On the relative strength of families of intersection cuts arising from pairs of tableau constraints in mixed integer programs

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March 5, 2013

Abstract

We compare the relative strength of valid inequalities for the integer hull of the feasible region of mixed integer linear programs with two equality constraints, two unrestricted integer variables and any number of nonnegative continuous variables. In particular, we prove that the closure of Type 2 triangle (resp. Type 3 triangle; quadrilateral) inequalities, are all within a factor of 1.5 of the integer hull, and provide examples showing that the approximation factor is not less than 1.125. There is no fixed approximation ratio for split or Type 1 triangle inequalities however.

*Research of this author was supported in part by a Mellon Fellowship.
†Research of this author was supported in part by NSF grant CMMI1024554 and ONR grant N00014-09-1-0033.
‡Research of this author was supported in part by a Discovery Grant from NSERC and ONR grant N00014-12-1-0049.
§Research of this author was supported in part by a Discovery Grant from NSERC and ONR grant N00014-12-1-0049.

1 Introduction

We consider mixed integer linear programs with n unrestricted-in-sign integer variables x and k non-negative continuous variables s. We assume that the n variables x are expressed in terms of the variables s as follows,

$$\begin{array}{rcl} x & = & f + \sum_{j=1}^{k} r^j s_j \\ x & \in & \mathbb{Z}^n \\ s & \in & \mathbb{R}_+^k. \end{array} \tag{1}$$

We assume $f \in \mathbb{Q}^n \setminus \mathbb{Z}^n$, $k \geq 1$, and $r^j \in \mathbb{Q}^n \setminus \{0\}$ for all $j \in [k]$ [1]. In particular, $s = 0$ is not a solution of (1). Denote by Γ the ordered set $r^1, \ldots, r^k$; we write $\mathrm{R}(f;\Gamma)$ for the convex hull of all vectors $s \in \mathbb{R}_+^k$ such that $f + \sum_{j=1}^{k} r^j s_j$ is integral. It follows from [14] that $\mathrm{R}(f;\Gamma)$ is an upper comprehensive polyhedron (a set $C \subseteq \mathbb{R}_+^k$ is *upper comprehensive* if for all $x \in C$, $x' \geq x$ implies $x' \in C$). Therefore, $\mathrm{R}(f;\Gamma)$ is defined by the inequalities $s \geq 0$ and a finite number of inequalities of the form $\sum_{j=1}^{k} \gamma_j s_j \geq 1$ where $\gamma \geq 0$. The study of $\mathrm{R}(f;\Gamma)$ when $n = 2$ was initiated in the seminal paper [1]. In this paper, we also mainly consider the case $n = 2$.

Given a pure integer linear program (IP) with a fractional optimal basic solution to its linear programming relaxation, we can construct a relaxation of (IP) of the form $\mathrm{R}(f;\Gamma)$ for $n = 2$ that does not contain the current basic solution. In particular, valid constraints for $\mathrm{R}(f;\Gamma)$ where $n = 2$ can be used in cutting plane algorithms. One can proceed as follows: (a) express the basic variables x as a linear combination of the non-basic variables s; (b) select a pair of constraints associated with a pair of basic variables x_i, x_j where x_i, x_j are not both integer; and finally, (c) relax the conditions that the basic variables be non-negative and that the non-basic variables be integer.

Following [1], [3], [8], and [10] we will use the classification of minimal valid constraints of $\mathrm{R}(f;\Gamma)$ for $n = 2$. Our goal in this paper is to compare the relative strength of these different classes of constraints. As in [3], our emphasis is on worst-case bounds. For a probabilistic analysis, see [12], [9], [4].

1.1 Classifying constraints of $\mathrm{R}(f;\Gamma)$

By a *lattice-free convex set* we mean a convex set with no integral point in its relative interior. A lattice-free convex set B is *maximal* if no proper superset of B is a lattice-free convex set. Given a lattice-free convex set $B \subseteq \mathbb{R}^n$ and $f \in \mathbb{Q}^n \setminus \mathbb{Z}^n$ that is an interior point of B we define the function, $\psi_{f;B} : \mathbb{R}^n \to \mathbb{R}$ as follows: for $r \in \mathbb{R}^n$, if there is a positive scalar λ such that the point $f + \lambda r$ is on the boundary of B, $\psi_{f;B}(r) := \frac{1}{\lambda}$. Otherwise, if there is not such λ, $\psi_{f;B}(r) := 0$. Whenever f is clear from the context, we may drop f from the notation for ψ and simply write ψ_B.

