A NOTE ON THE EXTENSION COMPLEXITY OF THE KNAPSACK POLYTOPE

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ABSTRACT. We show that there are 0-1 and unbounded knapsack polytopes with super-polynomial extension complexity. More specifically, for each $n \in \mathbb{N}$ we exhibit 0-1 and unbounded knapsack polyhedra in dimension n with extension complexity $\Omega(2^{\sqrt{n}})$.

1. INTRODUCTION

There has been considerable success in using linear programming techniques to devise theoretically efficient approximation algorithms (see for example Vazirani [2004] or Williamson and Shmoys [2011]) or practically efficient exact algorithms for difficult combinatorial optimization problems (see for example Pochet and Wolsey [2006] or Applegate et al. [2007]). In this approach, the feasible solutions are encoded as points in the euclidian space, and the theoretical object of interest becomes the polyhedron formed by the convex hull of the feasible solutions. Several natural questions arise in this context. The first one is about the extension complexity: is it possible to give a linear extension (or linear extended formulation) with a small number of facets for a given polyhedron? In case of a negative answer, the next question is about approximability: is there a polyhedron with small extension complexity that approximates provably well a given polyhedron? Here the measure of approximation is typically the worst ratio across a set of objective functions between the solution values when optimizing over the polyhedron and its approximation.

It has been recently shown by Fiorini et al. [2012a] that the extension complexity of the following polytopes is high (i.e., super-polynomial in the dimension): the correlation polytope, the cut polytope, the stable set polytope, and the TSP polytope. Moreover the correlation polytope cannot be approximated within a factor of $n^{1-\epsilon}$ by other polytopes with polynomial extension complexity with respect to certain objective functions (Braun et al. [2012], Braverman and Moitra [2012], Braun and Pokutta [2013]). A consequence of this results is that any approximate extended formulation (of a natural encoding) of the CLIQUE problem with guarantee $n^{1-\epsilon}$ has super-polynomial size.

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The 0-1 knapsack problem is arguably the easiest NP-Hard optimization problem with general objective function: it admits a pseudo-polynomial time algorithm, a simple FPTAS, and it can be modelled as an integer program with a single non-trivial linear constraint. The max 0-1 knapsack polytope associated with weights $a \in \mathbb{R}^n_+$ and right-hand side $b > 0$ is

$$\mathrm{MAXK}(n,a,b) := \mathrm{conv}\left(\left\{x \in \{0,1\}^n \mid a^T x \leq b\right\}\right) \subseteq \mathbb{R}^n,$$

while the min version is defined as:

$$\mathrm{MINK}(n,a,b) := \mathrm{conv}\left(\left\{x \in \{0,1\}^n \mid a^T x \geq b\right\}\right) \subseteq \mathbb{R}^n.$$

Note that when discussing the extension complexity, $\mathrm{MAXK}(n,a,b)$ and $\mathrm{MINK}(n,a,b)$ are equivalent since one is obtained from the other by complementing variables (a linear map). However, when discussing approximability, there is no equivalence because the objective function plays a role in the measure of approximability. In particular, complementing variables introduces a constant term in the objective.

It is known that any max 0-1 knapsack polytope $\mathrm{MAXK}(n,a,b)$ can be arbitrarily well approximated by a polytope with polynomial extension complexity, see Bienstock [2008]. Specifically, for any knapsack polytope $\mathrm{MAXK}(n,a,b)$ and $\epsilon > 0$, there is a linear program with $O(n^{1+\frac{1}{\epsilon}})$ variables and constraints whose optimal solution value for *any* objective function is at most $1+\epsilon$ the optimal solution of the knapsack problem. That is, the 0-1 knapsack polytope admits a Polynomial-Size Relaxation Scheme (PSRS). The existence of a *fully* PSRS (of extension complexity polynomial in n and $\frac{1}{\epsilon}$) remains open. Little is known for the min 0-1 knapsack polytope $\mathrm{MINK}(n,a,b)$: no polytope with polynomial extension complexity is known to achieve a constant approximation ratio, and no lower bound on this approximation ratio is known either.

