} = 0, \tag{2.73}$$

$$\max_{\widehat{X}\in\mathbb{C}^{m\times n},\, Y\succcurlyeq 0} i_+\begin{bmatrix} 0 & \widehat{X}_3 \\ \widehat{X}_3^* & Y - \widehat{X}_1^*\widehat{X}_1 + \widehat{X}_2^*\widehat{X}_2 \end{bmatrix} = n, \tag{2.74}$$

$$\max_{\widehat{X}\in\mathbb{C}^{m\times n},\, Y\succcurlyeq 0} i_-\begin{bmatrix} 0 & \widehat{X}_3 \\ \widehat{X}_3^* & Y - \widehat{X}_1^*\widehat{X}_1 + \widehat{X}_2^*\widehat{X}_2 \end{bmatrix} = \min\{\, m - t, \ \ n \,\}, \tag{2.75}$$

$$\min_{\widehat{X}\in\mathbb{C}^{m\times n},\, Y\succcurlyeq 0} i_\pm\begin{bmatrix} 0 & \widehat{X}_3 \\ \widehat{X}_3^* & Y - \widehat{X}_1^*\widehat{X}_1 + \widehat{X}_2^*\widehat{X}_2 \end{bmatrix} = 0. \tag{2.76}$$

Substituting (2.72)–(2.76) into (2.65)–(2.67) yields (2.43)–(2.48).

Applying Lemma 2.5(e)—(g) to (2.43)–(2.48) leads to (f)–(h).

Setting $A \succcurlyeq 0$ in (2.39)–(2.48) leads to (2.49)–(2.60).

Results (j)—(o) follow from (2.49)–(2.60).

It is easy to verify that under $A \succcurlyeq 0$ with $r(A) = s$,

$$M(X,Y) = \begin{bmatrix} I_s & 0 & \widehat{X}_1 \\ 0 & 0 & \widehat{X}_3 \\ \widehat{X}_1^* & \widehat{X}_3^* & Y \end{bmatrix} \succcurlyeq 0 \Leftrightarrow \widehat{X}_3 = 0 \text{ and } Y \succcurlyeq \widehat{X}_1^*\widehat{X}_1. \tag{2.77}$$

So that

$$\max_{\phi(X,\,Y)\succcurlyeq 0} r[\phi(X,\,Y)] = \max_{\phi(X,\,Y)\succcurlyeq 0} r\begin{bmatrix} I_s & \widehat{X}_1 \\ \widehat{X}_1^* & Y \end{bmatrix} = s + \max_{Y \succcurlyeq \widehat{X}_1^*\widehat{X}_1} r(\,Y - \widehat{X}_1^*\widehat{X}_1\,) = r(A) + n, \tag{2.78}$$

$$\min_{\phi(X,\,Y)\succcurlyeq 0} r[\phi(X,\,Y)] = \min_{\phi(X,\,Y)\succcurlyeq 0} r\begin{bmatrix} I_s & \widehat{X}_1 \\ \widehat{X}_1^* & Y \end{bmatrix} = s + \min_{Y \succcurlyeq \widehat{X}_1^*\widehat{X}_1} r(\,Y - \widehat{X}_1^*\widehat{X}_1\,) = r(A), \tag{2.79}$$

establishing (2.61) and (2.62). Result (q) follows from (2.78) and (2.79). □

Lemma 2.8 *Let $A \in \mathbb{C}_{\rm H}^{m}$ and $B \in \mathbb{C}^{m\times n}$ be given. Then, there exist a unitary matrix $U \in \mathbb{C}^{n\times n}$ and a nonsingular matrix $P \in \mathbb{C}^{m\times m}$ such that*

$$A = P^* D_A P \quad \text{and} \quad B = P^* D_B U, \tag{2.80}$$

where

$$D_A = \begin{array}{c} \begin{bmatrix} 0 & I & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Delta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} f \\ f \\ t \\ k \\ l-t \\ m-(2f+l+k) \end{matrix} \\ \begin{matrix} f & f & t & k & l-t & m-(2f+l+k) \end{matrix} \end{array}, \tag{2.81}$$

$$D_B = \begin{array}{c} \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \Sigma & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} f \\ f \\ t \\ k \\ l-t \\ m-(2f+l+k) \end{matrix} \\ \begin{matrix} f & t & l-t & n-(f+l) \end{matrix} \end{array}, \tag{2.82}$$

where

$$\Delta_1 = \begin{bmatrix} I_{s_1} & 0 \\ 0 & -I_{t-s_1} \end{bmatrix}, \quad \Delta_2 = \begin{bmatrix} I_{s_2} & 0 \\ 0 & -I_{k-s_2} \end{bmatrix}, \tag{2.83}$$

$$\Sigma = {\rm diag}\{\sigma_1, \cdots, \sigma_{s_1}, \lambda_{s_1+1}, \cdots, \lambda_t\}, \quad \sigma_1 \geqslant \cdots \geqslant \sigma_{s_1} > 0, \quad \lambda_{s_1+1} \geqslant \cdots \geqslant \lambda_t > 0, \tag{2.84}$$

and

$$f = r(B) + r[\,A,\,B\,] - r\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}, \tag{2.85}$$

$$k = r\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} - 2r(B), \tag{2.86}$$

$$l = r\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} - r[\,A,\,B\,], \tag{2.87}$$

$$t = r(A) + r\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} - 2r[\,A,\,B\,], \tag{2.88}$$

$$f + s_1 + s_2 = i_+(A), \tag{2.89}$$

$$f + t + k - s_1 - s_2 = i_-(A), \tag{2.90}$$

$$f + l + s_2 = i_+\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}, \tag{2.91}$$

$$f + l + k - s_2 = i_-\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}. \tag{2.92}$$

The structures of the blocks in (2.80)–(2.84) and the parameters in (2.85)–(2.92) are formulated more explicitly, which improve the results in [22] and [4]. In fact, the blocks in (2.80)–(2.84) can be derived from those of Lemma 2.1 in [4] through a series of elementary transformations, and (2.89)–(2.92) follow from applying Lemma 2.3 to (2.81)–(2.84).

3 The rank and inertia of $A + X$ subject to rank and semi-definite restrictions

One of the special cases in (1.1) is the ordinary sum $A + X$. Many results on ranks and inertias of sum of two Hermitian matrices were established; see, e.g., [1, 3, 13]. Note that the rank and inertia of $A+X$ may vary with respect to the choice of the Hermitian matrix X. In this section, we derive explicit formulas for calculating the extremal ranks and inertias $A + X$ subject to $X \in \mathbb{C}_{\mathrm{H}}^{m}$ and $X \succcurlyeq 0$, respectively. The formulas obtained will be used in Sections 4 and 5.

Theorem 3.1 *Let $A \in \mathbb{C}_{\mathrm{H}}^{m}$ be given, $X \in \mathbb{C}_{\mathrm{H}}^{m}$ be a variable matrix, and assume that p and q are two integers satisfying*

$$0 \leqslant p \leqslant q \leqslant m. \tag{3.1}$$

Then,

(a) *The following equalities hold*

$$\max_{X\in\mathbb{C}_{\mathrm{H}}^{m},\, r(X)=q} r(\, A + X\,) = \min\{\, m, \quad r(A) + q\,\}, \tag{3.2}$$

$$\min_{X\in\mathbb{C}_{\mathrm{H}}^{m},\, r(X)=q} r(\, A + X\,) = |\, r(A) - q\,|, \tag{3.3}$$

$$\max_{X\in\mathbb{C}_{\mathrm{H}}^{m},\, r(X)=q} i_{+}(\, A + X\,) = \min\{\, m, \quad i_{+}(A) + q\,\}, \tag{3.4}$$

$$\min_{X\in\mathbb{C}_{\mathrm{H}}^{m},\, r(X)=q} i_{+}(\, A + X\,) = \max\{\, 0, \quad i_{+}(A) - q\}, \tag{3.5}$$

$$\max_{X\in\mathbb{C}_{\mathrm{H}}^{m},\, r(X)=q} i_{-}(\, A + X\,) = \min\{\, m, \quad i_{-}(A) + q\}, \tag{3.6}$$

$$\min_{X\in\mathbb{C}_{\mathrm{H}}^{m},\, r(X)=q} i_{-}(\, A + X\,) = \max\{\, 0, \quad i_{-}(A) - q\,\}. \tag{3.7}$$

(b) *The following statements hold.*

(i) *For any integer t_1 between the two quantities on the right-hand sides of* (3.2) *and* (3.3), *there exists an $X \in \mathbb{C}_{H}^{m\times m}$ with $r(X) = q$ such that $r(\, A + X\,) = t_1$.*

(ii) *For any integer t_2 between the two quantities on the right-hand sides of* (3.4) *and* (3.5), *there exists an $X \in \mathbb{C}_{H}^{m\times m}$ with $r(X) = q$ such that $i_{+}(\, A + X\,) = t_2$.*

(iii) *For any integer t_3 between the two quantities on the right-hand sides of* (3.6) *and* (3.7), *there exists an $X \in \mathbb{C}_{H}^{m\times m}$ with $r(X) = q$ such that $i_{-}(\, A + X\,) = t_3$.*

(iv) *There exists an $X \in \mathbb{C}_{\mathrm{H}}^{m}$ with $r(X) = q$ such that $A + X$ is nonsingular if and only if $r(A) \geqslant m - q$.*

(v) *There exists an $X \in \mathbb{C}_{\mathrm{H}}^{m}$ with $r(X) = q$ such that $A + X = 0$ if and only if $r(A) = q$.*

(vi) *There exists an $X \in \mathbb{C}_{\mathrm{H}}^{m}$ with $r(X) = q$ such that $A + X \succ 0$ if and only if $i_{+}(A) \geqslant m - q$.*

(vii) *There exists an $X \in \mathbb{C}_{\mathrm{H}}^{m}$ with $r(X) = q$ such that $A + X \succcurlyeq 0$ if and only if $i_{-}(A) \leqslant q$.*

(viii) *There exists an $X \in \mathbb{C}_{\mathrm{H}}^{m}$ with $r(X) = q$ such that $A + X \prec 0$ if and only if $i_{-}(A) \geqslant m - q$.*

(ix) *There exists an $X \in \mathbb{C}_{\mathrm{H}}^{m}$ with $r(X) = q$ such that $A + X \preccurlyeq 0$ if and only if $i_{+}(A) \leqslant q$.*

(b) *The following equalities hold*

$$\max_{X\in\mathbb{C}_{\mathrm{H}}^{m},\, p\leqslant r(X)\leqslant q} r(\, A + X\,) = \min\{\, m, \quad r(A) + q\,\}, \tag{3.8}$$

$$\min_{X\in\mathbb{C}_{\mathrm{H}}^{m},\, p\leqslant r(X)\leqslant q} r(\, A + X\,) = \max\{\, 0,\ p - r(A),\ r(A) - q\}, \tag{3.9}$$

$$\max_{X\in\mathbb{C}_{\mathrm{H}}^{m},\, p\leqslant r(X)\leqslant q} i_{+}(\, A + X\,) = \min\{\, m, \quad i_{+}(A) + q\,\}, \tag{3.10}$$

$$\min_{X\in\mathbb{C}_{\mathrm{H}}^{m},\, p\leqslant r(X)\leqslant q} i_{+}(\, A + X\,) = \max\{\, 0, \quad i_{+}(A) - q\}, \tag{3.11}$$

$$\max_{X\in\mathbb{C}_{\mathrm{H}}^{m},\, p\leqslant r(X)\leqslant q} i_{-}(\, A + X\,) = \min\{\, m, \quad i_{-}(A) + q\}, \tag{3.12}$$

$$\min_{X\in\mathbb{C}_{\mathrm{H}}^{m},\, p\leqslant r(X)\leqslant q} i_{-}(\, A + X\,) = \max\{\, 0, \quad i_{-}(A) - q\,\}. \tag{3.13}$$

(b) *The following statements hold.*

(i) *For any integer* t_1 *between the two quantities on the right-hand sides of* (3.8) *and* (3.9), *there exists an* $X \in \mathbb{C}_H^{m\times m}$ *with* $p \leqslant r(X) \leqslant q$ *such that* $r(\,A + X\,) = t_1$.

(ii) *For any integer* t_2 *between the two quantities on the right-hand sides of* (3.10) *and* (3.11), *there exists an* $X \in \mathbb{C}_H^{m\times m}$ *with* $p \leqslant r(X) \leqslant q$ *such that* $i_+(\,A + X\,) = t_2$.

(iii) *For any integer* t_3 *between the two quantities on the right-hand sides of* (3.12) *and* (3.13), *there exists an* $X \in \mathbb{C}_H^{m\times m}$ *with* $p \leqslant r(X) \leqslant q$ *such that* $i_-(\,A + X\,) = t_3$.

(iv) *There exists an* $X \in \mathbb{C}_{\mathrm{H}}^{m}$ *with* $p \leqslant r(X) \leqslant q$ *such that* $A+X$ *is nonsingular if and only if* $r(A) \geqslant m-q$.

(v) *There exists an* $X \in \mathbb{C}_{\mathrm{H}}^{m}$ *with* $p \leqslant r(X) \leqslant q$ *such that* $A + X = 0$ *if and only if* $p \leqslant r(A) \leqslant q$.

(vi) *There exists an* $X \in \mathbb{C}_{\mathrm{H}}^{m}$ *with* $p \leqslant r(X) \leqslant q$ *such that* $A + X \succ 0$ *if and only if* $i_+(A) \geqslant m - q$.

(vii) *There exists an* $X \in \mathbb{C}_{\mathrm{H}}^{m}$ *with* $p \leqslant r(X) \leqslant q$ *such that* $A + X \succcurlyeq 0$ *if and only if* $i_-(A) \leqslant q$.

(viii) *There exists an* $X \in \mathbb{C}_{\mathrm{H}}^{m}$ *with* $p \leqslant r(X) \leqslant q$ *such that* $A + X \prec 0$ *if and only if* $i_-(A) \geqslant m - q$.

(ix) *There exists an* $X \in \mathbb{C}_{\mathrm{H}}^{m}$ *with* $p \leqslant r(X) \leqslant q$ *such that* $A + X \preccurlyeq 0$ *if and only if* $i_+(A) \leqslant q$.

(c) *The following equalities hold*

$$\max_{X\in\mathbb{C}_{\mathrm{H}}^{m},\, 0\leqslant r(X)\leqslant q} r(\,A + X\,) = \min\{\,m, \quad r(A) + q\,\}, \tag{3.14}$$

$$\min_{X\in\mathbb{C}_{\mathrm{H}}^{m},\, 0\leqslant r(X)\leqslant q} r(\,A + X\,) = \max\{\,0, \quad r(A) - q\}, \tag{3.15}$$

$$\max_{X\in\mathbb{C}_{\mathrm{H}}^{m},\, 0\leqslant r(X)\leqslant q} i_+(\,A + X\,) = \min\{\,m, \quad i_+(A) + q\,\}, \tag{3.16}$$

$$\min_{X\in\mathbb{C}_{\mathrm{H}}^{m},\, 0\leqslant r(X)\leqslant q} i_+(\,A + X\,) = \max\{\,0, \quad i_+(A) - q\,\}, \tag{3.17}$$

$$\max_{X\in\mathbb{C}_{\mathrm{H}}^{m},\, 0\leqslant r(X)\leqslant q} i_-(\,A + X\,) = \min\{\,m, \quad i_-(A) + q\,\}, \tag{3.18}$$

$$\min_{X\in\mathbb{C}_{\mathrm{H}}^{m},\, 0\leqslant r(X)\leqslant q} i_-(\,A + X\,) = \max\{\,0, \quad i_-(A) - q\,\}. \tag{3.19}$$

(d) *The following equalities hold*

$$\max_{X\in\mathbb{C}_{\mathrm{H}}^{m},\, p\leqslant r(X)\leqslant m} r(\,A + X\,) = m, \tag{3.20}$$

$$\min_{X\in\mathbb{C}_{\mathrm{H}}^{m},\, p\leqslant r(X)\leqslant m} r(\,A + X\,) = \max\{\,0, \ p - r(A)\}, \tag{3.21}$$

$$\max_{X\in\mathbb{C}_{\mathrm{H}}^{m},\, p\leqslant r(X)\leqslant m} i_+(\,A + X\,) = m, \tag{3.22}$$

$$\min_{X\in\mathbb{C}_{\mathrm{H}}^{m},\, p\leqslant r(X)\leqslant m} i_+(\,A + X\,) = 0, \tag{3.23}$$

$$\max_{X\in\mathbb{C}_{\mathrm{H}}^{m},\, p\leqslant r(X)\leqslant m} i_-(\,A + X\,) = m, \tag{3.24}$$

$$\min_{X\in\mathbb{C}_{\mathrm{H}}^{m},\, p\leqslant r(X)\leqslant m} i_-(\,A + X\,) = 0. \tag{3.25}$$

The Hermitian matrices X*s satisfying these equalities can be formulated from the canonical form of certain operations* A *and* B *under the Hermitian congruence.*

Proof. It is easy to see from (2.1)–(2.4) that the right-hand sides of (3.2), (3.4) are (3.6) are upper bounds, while the right-hand sides of (3.3), (3.5) are (3.7) are lower bounds. Without loss of generality, we assume that A is of the form

$$A_1 = \mathrm{diag}\{\,I_s, \ -I_t, \ 0\,\} \quad \text{or} \quad A_2 = \mathrm{diag}\{\,-I_t, \ I_s, \ 0\,\}. \tag{3.26}$$

The establishments of (3.2)–(3.7) are based on the following assertions:

(i) Let $X = \mathrm{diag}\{\,0_{n-q}, \, 2I_q\,\}$. If $m \leqslant r(A) + q$, then $r(\,A_1 + X\,) = m$; if $m > r(A) + q$, then $r(\,A_1 + X\,) = r(A) + r(X) = r(A) + q$, so that (3.2) holds.