Let $f \in \mathbb{Q}^n \setminus \mathbb{Z}^n$ and Γ be an ordered set $r^1, \ldots, r^k \in \mathbb{Q}^n \setminus \{0\}$. For any lattice-free convex set B containing f in its interior the inequality

$$\sum_{j=1}^{k} \psi_{f;B}(r^j) s_j \geq 1, \tag{2}$$

[1] $[k] := \{1, 2, \ldots, k\}$.

is a valid constraint for $\mathrm{R}(f;\Gamma)$, called *intersection cut* [2]. We say that inequality (2) is the *intersection cut of* B, f, Γ. The *trivial* inequalities $s_j \geq 0$ are also valid for $\mathrm{R}(f;\Gamma)$. Given an upper comprehensive closed convex set $C \in \mathbb{R}^n_+ \setminus \{0\}$, a nontrivial valid inequality $\sum_{j=1}^k \gamma_j s_j \geq 1$ for C is said to be a *minimal constraint* if, for any vector $\gamma' \leq \gamma$ distinct from γ, the inequality $\sum_{j=1}^k \gamma'_j s_j \geq 1$ is violated by some $s \in C$. It is proved in [6] that all the nontrivial minimal constraints for $\mathrm{R}(f;\Gamma)$ are intersection cuts.

Theorem 1.1. *Let $f \in \mathbb{Q}^n \setminus \mathbb{Z}^n$ and Γ be an ordered set $r^1, \ldots, r^k \in \mathbb{Q}^n \setminus \{0\}$. If $\mathrm{R}(f;\Gamma) \neq \emptyset$, all nontrivial minimal constraints for $\mathrm{R}(f;\Gamma)$ are intersection cuts.*

Thus, by characterizing maximal lattice-free convex sets, we can obtain a classification of the minimal constraints for $\mathrm{R}(f;\Gamma)$. Lovász [13] classified maximal lattice-free convex sets. The classification for the case $n = 2$ is as follows:

Theorem 1.2. *In the plane, a maximal lattice-free convex set with nonempty interior is one of the following:*

1. *A* split $\left\{x \in \mathbb{R}^2 : c \leq ax_1 + bx_2 \leq c + 1\right\}$ *where a and b are coprime integers and c is an integer;*
2. *A triangle with at least one integral point in the interior of each of its edges;*
3. *A quadrilateral containing exactly four integral points, with exactly one of them in the interior of each of its edges; Moreover, these four integral points are vertices of a parallelogram of area 1.*

Moreover, see [10], the maximal lattice-free triangles are of one of three possible types,

Type 1: triangles with integral vertices and exactly one integral point in the relative interior of each edge;

Type 2: triangles with at least one fractional vertex v, exactly one integral point in the relative interior of the two edges incident to v and at least two integral points on the third edge;

Type 3: triangles with exactly three integral points on the boundary, one in the relative interior of each edge.

We illustrate the three types of triangles in Figure 1. Consider $f \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$ and Γ an ordered set $r^1, \ldots, r^k \in \mathbb{Q}^2 \setminus \{0\}$. We define $\mathrm{S}(f;\Gamma)$ to be the set of points $s \in \mathbb{R}^k_+$ which satisfy all the intersection cuts for B, f, Γ where B is a maximal lattice-free split. Similarly, for $i \in [3]$ we denote by $\triangle_i(f;\Gamma)$ the set of points which satisfy all the intersection cuts for B, f, Γ where B is a Type i triangle. Finally, $\Box(f;\Gamma)$ denotes the set of points which satisfy all the intersection cuts for B, f, Γ where B is maximal lattice-free quadrilateral.

Then, Theorem 1.1 together with Theorem 1.2 imply that

$$\mathrm{R}(f;\Gamma) = \mathrm{S}(f;\Gamma) \cap \triangle(f;\Gamma) \cap \Box(f;\Gamma),$$

where

$$\triangle(f;\Gamma) := \triangle_1(f;\Gamma) \cap \triangle_2(f;\Gamma) \cap \triangle_3(f;\Gamma).$$

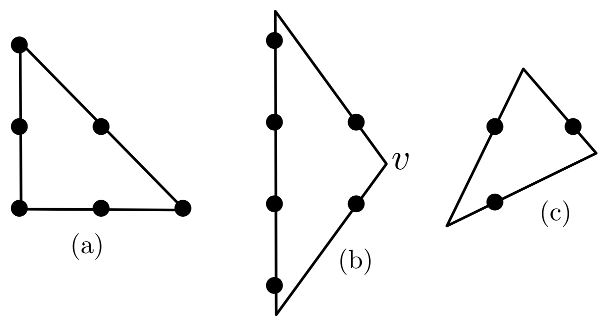

Figure 1: (a) Type 1, (b) Type 2, (c) Type 3.

One of our goals is to compare $\mathrm{R}(f;\Gamma)$ to each of $\mathrm{S}(f;\Gamma)$, $\triangle_1(f;\Gamma)$, $\triangle_2(f;\Gamma)$, $\triangle_3(f;\Gamma)$, $\square(f;\Gamma)$. Before we can state our results, we need to review some tools to compare the strength of different relaxations.