We show here that there are knapsack polytopes with super-polynomial extension complexity. More specifically, for each $n \in \mathbb{N}$ we exhibit a knapsack polytope $\mathrm{MAXK}(n,a,b)$ with extension complexity $xc(\mathrm{MAXK}(n,a,b)) = \Omega(2^{\sqrt{n}})$. Note that this is the first example of a family of polytopes with high extension complexity that admits a PSRS. Moreover the same construction and result applies to the unbounded knapsack polyhedron. Our result yields an easy route to show high extension complexity of other families of polyhedra: it suffices to show that any 0-1 (or unbounded) knapsack polyhedron can be embedded into a polyhedron of interest of suitably small dimension.

2. Preliminaries and basic notation

Definition 2.1 (Extension of a polyhedron)**.** Let $P \subseteq \mathbb{R}^n$ be a polyhedron. A polyhedron $Q \subseteq \mathbb{R}^d$ is an *extension of* P, if there exists a linear map $\pi : \mathbb{R}^d \to \mathbb{R}^n$ so that $\pi(Q) = P$.

The *size* of an extension Q denoted by $\mathrm{size}(Q)$ is the number of its facets.

The reason to consider extensions is due to the fact that it can have a significantly smaller number of inequalities than P. In fact we are interested in the minimal representation of P by an extension Q

Definition 2.2 (Extension complexity)**.** Let $P \subseteq \mathbb{R}^n$ be a polyhedron. The *extension complexity* of P is given by

$$xc(P) := \min\left\{\mathrm{size}(Q) \mid Q \text{ is an extension of } P\right\}.$$

In order to establish a lower bound on extension complexity of the knapsack polytope we will embed the correlation polytope (which is isomorphic to the cut polytope and sometimes also called boolean quadric polytope) into a face of a suitably chosen knapsack polytope.

Definition 2.3 (Correlation polytope)**.** Let $n \in \mathbb{N}$. The *correlation polytope* on n bits is defined as

$$COR(n) := \mathrm{conv}\left(\left\{bb^T \mid b \in \{0,1\}^n\right\}\right) \subseteq \mathbb{R}^{n\times n}.$$

In Fiorini et al. [2012a] the following result was established

Theorem 2.4 (Lower bound for the correlation polytope)**.** *For $n \in \mathbb{N}$ it holds*

$$xc(COR(n)) = 2^{\Omega(n)}.$$

3. THE EXTENSION COMPLEXITY OF THE KNAPSACK POLYTOPE IS LARGE

In this section we will show that the extension complexity of the knapsack polytope is super-polynomial in the dimension. For this we use the reduction mechanism established in Fiorini et al. [2012a] and we will embed 3SAT, similar to the construction in Fiorini et al. [2012b].

Lemma 3.1 (Monotonicity). *Let P, Q, and F be polytopes. Then:*

(1) if F is an extension of P, then $xc(F) \geq xc(P)$;
(2) if F is a face of Q, then $xc(Q) \geq xc(F)$.

Proof. See [Fiorini et al., 2012b, Lemma 9]. □

We will construct a knapsack polytope so that the integral solutions of a particular face of this polytope are in one-to-one correspondence with the $n \times n$ matrices bb^T where $b \in \{0,1\}^n$. We will then show that this face is an extension of the correlation polytope and invoke Lemma 3.1 More precisely, the construction consists of two steps:

(1) Construct a 3SAT formula φ_n, so that the satisfying assignments of this formula are in one-to-one correspondence with the matrices $bb^\intercal$ with $b \in \{0,1\}^n$.
(2) Define a knapsack polytope whose 0/1 solutions *contain* the satisfying assignments for φ_n. Then take a face of this polytope, so that we obtain a one-to-one correspondence.