(ii) If $r(A) \leqslant q$, then setting $X = \mathrm{diag}\{\,-I_s, \, I_t, \, I_{q-r(A)}, \, 0_{n-q}\,\}$ gives $r(\,A + X\,) = q - r(A)$; if $r(A) > q$ and $s > q$, then setting $X = \mathrm{diag}\{\,-I_q, \, 0_{n-q}\,\}$ gives $r(\,A+X\,) = r(A)-q$, so that (3.3) holds; if $r(A) > q \geqslant s$, then setting $X = \mathrm{diag}\{\,-I_s, \, I_{q-s}, \, 0_{n-q}\,\}$ gives $r(\,A + X\,) = r(A) - q$, so that (3.3) holds.

(iii) If $m \leqslant i_+(A) + q$, then setting $X = \mathrm{diag}\{\,0_{n-q}, \, 2I_q\,\}$ gives $r(\,A + X\,) = m$; if $m > i_+(A) + q$, then setting $X = \mathrm{diag}\{\,0_{n-q}, \, 2I_q\,\}$ gives $i_+(\,A + X\,) = i_+(A) + r(X) = i_+(A) + q$, so that (3.4) holds.

(iv) If $i_+(A) \leqslant q$, then setting $X = \mathrm{diag}\{ -2I_q, 0_{n-q} \}$ gives $i_-(A + X) = 0$; if $i_+(A) > q$, then setting $X = \mathrm{diag}\{ -I_q, 0_{n-q} \}$ gives $i_+(A + X) = i_+(A) - q$, so that (3.5) holds.

(v) If $m \leqslant i_-(A) + q$, then setting $X = \mathrm{diag}\{ -2I_s,\ 0_{n-q},\ -2I_{q-s} \}$ gives $i_-(A_2 + X) = m$; if $m > i_-(A) + q$, then setting $X = \mathrm{diag}\{ -2I_{s-k}, 0_{n-q},\ -2I_{q-s+k} \}$ gives $i_-(A + X) = i_-(A) + r(X) = i_-(A) + q$, so that (3.6) holds.

(vi) If $i_-(A) \leqslant q$, then setting $X = \mathrm{diag}\{ 0_s, I_q, 0_{n-s-q} \}$ gives $i_-(A_2 + X) = 0$; if $i_-(A) > q$, then setting $X = \mathrm{diag}\{ 0_s, I_q, 0_{n-s-q} \}$ gives $i_+(A + X) = i_-(A) - q$, so that (3.7) holds.

It can be seen from (1.19) that

$$\max_{X\in\mathbb{C}_{\mathrm{H}}^m,\, p\leqslant r(X)\leqslant q} r(A + X) = \max\left\{ \max_{X\in\mathbb{C}_{\mathrm{H}}^m,\, r(X)=p} r(A + X),\ \max_{X\in\mathbb{C}_{\mathrm{H}}^m,\, r(X)=p+1} r(A + X), \ldots,\ \max_{X\in\mathbb{C}_{\mathrm{H}}^m,\, r(X)=q} r(A + X) \right\}, \tag{3.27}$$

$$\min_{X\in\mathbb{C}_{\mathrm{H}}^m,\, p\leqslant r(X)\leqslant q} r(A + X) = \min\left\{ \min_{X\in\mathbb{C}_{\mathrm{H}}^m,\, r(X)=p} r(A + X),\ \min_{X\in\mathbb{C}_{\mathrm{H}}^m,\, r(X)=p+1} r(A + X), \ldots,\ \min_{X\in\mathbb{C}_{\mathrm{H}}^m,\, r(X)=q} r(A + X) \right\}, \tag{3.28}$$

$$\max_{X\in\mathbb{C}_{\mathrm{H}}^m,\, p\leqslant r(X)\leqslant q} i_+(A + X) = \max\left\{ \max_{X\in\mathbb{C}_{\mathrm{H}}^m,\, r(X)=p} i_+(A + X),\ \max_{r(X)=p+1} i_+(A + X), \ldots,\ \max_{X\in\mathbb{C}_{\mathrm{H}}^m,\, r(X)=q} i_+(A + X) \right\}, \tag{3.29}$$

$$\min_{X\in\mathbb{C}_{\mathrm{H}}^m,\, p\leqslant r(X)\leqslant q} i_+(A + X) = \min\left\{ \min_{X\in\mathbb{C}_{\mathrm{H}}^m,\, r(X)=p} i_+(A + X),\ \min_{X\in\mathbb{C}_{\mathrm{H}}^m,\, r(X)=p+1} i_+(A + X), \ldots,\ \min_{X\in\mathbb{C}_{\mathrm{H}}^m,\, r(X)=q} i_+(A + X) \right\}, \tag{3.30}$$

$$\max_{X\in\mathbb{C}_{\mathrm{H}}^m,\, p\leqslant r(X)\leqslant q} i_-(A + X) = \max\left\{ \max_{X\in\mathbb{C}_{\mathrm{H}}^m,\, r(X)=p} i_-(A + X),\ \max_{X\in\mathbb{C}_{\mathrm{H}}^m,\, r(X)=p+1} i_-(A + X), \ldots,\ \max_{X\in\mathbb{C}_{\mathrm{H}}^m,\, r(X)=q} i_-(A + X) \right\}, \tag{3.31}$$

$$\min_{X\in\mathbb{C}_{\mathrm{H}}^m,\, p\leqslant r(X)\leqslant q} i_-(A + X) = \min\left\{ \min_{X\in\mathbb{C}_{\mathrm{H}}^m,\, r(X)=p} i_-(A + X),\ \min_{X\in\mathbb{C}_{\mathrm{H}}^m,\, r(X)=p+1} i_-(A + X), \ldots,\ \min_{X\in\mathbb{C}_{\mathrm{H}}^m,\, r(X)=q} i_-(A + X) \right\}. \tag{3.32}$$

Substituting (3.2)–(3.7) for $r(X) = p,\ p+1, \ldots, q$ into (3.27)–(3.32) and making the max-min comparison, we obtain

$$\max_{X\in\mathbb{C}_{\mathrm{H}}^m,\, p\leqslant r(X)\leqslant q} r(A + X) = \max\{\min\{ m,\ r(A) + p \},\ \min\{ m,\ r(A) + p + 1 \}, \ldots, \min\{ m,\ r(A) + q \}\} = \min\{ m,\ r(A) + q \}, \tag{3.33}$$

$$\min_{X\in\mathbb{C}_{\mathrm{H}}^m,\, p\leqslant r(X)\leqslant q} r(A + X) = \min\{|\, r(A) - p\,|,\ |\, r(A) - p - 1\,|, \ldots, |\, r(A) - q\,|\} = \max\{ 0,\ p - r(A),\ r(A) - q\}, \tag{3.34}$$

$$\max_{X\in\mathbb{C}_{\mathrm{H}}^m,\, p\leqslant r(X)\leqslant q} i_+(A + X) = \max\{\min\{ m,\ i_+(A) + p \},\ \min\{ m,\ i_+(A) + p + 1 \}, \ldots, \min\{ m,\ i_+(A) + q \}\} = \max\{ m,\ i_+(A) + q \}, \tag{3.35}$$

$$\min_{X\in\mathbb{C}_{\mathrm{H}}^m,\, p\leqslant r(X)\leqslant q} i_+(A + X) = \min\{\min\{ 0,\ i_+(A) - p \},\ \min\{ 0,\ i_+(A) - p - 1 \}, \ldots, \min\{ 0,\ i_+(A) - q \}\} = \min\{ 0,\ i_+(A) - q \}, \tag{3.36}$$

$$\max_{X\in\mathbb{C}_{\mathrm{H}}^m,\, p\leqslant r(X)\leqslant q} i_-(A + X) = \max\{\min\{ m,\ i_-(A) + p \},\ \min\{ m,\ i_-(A) + p + 1 \}, \ldots, \min\{ m,\ i_-(A) + q \}\} = \min\{ m,\ i_-(A) + q \}, \tag{3.37}$$

$$\min_{X\in\mathbb{C}_{\mathrm{H}}^m,\, p\leqslant r(X)\leqslant q} i_-(A + X) = \min\{\min\{ 0,\ i_-(A) - p \},\ \min\{ 0,\ i_-(A) - p - 1 \}, \ldots, \min\{ 0,\ i_-(A) - q \}\} = \max\{ 0,\ i_-(A) - q \}, \tag{3.38}$$

as required for (3.8)–(3.13). Results (c) and (d) follow directly from (b). □

It is easy to see that (3.8)–(3.13) also hold for any $p < 0$ and $q > m$. So that we can use (3.8)–(3.13) under the condition

$$p \leqslant m, \quad 0 \leqslant q, \quad p \leqslant q. \tag{3.39}$$

Theorem 3.2 *Let $A \in \mathbb{C}_{\mathrm{H}}^{m}$ be given, $X \in \mathbb{C}_{\mathrm{H}}^{m}$ be a variable matrix, and assume that p and q are two integers satisfying $0 \leqslant p \leqslant q \leqslant m$. Then,*

(a) *The following equalities hold*

$$\max_{0 \preccurlyeq X,\, r(X)=q} r(\, A + X\,) = \min\{m, \quad r(A) + q\}\,, \tag{3.40}$$

$$\min_{0 \preccurlyeq X,\, r(X)=q} r(\, A + X\,) = \max\{\, i_{+}(A), \quad q - i_{-}(A), \quad r(A) - q\}\,, \tag{3.41}$$

$$\max_{0 \preccurlyeq X,\, r(X)=q} i_{+}(\, A + X\,) = \min\{m, \quad i_{+}(A) + q\}\,, \tag{3.42}$$

$$\min_{0 \preccurlyeq X,\, r(X)=q} i_{+}(\, A + X\,) = \max\{i_{+}(A), \quad q - i_{-}(A)\}\,, \tag{3.43}$$

$$\max_{0 \preccurlyeq X,\, r(X)=q} i_{-}(\, A + X\,) = i_{-}(A), \tag{3.44}$$

$$\min_{0 \preccurlyeq X,\, r(X)=q} i_{-}(\, A + X\,) = \max\{0, \quad i_{-}(A) - q\,\}\,, \tag{3.45}$$

$$\max_{0 \preccurlyeq X,\, r(X)=q} r(\, A - X\,) = \min\{m, \quad r(A) + q\}\,, \tag{3.46}$$

$$\min_{0 \preccurlyeq X,\, r(X)=q} r(\, A - X\,) = \max\{\, i_{-}(A), \quad q - i_{+}(A), \quad r(A) - q\}\,, \tag{3.47}$$

$$\max_{0 \preccurlyeq X,\, r(X)=q} i_{+}(\, A - X\,) = i_{+}(A), \tag{3.48}$$

$$\min_{0 \preccurlyeq X,\, r(X)=q} i_{+}(\, A - X\,) = \max\{0, \quad i_{+}(A) - q\,\}\,, \tag{3.49}$$

$$\max_{0 \preccurlyeq X,\, r(X)=q} i_{-}(\, A - X\,) = \min\{m, \quad i_{-}(A) + q\}\,, \tag{3.50}$$

$$\min_{0 \preccurlyeq X,\, r(X)=q} i_{-}(\, A - X\,) = \max\{i_{-}(A), \quad q - i_{+}(A)\}\,. \tag{3.51}$$

(b) *The following statements hold.*

(i) *For any integer t_1 between the two quantities on the right-hand sides of* (3.40) *and* (3.41), *there exists a $0 \preccurlyeq X \in \mathbb{C}_{H}^{m \times m}$ with $r(X) = q$ such that $r(\, A + X\,) = t_1$.*

(ii) *For any integer t_2 between the two quantities on the right-hand sides of* (3.42) *and* (3.43), *there exists a $0 \preccurlyeq X \in \mathbb{C}_{H}^{m \times m}$ with $r(X) = q$ such that $i_{+}(\, A + X\,) = t_2$.*

(iii) *For any integer t_3 between the two quantities on the right-hand sides of* (3.44) *and* (3.45), *there exists a $0 \preccurlyeq X \in \mathbb{C}_{H}^{m \times m}$ with $r(X) = q$ such that $i_{-}(\, A + X\,) = t_3$.*

(iv) *For any integer t_4 between the two quantities on the right-hand sides of* (3.46) *and* (3.47), *there exists a $0 \preccurlyeq X \in \mathbb{C}_{H}^{m \times m}$ with $r(X) = q$ such that $r(\, A - X\,) = t_4$.*

(v) *For any integer t_5 between the two quantities on the right-hand sides of* (3.48) *and* (3.49), *there exists a $0 \preccurlyeq X \in \mathbb{C}_{H}^{m \times m}$ with $r(X) = q$ such that $i_{+}(\, A - X\,) = t_5$.*

(vi) *For any integer t_3 between the two quantities on the right-hand sides of* (3.50) *and* (3.51), *there exists a $0 \preccurlyeq X \in \mathbb{C}_{H}^{m \times m}$ with $r(X) = q$ such that $i_{-}(\, A - X\,) = t_6$.*

(vii) *There exists a $0 \preccurlyeq X \in \mathbb{C}_{\mathrm{H}}^{m}$ with $r(X) = q$ such that $A + X$ is nonsingular if and only if $r(A) \geqslant m - q$.*

(viii) *$A + X$ is nonsingular for all $0 \preccurlyeq X \in \mathbb{C}_{\mathrm{H}}^{m}$ with $r(X) = q$ such if and only if $A \succ 0$, or $A \succcurlyeq 0$ and $q = m$, or $r(A) = m$ and $q = 0$.*

(ix) *There exists a $0 \preccurlyeq X \in \mathbb{C}_{\mathrm{H}}^{m}$ with $r(X) = q$ such that $A + X = 0$ if and only if $A \preccurlyeq 0$ and $r(A) = q$.*

(x) *There exists a $0 \preccurlyeq X \in \mathbb{C}_{\mathrm{H}}^{m}$ with $r(X) = q$ such that $A + X \succ 0$ if and only if $i_{+}(A) \geqslant m - q$.*

(xi) *$A + X \succ 0$ holds for all $0 \preccurlyeq X \in \mathbb{C}_{\mathrm{H}}^{m}$ with $r(X) = q$ such if and only if $A \succ 0$, or $A \succcurlyeq 0$ and $q = m$.*

(xii) *There exists a $0 \preccurlyeq X \in \mathbb{C}_{\mathrm{H}}^{m}$ with $r(X) = q$ such that $A + X \succcurlyeq 0$ if and only if $i_{-}(A) \leqslant q$.*

(xiii) *$A + X \succcurlyeq 0$ holds for all $0 \preccurlyeq X \in \mathbb{C}_{\mathrm{H}}^{m}$ with $r(X) = q$ such if and only if $A \succcurlyeq 0$.*

(xiv) *There exists a $0 \preccurlyeq X \in \mathbb{C}_{\mathrm{H}}^{m}$ with $r(X) = q$ such that $A + X \prec 0$ if and only if $A \prec 0$.*

(xv) *$A + X \prec 0$ holds for all $0 \preccurlyeq X \in \mathbb{C}_{\mathrm{H}}^{m}$ with $r(X) = q$ such if and only if $A \prec 0$ and $q = 0$.*

(xvi) *There exists a $0 \preccurlyeq X \in \mathbb{C}_{\mathrm{H}}^{m}$ with $r(X) = q$ such that $A + X \preccurlyeq 0$ if and only if $A \preccurlyeq 0$ and $r(A) \geqslant q$.*

(xvii) $A + X \preccurlyeq 0$ *holds for all* $0 \preccurlyeq X \in \mathbb{C}_{\rm H}^{m}$ *with* $r(X) = q$ *such if and only if* $A \preccurlyeq 0$ *and* $q = 0$.