1.2 Comparing relaxations

Let $\alpha > 0$ be a scalar and let $C \subseteq \mathbb{R}^n_+$ be an upper comprehensive convex set. We denote by αC the set

$$\{\alpha x : x \in C\}.$$

Note that since C is upper comprehensive, if $\alpha \leq 1$ then $\alpha C \supseteq C$. Consider now a pair $C_1, C_2 \subseteq \mathbb{R}^n_+$ that are both convex and upper comprehensive. We define,

$$\rho[C_1, C_2] := \inf \left\{ \frac{1}{\alpha} : \alpha C_2 \supseteq C_1 \right\}.$$

I.e., $\rho[C_1, C_2]$ indicates by how much we need to inflate C_2 to contain C_1. Thus C_2 is a relaxation of C_1 if and only if $\rho[C_1, C_2] \leq 1$. If no $\alpha > 0$ exists such that $\alpha C_2 \supseteq C_1$, the value of $\rho[C_1, C_2]$ is defined to be $+\infty$. In this context, for nonempty upper comprehensive convex sets $C \subseteq \mathbb{R}^n_+$, for consistency and convenience, we define $0 \cdot C := \mathbb{R}^n_+$.

For $i \in \{1, 2\}$, let $\mathcal{L}_i$ denote a family of maximal lattice-free convex sets of $\mathbb{R}^n$. Consider $f \in \mathbb{Q}^n \setminus \mathbb{Z}^n, \Gamma = r^1, \ldots, r^k \in \mathbb{Q}^n \setminus \{0\}$ for $k \geq 1$. Then, denote by $\mathcal{L}_i(f, \Gamma)$ the set of points $s \in \mathbb{R}^k_+$ which satisfy all the intersection cuts of B, f, Γ for all $B \in \mathcal{L}_i$ that contain f in their interior. Note that $\mathcal{L}_i(f, \Gamma)$ is an upper comprehensive convex set, but it need not be a polyhedron. The following parameter gives a worst-case measure for pairs of classes of sets defined by intersection cuts,

$$\begin{aligned} &\rho\left[\mathcal{L}_1, \mathcal{L}_2\right] \\ &= \quad \sup \left\{ \rho\left[\mathcal{L}_1(f, \Gamma), \mathcal{L}_2(f, \Gamma)\right] : f \in \mathbb{Q}^n \setminus \mathbb{Z}^n,\ \Gamma = r^1, \ldots, r^k \in \mathbb{Q}^n \setminus \{0\}, k \geq 1 \right\}. \end{aligned} \tag{3}$$

Denote by S the set of all (lattice-free) splits in $\mathbb{R}^2$; denote by $\triangle_i$ the set of all Type i triangles ($i \in \{1, 2, 3\}$); denote by $\square$ the set of all quadrilaterals that are maximal lattice-free; denote by $\triangle$ the set $\triangle_1 \cup \triangle_2 \cup \triangle_3$, and denote by R the set $\mathrm{S} \cup \triangle \cup \square$, i.e., the set of all full-dimensional, maximal lattice-free convex sets in $\mathbb{R}^2$. Then we can define,

$$\rho[\mathrm{S}, \mathrm{R}], \rho[\triangle_i, \mathrm{R}], \rho[\square, \mathrm{R}], \rho[\mathrm{S}, \triangle_i], \rho[\triangle_i, \mathrm{S}], \rho[\triangle_i, \triangle_j], \rho[\square, \triangle_i], \rho[\triangle_i, \square],$$

where $i, j \in [3]$, and $i \neq j$. For instance, $\rho[\triangle_2, \mathrm{R}]$ measures by how much, in the worst case, we have to inflate $\mathrm{R}(f;\Gamma)$ to contain $\triangle_2(f;\Gamma)$. The value is at least 1 as the latter set is a relaxation of the former set. Basu et al. [3] proved $\rho[\mathrm{S}, \triangle_1] = 2$, $\rho[\triangle_2, \square] \leq 2$, $\rho[\triangle, \mathrm{R}] \leq 2$ and $\rho[\square, \mathrm{R}] \leq 2$.

1.3 A summary of the main results

In this section, we present the main results of the paper (all these results are for $n = 2$). The proofs are given in the subsequent sections. We will use the following convention throughout the paper: unless specified otherwise, f will denote a vector of $\mathbb{Q}^2 \setminus \mathbb{Z}^2$ and Γ a sequence $r^1, \ldots, r^k \in \mathbb{Q}^2 \setminus \{0\}$ for some $k \geq 1$.

First, we show inclusions between various relaxations of $\mathrm{R}(f;\Gamma)$.