We start with encoding the vertices of the correlation polytope in a 3SAT formula. Obtaining such a formula is rather straightforward. In fact, the entry $(bb^\intercal)_{ij} = b_i b_j$ and, as $b \in \{0,1\}^n$, this is the boolean function $b_i \wedge b_j$. In particular $(bb^\intercal)_{ii} = b_i$. We encode the graph of this boolean function, i.e., we define a set of tuples (x, y, z) where the tuples adhere to the structural condition $(b_i, b_j, b_i \wedge b_j)$ with $b \in \{0,1\}^n$. We use Boolean variables $C_{ij} \in \{0,1\}$ for $i, j \in [n]$ and let

$$(C_{ii} \vee C_{jj} \vee \overline{C_{ij}}) \wedge (C_{ii} \vee \overline{C_{jj}} \vee \overline{C_{ij}}) \wedge (\overline{C_{ii}} \vee C_{jj} \vee \overline{C_{ij}}) \wedge (\overline{C_{ii}} \vee \overline{C_{jj}} \vee C_{ij}).$$

The four clauses $(C_{ii} \vee C_{jj} \vee \overline{C_{ij}})$, $(C_{ii} \vee \overline{C_{jj}} \vee \overline{C_{ij}})$, $(\overline{C_{ii}} \vee C_{jj} \vee \overline{C_{ij}})$ and $(\overline{C_{ii}} \vee \overline{C_{jj}} \vee C_{ij})$ model the equation $C_{ij} = C_{ii} \wedge C_{jj}$. We obtain φ_n by taking all these formulas for $i, j \in [n]$ and we define.

$$\varphi_n := \bigwedge_{\substack{i,j \in [n] \\ i \neq j}} \left[(C_{ii} \vee C_{jj} \vee \overline{C_{ij}}) \wedge (C_{ii} \vee \overline{C_{jj}} \vee \overline{C_{ij}}) \wedge (\overline{C_{ii}} \vee C_{jj} \vee \overline{C_{ij}}) \wedge (\overline{C_{ii}} \vee \overline{C_{jj}} \vee C_{ij}) \right].$$

Hence, $C \in \{0,1\}^{n \times n}$ satisfies φ_n if and only if there exists $b \in \{0,1\}^n$ such that $C_{ij} = b_i \wedge b_j$ for all $i, j \in [n]$, or as matrices, $C = bb^\intercal$. Observe that φ_n has $4n^2$ clauses.

Remark 3.2. The formula φ_n can be even further reduced using only three clauses

$$(C_{ii} \vee \overline{C_{ij}}) \wedge (C_{jj} \vee \overline{C_{ij}}) \wedge (\overline{C_{ii}} \vee \overline{C_{jj}} \wedge C_{ij})$$

for each pair $i, j \in [n]$, however we keep the slightly longer version for the sake of exposition as those are already in 3SAT form.

We will now turn the formula φ_n into a family of inequalities that will then, in a later step, turned into a single knapsack inequality. We will choose a reformulation of the clauses that suits our later needs. Abusing notation, we will use the same notation C_{ij} referring to the SAT variables as well as the variables in the inequalities. It is important to note that in the linear programming formulation we will have separate variables for C_{ij} and its negation $\overline{C_{ij}}$ in contrast to the SAT formula. The family of linear inequalities will be in the variables $C_{ij}, \overline{C_{ij}}$ for all $i, j \in [n]$ and $s_1^{\mathscr{C}}, s_2^{\mathscr{C}}, d_1^{\mathscr{C}}, d_2^{\mathscr{C}}$ for all clauses $\mathscr{C} \in \varphi_n$. In the following let $k := 2n^2 + 16n^2 = 18n^2$ denote the number of variables in the polyhedral description. We define the following family of inequalities (they will be enforced as equalities later, so that their meaning is best understood by assuming equality).

(1) (*Enforce clause satisfaction*): Let $\mathscr{C}$ be a clause of φ_n written as $\mathscr{C} = \alpha \vee \beta \vee \gamma$ where α, β, γ are literals, i.e., variables or their negation. We define the inequality

$$\overline{\alpha} + \overline{\beta} + \overline{\gamma} + s_1^{\mathscr{C}} + s_2^{\mathscr{C}} \leq 2.$$

where the variables $s_1^{\mathscr{C}}, s_2^{\mathscr{C}}$ are clause specific slacks. Observe that the inequality prevents us from choosing $\overline{\alpha} = \overline{\beta} = \overline{\gamma} = 1$ which would be equivalent (on the later defined face) to $\alpha = \beta = \gamma = 0$ and thus $\mathscr{C}$ woud not be satisfied. We might think of this inequality as preventing us from not satisfying $\mathscr{C}$.