(xviii) *There exists a* $0 \preccurlyeq X \in \mathbb{C}_{\rm H}^{m}$ *with* $r(X) = q$ *such that* $A - X$ *is nonsingular if and only if* $r(A) \geqslant m - q$.

(xix) $A - X$ *is nonsingular for all* $X \in \mathbb{C}_{\rm H}^{m}$ *with* $r(X) = q$ *and* $X \succcurlyeq 0$ *such if and only if* $A \prec 0$, *or* $A \preccurlyeq 0$ *and* $q = m$, *or* $r(A) = m$ *and* $q = 0$.

(xx) *There exists a* $0 \preccurlyeq X \in \mathbb{C}_{\rm H}^{m}$ *with* $r(X) = q$ *such that* $A - X = 0$ *if and only if* $A \succcurlyeq 0$ *and* $r(A) = q$.

(xx) *There exists a* $0 \preccurlyeq X \in \mathbb{C}_{\rm H}^{m}$ *with* $r(X) = q$ *such that* $A - X \succ 0$ *if and only if* $A \succ 0$.

(xxii) $A - X \succ 0$ *holds for all* $0 \preccurlyeq X \in \mathbb{C}_{\rm H}^{m}$ *with* $r(X) = q$ *such if and only if* $A \succ 0$ *and* $q = 0$.

(xxiii) *There exists a* $0 \preccurlyeq X \in \mathbb{C}_{\rm H}^{m}$ *with* $r(X) = q$ *such that* $A - X \succcurlyeq 0$ *if and only if* $A \succcurlyeq 0$ *and* $r(A) \geqslant q$.

(xxvi) $A - X \succcurlyeq 0$ *holds for all* $0 \preccurlyeq X \in \mathbb{C}_{\rm H}^{m}$ *with* $r(X) = q$ *such that if and only if* $A \succcurlyeq 0$ *and* $q = 0$.

(xxv) *There exists a* $0 \preccurlyeq X \in \mathbb{C}_{\rm H}^{m}$ *with* $r(X) = q$ *such that* $A - X \prec 0$ *if and only if* $i_{-}(A) \geqslant m - q$.

(xxvi) $A - X \prec 0$ *holds for all* $0 \preccurlyeq X \in \mathbb{C}_{\rm H}^{m}$ *with* $r(X) = q$ *such that if and only if* $A \prec 0$ *and* $q = 0$.

(xxvii) *There exists a* $0 \preccurlyeq X \in \mathbb{C}_{\rm H}^{m}$ *with* $r(X) = q$ *such that* $A - X \preccurlyeq 0$ *if and only if* $i_{+}(A) \leqslant q$.

(xxviii) $A - X \preccurlyeq 0$ *holds for all* $0 \preccurlyeq X \in \mathbb{C}_{\rm H}^{m}$ *with* $r(X) = q$ *if and only if* $A \preccurlyeq 0$.

(c) *The following equalities hold*

$$\max_{0 \preccurlyeq X,\, p \leqslant r(X) \leqslant q} r(\, A + X\,) = \min\{\, m, \quad r(A) + q\,\}, \tag{3.52}$$

$$\min_{0 \preccurlyeq X,\, p \leqslant r(X) \leqslant q} r(\, A + X\,) = \max\{\, i_{+}(A), \ p - i_{-}(A), \ r(A) - q\,\}, \tag{3.53}$$

$$\max_{0 \preccurlyeq X,\, p \leqslant r(X) \leqslant q} i_{+}(\, A + X\,) = \min\{\, m, \quad i_{+}(A) + q\,\}, \tag{3.54}$$

$$\min_{0 \preccurlyeq X,\, p \leqslant r(X) \leqslant q} i_{+}(\, A + X\,) = \max\{\, i_{+}(A), \quad p - i_{-}(A)\}, \tag{3.55}$$

$$\max_{0 \preccurlyeq X,\, p \leqslant r(X) \leqslant q} i_{-}(\, A + X\,) = i_{-}(A), \tag{3.56}$$

$$\min_{0 \preccurlyeq X,\, p \leqslant r(X) \leqslant q} i_{-}(\, A + X\,) = \max\{\, 0, \quad i_{-}(A) - q\,\}, \tag{3.57}$$

$$\max_{0 \preccurlyeq X,\, p \leqslant r(X) \leqslant q} r(\, A - X\,) = \min\{\, m, \quad r(A) + q\,\}, \tag{3.58}$$

$$\min_{0 \preccurlyeq X,\, p \leqslant r(X) \leqslant q} r(\, A - X\,) = \max\{\, i_{-}(A), \ p - i_{+}(A), \ r(A) - q\,\}, \tag{3.59}$$

$$\max_{0 \preccurlyeq X,\, p \leqslant r(X) \leqslant q} i_{+}(\, A - X\,) = i_{+}(A), \tag{3.60}$$

$$\min_{0 \preccurlyeq X,\, p \leqslant r(X) \leqslant q} i_{+}(\, A - X\,) = \max\{\, 0, \quad i_{+}(A) - q\,\}, \tag{3.61}$$

$$\max_{0 \preccurlyeq X,\, p \leqslant r(X) \leqslant q} i_{-}(\, A - X\,) = \min\{\, m, \quad i_{-}(A) + q\,\}, \tag{3.62}$$

$$\min_{0 \preccurlyeq X,\, p \leqslant r(X) \leqslant q} i_{-}(\, A - X\,) = \max\{\, i_{-}(A), \quad p - i_{+}(A)\}. \tag{3.63}$$

In consequence, the following hold.

(i) *There exists a* $0 \preccurlyeq X \in \mathbb{C}_{\rm H}^{m}$ *with* $p \leqslant r(X) \leqslant q$ *such that* $A + X$ *is nonsingular if and only if* $r(A) \geqslant m - q$.

(ii) $A + X$ *is nonsingular for all* $0 \preccurlyeq X \in \mathbb{C}_{\rm H}^{m}$ *with* $p \leqslant r(X) \leqslant q$ *such if and only if* $A \succ 0$, *or* $A \succcurlyeq 0$ *and* $q = m$, *or* $r(A) = m$ *and* $q = 0$.

(iii) *There exists a* $0 \preccurlyeq X \in \mathbb{C}_{\rm H}^{m}$ *with* $p \leqslant r(X) \leqslant q$ *such that* $A + X = 0$ *if and only if* $A \preccurlyeq 0$ *and* $p \leqslant r(A) \leqslant q$.

(iv) *There exists a* $0 \preccurlyeq X \in \mathbb{C}_{\rm H}^{m}$ *with* $p \leqslant r(X) \leqslant q$ *such that* $A + X \succ 0$ *if and only if* $i_{+}(A) \geqslant m - q$.

(v) $A + X \succ 0$ *holds for all* $0 \preccurlyeq X \in \mathbb{C}_{\rm H}^{m}$ *with* $p \leqslant r(X) \leqslant q$ *if and only if* $A \succ 0$, *or* $A \succcurlyeq 0$ *and* $p = q = m$.

(vi) *There exists a* $0 \preccurlyeq X \in \mathbb{C}_{\rm H}^{m}$ *with* $p \leqslant r(X) \leqslant q$ *such that* $A + X \succcurlyeq 0$ *if and only if* $i_{-}(A) \leqslant q$.

(vii) $A + X \succcurlyeq 0$ *holds for all* $0 \preccurlyeq X \in \mathbb{C}_{\rm H}^{m}$ *with* $p \leqslant r(X) \leqslant q$ *if and only if* $A \succcurlyeq 0$.

(viii) *There exists a* $0 \preccurlyeq X \in \mathbb{C}_{\rm H}^{m}$ *with* $p \leqslant r(X) \leqslant q$ *such that* $A + X \prec 0$ *if and only if* $A \prec 0$.

(ix) $A + X \prec 0$ *holds for all* $0 \preccurlyeq X \in \mathbb{C}_{\rm H}^{m}$ *with* $p \leqslant r(X) \leqslant q$ *such if and only if* $A \prec 0$ *and* $q = 0$.

(x) *There exists a* $0 \preccurlyeq X \in \mathbb{C}_{\rm H}^{m}$ *with* $p \leqslant r(X) \leqslant q$ *such that* $A + X \preccurlyeq 0$ *if and only if* $A \preccurlyeq 0$ *and* $r(A) \geqslant p$.

(xi) *$A + X \preccurlyeq 0$ holds for all $0 \preccurlyeq X \in \mathbb{C}_{\mathrm{H}}^{m}$ with $p \leqslant r(X) \leqslant q$ if and only if $A \preccurlyeq 0$ and $q = 0$.*

(xii) *There exists a $0 \preccurlyeq X \in \mathbb{C}_{\mathrm{H}}^{m}$ with $p \leqslant r(X) \leqslant q$ such that $A - X$ is nonsingular if and only if $r(A) \geqslant m - q$.*

(xiii) *$A - X$ is nonsingular for all $0 \preccurlyeq X \in \mathbb{C}_{\mathrm{H}}^{m}$ with $p \leqslant r(X) \leqslant q$ if and only if $A \prec 0$, or $A \preccurlyeq 0$ and $q = m$, or $r(A) = m$ and $q = 0$.*

(xiv) *There exists a $0 \preccurlyeq X \in \mathbb{C}_{\mathrm{H}}^{m}$ with $p \leqslant r(X) \leqslant q$ such that $A - X = 0$ if and only if $A \succcurlyeq 0$ and $r(A) = q$.*

(xv) *There exists a $0 \preccurlyeq X \in \mathbb{C}_{\mathrm{H}}^{m}$ with $p \leqslant r(X) \leqslant q$ such that $A - X \succ 0$ if and only if $A \succ 0$.*

(xvi) *$A - X \succ 0$ holds for all $0 \preccurlyeq X \in \mathbb{C}_{\mathrm{H}}^{m}$ with $p \leqslant r(X) \leqslant q$ if and only if $A \succ 0$ and $q = 0$.*

(xvii) *There exists a $0 \preccurlyeq X \in \mathbb{C}_{\mathrm{H}}^{m}$ with $p \leqslant r(X) \leqslant q$ such that $A - X \succcurlyeq 0$ if and only if $A \succcurlyeq 0$ and $r(A) \geqslant p$.*

(xviii) *$A - X \succcurlyeq 0$ holds for all $0 \preccurlyeq X \in \mathbb{C}_{\mathrm{H}}^{m}$ with $p \leqslant r(X) \leqslant q$ if and only if $A \succcurlyeq 0$ and $q = 0$.*

(xix) *There exists a $0 \preccurlyeq X \in \mathbb{C}_{\mathrm{H}}^{m}$ with $p \leqslant r(X) \leqslant q$ such that $A - X \prec 0$ if and only if $i_{-}(A) \geqslant m - q$.*

(xx) *$A - X \prec 0$ holds for all $0 \preccurlyeq X \in \mathbb{C}_{\mathrm{H}}^{m}$ with $p \leqslant r(X) \leqslant q$ if and only if $A \prec 0$ and $p = q = m$.*

(xxi) *There exists a $0 \preccurlyeq X \in \mathbb{C}_{\mathrm{H}}^{m}$ with $p \leqslant r(X) \leqslant q$ such that $A - X \preccurlyeq 0$ if and only if $i_{+}(A) \leqslant q$.*

(xxii) *$A - X \preccurlyeq 0$ holds for all $0 \preccurlyeq X \in \mathbb{C}_{\mathrm{H}}^{m}$ with $p \leqslant r(X) \leqslant q$ if and only if $A \preccurlyeq 0$.*

(c) *The following equalities hold*

$$\max_{0 \preccurlyeq X,\, 0 \leqslant r(X) \leqslant q} r(\, A + X\,) = \min\{\, m, \quad r(A) + q\,\}, \tag{3.64}$$

$$\min_{0 \preccurlyeq X,\, 0 \leqslant r(X) \leqslant q} r(\, A + X\,) = \max\{\, i_{+}(A), \ r(A) - q\,\}, \tag{3.65}$$

$$\max_{0 \preccurlyeq X,\, 0 \leqslant r(X) \leqslant q} i_{+}(\, A + X\,) = \min\{\, m, \quad i_{+}(A) + q\,\}, \tag{3.66}$$

$$\min_{0 \preccurlyeq X,\, 0 \leqslant r(X) \leqslant q} i_{+}(\, A + X\,) = i_{+}(A), \tag{3.67}$$

$$\max_{0 \preccurlyeq X,\, 0 \leqslant r(X) \leqslant q} i_{-}(\, A + X\,) = i_{-}(A), \tag{3.68}$$

$$\min_{0 \preccurlyeq X,\, 0 \leqslant r(X) \leqslant q} i_{-}(\, A + X\,) = \max\{\, 0, \quad i_{-}(A) - q\,\}, \tag{3.69}$$

$$\max_{0 \preccurlyeq X,\, 0 \leqslant r(X) \leqslant q} r(\, A - X\,) = \min\{\, m, \quad r(A) + q\,\}, \tag{3.70}$$

$$\min_{0 \preccurlyeq X,\, 0 \leqslant r(X) \leqslant q} r(\, A - X\,) = \max\{\, i_{-}(A), \quad r(A) - q\,\}, \tag{3.71}$$

$$\max_{0 \preccurlyeq X,\, 0 \leqslant r(X) \leqslant q} i_{+}(\, A - X\,) = i_{+}(A), \tag{3.72}$$

$$\min_{0 \preccurlyeq X,\, 0 \leqslant r(X) \leqslant q} i_{+}(\, A - X\,) = \max\{\, 0, \quad i_{+}(A) - q\,\}, \tag{3.73}$$

$$\max_{0 \preccurlyeq X,\, 0 \leqslant r(X) \leqslant q} i_{-}(\, A - X\,) = \min\{\, m, \quad i_{-}(A) + q\,\}, \tag{3.74}$$

$$\min_{0 \preccurlyeq X,\, 0 \leqslant r(X) \leqslant q} i_{-}(\, A - X\,) = i_{-}(A). \tag{3.75}$$

(d) *The following equalities hold*

$$\max_{0\preccurlyeq X,\, p\leqslant r(X)\leqslant m} r(\,A+X\,) = m, \tag{3.76}$$

$$\min_{0\preccurlyeq X,\, p\leqslant r(X)\leqslant m} r(\,A+X\,) = \max\{\, i_{+}(A),\;\, p-i_{-}(A)\,\}, \tag{3.77}$$

$$\max_{0\preccurlyeq X,\, p\leqslant r(X)\leqslant m} i_{+}(\,A+X\,) = m, \tag{3.78}$$

$$\min_{0\preccurlyeq X,\, p\leqslant r(X)\leqslant m} i_{+}(\,A+X\,) = \max\,\{\, i_{+}(A), \quad p-i_{-}(A)\}\,, \tag{3.79}$$

$$\max_{0\preccurlyeq X,\, p\leqslant r(X)\leqslant m} i_{-}(\,A+X\,) = i_{-}(A), \tag{3.80}$$

$$\min_{0\preccurlyeq X,\, p\leqslant r(X)\leqslant m} i_{-}(\,A+X\,) = 0, \tag{3.81}$$

$$\max_{0\preccurlyeq X,\, p\leqslant r(X)\leqslant m} r(\,A-X\,) = m, \tag{3.82}$$

$$\min_{0\preccurlyeq X,\, p\leqslant r(X)\leqslant m} r(\,A-X\,) = \max\{\, i_{-}(A),\;\, p-i_{+}(A)\,\}, \tag{3.83}$$

$$\max_{0\preccurlyeq X,\, p\leqslant r(X)\leqslant m} i_{+}(\,A-X\,) = i_{+}(A), \tag{3.84}$$

$$\min_{0\preccurlyeq X,\, p\leqslant r(X)\leqslant m} i_{+}(\,A-X\,) = 0, \tag{3.85}$$

$$\max_{0\preccurlyeq X,\, p\leqslant r(X)\leqslant m} i_{-}(\,A-X\,) = m, \tag{3.86}$$

$$\min_{0\preccurlyeq X,\, p\leqslant r(X)\leqslant m} i_{-}(\,A-X\,) = \max\,\{\, i_{-}(A), \quad p-i_{+}(A)\}\,. \tag{3.87}$$

Proof. Without loss of generality, we assume that A is of the form (3.26). It is easy to see from (2.1)–(2.6) that the right-hand sides of (3.40), (3.42) are (3.44) are upper bounds, while the right-hand sides of (3.41), (3.43) are (3.45) are lower bounds. The establishments of (3.40)–(3.45) are based on the following assertions:

(i) Set $X = \mathrm{diag}\{\, 0_{n-q},\, 2I_q\,\}$. If $m \leqslant r(A)+q$, then $r(\,A_1+X\,) = m$; if $m > r(A)+q$, then $r(\,A_1+X\,) = r(A)+r(X) = r(A)+q$, so that (3.40) holds.