Theorem 1.3. *For all $f \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$ and $\Gamma = r^1, \ldots, r^k \in \mathbb{Q}^2 \setminus \{0\}$. We have,*

1. $\triangle_2(f;\Gamma) \subseteq \triangle_1(f;\Gamma)$ *and*
2. $\triangle_2(f;\Gamma) \subseteq \mathrm{S}(f;\Gamma)$.

Theorem 1.4. *For all $f \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$ and $\Gamma = r^1, \ldots, r^k \in \mathbb{Q}^2 \setminus \{0\}$. We have,*

1. $\square(f;\Gamma) \subseteq \triangle_2(f;\Gamma)$ *and*
2. $\triangle_3(f;\Gamma) \subseteq \triangle_2(f;\Gamma)$.

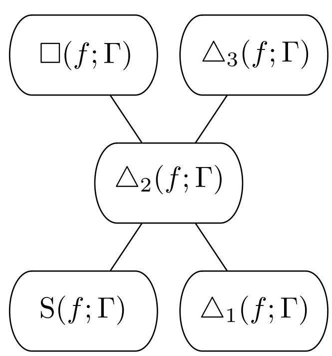

Figure 2: Inclusion lattice for all families of inequalities

Theorem 1.4 in [3], states that $\triangle(f;\Gamma) \subseteq \mathrm{S}(f;\Gamma)$. However, their proof in fact establishes the stronger statement Theorem 1.3(2). We summarize the content of Theorems 1.3 and 1.4 by drawing the associated subset inclusion lattice in Figure 2. Note, in that figure, sets that appear higher are included in sets that appear lower. In other words, the higher the set in the figure the tighter the relaxation. The next two theorems show that, in general, both the pairs $\mathrm{S}(f;\Gamma), \triangle_1(f;\Gamma)$ and $\square(f;\Gamma), \triangle_3(f;\Gamma)$ are incomparable.

Theorem 1.5. *(1) $\rho[\triangle_1, \mathrm{S}] = +\infty$ and (2) $\rho[\mathrm{S}, \triangle_1] = 2$.*

Theorem 1.6. *(1) $\rho[\triangle_3, \square] \geq 1.125$ and (2) $\rho[\square, \triangle_3] \geq 1.125$.*

	S	$\triangle_1$	$\triangle_2$	$\triangle_3$	$\square$
Lower bound	$+\infty$ ⓐ	$+\infty$ ⓑ	1.125 ⓒ	1.125 ⓓ	1.125 ⓔ
Upper bound	$+\infty$ ⓕ	$+\infty$ ⓖ	1.5 ⓗ	1.5 ⓘ	1.5 ⓙ

Table 1: Strength of different relaxations of $\mathrm{R}(f;\Gamma)$

Next we wish to compare the strength of the following relaxations of $\mathrm{R}(f;\Gamma)$, namely: $\mathrm{S}(f;\Gamma)$, $\triangle_1(f;\Gamma)$, $\triangle_2(f;\Gamma)$, $\triangle_3(f;\Gamma)$, and $\square(f;\Gamma)$. The next table summarizes these results. We give upper and lower bounds for $\rho[\#,\mathrm{R}]$ where $\#$ denotes one of $\mathrm{S},\triangle_1,\triangle_2,\triangle_3,\square$. Entries ⓐ and ⓑ indicate that there are instances where the relaxations $\mathrm{S}(f;\Gamma)$ and $\triangle_1(f;\Gamma)$ of $\mathrm{R}(f;\Gamma)$ can be arbitrarily poor. In other words, only using intersection cuts arising from Splits and Type 1 triangles does not always give a good approximation of $\mathrm{R}(f;\Gamma)$. Thus, it is sometimes necessary to use intersection cuts arising from either: Type 2 triangles, Type 3 triangles or quadrilaterals. Entries ⓒ, ⓓ, ⓔ indicate that if we are using only one class of these cuts, then there are instances where we will have at least a gap of 12% between the relaxation and $\mathrm{R}(f;\Gamma)$. Entries ⓗ, ⓘ, ⓙ indicate that by using a single class of cuts arising from Type 2 triangles, Type 3 triangles or quadrilaterals, we will be able to guarantee that the associated relaxation of $\mathrm{R}(f;\Gamma)$ is within a factor of $\frac{3}{2}$.

In light of the results in Table 1 and Figure 2, intersection cuts arising from triangles of Type 2 are a natural class of cuts to investigate from an implementation point of view. Moreover, these cuts are simpler to describe than the classes of cuts arising from Type 3 triangles as well as those arising from quadrilaterals. This motivates further investigation of the parameter $\rho[\triangle_2,\mathrm{R}]$. Entries ⓒ and ⓗ imply that $1.125 \leq \rho[\triangle_2,\mathrm{R}] \leq 1.5$. The problem of finding the exact value of $\rho[\triangle_2,\mathrm{R}]$ remains open.