(2) (*SAT variables cannot be true and false simultaneously*): $C_{ij} + \overline{C_{ij}} \leq 1$ for all $i, j \in [n]$.

(3) (*Bounds for slacks*): $s_i^{\mathscr{C}} + d_i^{\mathscr{C}} \leq 1$ for $i \in \{1, 2\}$ and clauses $\mathscr{C} \in \varphi_n$.

(4) (*Nonnegativity of variables*): $C_{ij}, \overline{C_{ij}}, s_1^{\mathscr{C}}, s_2^{\mathscr{C}}, d_1^{\mathscr{C}}, d_2^{\mathscr{C}} \geq 0$ for all clauses $\mathscr{C} \in \varphi_n$ and $i, j \in [n]$.

The artificial split-up of the slack variable for the clause inequalities in (1) into two variables $s_1^{\mathscr{C}}, s_2^{\mathscr{C}}$ and enforcing the upper bounds in (3) and adding yet another set $d_1^{\mathscr{C}}, d_2^{\mathscr{C}}$ of slack variables serves the sole purpose of ensuring that the knapsack polytope will be contained in $[0, 1]^k$. However it does not matter for the actual construction of the face. We will recombine these inequalities into a single knapsack inequality. For this, let the above set of inequalities be enumerated by some index set I, and be written as $\{a_i x \leq b_i \mid i \in I\}$. We define the knapsack inequality $a = (\sum \lambda_i a_i)$ and $b = (\sum \lambda_i b_i)$ where the $\lambda_i \geq 0$ are algebraically independent reals. Our final knapsack polytope is given by

$$P := \operatorname{conv}\left(\left\{x \in \{0,1\}^k \mid ax \leq b\right\}\right)$$

and we consider the face

$$F := P \cap \{ax = b\}.$$

First note that by aggregating the inequalities into one we lose a lot of the information of each single inequality due to generating an implied inequality only. However, the algebraically independent numbers allow us to recover crucial information:

Observation 3.3. *Let $x_I \in \{0,1\}^k$ satisfy $ax_I = b$. Then*

$$a_i x_I = b_i$$

for all $i \in I$.

Proof. We write

$$0 = b - ax_I = (\sum \lambda_i b_i) - (\sum \lambda_i a_i) x_I = \sum \lambda_i (b_i - a_i x_I)$$

and by algebraic independence it follows that $b_i - a_i x_I$ for all $i \in I$. □

Remark 3.4 (Encoding length)**.** As we only care for the extension complexity we do not have to adhere to any encoding length issues. In fact a reduction with a small encoding length is unlikely to be possible as 3SAT is inapproximable within a factor of 7/8 whereas the knapsack problem has an FPTAS.

We could also have taken some other exponentially large coefficients like 4^i in Observation 3.3 that would ensure a similar property. However, for simplicity of construction and in order to highlight the independence of the encoding length we choose the algebraically independent numbers with infinite encoding length.

We claim that F is an extended formulation of the correlation polytope. This implies

$$xc(P) \geq xc(F) \geq xc(COR(n)) \in 2^{\Omega(n)}.$$

by Lemma 3.1 and Theorem 2.4.

Theorem 3.5. *The face F of P is an extension of the correlation polytope.*

Proof. First, we will establish a one-to-one correspondence between the integral solutions in the face F and the satisfying assignments of the formula. Let $x_I = (C_{ij}, \overline{C_{ij}} : i, j \in [n], s_1^{\mathscr{C}}, s_2^{\mathscr{C}}, d_1^{\mathscr{C}}, d_2^{\mathscr{C}} : \mathscr{C} \in \varphi_n)$ so that $x_I \in F \cap \{0,1\}^k$. Then x_I satisfies all inequalities from the definition of P (except for the nonnegativity constraints) with equality by Observation 3.3. Therefore each variable x_i is set to 0 or 1 as by constraints (2) and (3) all variables in the system are bounded by 1. Moreover, for each