(ii) Set $X = \mathrm{diag}\{\, I_q,\; 0_{n-q}\,\}$. If $q \leqslant i_{-}(A)$, then $r(\,A_2+X\,) = t-q+s = r(A)-q$; if $i_{-}(A) \leqslant q \leqslant r(A)$, then $r(\,A_2+X\,) = s = i_{+}(A)$; if $q \geqslant r(A)$, then $r(\,A_2+X\,) = q-t = q-i_{-}(A)$, establishing (3.41).

(iii) Set $X = \mathrm{diag}\{\, 0_{n-q},\; 2I_q\,\}$. If $m \leqslant i_{+}(A)+q$, then $i_{+}(\,A_1+X\,) = m$; if $m > i_{+}(A)+q$, then $i_{+}(\,A_1+X\,) = s+q = i_{+}(A)+q$, establishing (3.42).

(iv) Set $X = \mathrm{diag}\{\, I_q,\; 0_{n-q}\,\}$. If $q \leqslant r(A)$, then $i_{+}(\,A_1+X\,) = s = i_{+}(A)$; if $q \geqslant r(A)$, then $r(\,A_1+X\,) = q-t = q-i_{-}(A)$, establishing (3.43).

(v) Set $X = \mathrm{diag}\{\, I_q/2,\; 0_{n-q}\,\}$. Then $i_{-}(\,A_1+X\,) = t = i_{-}(A)$, establishing (3.44).

(vi) Set $X = \mathrm{diag}\{\, I_q,\, 0_{n-q}\,\}$. If $q \leqslant i_{-}(A)$, then $i_{-}(\,A_2+X\,) = t-q = i_{-}(A)-q$; if $q > i_{-}(A)$, then $i_{-}(\,A_2+X\,) = 0$, establishing (3.45).

Eqs. (3.46)–(3.51) can be shown similarly.

Note that the following decomposition of the cone of positive semi-definite matrices holds

$$\begin{aligned}&\{0 \preccurlyeq X \in \mathbb{C}_{\mathrm{H}}^{m} \,|\, p \leqslant r(X) \leqslant q\}\\ &= \{0 \preccurlyeq X \in \mathbb{C}_{\mathrm{H}}^{m} \,|\, r(X) = p\,\} \cup \{\, 0 \preccurlyeq X \in \mathbb{C}_{\mathrm{H}}^{m} \,|\, r(X) = p+1\,\} \cup \cdots \cup \{\, 0 \preccurlyeq X \in \mathbb{C}_{\mathrm{H}}^{m} \,|\, r(X) = q\,\}\,.\end{aligned} \tag{3.88}$$

So that

$$\max_{0\preccurlyeq X,\, p\leqslant r(X)\leqslant q} r(\, A+X\,)$$
$$= \max\left\{ \max_{0\preccurlyeq X,\, r(X)=p} r(\, A+X\,),\ \max_{0\preccurlyeq X,\, r(X)=p+1} r(\, A+X\,),\, \ldots,\, \max_{0\preccurlyeq X,\, r(X)=q} r(\, A+X\,) \right\}, \tag{3.89}$$

$$\min_{0\preccurlyeq X,\, p\leqslant r(X)\leqslant q} r(\, A+X\,)$$
$$= \min\left\{ \min_{0\preccurlyeq X,\, r(X)=p} r(\, A+X\,),\ \min_{0\preccurlyeq X,\, r(X)=p+1} r(\, A+X\,),\, \ldots,\, \min_{0\preccurlyeq X,\, r(X)=q} r(\, A+X\,) \right\}, \tag{3.90}$$

$$\max_{0\preccurlyeq X,\, p\leqslant r(X)\leqslant q} i_{+}(\, A+X\,)$$
$$= \max\left\{ \max_{0\preccurlyeq X,\, r(X)=p} i_{+}(\, A+X\,),\ \max_{0\preccurlyeq X,\, r(X)=p+1} i_{+}(\, A+X\,),\, \ldots,\, \max_{0\preccurlyeq X,\, r(X)=q} i_{+}(\, A+X\,) \right\}, \tag{3.91}$$

$$\min_{0\preccurlyeq X,\, p\leqslant r(X)\leqslant q} i_{+}(\, A+X\,)$$
$$= \min\left\{ \min_{0\preccurlyeq X,\, r(X)=p} i_{+}(\, A+X\,),\ \min_{0\preccurlyeq X,\, r(X)=p+1} i_{+}(\, A+X\,),\, \ldots,\, \min_{0\preccurlyeq X,\, r(X)=q} i_{+}(\, A+X\,) \right\}, \tag{3.92}$$

$$\max_{0\preccurlyeq X,\, p\leqslant r(X)\leqslant q} i_{-}(\, A+X\,)$$
$$= \max\left\{ \max_{0\preccurlyeq X,\, r(X)=p} i_{-}(\, A+X\,),\ \max_{0\preccurlyeq X,\, r(X)=p+1} i_{-}(\, A+X\,),\, \ldots,\, \max_{0\preccurlyeq X,\, r(X)=q} i_{-}(\, A+X\,) \right\}, \tag{3.93}$$

$$\min_{0\preccurlyeq X,\, p\leqslant r(X)\leqslant q} i_{-}(\, A+X\,)$$
$$= \min\left\{ \min_{0\preccurlyeq X,\, r(X)=p} i_{-}(\, A+X\,),\ \min_{0\preccurlyeq X,\, r(X)=p+1} i_{-}(\, A+X\,),\, \ldots,\, \min_{0\preccurlyeq X,\, r(X)=q} i_{-}(\, A+X\,) \right\}. \tag{3.94}$$

Substituting (3.40)–(3.45) for $r(X) = p,\ p+1, \ldots, q$ into (3.89)–(3.94) and making the max-min comparison, we obtain (3.52)–(3.57). Eqs. (3.58)–(3.63) hold by symmetry. Results (c) and (d) follow directly from (b). □

It is easy to see that (3.40)–(3.51) also hold for any $p < 0$ and $q > m$. So that we can use (3.40)–(3.51) under the condition

$$p \leqslant m, \quad 0 \leqslant q, \quad p \leqslant q. \tag{3.95}$$

Observe that all the results in this section are derived from the canonical forms with ± 1 and 0 of A, while the variable matrix X are taken as ± 1, $\pm j$ and 0. So that the completions of $A + X$ with extremal ranks and inertias are in fact some special types of discrete optimization problems with both variables and objective functions being integers.

4 Rank and inertia formulas of $A+BXB^{*}$ when X is Hermitian with a fixed rank

Based on the results in the previous two sections, we are able to derive explicit solutions to Problems 1.1 and 1.2. To do so, we need the following known result on the canonical form of $A + BXB^{*}$ (see [8]).

Lemma 4.1 *Let $A \in \mathbb{C}_{\mathrm{H}}^{m}$ and $B \in \mathbb{C}^{m\times n}$ be given, and $X \in \mathbb{C}_{\mathrm{H}}^{n}$ be a variable matrix. Then,*

(i) *$A + BXB^{*}$ can be decomposed as*

$$A + BXB^{*} = P^{*}D_{A}P + P^{*}D_{B}UXU^{*}D_{B}^{T}P = P^{*}(\, D_{A} + D_{B}YD_{B}^{T}\,)P, \tag{4.1}$$

where P, D_A and D_B are as given in (2.80), *and $Y = UXU^{*}$ satisfies*

$$X \in \mathbb{C}_{\mathrm{H}}^{n} \quad \Leftrightarrow \quad Y \in \mathbb{C}_{\mathrm{H}}^{n}. \tag{4.2}$$

(ii) *The inertia and rank of $A + BXB^{*}$ satisfy*

$$r(\, A + BXB^{*}\,) = r(\, D_{A} + D_{B}YD_{B}^{T}\,), \tag{4.3}$$
$$i_{\pm}(\, A + BXB^{*}\,) = i_{\pm}(\, D_{A} + D_{B}YD_{B}^{T}\,) \tag{4.4}$$

for $Y = UXU^{*}$ and $X \in \mathbb{C}_{\mathrm{H}}^{n}$.

(iii) *Partition the Hermitian matrix Y as*

$$Y = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ Y_{12}^* & Y_{22} & Y_{23} & Y_{24} \\ Y_{13}^* & Y_{23}^* & Y_{33} & Y_{34} \\ Y_{14}^* & Y_{24}^* & Y_{34}^* & Y_{44} \end{bmatrix} \begin{matrix} f \\ t \\ l-t \\ n-(f+l) \end{matrix}, \quad \begin{matrix} f & t & l-t & n-(f+l) \end{matrix} \tag{4.5}$$

where $Y_{ii} = Y_{ii}^*,\ i = 1, 2, 3, 4$. Then,

$$D_A + D_B Y D_B^T = \begin{bmatrix} Y_{11} & I & Y_{12}\Sigma & 0 & Y_{13} & 0 \\ I & 0 & 0 & 0 & 0 & 0 \\ \Sigma Y_{12}^* & 0 & \Delta_1 + \Sigma Y_{22}\Sigma & 0 & \Sigma Y_{23} & 0 \\ 0 & 0 & 0 & \Delta_2 & 0 & 0 \\ Y_{13}^* & 0 & Y_{23}^*\Sigma & 0 & Y_{33} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{4.6}$$

and following expansion formulas hold

$$r(\, A + BXB^* \,) = 2f + k + r\begin{bmatrix} Y_{22} + \Sigma^{-1}\Delta_1\Sigma^{-1} & Y_{23} \\ Y_{23}^* & Y_{33} \end{bmatrix} = 2f + k + r(\, \widehat{A} + \widehat{Y} \,), \tag{4.7}$$

$$i_+(\, A + BXB^* \,) = f + s_2 + i_+\begin{bmatrix} Y_{22} + \Sigma^{-1}\Delta_1\Sigma^{-1} & Y_{23} \\ Y_{23}^* & Y_{33} \end{bmatrix} = f + s_2 + i_+(\, \widehat{A} + \widehat{Y} \,), \tag{4.8}$$

$$i_-(\, A + BXB^* \,) = f + k - s_2 + i_-\begin{bmatrix} Y_{22} + \Sigma^{-1}\Delta_1\Sigma^{-1} & Y_{23} \\ Y_{23}^* & Y_{33} \end{bmatrix} = f + k - s_2 + i_-(\, \widehat{A} + \widehat{Y} \,), \tag{4.9}$$

where

$$\widehat{A} = \begin{bmatrix} \Sigma^{-1}\Delta_1\Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \quad \widehat{Y} = \begin{bmatrix} Y_{22} & Y_{23} \\ Y_{23}^* & Y_{33} \end{bmatrix}. \tag{4.10}$$

It can be seen from (4.7)–(4.9) that

$$\max_{X \in \mathbb{C}_{\mathrm{H}}^n} r(\, A + BXB^* \,) = 2f + k + \max_{\widehat{Y} \in \mathbb{C}_{\mathrm{H}}^l} r(\, \widehat{A} + \widehat{Y} \,), \tag{4.11}$$

$$\min_{X \in \mathbb{C}_{\mathrm{H}}^n} r(\, A + BXB^* \,) = 2f + k + \min_{\widehat{Y} \in \mathbb{C}_{\mathrm{H}}^l} r(\, \widehat{A} + \widehat{Y} \,), \tag{4.12}$$

$$\max_{X \in \mathbb{C}_{\mathrm{H}}^n} i_+(\, A + BXB^* \,) = f + s_2 + \max_{\widehat{Y} \in \mathbb{C}_{\mathrm{H}}^l} i_+(\, \widehat{A} + \widehat{Y} \,), \tag{4.13}$$

$$\min_{X \in \mathbb{C}_{\mathrm{H}}^n} i_+(\, A + BXB^* \,) = f + s_2 + \min_{\widehat{Y} \in \mathbb{C}_{\mathrm{H}}^l} i_+(\, \widehat{A} + \widehat{Y} \,), \tag{4.14}$$

$$\max_{X \in \mathbb{C}_{\mathrm{H}}^n} i_-(\, A + BXB^* \,) = f + k - s_2 + \max_{\widehat{Y} \in \mathbb{C}_{\mathrm{H}}^l} i_-(\, \widehat{A} + \widehat{Y} \,), \tag{4.15}$$

$$\min_{X \in \mathbb{C}_{\mathrm{H}}^n} i_-(\, A + BXB^* \,) = f + k - s_2 + \min_{\widehat{Y} \in \mathbb{C}_{\mathrm{H}}^l} i_-(\, \widehat{A} + \widehat{Y} \,). \tag{4.16}$$

So that the following result is obvious.

Lemma 4.2 ([13]) *Let $A \in \mathbb{C}_{\mathrm{H}}^m$ and $B \in \mathbb{C}^{m \times n}$ be given, and $X \in \mathbb{C}_{\mathrm{H}}^n$ be a variable matrix. Also define*

$$M = \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}. \tag{4.17}$$

Then, the global maximal and minimal rank and inertias of $A + BXB^*$ are given by

$$\max_{X \in \mathbb{C}_{\mathrm{H}}^n} r(\, A + BXB^* \,) = r[\, A,\ B \,], \tag{4.18}$$

$$\min_{X \in \mathbb{C}_{\mathrm{H}}^n} r(\, A + BXB^* \,) = 2r[\, A,\ B \,] - r(M), \tag{4.19}$$

$$\max_{X \in \mathbb{C}_{\mathrm{H}}^n} i_{\pm}(\, A + BXB^* \,) = i_{\pm}(M), \tag{4.20}$$

$$\min_{X \in \mathbb{C}_{\mathrm{H}}^n} i_{\pm}(\, A + BXB^* \,) = r[\, A,\ B \,] - i_{\mp}(M). \tag{4.21}$$

In what follows, we derive from (4.7)–(4.9) the global maximal and minimal rank and inertias of $A + BXB^*$ with rank restrictions on the rank of X.