1.4 Organization of the remainder of the paper

Section 2 proves that the inclusion lattice is as given in Figure 2, i.e. it proves Theorem 1.3 and Theorem 1.4. Section 3 expresses the quantities $\rho[\#_1,\#_2]$, for various parameters $\#_1,\#_2$, as the infimum of a semi-infinite linear program. The derivation is done for arbitrary dimension n and general families of maximal lattice-free convex sets.

Section 4 proves the following result,

Theorem 1.7. $\rho[\triangle_2,\square] \leq 1.5$.

As we mentioned above, Basu et al. [3] proved the weaker bound $\rho[\triangle_2,\square] \leq 2$. Section 5 proves the following result,

Theorem 1.8. $\rho[\triangle_2,\triangle_3] \leq 1.5$.

Section 6 proves Theorems 1.5(1) and 1.6. Theorem 1.5(2) follows from Theorem 1.6 in [3].

We conclude this section by showing (assuming Theorems 1.3, 1.4, 1.5, 1.6, 1.7 and 1.8) that Table 1 is correct. Before we proceed, observe that if for a pair of sets $C_1, C_2 \subseteq \mathbb{R}^n_+$ and a scalar $\alpha > 0$ we have $\alpha C_2 \supseteq C_1$, then for every $C_1' \subseteq C_1$ and $C_2' \supseteq C_2$, $\alpha C_2' \supseteq C_1'$. It follows in particular that,

Remark 1.9. *Let $C_1, C_2, C_1', C_2' \subseteq \mathbb{R}^n_+$ be upper comprehensive convex sets where $C_1' \subseteq C_1$ and $C_2' \supseteq C_2$. Then*

$$\rho[C_1', C_2'] \leq \rho[C_1, C_2].$$

Theorem 1.10. *Table 1 is correct.*

Proof. Theorem 1.8 in [3] says that $\rho[\mathrm{S}, \mathrm{R}] = +\infty$, i.e. entry ⓐ is correct. Theorem 1.5(1) states that $\rho[\triangle_1, \mathrm{S}] = \infty$. As $\mathrm{S}(f;\Gamma) \supseteq \mathrm{R}(f;\Gamma)$, it follows from Remark 1.9 that $\rho[\triangle_1, \mathrm{R}] = \infty$. Thus, entry ⓑ holds. Clearly, ⓐ and ⓑ imply respectively that ⓕ and ⓖ are correct. Theorem 1.6(1) states $\rho[\triangle_3, \square] \geq 1.125$. As $\square(f;\Gamma) \supseteq \mathrm{R}(f;\Gamma)$, Remark 1.9 implies that $\rho[\triangle_3, \mathrm{R}] \geq 1.125$. This proves entry ⓓ. Theorem 1.4(2) states that $\triangle_3(f;\Gamma) \subseteq \triangle_2(f;\Gamma)$. Thus, Remark 1.9 and entry ⓓ imply entry ⓒ. Theorem 1.6(2) states $\rho[\square, \triangle_3] \geq 1.125$. As $\triangle_3(f;\Gamma) \supseteq \mathrm{R}(f;\Gamma)$, Remark 1.9 implies ⓔ.

Consider entry ⓗ. Consider a fixed pair of f and Γ. Theorem 1.7 implies that $\frac{2}{3}\square(f;\Gamma) \supseteq \triangle_2(f;\Gamma)$. Theorem 1.8 implies that $\frac{2}{3}\triangle_3(f;\Gamma) \supseteq \triangle_2(f;\Gamma)$. By Theorem 1.3(1), $\triangle_1(f;\Gamma) \supseteq \triangle_2(f;\Gamma)$, in particular, $\frac{2}{3}\triangle_1(f;\Gamma) \supseteq \triangle_2(f;\Gamma)$. Similarly, Theorem 1.3(2) implies that $\frac{2}{3}\mathrm{S}(f;\Gamma) \supseteq \triangle_2(f;\Gamma)$. It follows that,

$$\frac{2}{3}\Big[\mathrm{S}(f;\Gamma) \cap \triangle_1(f;\Gamma) \cap \triangle_2(f;\Gamma) \cap \triangle_3(f;\Gamma) \cap \square(f;\Gamma)\Big] \supseteq \triangle_2(f;\Gamma).$$

Hence, $\frac{2}{3}\mathrm{R}(f;\Gamma) \supseteq \triangle_2(f;\Gamma)$, proving ⓗ. Finally, ⓘ, ⓙ follow from ⓗ, Remark 1.9 and the facts that $\triangle_2(f;\Gamma) \supseteq \triangle_3(f;\Gamma)$, $\triangle_2(f;\Gamma) \supseteq \square(f;\Gamma)$ (see Theorem 1.4). □

2 The inclusion lattice

For $B \subset \mathbb{R}^n$, we define its *ϵ-relaxation* by

$$\mathrm{relax}(B;\epsilon) := \{s \in \mathbb{R}^n : \|s - \bar{s}\| \leq \epsilon, \text{ for some } \bar{s} \in B\}.$$

In this section, we shall derive Theorems 1.3 and 1.4 from the following result.