$i,j \in [n]$ either C_{ij} or $\overline{C_{ij}}$ is set to 1 by constraint (2). Then, as for each clause $\mathscr{C} \in \varphi_n$ of the form $\mathscr{C} = \alpha \vee \beta \vee \gamma$ we have

$$\overline{\alpha} + \overline{\beta} + \overline{\gamma} + s_1^{\mathscr{C}} + s_2^{\mathscr{C}} = 2,$$

by constraint (1) and so at most two of three variables $\overline{\alpha}, \overline{\beta}, \overline{\gamma}$ can be set to 1. Thus at least one of the variables α, β, γ is 1 and hence the formula $\mathscr{C}$ is satisfied. For the converse, observe that any feasible assignment τ to φ_n implies a feasible assignment $x_I \in F \cap \{0,1\}^n$ by putting $C_{ij} = 1, \overline{C_{ij}} = 0$ if τ_{ij} is assignment *true* and otherwise putting $C_{ij} = 0, \overline{C_{ij}} = 1$. We choose the $s_i^{\mathscr{C}}$ so that the inequalities (1) are satisfied with equality and then pick $d_1^{\mathscr{C}}, d_2^{\mathscr{C}}$ to satisfy (3) with equality. Thus $(C_{ij}, \overline{C_{ij}} : i,j \in [n], s_1^{\mathscr{C}}, s_2^{\mathscr{C}}, d_1^{\mathscr{C}}, d_2^{\mathscr{C}} : \mathscr{C} \in \varphi_n) \in F \cap \{0,1\}^k$.

It remains to define the linear projection. For this let $\pi : F \to COR(n)$ via $x \mapsto y$ by defining $y_i = C_{ii}$. Observe that $C_{ij} = C_{ii}C_{jj} = C_{ji}$ by the structure enforced by φ_n which in turn is encoded in F. Hence $C = yy^\intercal$. □

Observe that P uses $O(k) = O(n^2)$ variables and $xc(P) = 2^{\Omega(n)}$. We summarize the main result in

Corollary 3.6. *For $n \in \mathbb{N}$ there exists a 0-1 knapsack polytope $P \subset \mathbb{R}^n$ so that $xc(P) \in \Omega(2^{\sqrt{n}})$.*

Corollary 3.6 provides an interesting contrast to the lower bounds results in Braun et al. [2012]. It is the first example of a polytope with high extension complexity that can, however, be approximated arbitrarily well by polytopes with polynomial extension complexity.

Consider now the alternative definition of P as an unbounded knapsack polytope:

$$P' := \operatorname{conv}\left(\left\{x \in \mathbb{N}^k \mid ax \le b\right\}\right)$$

and the similar face

$$F' := P' \cap \{ax = b\}.$$

The analog of Observation 3.3 stills holds:

Observation 3.7. *Let $x_I \in \mathbb{Z}^k$ satisfy $ax_I = b$. Then*

$$a_i x_I = b_i$$

for all $i \in I$.

Because of constraints (2) and (3) all the integer solutions on the face F' will be 0 or 1. Therefore we obtain the same result for the unbounded knapsack polytope.

Corollary 3.8. *For $n \in \mathbb{N}$ there exists an unbounded knapsack polytope $P \subset \mathbb{R}^n$ so that $xc(P) \in \Omega(2^{\sqrt{n}})$.*

It should be pointed out that, although the construction of Bienstock [2008] applies to the 0-1 case only, it is easily extended to the unbounded case. Indeed, using a classical construction described in Kellerer et al. [2004], any unbounded knapsack (or even with general bounds) can be formulated as a 0-1 knapsack problem by inserting $k = 0, \ldots, \log\lfloor \frac{b}{a_i} \rfloor$ copies of each item i with weights 2^k. Using Bienstock's construction applied to this extension, we get a PSRS for the unbounded knapsack of extension complexity $O((n \log b)^{1+\frac{1}{\epsilon}})$. It is an open question whether the dependency in $\log b$ can be eliminated.

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We learned that recently a result similar to ours was independently established in Avis and Tiwary [2013].

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