Theorem 4.3 *Let $A \in \mathbb{C}_{\rm H}^{m}$ and $B \in \mathbb{C}^{m\times n}$ be given, $X \in \mathbb{C}_{\rm H}^{n}$ be a variable matrix, M be the matrix in (4.17), and assume that q is an integer satisfying $0 \leqslant q \leqslant n$. Then, the following equalities hold*

$$\max_{X\in\mathbb{C}_{\rm H}^{n},\, r(X)=q} r(\,A + BXB^*\,) = \min\{\, r[\,A,\, B\,],\quad r(A) + q\,\}, \tag{4.22}$$

$$\min_{X\in\mathbb{C}_{\rm H}^{n},\, r(X)=q} r(\,A + BXB^*\,) = \max\{\, 2r[\,A,\ B\,] - r(M),\quad 2r[\,A,\, B\,] - r(A) + q - 2n,\quad r(A) - q\,\}, \tag{4.23}$$

$$\max_{X\in\mathbb{C}_{\rm H}^{n},\, r(X)=q} i_{+}(\,A + BXB^*\,) = \min\{\, i_{+}(M),\quad i_{+}(A) + q\,\}, \tag{4.24}$$

$$\min_{X\in\mathbb{C}_{\rm H}^{n},\, r(X)=q} i_{+}(\,A + BXB^*\,) = \max\{\, r[\,A,\ B\,] - i_{-}(M),\quad i_{+}(A) - q\,\}, \tag{4.25}$$

$$\max_{X\in\mathbb{C}_{\rm H}^{n},\, r(X)=q} i_{-}(\,A + BXB^*\,) = \min\{\, i_{-}(M),\quad i_{-}(A) + q\,\}, \tag{4.26}$$

$$\min_{X\in\mathbb{C}_{\rm H}^{n},\, r(X)=q} i_{-}(\,A + BXB^*\,) = \max\{\, r[\,A,\ B\,] - i_{+}(M),\quad i_{-}(A) - q\,\}. \tag{4.27}$$

In particular,

$$\max_{X\in\mathbb{C}_{\rm H}^{n},\, r(X)=n} r(\,A + BXB^*\,) = r[\,A,\, B\,], \tag{4.28}$$

$$\min_{X\in\mathbb{C}_{\rm H}^{n},\, r(X)=n} r(\,A + BXB^*\,) = \max\{\, 2r[\,A,\ B\,] - r(M),\quad 2r[\,A,\, B\,] - r(A) - n\,\}, \tag{4.29}$$

$$\max_{X\in\mathbb{C}_{\rm H}^{n},\, r(X)=n} i_{+}(\,A + BXB^*\,) = i_{+}(M), \tag{4.30}$$

$$\min_{X\in\mathbb{C}_{\rm H}^{n},\, r(X)=n} i_{+}(\,A + BXB^*\,) = \max\{\, r[\,A,\ B\,] - i_{-}(M),\quad i_{+}(A) - n\,\}, \tag{4.31}$$

$$\max_{X\in\mathbb{C}_{\rm H}^{n},\, r(X)=n} i_{-}(\,A + BXB^*\,) = i_{-}(M), \tag{4.32}$$

$$\min_{X\in\mathbb{C}_{\rm H}^{n},\, r(X)=n} i_{-}(\,A + BXB^*\,) = \max\{\, r[\,A,\ B\,] - i_{+}(M),\quad i_{-}(A) - n\,\}. \tag{4.33}$$

In consequence, the following hold.

(i) *There exists an $X \in \mathbb{C}_{\rm H}^{n}$ with $r(X) = q$ such that $A + BXB^*$ is nonsingular if and only if $r[\,A,\, B\,] = m$ and $r(A) \geqslant m - q$.*

(ii) *$A + BXB^*$ is nonsingular for all $X \in \mathbb{C}_{\rm H}^{n}$ with $r(X) = q$ if and only if $r(A) = m$ and $B^*A^{-1}B = 0$, or $2r[\,A,\, B\,] - r(A) + q - 2n = m$, or $r(A) = m$ and $q = 0$.*

(iii) *There exists an $X \in \mathbb{C}_{\rm H}^{n}$ with $r(X) = q$ such that $A + BXB^* = 0$ if and only if $\mathscr{R}(A) \subseteq \mathscr{R}(B)$ and $r(A) \leqslant q \leqslant r(A) - 2r(B) + 2n$.*

(iv) *There exists an $X \in \mathbb{C}_{\rm H}^{n}$ with $r(X) = q$ such that $A + BXB^* \succ 0$ if and only if $i_{+}(M) = m$ and $i_{+}(A) \geqslant m - q$, or $i_{+}(M) \geqslant m$ and $i_{+}(A) = m - q$.*

(v) *$A + BXB^* \succ 0$ for all $X \in \mathbb{C}_{\rm H}^{n}$ with $r(X) = q$ if and only if $A \succ 0$ and $q = 0$.*

(vi) *There exists an $X \in \mathbb{C}_{\rm H}^{n}$ with $r(X) = q$ such that $A + BXB^* \succcurlyeq 0$ if and only if $i_{+}(M) = r[\,A,\, B\,]$ and $i_{-}(A) \leqslant q$.*

(vii) *$A + BXB^* \succcurlyeq 0$ for all $X \in \mathbb{C}_{\rm H}^{n}$ with $r(X) = q$ if and only if $A \succcurlyeq 0$ and $B = 0$, or $A \succcurlyeq 0$ and $q = 0$.*

(viii) *There exists an $X \in \mathbb{C}_{\rm H}^{n}$ with $r(X) = q$ such that $A + BXB^* \prec 0$ if and only if $i_{-}(M) = m$ and $i_{-}(A) \geqslant m - q$, or $i_{-}(M) \geqslant m$ and $i_{-}(A) = m - q$.*

(ix) *$A + BXB^* \prec 0$ for all $X \in \mathbb{C}_{\rm H}^{n}$ with $r(X) = q$ if and only if $A \prec 0$ and $q = 0$.*

(x) *There exists an $X \in \mathbb{C}_{\rm H}^{n}$ with $r(X) = q$ such that $A + BXB^* \preccurlyeq 0$ if and only if $i_{-}(M) = r[\,A,\, B\,]$ and $i_{+}(A) \leqslant q$.*

(xi) *$A + BXB^* \preccurlyeq 0$ for all $X \in \mathbb{C}_{\rm H}^{n}$ with $r(X) = q$ if and only if $A \preccurlyeq 0$ and $B = 0$, or $A \preccurlyeq 0$ and $q = 0$.*

Proof. It can be seen from (4.1) and (4.2) that

$$X \in \mathbb{C}_{\rm H}^{n} \ \text{ and } \ r(X) = q \Leftrightarrow Y \in \mathbb{C}_{\rm H}^{n} \ \text{ and } \ r(Y) = q. \tag{4.34}$$

In this case, we derive from (4.7)–(4.9) that

$$\max_{X\in\mathbb{C}_{\mathrm{H}}^{n},\, r(X)=q} r(\,A+BXB^{*}\,) = 2f+k+\max_{Y\in\mathbb{C}_{\mathrm{H}}^{n},\, r(Y)=q} r(\,\widehat{A}+\widehat{Y}\,), \tag{4.35}$$

$$\min_{X\in\mathbb{C}_{\mathrm{H}}^{n},\, r(X)=q} r(\,A+BXB^{*}\,) = 2f+k+\min_{Y\in\mathbb{C}_{\mathrm{H}}^{n},\, r(Y)=q} r(\,\widehat{A}+\widehat{Y}\,), \tag{4.36}$$

$$\max_{X\in\mathbb{C}_{\mathrm{H}}^{n},\, r(X)=q} i_{+}(\,A+BXB^{*}\,) = f+s_2+\max_{Y\in\mathbb{C}_{\mathrm{H}}^{n},\, r(Y)=q} i_{+}(\,\widehat{A}+\widehat{Y}\,), \tag{4.37}$$

$$\min_{X\in\mathbb{C}_{\mathrm{H}}^{n},\, r(X)=q} i_{+}(\,A+BXB^{*}\,) = f+s_2+\min_{Y\in\mathbb{C}_{\mathrm{H}}^{n},\, r(Y)=q} i_{+}(\,\widehat{A}+\widehat{Y}\,), \tag{4.38}$$

$$\max_{X\in\mathbb{C}_{\mathrm{H}}^{n},\, r(X)=q} i_{-}(\,A+BXB^{*}\,) = f+k-s_2+\max_{Y\in\mathbb{C}_{\mathrm{H}}^{n},\, r(Y)=q} i_{-}(\,\widehat{A}+\widehat{Y}\,), \tag{4.39}$$

$$\min_{X\in\mathbb{C}_{\mathrm{H}}^{n},\, r(X)=q} i_{-}(\,A+BXB^{*}\,) = f+k-s_2+\min_{Y\in\mathbb{C}_{\mathrm{H}}^{n},\, r(Y)=q} i_{-}(\,\widehat{A}+\widehat{Y}\,). \tag{4.40}$$

Also note from Lemma 2.7(b) and (4.5) that

$$Y\in\mathbb{C}_{\mathrm{H}}^{n} \ \text{ and } \ r(Y)=q \ \Leftrightarrow\ Y_{11}=Y_{11}^{*}, \ \ Y_{44}=Y_{44}^{*}, \ \ \widehat{Y}\in\mathbb{C}_{\mathrm{H}}^{l} \ \text{ and } \ q-2(n-l)\leqslant r(\widehat{Y})\leqslant q. \tag{4.41}$$

Under $0\leqslant q-2(n-l)\leqslant q\leqslant l$, applying Lemma 2.7(b) to $\widehat{A}+\widehat{Y}$ in (4.35)–(4.39) and simplifying by (2.81) and (2.83), we obtain

$$\max_{Y\in\mathbb{C}_{\mathrm{H}}^{n},\, r(Y)=q} r(\,\widehat{A}+\widehat{Y}\,) = \max_{q-2(n-l)\leqslant r(\widehat{Y})\leqslant q} r(\,\widehat{A}+\widehat{Y}\,) = \min\{\,l, \ \ r(\widehat{A})+q\,\} = \min\{\,l, \ \ t+q\,\}, \tag{4.42}$$

$$\begin{aligned}\min_{Y\in\mathbb{C}_{\mathrm{H}}^{n},\, r(Y)=q} r(\,\widehat{A}+\widehat{Y}\,) = \min_{q-2(n-l)\leqslant r(\widehat{Y})\leqslant q} r(\,\widehat{A}+\widehat{Y}\,) &= \max\{\,0, \ q-2(n-l)-r(\widehat{A}), \ \ r(\widehat{A})-q\,\}\\ &= \max\{\,0, \ 2l-t+q-2n, \ \ t-q\,\},\end{aligned} \tag{4.43}$$

$$\max_{Y\in\mathbb{C}_{\mathrm{H}}^{n},\, r(Y)=q} i_{+}(\,\widehat{A}+\widehat{Y}\,) = \max_{q-2(n-l)\leqslant r(\widehat{Y})\leqslant q} i_{+}(\,\widehat{A}+\widehat{Y}\,) = \min\{\,l, \ \ i_{+}(\widehat{A})+q\,\} = \min\{\,l, \ \ s_1+q\,\}, \tag{4.44}$$

$$\min_{Y\in\mathbb{C}_{\mathrm{H}}^{n},\, r(Y)=q} i_{+}(\,\widehat{A}+\widehat{Y}\,) = \min_{q-2(n-l)\leqslant r(\widehat{Y})\leqslant q} i_{+}(\,\widehat{A}+\widehat{Y}\,) = \max\{\,0, \ \ i_{+}(\widehat{A})-q\} = \max\{\,0, \ \ s_1-q\,\}, \tag{4.45}$$

$$\max_{Y\in\mathbb{C}_{\mathrm{H}}^{n},\, r(Y)=q} i_{-}(\,\widehat{A}+\widehat{Y}\,) = \max_{q-2(n-l)\leqslant r(\widehat{Y})\leqslant q} i_{-}(\,\widehat{A}+\widehat{Y}\,) = \min\{\,l, \ \ i_{-}(\widehat{A})+q\} = \min\{l, \ \ t-s_1+q\,\}, \tag{4.46}$$

$$\min_{Y\in\mathbb{C}_{\mathrm{H}}^{n},\, r(Y)=q} i_{-}(\,\widehat{A}+\widehat{Y}\,) = \min_{q-2(n-l)\leqslant r(\widehat{Y})\leqslant q} i_{-}(\,\widehat{A}+\widehat{Y}\,) = \max\{\,0, \ \ i_{-}(\widehat{A})-q\,\} = \max\{\,0, \ \ t-s_1-q\,\}. \tag{4.47}$$

Substituting (4.42)–(4.47) into (4.35)–(4.40) and simplifying by (2.85)–(2.92), we obtain (4.22)–(4.27).

Under $q-2(n-l)\leqslant 0$, applying Lemma 3.1(c) to $\widehat{A}+\widehat{Y}$ in (4.35)–(4.39) and simplifying by (2.81) and (2.83), we obtain

$$\max_{Y\in\mathbb{C}_{\mathrm{H}}^{n},\, r(Y)=q} r(\,\widehat{A}+\widehat{Y}\,) = \max_{0\leqslant r(\widehat{Y})\leqslant q} r(\,\widehat{A}+\widehat{Y}\,) = \min\{\,l, \ \ r(\widehat{A})+q\,\} = \min\{\,l, \ \ t+q\,\}, \tag{4.48}$$

$$\min_{Y\in\mathbb{C}_{\mathrm{H}}^{n},\, r(Y)=q} r(\,\widehat{A}+\widehat{Y}\,) = \min_{0\leqslant r(\widehat{Y})\leqslant q} r(\,\widehat{A}+\widehat{Y}\,) = \max\{\,0, \ r(\widehat{A})-q\,\} = \max\{\,0, \ \ t-q\,\}, \tag{4.49}$$

$$\max_{Y\in\mathbb{C}_{\mathrm{H}}^{n},\, r(Y)=q} i_{+}(\,\widehat{A}+\widehat{Y}\,) = \max_{0\leqslant r(\widehat{Y})\leqslant q} i_{+}(\,\widehat{A}+\widehat{Y}\,) = \min\{\,l, \ \ i_{+}(\widehat{A})+q\,\} = \min\{\,l, \ \ s_1+q\,\}, \tag{4.50}$$

$$\min_{Y\in\mathbb{C}_{\mathrm{H}}^{n},\, r(Y)=q} i_{+}(\,\widehat{A}+\widehat{Y}\,) = \min_{0\leqslant r(\widehat{Y})\leqslant q} i_{+}(\,\widehat{A}+\widehat{Y}\,) = \max\{\,0, \ \ i_{+}(\widehat{A})-q\} = \max\{\,0, \ \ s_1-q\,\}, \tag{4.51}$$

$$\max_{Y\in\mathbb{C}_{\mathrm{H}}^{n},\, r(Y)=q} i_{-}(\,\widehat{A}+\widehat{Y}\,) = \max_{0\leqslant r(\widehat{Y})\leqslant q} i_{-}(\,\widehat{A}+\widehat{Y}\,) = \min\{\,l, \ \ i_{-}(\widehat{A})+q\} = \min\{l, \ \ t-s_1+q\,\}, \tag{4.52}$$

$$\min_{Y\in\mathbb{C}_{\mathrm{H}}^{n},\, r(Y)=q} i_{-}(\,\widehat{A}+\widehat{Y}\,) = \min_{0\leqslant r(\widehat{Y})\leqslant q} i_{-}(\,\widehat{A}+\widehat{Y}\,) = \max\{\,0, \ \ i_{-}(\widehat{A})-q\,\} = \max\{\,0, \ \ t-s_1-q\,\}. \tag{4.53}$$

Under $l \leqslant q$, applying Lemma 3.1(d) to $\widehat{A} + \widehat{Y}$ in (4.35)–(4.39) and simplifying by (2.81) and (2.83), we obtain

$$\max_{Y \in \mathbb{C}_{\rm H}^{n},\, r(Y)=q} r(\, \widehat{A} + \widehat{Y}\,) = \max_{q-2(n-l) \leqslant r(\widehat{Y}) \leqslant l} r(\, \widehat{A} + \widehat{Y}\,) = l, \tag{4.54}$$

$$\min_{Y \in \mathbb{C}_{\rm H}^{n},\, r(Y)=q} r(\, \widehat{A} + \widehat{Y}\,) = \min_{q-2(n-l) \leqslant r(\widehat{Y}) \leqslant l} r(\, \widehat{A} + \widehat{Y}\,) = \max\{\, 0,\ q - 2(n-l) - r(\widehat{A})\,\} = \max\{\, 0,\ 2l - t + q - 2n\,\}, \tag{4.55}$$

$$\max_{Y \in \mathbb{C}_{\rm H}^{n},\, r(Y)=q} i_{+}(\, \widehat{A} + \widehat{Y}\,) = \max_{q-2(n-l) \leqslant r(\widehat{Y}) \leqslant l} i_{+}(\, \widehat{A} + \widehat{Y}\,) = l, \tag{4.56}$$

$$\min_{Y \in \mathbb{C}_{\rm H}^{n},\, r(Y)=q} i_{+}(\, \widehat{A} + \widehat{Y}\,) = \min_{q-2(n-l) \leqslant r(\widehat{Y}) \leqslant l} i_{+}(\, \widehat{A} + \widehat{Y}\,) = 0, \tag{4.57}$$

$$\max_{Y \in \mathbb{C}_{\rm H}^{n},\, r(Y)=q} i_{-}(\, \widehat{A} + \widehat{Y}\,) = \max_{q-2(n-l) \leqslant r(\widehat{Y}) \leqslant l} i_{-}(\, \widehat{A} + \widehat{Y}\,) = l, \tag{4.58}$$

$$\min_{Y \in \mathbb{C}_{\rm H}^{n},\, r(Y)=q} i_{-}(\, \widehat{A} + \widehat{Y}\,) = \min_{q-2(n-l) \leqslant r(\widehat{Y}) \leqslant l} i_{-}(\, \widehat{A} + \widehat{Y}\,) = 0. \tag{4.59}$$