Proposition 2.1. *Let $\mathcal{L}, \mathcal{L}'$ denote families of lattice-free convex sets in $\mathbb{R}^n$. Let $f \in \mathbb{Q}^n \setminus \mathbb{Z}^n$ and let $\Gamma = r^1, \ldots, r^k \in \mathbb{Q}^n \setminus \{0\}$. Suppose that for every $\epsilon > 0$ and every $B \in \mathcal{L}$, there exists $B' \in \mathcal{L}'$ such that $B \subseteq \mathrm{relax}(B';\epsilon)$. Then $\mathcal{L}'(f,\Gamma) \subseteq \mathcal{L}(f,\Gamma)$.*

Proof. Suppose the assumption holds. Let $\bar{s} \in \mathbb{R}^n \setminus \mathcal{L}(f,\Gamma)$, we will prove that $\bar{s} \notin \mathcal{L}'(f,\Gamma)$. Since $\bar{s} \notin \mathcal{L}(f,\Gamma)$, there exists $B \in \mathcal{L}$ such that the intersection cut

$$\sum_{i=1}^{k} \psi_B(r^i) s_i \geq 1$$

is violated by $\bar{s}$. Let $\delta := 1 - \sum_{i=1}^{k} \psi_B(r^i)\bar{s}_i > 0$. For

$$\epsilon := \frac{\delta}{2 \max\limits_{i \in [k]} \left\{ \dfrac{\psi_B(r^i)}{\|r^i\|} \right\}} > 0,$$

let $B'_\epsilon \in \mathcal{L}'$ satisfy $B \subseteq \text{relax}(B'_\epsilon; \epsilon)$. Then, coefficients of the intersection cut (see Section 1.1) generated by B'_ϵ satisfy

$$\frac{1}{\psi_{B'_\epsilon}(r^i)} \geq \frac{1}{\psi_B(r^i)} - \frac{\epsilon}{\|r^i\|}, \quad \forall i \in [k].$$

Hence,

$$\begin{aligned}
\sum_{i=1}^{k} \psi_{B'_\epsilon}(r^i)\bar{s}_i &\leq \sum_{i=1}^{k} \frac{\psi_B(r^i)}{1 - \epsilon \left[\frac{\psi_B(r^i)}{\|r^i\|}\right]} \bar{s}_i \\
&\leq \frac{1}{1 - \epsilon \max_{i \in [k]} \left\{\frac{\psi_B(r^i)}{\|r^i\|}\right\}} \sum_{i=1}^{k} \psi_B(r^i)\bar{s}_i \\
&= \frac{1-\delta}{1 - \epsilon \max_{i \in [k]} \left\{\frac{\psi_B(r^i)}{\|r^i\|}\right\}} < 1,
\end{aligned}$$

where we used the definition of ϵ in the last two inequalities. Thus, $\bar{s} \notin \mathcal{L}'(f, \Gamma)$, as desired. $\square$

A function $\phi : \mathbb{R}^n \to \mathbb{R}$ is an *affine unimodular transformation* if $\phi(x) = c + Mx$ where $c \in \mathbb{Z}^n$, $M \in \mathbb{Z}^{n \times n}$ and M is unimodular, i.e. $\det(M) = \pm 1$. It can be readily checked that x is integral if and only if $\phi(x)$ is integral.

A Type 1 triangle is *normalized* if it has corners $(0,0)^T$, $(2,0)^T$ and $(0,2)^T$. A Type 2 triangle is *normalized* if one of its edges contains the points $(0,0)^T$, $(0,1)^T$ and the other two edges contain in their interior the points $(1,1)^T$ and $(1,0)^T$ respectively. See Figure 3 (a) and (b) for examples of normalized triangles of Type 1 and Type 2 respectively (consider the triangles where the boundary is indicated by a thick line).

We leave the following to the reader.

Remark 2.2. *Let T be a triangle of Type i where $i \in [2]$. Then, there exists an affine unimodular transformation $\phi : \mathbb{R}^2 \to \mathbb{R}$ with the property that $\phi(T)$ is a normalized triangle of Type i.*

The next remark shows that Type 1 triangles can be approximated by Type 2 triangles, and that Type 2 triangles can be approximated by both Type 3 triangles and quadrilaterals.