These two groups of formulas are special cases (4.41)–(4.46), so that (4.22)–(4.27) hold for any q satisfying $0 \leqslant q \leqslant n$. $\square$

Corollary 4.4 *Let $\phi(X)$ and M be as given in* (1.1) *and* (4.17), *and assume that p and q are two integer satisfying $0 \leqslant p \leqslant q \leqslant n$. Then, the following equalities hold*

$$\max_{X \in \mathbb{C}_{\rm H}^{n},\, p \leqslant r(X) \leqslant q} r(\, A + BXB^{*}\,) = \min\{\, r[\, A,\, B\,],\ \ r(A) + q\,\}, \tag{4.60}$$

$$\min_{X \in \mathbb{C}_{\rm H}^{n},\, p \leqslant r(X) \leqslant q} r(\, A + BXB^{*}\,) = \min\{ u_{p},\ u_{p+1},\ \ldots,\ u_{q}\,\}, \tag{4.61}$$

$$\max_{X \in \mathbb{C}_{\rm H}^{n},\, p \leqslant r(X) \leqslant q} i_{+}(\, A + BXB^{*}\,) = \min\{\, i_{+}(M),\ \ i_{+}(A) + q\,\}, \tag{4.62}$$

$$\min_{X \in \mathbb{C}_{\rm H}^{n},\, p \leqslant r(X) \leqslant q} i_{+}(\, A + BXB^{*}\,) = \max\{\, r[\, A,\ B\,] - i_{-}(M),\ \ i_{+}(A) - q\,\}, \tag{4.63}$$

$$\max_{X \in \mathbb{C}_{\rm H}^{n},\, p \leqslant r(X) \leqslant q} i_{-}(\, A + BXB^{*}\,) = \min\{\, i_{-}(M),\ \ i_{-}(A) + q\,\}, \tag{4.64}$$

$$\min_{X \in \mathbb{C}_{\rm H}^{n},\, p \leqslant r(X) \leqslant q} i_{-}(\, A + BXB^{*}\,) = \max\{\, r[\, A,\ B\,] - i_{+}(M),\ \ i_{-}(A) - q\,\}, \tag{4.65}$$

where

$$\begin{aligned} u_{p} &= \max\{\, 2r[\, A,\ B\,] - r(M),\ 2r[\, A,\, B\,] - r(A) + p - 2n,\ r(A) - p\},\\ u_{p+1} &= \max\{\, 2r[\, A,\ B\,] - r(M),\ 2r[\, A,\, B\,] - r(A) + p + 1 - 2n,\ r(A) - p - 1\},\\ &\ \ \vdots\\ u_{q} &= \max\{\, 2r[\, A,\ B\,] - r(M),\ 2r[\, A,\, B\,] - r(A) + q - 2n,\ r(A) - q\,\}. \end{aligned}$$

In consequence, the following hold.

(i) *There exists an $X \in \mathbb{C}_{\rm H}^{n}$ with $p \leqslant r(X) \leqslant q$ such that $A + BXB^{*}$ is nonsingular if and only if $r[\, A,\, B\,] = m$ and $r(A) \geqslant m - q$.*

(ii) *$A + BXB^{*}$ is nonsingular for all $X \in \mathbb{C}_{\rm H}^{n}$ with $p \leqslant r(X) \leqslant q$ if and only if $u_{p} \geqslant m,\ u_{p+1} \geqslant m,\ \ldots,\ u_{q} \geqslant m$.*

(iii) *There exists an $X \in \mathbb{C}_{\rm H}^{n}$ with $p \leqslant r(X) \leqslant q$ such that $A + BXB^{*} = 0$ if and only if $\mathscr{R}(A) \subseteq \mathscr{R}(B)$ and $r(A) \leqslant p \leqslant q \leqslant r(A) - 2r(B) + 2n$.*

(iv) *There exists an $X \in \mathbb{C}_{\rm H}^{n}$ with $p \leqslant r(X) \leqslant q$ such that $A + BXB^{*} \succ 0$ if and only if $i_{+}(M) = m$ and $i_{+}(A) \geqslant m - q$, or $i_{+}(M) \geqslant m$ and $i_{+}(A) = m - q$.*

(v) *$A + BXB^{*} \succ 0$ for all $X \in \mathbb{C}_{\rm H}^{n}$ with $p \leqslant r(X) \leqslant q$ if and only if $A \succ 0$ and $p = q = 0$.*

(vi) *There exists an $X \in \mathbb{C}_{\rm H}^{n}$ with $p \leqslant r(X) \leqslant q$ such that $A + BXB^{*} \succcurlyeq 0$ if and only if $i_{+}(M) = r[\, A,\, B\,]$ and $i_{-}(A) \leqslant q$.*

(vii) *$A + BXB^{*} \succcurlyeq 0$ for all $X \in \mathbb{C}_{\rm H}^{n}$ with $p \leqslant r(X) \leqslant q$ if and only if $A \succcurlyeq 0$ and $B = 0$, or $A \succcurlyeq 0$ and $p = q = 0$.*

(viii) *There exists an* $X \in \mathbb{C}_{\rm H}^{n}$ *with* $p \leqslant r(X) \leqslant q$ *such that* $A + BXB^{*} \prec 0$ *if and only if* $i_{-}(M) = m$ *and* $i_{-}(A) \geqslant m - q$, *or* $i_{-}(M) \geqslant m$ *and* $i_{-}(A) = m - q$.

(ix) $A + BXB^{*} \prec 0$ *for all* $X \in \mathbb{C}_{\rm H}^{n}$ *with* $p \leqslant r(X) \leqslant q$ *if and only if* $A \prec 0$ *and* $p = q = 0$.

(x) *There exists an* $X \in \mathbb{C}_{\rm H}^{n}$ *with* $p \leqslant r(X) \leqslant q$ *such that* $A + BXB^{*} \preccurlyeq 0$ *if and only if* $i_{-}(M) = r[\,A,\,B\,]$ *and* $i_{+}(A) \leqslant q$.

(xi) $A + BXB^{*} \preccurlyeq 0$ *for all* $X \in \mathbb{C}_{\rm H}^{n}$ *with* $p \leqslant r(X) \leqslant q$ *if and only if* $A \preccurlyeq 0$ *and* $B = 0$, *or* $A \preccurlyeq 0$ *and* $p = q = 0$.

Proof. Note from (1.19) that

$$\max_{X\in\mathbb{C}_{\rm H}^{n},\, p\leqslant r(X)\leqslant q} r(\,A+BXB^{*}\,) = \max\left\{\max_{X\in\mathbb{C}_{\rm H}^{n},\, r(X)=p} r(\,A+BXB^{*}\,),\ \max_{X\in\mathbb{C}_{\rm H}^{n},\, r(X)=p+1} r(\,A+BXB^{*}\,), \ldots, \max_{X\in\mathbb{C}_{\rm H}^{n},\, r(X)=q} r(\,A+BXB^{*}\,)\right\}, \tag{4.66}$$

$$\min_{X\in\mathbb{C}_{\rm H}^{n},\, p\leqslant r(X)\leqslant q} r(\,A+BXB^{*}\,) = \min\left\{\min_{X\in\mathbb{C}_{\rm H}^{n},\, r(X)=p} r(\,A+BXB^{*}\,),\ \min_{X\in\mathbb{C}_{\rm H}^{n},\, r(X)=p+1} r(\,A+BXB^{*}\,), \ldots, \max_{X\in\mathbb{C}_{\rm H}^{n},\, r(X)=q} r(\,A+BXB^{*}\,)\right\}, \tag{4.67}$$

$$\max_{X\in\mathbb{C}_{\rm H}^{n},\, p\leqslant r(X)\leqslant q} i_{+}(\,A+BXB^{*}\,) = \max\left\{\max_{X\in\mathbb{C}_{\rm H}^{n},\, r(X)=p} i_{+}(\,A+BXB^{*}\,),\ \max_{X\in\mathbb{C}_{\rm H}^{n},\, r(X)=p+1} i_{+}(\,A+BXB^{*}\,), \ldots, \max_{X\in\mathbb{C}_{\rm H}^{n},\, r(X)=q} i_{+}(\,A+BXB^{*}\,)\right\}, \tag{4.68}$$

$$\min_{X\in\mathbb{C}_{\rm H}^{n},\, p\leqslant r(X)\leqslant q} i_{+}(\,A+BXB^{*}\,) = \min\left\{\min_{X\in\mathbb{C}_{\rm H}^{n},\, r(X)=p} i_{+}(\,A+BXB^{*}\,),\ \min_{X\in\mathbb{C}_{\rm H}^{n},\, r(X)=p+1} i_{+}(\,A+BXB^{*}\,), \ldots, \min_{X\in\mathbb{C}_{\rm H}^{n},\, r(X)=q} i_{+}(\,A+BXB^{*}\,)\right\}, \tag{4.69}$$

$$\max_{X\in\mathbb{C}_{\rm H}^{n},\, p\leqslant r(X)\leqslant q} i_{-}(\,A+BXB^{*}\,) = \max\left\{\max_{X\in\mathbb{C}_{\rm H}^{n},\, r(X)=p} i_{-}(\,A+BXB^{*}\,),\ \max_{X\in\mathbb{C}_{\rm H}^{n},\, r(X)=p+1} i_{-}(\,A+BXB^{*}\,), \ldots, \max_{X\in\mathbb{C}_{\rm H}^{n},\, r(X)=q} i_{-}(\,A+BXB^{*}\,)\right\}, \tag{4.70}$$

$$\min_{X\in\mathbb{C}_{\rm H}^{n},\, p\leqslant r(X)\leqslant q} i_{-}(\,A+BXB^{*}\,) = \min\left\{\min_{X\in\mathbb{C}_{\rm H}^{n},\, r(X)=p} i_{-}(\,A+BXB^{*}\,),\ \min_{X\in\mathbb{C}_{\rm H}^{n},\, r(X)=p+1} i_{-}(\,A+BXB^{*}\,), \ldots, \min_{X\in\mathbb{C}_{\rm H}^{n},\, r(X)=q} i_{-}(\,A+BXB^{*}\,)\right\}. \tag{4.71}$$

Substituting (4.22)–(4.27) for $r(X) = p,\, p+1, \ldots,\, q$ into (4.66)–(4.71) and making max-min comparisons, we obtain (4.60)–(4.65). □

Corollary 4.5 *Let* $\phi(X)$ *and* M *be as given in* (1.1) *and* (4.17), *and assume that* p *is an integer satisfying* $0 \leqslant p \leqslant n$. *Then, the following equalities hold*

$$\max_{X\in\mathbb{C}_{\rm H}^{n},\, p\leqslant r(X)\leqslant n} r(\,A+BXB^{*}\,) = r[\,A,\,B\,], \tag{4.72}$$

$$\min_{X\in\mathbb{C}_{\rm H}^{n},\, p\leqslant r(X)\leqslant n} r(\,A+BXB^{*}\,) = \min\{\max\{2r[\,A,\ B\,] - r(M),\ 2r[\,A,\,B\,] - r(A) + p - 2n,\ r(A) - p\}, \ldots, \max\{2r[\,A,\ B\,] - r(M),\ 2r[\,A,\,B\,] - r(A) - n\,\}\}, \tag{4.73}$$

$$\max_{X\in\mathbb{C}_{\rm H}^{n},\, p\leqslant r(X)\leqslant n} i_{+}(\,A+BXB^{*}\,) = i_{+}(M), \tag{4.74}$$

$$\min_{X\in\mathbb{C}_{\rm H}^{n},\, p\leqslant r(X)\leqslant n} i_{+}(\,A+BXB^{*}\,) = \max\{\,r[\,A,\ B\,] - i_{-}(M),\quad i_{+}(A) - n\,\}, \tag{4.75}$$

$$\max_{X\in\mathbb{C}_{\rm H}^{n},\, p\leqslant r(X)\leqslant n} i_{-}(\,A+BXB^{*}\,) = i_{-}(M), \tag{4.76}$$

$$\min_{X\in\mathbb{C}_{\rm H}^{n},\, p\leqslant r(X)\leqslant n} i_{-}(\,A+BXB^{*}\,) = \max\{\,r[\,A,\ B\,] - i_{+}(M),\quad i_{-}(A) - n\,\}. \tag{4.77}$$

Corollary 4.6 ([17]) *Let $\phi(X)$ and M be as given in* (1.1) *and* (4.17). *Then,*

$$\max_{X\in\mathbb{C}^n_{\rm H},\, 0\leqslant r(X)\leqslant q} r(\,A + BXB^*\,) = \min\{\, r[\,A,\, B\,],\quad r(A) + q\,\}, \tag{4.78}$$

$$\min_{X\in\mathbb{C}^n_{\rm H},\, 0\leqslant r(X)\leqslant q} r(\,A + BXB^*\,) = \max\{\, 2r[\,A,\, B\,] - r(M),\quad r(A) - q\}\,, \tag{4.79}$$

$$\max_{X\in\mathbb{C}^n_{\rm H},\, 0\leqslant r(X)\leqslant q} i_+(\,A + BXB^*\,) = \min\{\, i_+(M),\quad i_+(A) + q\,\}\,, \tag{4.80}$$

$$\min_{X\in\mathbb{C}^n_{\rm H},\, 0\leqslant r(X)\leqslant q} i_+(\,A + BXB^*\,) = \max\{\, r[\,A,\, B\,] - i_-(M),\quad i_+(A) - q\}\,, \tag{4.81}$$

$$\max_{X\in\mathbb{C}^n_{\rm H},\, 0\leqslant r(X)\leqslant q} i_-(\,A + BXB^*\,) = \min\{\, i_-(M),\quad i_-(A) + q\,\}\,, \tag{4.82}$$

$$\min_{X\in\mathbb{C}^n_{\rm H},\, 0\leqslant r(X)\leqslant q} i_-(\,A + BXB^*\,) = \max\{\, r[\,A,\, B\,] - i_+(M),\quad i_-(A) - q\,\}. \tag{4.83}$$

5 Rank and inertia formulas of $A \pm BXB^*$ when X is positive semi-definite

Theorem 5.1 *Let $A \in \mathbb{C}^m_{\rm H}$, $B \in \mathbb{C}^{m\times n}$ be given, M the matrix in* (4.17), *$X \in \mathbb{C}^n_{\rm H}$ a variable matrix, and assume that q is an integer satisfying $0 \leqslant q \leqslant n$. Then,*

(a) *The following equalities hold*

$$\max_{0\preccurlyeq X,\, r(X)=q} r(\,A + BXB^*\,) = \min\{\, r[\,A,\, B\,],\quad r(A) + q\,\}, \tag{5.1}$$

$$\min_{0\preccurlyeq X,\, r(X)=q} r(\,A + BXB^*\,) = \max\{\, i_+(A) + r[\,A,\ B\,] - i_+(M),\ 2r[\,A,\ B\,] - i_-(A) - i_+(M) - n + q,\ r(A) - q\,\}\,, \tag{5.2}$$

$$\max_{0\preccurlyeq X,\, r(X)=q} i_+(\,A + BXB^*\,) = \min\{\, i_+(M),\quad i_+(A) + q\,\}, \tag{5.3}$$

$$\min_{0\preccurlyeq X,\, r(X)=q} i_+(\,A + BXB^*\,) = \max\{\, r[\,A,\ B\,] - i_-(M),\quad i_+(A) - q\,\}, \tag{5.4}$$

$$\max_{0\preccurlyeq X,\, r(X)=q} i_-(\,A + BXB^*\,) = i_-(A), \tag{5.5}$$

$$\min_{0\preccurlyeq X,\, r(X)=q} i_-(\,A + BXB^*\,) = \max\{\, r[\,A,\ B\,] - i_+(M),\quad i_-(A) - q\,\}. \tag{5.6}$$

In particular,

$$\max_{X\succ 0} r(\,A + BXB^*\,) = r[\,A,\, B\,], \tag{5.7}$$

$$\min_{X\succ 0} r(\,A + BXB^*\,) = \max\{\, 2r[\,A,\ B\,] - i_-(A) - i_+(M),\ r(A) - n\,\}\,, \tag{5.8}$$

$$\max_{X\succ 0} i_+(\,A + BXB^*\,) = \min\{\, i_+(M),\quad i_+(A) + n\,\}, \tag{5.9}$$

$$\min_{X\succ 0} i_+(\,A + BXB^*\,) = \max\{\, r[\,A,\ B\,] - i_-(M),\quad i_+(A) - n\,\}, \tag{5.10}$$

$$\max_{X\succ 0} i_-(\,A + BXB^*\,) = i_-(A), \tag{5.11}$$

$$\min_{X\succ 0} i_-(\,A + BXB^*\,) = \max\{\, r[\,A,\ B\,] - i_+(M),\quad i_-(A) - n\,\}. \tag{5.12}$$

In consequence, the following hold.

(i) *There exists a $0 \preccurlyeq X \in \mathbb{C}^n_{\rm H}$ with $r(X) = q$ such that $A + BXB^*$ is nonsingular if and only if $r[\,A,\, B\,] = m$ and $r(A) \geqslant m - q$.*

(ii) *$A + BXB^*$ is nonsingular for all $0 \preccurlyeq X \in \mathbb{C}^n_{\rm H}$ with $r(X) = q$ if and only if $i_+(M) = i_+(A) + r[\,A,\ B\,] - m$, or $2r[\,A,\ B\,] = i_-(A) + i_+(M) - q + n + m$, or $r(A) = m$ and $q = 0$.*

(iii) *There exists a $0 \preccurlyeq X \in \mathbb{C}^m_{\rm H}$ with $r(X) = q$ such that $A + BXB^* = 0$ if and only if $A \preccurlyeq 0$, $\mathscr{R}(A) \subseteq \mathscr{R}(B)$ and $n - r(B) \geqslant q - r(A) \geqslant 0$.*

(iv) *There exists a $0 \preccurlyeq X \in \mathbb{C}^m_{\rm H}$ with $r(X) = q$ such that $A + BXB^* \succ 0$ if and only if $i_+(M) = m$ and $i_+(A) \geqslant m - q$, or $i_+(M) \geqslant m$ and $i_+(A) = m - q$.*

(v) *$A + BXB^* \succ 0$ holds for all $0 \preccurlyeq X \in \mathbb{C}^m_{\rm H}$ with $r(X) = q$ if and only if $r[\,A,\ B\,] = m$ and $M \succcurlyeq 0$, or $A \succ 0$ and $q = 0$.*

(vi) *There exists a $0 \preccurlyeq X \in \mathbb{C}^m_{\rm H}$ with $r(X) = q$ such that $A + BXB^* \succcurlyeq 0$ if and only if $r[,A,\ B\,] \leqslant i_+(M)$ and $i_-(A) = q$, or $r[\,A,\ B\,] = i_+(M)$ and $i_-(A) \leqslant q$.*

(vii) *$A + BXB^{*} \succcurlyeq 0$ holds for all $0 \preccurlyeq X \in \mathbb{C}_{\mathrm{H}}^{m}$ with $r(X) = q$ if and only if $A \succcurlyeq 0$.*

(viii) *There exists a $0 \preccurlyeq X \in \mathbb{C}_{\mathrm{H}}^{m}$ with $r(X) = q$ such that $A + BXB^{*} \prec 0$ if and only if $A \prec 0$.*

(ix) *$A + BXB^{*} \prec 0$ holds for all $0 \preccurlyeq X \in \mathbb{C}_{\mathrm{H}}^{m}$ with $r(X) = q$ if and only if $r[\,A,\ B\,] = m$ and $M \preccurlyeq 0$, or $A \prec 0$ and $q = 0$.*

(x) *There exists a $0 \preccurlyeq X \in \mathbb{C}_{\mathrm{H}}^{m}$ with $r(X) = q$ such that $A + BXB^{*} \preccurlyeq 0$ if and only if $r[\,A,\ B\,] = i_{-}(M)$ and $i_{+}(A) \leqslant q$, or $r[\,A,\ B\,] \leqslant i_{-}(M)$ and $i_{+}(A) = q$.*

(xi) *$A + BXB^{*} \preccurlyeq 0$ holds for all $0 \preccurlyeq X \in \mathbb{C}_{\mathrm{H}}^{m}$ with $r(X) = q$ if and only if $A \preccurlyeq 0$ and $q = 0$.*

(b) *The following equalities hold*

$$\max_{0 \preccurlyeq X,\, r(X)=q} r(\,A - BXB^{*}\,) = \min\{\, r[\,A,\, B\,],\quad r(A) + q\,\}, \tag{5.13}$$

$$\min_{0 \preccurlyeq X,\, r(X)=q} r(\,A - BXB^{*}\,) = \max\{\, i_{-}(A) + r[\,A,\ B\,] - i_{-}(M),\quad 2r[\,A,\ B\,] - i_{+}(A) - i_{-}(M) - n + q,\quad r(A) - q\,\}, \tag{5.14}$$

$$\max_{0 \preccurlyeq X,\, r(X)=q} i_{+}(\,A - BXB^{*}\,) = i_{+}(A), \tag{5.15}$$

$$\min_{0 \preccurlyeq X,\, r(X)=q} i_{+}(\,A - BXB^{*}\,) = \max\{\, r[\,A,\ B\,] - i_{-}(M),\quad i_{+}(A) - q\,\}, \tag{5.16}$$

$$\max_{0 \preccurlyeq X,\, r(X)=q} i_{-}(\,A - BXB^{*}\,) = \min\{\, i_{-}(M),\quad i_{-}(A) + q\,\}, \tag{5.17}$$

$$\min_{0 \preccurlyeq X,\, r(X)=q} i_{-}(\,A - BXB^{*}\,) = \max\{\, r[\,A,\ B\,] - i_{+}(M),\quad i_{-}(A) - q\,\}. \tag{5.18}$$

In particular,

$$\max_{X \succ 0} r(\,A - BXB^{*}\,) = r[\,A,\, B\,], \tag{5.19}$$

$$\min_{X \succ 0} r(\,A - BXB^{*}\,) = \max\{\, 2r[\,A,\ B\,] - i_{+}(A) - i_{-}(M),\quad r(A) - n\,\}, \tag{5.20}$$

$$\max_{X \succ 0} i_{+}(\,A - BXB^{*}\,) = i_{+}(A), \tag{5.21}$$

$$\min_{X \succ 0} i_{+}(\,A - BXB^{*}\,) = \max\{\, r[\,A,\ B\,] - i_{-}(M),\quad i_{+}(A) - n\,\}, \tag{5.22}$$

$$\max_{X \succ 0} i_{-}(\,A - BXB^{*}\,) = \min\{\, i_{-}(M),\quad i_{-}(A) + n\,\}, \tag{5.23}$$

$$\min_{X \succ 0} i_{-}(\,A - BXB^{*}\,) = \max\{\, r[\,A,\ B\,] - i_{+}(M),\quad i_{-}(A) - n\,\}. \tag{5.24}$$

In consequence, the following hold.

(i) *There exists a $0 \preccurlyeq X \in \mathbb{C}_{\mathrm{H}}^{n}$ with $r(X) = q$ such that $A - BXB^{*}$ is nonsingular if and only if $r[\,A,\, B\,] = m$ and $r(A) \geqslant m - q$.*

(ii) *$A - BXB^{*}$ is nonsingular for all $0 \preccurlyeq X \in \mathbb{C}_{\mathrm{H}}^{n}$ with $r(X) = q$ if and only if $i_{-}(M) = i_{-}(A) + r[\,A,\ B\,] - m$, or $2r[\,A,\ B\,] = i_{+}(A) + i_{-}(M) - q + n + m$, or $r(A) = m$ and $q = 0$.*

(iii) *There exists a $0 \preccurlyeq X \in \mathbb{C}_{\mathrm{H}}^{m}$ with $r(X) = q$ such that $A - BXB^{*} = 0$ if and only if $A \succcurlyeq 0$, $\mathscr{R}(A) \subseteq \mathscr{R}(B)$ and $n - r(B) \geqslant q - r(A) \geqslant 0$.*

(iv) *There exists a $0 \preccurlyeq X \in \mathbb{C}_{\mathrm{H}}^{m}$ with $r(X) = q$ such that $A - BXB^{*} \succ 0$ if and only if $A \succ 0$.*

(v) *$A - BXB^{*} \succ 0$ holds for all $0 \preccurlyeq X \in \mathbb{C}_{\mathrm{H}}^{m}$ with $r(X) = q$ if and only if $r[\,A,\ B\,] = m$ and $M \succcurlyeq 0$, or $A \succ 0$ and $q = 0$.*

(vi) *There exists a $0 \preccurlyeq X \in \mathbb{C}_{\mathrm{H}}^{m}$ with $r(X) = q$ such that $A - BXB^{*} \succcurlyeq 0$ if and only if $r[,A,\ B\,] \leqslant i_{+}(M)$ and $i_{-}(A) = q$, or $r[\,A,\ B\,] = i_{+}(M)$ and $i_{-}(A) \leqslant q$.*

(vii) *$A - BXB^{*} \succcurlyeq 0$ holds for all $0 \preccurlyeq X \in \mathbb{C}_{\mathrm{H}}^{m}$ with $r(X) = q$ if and only if $M \succcurlyeq 0$, or $A \succcurlyeq 0$ and $q = 0$.*

(viii) *There exists a $0 \preccurlyeq X \in \mathbb{C}_{\mathrm{H}}^{m}$ with $r(X) = q$ such that $A - BXB^{*} \prec 0$ if and only if $i_{-}(M) = m$ and $i_{-}(A) \geqslant m - q$, or $i_{-}(M) \geqslant m$ and $i_{-}(A) = m - q$.*

(ix) *$A - BXB^{*} \prec 0$ holds for all $0 \preccurlyeq X \in \mathbb{C}_{\mathrm{H}}^{m}$ with $r(X) = q$ if and only if $r[\,A,\ B\,] = m$ and $M \preccurlyeq 0$, or $A \prec 0$ and $q = 0$.*

(x) *There exists a $0 \preccurlyeq X \in \mathbb{C}_{\mathrm{H}}^{m}$ with $r(X) = q$ such that $A - BXB^{*} \preccurlyeq 0$ if and only if $r[\,A,\ B\,] = i_{-}(M)$ and $i_{+}(A) \leqslant q$, or $r[\,A,\ B\,] \leqslant i_{-}(M)$ and $i_{+}(A) = q$.*

(xi) *$A - BXB^{*} \preccurlyeq 0$ holds for all $0 \preccurlyeq X \in \mathbb{C}_{\mathrm{H}}^{m}$ with $r(X) = q$ if and only if $A \preccurlyeq 0$.*

Proof. It is obvious from (4.1) and (4.2) that

$$0 \preccurlyeq X \ \text{ and } \ r(X) = q \Leftrightarrow 0 \preccurlyeq Y \ \text{ and } \ r(Y) = q. \tag{5.25}$$

In this case, we derive from (4.7)–(4.9) that

$$\max_{0 \preccurlyeq X,\, r(X)=q} r(\, A + BXB^* \,) = 2f + k + \max_{0 \preccurlyeq Y,\, r(Y)=q} r(\, \widehat{A} + \widehat{Y} \,), \tag{5.26}$$

$$\min_{0 \preccurlyeq X,\, r(X)=q} r(\, A + BXB^* \,) = 2f + k + \min_{0 \preccurlyeq Y,\, r(Y)=q} r(\, \widehat{A} + \widehat{Y} \,), \tag{5.27}$$

$$\max_{0 \preccurlyeq X,\, r(X)=q} i_+(\, A + BXB^* \,) = f + s_2 + \max_{0 \preccurlyeq Y,\, r(Y)=q} i_+(\, \widehat{A} + \widehat{Y} \,), \tag{5.28}$$

$$\min_{0 \preccurlyeq X,\, r(X)=q} i_+(\, A + BXB^* \,) = f + s_2 + \min_{0 \preccurlyeq Y,\, r(Y)=q} i_+(\, \widehat{A} + \widehat{Y} \,), \tag{5.29}$$

$$\max_{0 \preccurlyeq X,\, r(X)=q} i_-(\, A + BXB^* \,) = f + k - s_2 + \max_{0 \preccurlyeq Y,\, r(Y)=q} i_-(\, \widehat{A} + \widehat{Y} \,), \tag{5.30}$$

$$\min_{0 \preccurlyeq X,\, r(X)=q} i_-(\, A + BXB^* \,) = f + k - s_2 + \min_{0 \preccurlyeq Y,\, r(Y)=q} i_-(\, \widehat{A} + \widehat{Y} \,). \tag{5.31}$$

Also note from Lemma 2.7(q) and (4.5) that

$$0 \preccurlyeq Y \text{ and } r(Y) = q \ \Leftrightarrow \ 0 \preccurlyeq \widehat{Y}, \ \ q - (n-l) \leqslant r(\widehat{Y}) \leqslant q,$$
$$\mathscr{R}\begin{bmatrix} Y_{12}^* & Y_{24} \\ Y_{13}^* & Y_{34} \end{bmatrix} \subseteq \mathscr{R}(\widehat{Y}) \text{ and } \begin{bmatrix} Y_{11} & Y_{14} \\ Y_{14}^* & Y_{44} \end{bmatrix} - \begin{bmatrix} Y_{12} & Y_{13} \\ Y_{24}^* & Y_{34}^* \end{bmatrix} \widehat{Y}^{\dagger} \begin{bmatrix} Y_{12}^* & Y_{24} \\ Y_{13}^* & Y_{34} \end{bmatrix} \succcurlyeq 0. \tag{5.32}$$

Applying (4.5) and Lemma 3.2(b) to $\widehat{A} + \widehat{Y}$ in (5.26)–(5.31) and simplifying by (2.83), we obtain

$$\max_{0 \preccurlyeq Y,\, r(Y)=q} r(\, \widehat{A} + \widehat{Y} \,) = \max_{0 \preccurlyeq \widehat{Y},\, q-(n-l) \leqslant r(\widehat{Y}) \leqslant q} r(\, \widehat{A} + \widehat{Y} \,) = \min\{\, l, \ \ r(\widehat{A}) + q \,\} = \min\{\, l, \ \ t + q \,\}, \tag{5.33}$$

$$\begin{aligned} \min_{0 \preccurlyeq Y,\, r(Y)=q} r(\, \widehat{A} + \widehat{Y} \,) &= \min_{0 \preccurlyeq \widehat{Y},\, q-(n-l) \leqslant r(\widehat{Y}) \leqslant q} r(\, \widehat{A} + \widehat{Y} \,) = \max\{\, i_+(\widehat{A}), \ q - (n-l) - i_-(\widehat{A}), \ r(\widehat{A}) - q \,\} \\ &= \max\{\, s_1, \ q - n + l - t + s_1, \ t - q \,\}, \end{aligned} \tag{5.34}$$

$$\max_{0 \preccurlyeq Y,\, r(Y)=q} i_+(\, \widehat{A} + \widehat{Y} \,) = \max_{0 \preccurlyeq \widehat{Y},\, q-(n-l) \leqslant r(\widehat{Y}) \leqslant q} i_+(\, \widehat{A} + \widehat{Y} \,) = \min\{\, l, \ \ i_+(\widehat{A}) + q \,\} = \min\{\, l, \ \ s_1 + q \,\}, \tag{5.35}$$

$$\min_{0 \preccurlyeq Y,\, r(Y)=q} i_+(\, \widehat{A} + \widehat{Y} \,) = \min_{0 \preccurlyeq \widehat{Y},\, q-(n-l) \leqslant r(\widehat{Y}) \leqslant q} i_+(\, \widehat{A} + \widehat{Y} \,) = \max\{\, 0, \ \ i_+(\widehat{A}) - q\} = \max\{\, 0, \ \ s_1 - q \,\}, \tag{5.36}$$

$$\max_{0 \preccurlyeq Y,\, r(Y)=q} i_-(\, \widehat{A} + \widehat{Y} \,) = \max_{0 \preccurlyeq \widehat{Y},\, q-(n-l) \leqslant r(\widehat{Y}) \leqslant q} i_-(\, \widehat{A} + \widehat{Y} \,) = i_-(\widehat{A}) = t - s_1, \tag{5.37}$$

$$\min_{0 \preccurlyeq Y,\, r(Y)=q} i_-(\, \widehat{A} + \widehat{Y} \,) = \min_{0 \preccurlyeq \widehat{Y},\, q-(n-l) \leqslant r(\widehat{Y}) \leqslant q} i_-(\, \widehat{A} + \widehat{Y} \,) = \max\{\, 0, \ \ i_-(\widehat{A}) - q \,\} = \max\{\, 0, \ \ t - s_1 - q \,\}. \tag{5.38}$$

Substituting (5.33)–(5.38) into (5.26)–(5.31) and simplifying (2.85)–(2.92), we obtain (5.1)–(5.6). Note that

$$r(\, A - BXB^* \,) = r(\, -A + BXB^* \,), \ \ i_+(\, A - BXB^* \,) = i_-(\, -A + BXB^* \,), \ \ i_-(\, A - BXB^* \,) = i_+(\, -A + BXB^* \,).$$

Thus, applying (5.1)–(5.6) to the right-hand sides of the three equalities gives (5.7)–(5.12).