Remark 2.3. *Let $B \subset \mathbb{R}^2$ be a normalized triangle of Type i for some $i \in [2]$. Then, for every $\epsilon > 0$,*

1. *If $i = 1$ there exists a Type 2 triangle B' such that $B \subseteq \text{relax}(B'; \epsilon)$.*
2. *If $i = 2$ there exists a Type 3 triangle B' such that $B \subseteq \text{relax}(B'; \epsilon)$.*
3. *If $i = 2$ there exists a quadrilateral B' such that $B \subseteq \text{relax}(B'; \epsilon)$.*

The proof is illustrated in Figure 3. Part (a) indicates how a Type 2 triangle can approximate a Type 1 triangle. The triangle B where the boundary is indicated by a thick line is the Type 1 triangle. The shaded triangle B' is the Type 2 triangle. It is obtained by tilting one of the edges of the Type 1 triangle around one of the lattice points in the interior of that edge.

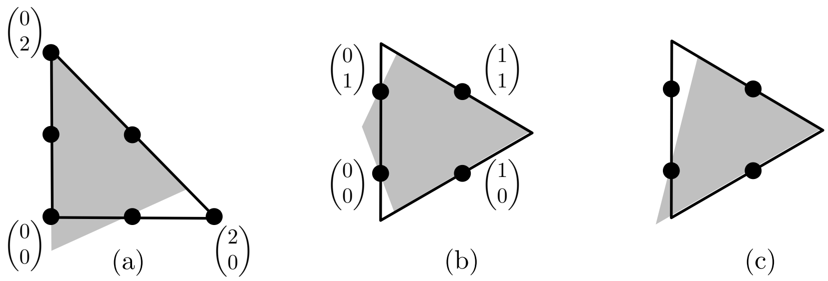

Figure 3: Approximating Type 1 and Type 2 triangles.

Similarly, part (b) indicates how a quadrilateral can approximate a Type 2 triangle, and part (c) shows how a Type 3 triangle can approximate a Type 2 triangle.

We are now ready to prove Theorem 1.3(1), i.e. that $\triangle_2(f;\Gamma) \subseteq \triangle_1(f;\Gamma)$ for any f, Γ. Because of Proposition 2.1, it suffices to show that for every $\epsilon > 0$ and every triangle B of Type 1, there exists a triangle B' of Type 2 such that $B \subseteq \text{relax}(B';\epsilon)$. Moreover, because of Remark 2.2, it suffices to prove the result for the case where B is normalized. This result follows from Remark 2.3(1). (Recall from Section 1.3 that Theorem 1.3(2) follows from Theorem 1.4 in [3].) Similarly, Theorem 1.4(1) and (2) follows from Proposition 2.1 and Remark 2.3 part (3) and (2) respectively.

3 Expressing $\rho[\#_1, \#_2]$ as an optimization problem

We write $B \sim B'$ for $B, B' \subseteq \mathbb{R}^n$ if B' can be obtained from B by some affine unimodular transformation. Let $\mathcal{L}$ be a family of lattice-free convex sets in $\mathbb{R}^n$. We say that $\mathcal{L}$ is *closed under unimodular transformations* if for all $B \in \mathcal{L}$ and $B' \sim B$, $B' \in \mathcal{L}$. As $\sim$ defines an equivalence relation, we can, in that case, partition $\mathcal{L}$ into equivalence classes. A *set of representatives* of $\mathcal{L}$ is a subset of $\mathcal{L}$ which consists of one lattice-free convex set for each equivalence class. Given a set $A \subseteq \mathbb{R}^n$ and $a \in \mathbb{R}^n$, we write $a + A$ for the set $\{a + a' : a' \in A\}$.

We are now ready to state the main result of this section.

Theorem 3.1. *For $i \in \{1,2\}$, $\mathcal{L}_i$ denotes a family of lattice-free convex sets in $\mathbb{R}^n$. Suppose that $\mathcal{L}_2$ is closed under unimodular transformations and let $\hat{\mathcal{L}}_2$ denote a set of representatives of $\mathcal{L}_2$. Suppose that all sets of $\mathcal{L}_2$ are polytopes with exactly ℓ extreme points. Then*

$$\frac{1}{\rho[\mathcal{L}_1,\mathcal{L}_2]} = \inf \quad \sum_{i=1}^{\ell} s_i$$

$$\text{subject to}$$

$$s \in \mathcal{L}_1(f; r^1,\ldots,r^\ell)$$
$$f + \text{conv}\left\{r^1,\ldots,r^\ell\right\} \in \hat{\mathcal{L}}_2$$
$$f \in \mathbb{Q}^n \setminus \mathbb{Z}^n; r^1,\ldots,r^\ell \in \mathbb{Q}^n \setminus \{0\}; \text{cone}\left\{r^1,\ldots,r^\ell\right\} = \mathbb{R}^n.$$

Note, this optimization problem has potentially an infinite number of constraints as we get a constraint of $\mathcal{L}_1\big(f; r^1, \ldots, r^\ell\big)$ for each $B \in \mathcal{L}_1$. The condition that cone $\big\{r^1, \ldots, r^\ell\big\} = \mathbb{R}^n$ ensures that f is in the interior of the polytope $f + \operatorname{conv}\big\{r^1, \ldots, r^\ell\big\}$.