Applying Lemma 2.1 to (5.1)–(5.6) leads to the results in (i)—(xi) of (a); and to (5.7)–(5.12) leads to the results in (i)—(xi) of (b). □

Corollary 5.2 *Let $A \in \mathbb{C}_{\mathrm{H}}^m$, $B \in \mathbb{C}^{m \times n}$ be given, M the matrix in (4.17), $X \in \mathbb{C}_{\mathrm{H}}^n$ a variable matrix, and assume that p and q are an two integers satisfying $0 \leqslant p \leqslant q \leqslant n$. Then,*

(a) *The following equalities hold*

$$\max_{0 \preccurlyeq X,\, p \leqslant r(X) \leqslant q} r(\, A + BXB^* \,) = \min\{\, r[\, A, \, B \,], \ \ r(A) + q \,\}, \tag{5.39}$$

$$\begin{aligned} \min_{0 \preccurlyeq X,\, p \leqslant r(X) \leqslant q} r(\, A + BXB^* \,) = \max \{\, & i_+(A) + r[\, A, \ B \,] - i_+(M), \ 2r[\, A, \ B \,] - i_-(A) - i_+(M) - n + p, \\ & r(A) - q \,\}, \end{aligned} \tag{5.40}$$

$$\max_{0 \preccurlyeq X,\, p \leqslant r(X) \leqslant q} i_+(\, A + BXB^* \,) = \min\{\, i_+(M), \ \ i_+(A) + q \,\}, \tag{5.41}$$

$$\min_{0 \preccurlyeq X,\, p \leqslant r(X) \leqslant q} i_+(\, A + BXB^* \,) = \max\{\, r[\, A, \ B \,] - i_-(M), \ \ i_+(A) - q \,\}, \tag{5.42}$$

$$\max_{0 \preccurlyeq X,\, p \leqslant r(X) \leqslant q} i_-(\, A + BXB^* \,) = i_-(A), \tag{5.43}$$

$$\min_{0 \preccurlyeq X,\, p \leqslant r(X) \leqslant q} i_-(\, A + BXB^* \,) = \max\{\, r[\, A, \ B \,] - i_+(M), \ \ i_-(A) - q \,\}, \tag{5.44}$$

and

$$\max_{0\preccurlyeq X,\, p\leqslant r(X)\leqslant q} r(\, A - BXB^{*}\,) = \min\{\, r[\, A,\, B\,],\quad r(A) + q\,\}, \tag{5.45}$$

$$\min_{0\preccurlyeq X,\, p\leqslant r(X)\leqslant q} r(\, A - BXB^{*}\,) = \max\{\, i_{-}(A) + r[\, A,\ B\,] - i_{-}(M),\quad 2r[\, A,\ B\,] - i_{+}(A) - i_{-}(M) - n + p,\ r(A) - q\,\}, \tag{5.46}$$

$$\max_{0\preccurlyeq X,\, p\leqslant r(X)\leqslant q} i_{+}(\, A - BXB^{*}\,) = i_{+}(A), \tag{5.47}$$

$$\min_{0\preccurlyeq X,\, p\leqslant r(X)\leqslant q} i_{+}(\, A - BXB^{*}\,) = \max\{\, r[\, A,\ B\,] - i_{-}(M),\quad i_{+}(A) - q\,\}, \tag{5.48}$$

$$\max_{0\preccurlyeq X,\, p\leqslant r(X)\leqslant q} i_{-}(\, A - BXB^{*}\,) = \min\{\, i_{-}(M),\quad i_{-}(A) + q\,\}, \tag{5.49}$$

$$\min_{0\preccurlyeq X,\, p\leqslant r(X)\leqslant q} i_{-}(\, A - BXB^{*}\,) = \max\{\, r[\, A,\ B\,] - i_{+}(M),\quad i_{-}(A) - q\,\}. \tag{5.50}$$

(b) [17] *The following equalities hold*

$$\max_{0\preccurlyeq X,\, 0\leqslant r(X)\leqslant q} r(\, A + BXB^{*}\,) = \min\{\, r[\, A,\, B\,],\quad r(A) + q\,\}, \tag{5.51}$$

$$\min_{0\preccurlyeq X,\, 0\leqslant r(X)\leqslant q} r(\, A + BXB^{*}\,) = \max\{\, i_{+}(A) + r[\, A,\, B\,] - i_{+}(M),\quad r(A) - q\}\,, \tag{5.52}$$

$$\max_{0\preccurlyeq X,\, 0\leqslant r(X)\leqslant q} i_{+}(\, A + BXB^{*}\,) = \min\{\, i_{+}(M),\quad i_{+}(A) + q\,\}\,, \tag{5.53}$$

$$\min_{0\preccurlyeq X,\, 0\leqslant r(X)\leqslant q} i_{+}(\, A + BXB^{*}\,) = i_{+}(A), \tag{5.54}$$

$$\max_{0\preccurlyeq X,\, 0\leqslant r(X)\leqslant q} i_{-}(\, A + BXB^{*}\,) = i_{-}(A), \tag{5.55}$$

$$\min_{0\preccurlyeq X,\, 0\leqslant r(X)\leqslant q} i_{-}(\, A + BXB^{*}\,) = \max\{\, r[\, A,\, B\,] - i_{+}(M),\quad i_{-}(A) - q\,\}, \tag{5.56}$$

and

$$\max_{0\preccurlyeq X,\, 0\leqslant r(X)\leqslant q} r(\, A - BXB^{*}\,) = \min\{\, r[\, A,\, B\,],\quad r(A) + q\,\}, \tag{5.57}$$

$$\min_{0\preccurlyeq X,\, 0\leqslant r(X)\leqslant q} r(\, A - BXB^{*}\,) = \max\{\, i_{-}(A) + r[\, A,\, B\,] - i_{-}(M),\quad r(A) - q\}\,, \tag{5.58}$$

$$\max_{0\preccurlyeq X,\, 0\leqslant r(X)\leqslant q} i_{+}(\, A - BXB^{*}\,) = i_{+}(A), \tag{5.59}$$

$$\min_{0\preccurlyeq X,\, 0\leqslant r(X)\leqslant q} i_{+}(\, A - BXB^{*}\,) = \max\{\, r[\, A,\, B\,] - i_{-}(M),\quad i_{+}(A) - q\,\}, \tag{5.60}$$

$$\max_{0\preccurlyeq X,\, 0\leqslant r(X)\leqslant q} i_{-}(\, A - BXB^{*}\,) = \min\{\, i_{-}(M),\quad i_{-}(A) + q\,\}\,, \tag{5.61}$$

$$\min_{0\preccurlyeq X,\, 0\leqslant r(X)\leqslant q} i_{-}(\, A - BXB^{*}\,) = i_{-}(A) \tag{5.62}$$

(c) *The following equalities hold*

$$\max_{0\preccurlyeq X,\, p\leqslant r(X)\leqslant n} r(\, A + BXB^{*}\,) = r[\, A,\, B\,], \tag{5.63}$$

$$\min_{0\preccurlyeq X,\, p\leqslant r(X)\leqslant n} r(\, A + BXB^{*}\,) = \max\{\, i_{+}(A) + r[\, A,\ B\,] - i_{+}(M),\ 2r[\, A,\ B\,] - i_{-}(A) - i_{+}(M) - n + p,\ r(A) - n\,\}\,, \tag{5.64}$$

$$\max_{0\preccurlyeq X,\, p\leqslant r(X)\leqslant q} i_{+}(\, A + BXB^{*}\,) = \min\{\, i_{+}(M),\quad i_{+}(A) + n\,\}, \tag{5.65}$$

$$\min_{0\preccurlyeq X,\, p\leqslant r(X)\leqslant q} i_{+}(\, A + BXB^{*}\,) = \max\{\, r[\, A,\ B\,] - i_{-}(M),\quad i_{+}(A) - n\,\}, \tag{5.66}$$

$$\max_{0\preccurlyeq X,\, p\leqslant r(X)\leqslant q} i_{-}(\, A + BXB^{*}\,) = i_{-}(A), \tag{5.67}$$

$$\min_{0\preccurlyeq X,\, p\leqslant r(X)\leqslant q} i_{-}(\, A + BXB^{*}\,) = \max\{\, r[\, A,\ B\,] - i_{+}(M),\quad i_{-}(A) - n\,\}, \tag{5.68}$$

and

$$\max_{0\preccurlyeq X,\, p\leqslant r(X)\leqslant n} r(\,A - BXB^*\,) = r[\,A,\, B\,], \tag{5.69}$$

$$\min_{0\preccurlyeq X,\, p\leqslant r(X)\leqslant n} r(\,A - BXB^*\,) = \max\{\, i_-(A) + r[\,A,\ B\,] - i_-(M),\quad 2r[\,A,\ B\,] - i_+(A) - i_-(M) - n + p,\quad r(A) - n\,\}, \tag{5.70}$$

$$\max_{0\preccurlyeq X,\, p\leqslant r(X)\leqslant n} i_+(\,A - BXB^*\,) = i_+(A), \tag{5.71}$$

$$\min_{0\preccurlyeq X,\, p\leqslant r(X)\leqslant n} i_+(\,A - BXB^*\,) = \max\{\, r[\,A,\ B\,] - i_-(M),\quad i_+(A) - n\,\}, \tag{5.72}$$

$$\max_{0\preccurlyeq X,\, p\leqslant r(X)\leqslant n} i_-(\,A - BXB^*\,) = \min\{\, i_-(M),\quad i_-(A) + n\,\}, \tag{5.73}$$

$$\min_{0\preccurlyeq X,\, p\leqslant r(X)\leqslant n} i_-(\,A - BXB^*\,) = \max\{\, r[\,A,\ B\,] - i_+(M),\quad i_-(A) - n\,\}. \tag{5.74}$$

$$\max_{X\in\mathbb{C}^{n\times n}} r(\,A + BXX^*B^*\,) = r[\,A,\, B\,], \qquad \min_{X\in\mathbb{C}^{n\times n}} r(\,A + BXX^*B^*\,) = i_+(A) + r[\,A,\, B\,] - i_+(M), \tag{5.75}$$

$$\max_{X\in\mathbb{C}^{n\times n}} i_+(\,A + BXX^*B^*\,) = i_+(M), \qquad \min_{X\in\mathbb{C}^{n\times n}} i_+(\,A + BXX^*B^*\,) = i_+(A), \tag{5.76}$$

$$\max_{X\in\mathbb{C}^{n\times n}} i_-(\,A + BXX^*B^*\,) = i_-(A), \qquad \min_{X\in\mathbb{C}^{n\times n}} i_-(\,A + BXX^*B^*\,) = r[\,A,\, B\,] - i_+(M), \tag{5.77}$$

$$\max_{X\in\mathbb{C}^{n\times n}} r(\,A - BXX^*B^*\,) = r[\,A,\, B\,], \qquad \min_{X\in\mathbb{C}^{n\times n}} r(\,A - BXX^*B^*\,) = i_-(A) + r[\,A,\, B\,] - i_-(M), \tag{5.78}$$

$$\max_{X\in\mathbb{C}^{n\times n}} i_+(\,A - BXX^*B^*\,) = i_+(A), \qquad \min_{X\in\mathbb{C}^{n\times n}} i_+(\,A - BXX^*B^*\,) = r[\,A,\, B\,] - i_-(M), \tag{5.79}$$

$$\max_{X\in\mathbb{C}^{n\times n}} i_-(\,A - BXX^*B^*\,) = i_-(M), \qquad \min_{X\in\mathbb{C}^{n\times n}} i_-(\,A - BXX^*B^*\,) = i_-(A). \tag{5.80}$$

Proof. Note from (1.20) that

$$\max_{0\preccurlyeq X,\, p\leqslant r(X)\leqslant q} r(\,A \pm BXB^*\,) = \max\left\{ \max_{0\preccurlyeq X,\, r(X)=p} r(\,A \pm BXB^*\,),\ \max_{0\preccurlyeq X,\, r(X)=p+1} r(\,A \pm BXB^*\,),\ \ldots,\ \max_{0\preccurlyeq X,\, r(X)=q} r(\,A \pm BXB^*\,) \right\}, \tag{5.81}$$

$$\min_{0\preccurlyeq X,\, p\leqslant r(X)\leqslant q} r(\,A \pm BXB^*\,) = \min\left\{ \min_{0\preccurlyeq X,\, r(X)=p} r(\,A \pm BXB^*\,),\ \min_{0\preccurlyeq X,\, r(X)=p+1} r(\,A \pm BXB^*\,),\ \ldots,\ \max_{0\preccurlyeq X,\, r(X)=q} r(\,A \pm BXB^*\,) \right\}, \tag{5.82}$$

$$\max_{0\preccurlyeq X,\, p\leqslant r(X)\leqslant q} i_+(\,A + X\,) = \max\left\{ \max_{0\preccurlyeq X,\, r(X)=p} i_+(\,A \pm BXB^*\,),\ \max_{0\preccurlyeq X,\, r(X)=p+1} i_+(\,A \pm BXB^*\,),\ \ldots,\ \max_{0\preccurlyeq X,\, r(X)=q} i_+(\,A \pm BXB^*\,) \right\}, \tag{5.83}$$

$$\min_{0\preccurlyeq X,\, p\leqslant r(X)\leqslant q} i_+(\,A + X\,) = \min\left\{ \min_{0\preccurlyeq X,\, r(X)=p} i_+(\,A \pm BXB^*\,),\ \min_{0\preccurlyeq X,\, r(X)=p+1} i_+(\,A \pm BXB^*\,),\ \ldots,\ \min_{0\preccurlyeq X,\, r(X)=q} i_+(\,A \pm BXB^*\,) \right\}, \tag{5.84}$$

$$\max_{0\preccurlyeq X,\, p\leqslant r(X)\leqslant q} i_-(\,A \pm BXB^*\,) = \max\left\{ \max_{0\preccurlyeq X,\, r(X)=p} i_-(\,A \pm BXB^*\,),\ \max_{0\preccurlyeq X,\, r(X)=p+1} i_-(\,A \pm BXB^*\,),\ \ldots,\ \max_{0\preccurlyeq X,\, r(X)=q} i_-(\,A \pm BXB^*\,) \right\}, \tag{5.85}$$

$$\min_{0\preccurlyeq X,\, p\leqslant r(X)\leqslant q} i_-(\,A \pm BXB^*\,) = \min\left\{ \min_{0\preccurlyeq X,\, r(X)=p} i_-(\,A \pm BXB^*\,),\ \min_{0\preccurlyeq X,\, r(X)=p+1} i_-(\,A \pm BXB^*\,),\ \ldots,\ \min_{0\preccurlyeq X,\, r(X)=q} i_-(\,A \pm BXB^*\,) \right\}. \tag{5.86}$$

Applying (5.1)–(5.12) to (5.81)–(5.86) and simplifying yields (5.39)–(5.50). $\square$

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