A *normalized quadrilateral* is a maximal lattice-free quadrilateral, where each of the points $(0,0)^T$, $(0,1)^T$, $(1,0)^T$, $(1,1)^T$ is in the interior of a different edge of the quadrilateral. A Type 3 triangle is *normalized* if each of the points $(0,0)^T$, $(1,0)^T$, $(0,1)^T$ is in the interior of a different edge of the triangle. Moreover, we require that $(0,0)^T$ and $(1,1)^T$ be on different sides of the line containing the edge of the triangle with the point $(1,0)^T$ in its interior. See Figure 6. We leave the following to the reader.

Remark 3.2. *Let B be a quadrilateral (resp. a triangle of Type 3). There exists an affine unimodular transformation $\phi : \mathbb{R}^2 \to \mathbb{R}$ with the property that $\phi(B)$ is a normalized quadrilateral (resp. triangle of Type 3).*

Observe that both $\square$ and $\triangle_3$ are families of lattice-free convex sets that are closed under affine unimodular transformations. We now obtain readily the following corollaries of Theorem 3.1 and Remark 3.2.

Corollary 3.3. *Let $\widehat{\square}$ be the set of all normalized quadrilaterals. Then, we have*

$$\begin{aligned} \frac{1}{\rho[\triangle_2, \square]} = \inf \quad & s_1 + s_2 + s_3 + s_4 \\ \text{subject to} \quad & \\ & s \in \triangle_2(f; r^1, r^2, r^3, r^4) \\ & f + \operatorname{conv}\big\{r^1, r^2, r^3, r^4\big\} \in \widehat{\square} \\ & f \in \mathbb{Q}^2 \setminus \mathbb{Z}^2; r^1, r^2, r^3, r^4 \in \mathbb{Q}^2 \setminus \{0\}; \operatorname{cone}\big\{r^1, r^2, r^3, r^4\big\} = \mathbb{R}^2. \end{aligned}$$

Corollary 3.4. *Let $\widehat{\triangle_3}$ be the set of all normalized triangles of Type 3. Then, we have*

$$\begin{aligned} \frac{1}{\rho[\triangle_2, \triangle_3]} = \inf \quad & s_1 + s_2 + s_3 \\ \text{subject to} \quad & \\ & s \in \square(f; r^1, r^2, r^3) \\ & f + \operatorname{conv}\big\{r^1, r^2, r^3\big\} \in \widehat{\triangle_3} \\ & f \in \mathbb{Q}^2 \setminus \mathbb{Z}^2; r^1, r^2, r^3 \in \mathbb{Q}^2 \setminus \{0\}; \operatorname{cone}\big\{r^1, r^2, r^3\big\} = \mathbb{R}^2. \end{aligned}$$

Corollary 3.3 will be used in the proof of Theorem 1.7, Corollary 3.4 will be utilized in the proof of Theorem 1.8. We present a number of preliminaries before we proceed with the proof of Theorem 3.1. Given a nonempty upper comprehensive convex set $C \subseteq \mathbb{R}^n_+$ and $a \in \mathbb{R}^n_+$ we define,

$$\tau[a; C] = \inf\big\{a^T x : x \in C\big\}.$$

Theorem 3.5. *Let $C_1, C_2 \subseteq \mathbb{R}^n_+$ be nonempty, upper comprehensive, closed convex sets. Assume $0 \notin C_2$. Then,*

$$\frac{1}{\rho[C_1, C_2]} = \inf\big\{\tau[a, C_1] : a^T x \geq 1 \textit{ is a minimal constraint for } C_2\big\}.$$

Here, we define $\rho[C_1, C_2]$ to be $+\infty$ when the infimum is 0.

The case where both C_1 and C_2 are polyhedra appeared in [11]. The case where only C_1 is required to be a polyhedron appeared in [3].

Proof. Note that every upper comprehensive closed convex set C in $\mathbb{R}^n_+ \setminus \{0\}$ can be expressed as

$$C = \left\{ x \in \mathbb{R}^n_+ : a^T x \geq 1, a \in \mathcal{A}(C) \right\},$$

where $\mathcal{A}(C) \subseteq \mathbb{R}^n_+$ is the set of all a such that $a^T x = 1$ defines a supporting hyperplane for C. The assumption $0 \notin C$ implies that $\mathcal{A}(C)$ is nonempty. Let

$$\alpha := \inf \left\{ \tau[a, C_1] : a \in \mathcal{A}(C_2) \right\}.$$

First, we prove $C_1 \subseteq \alpha C_2$. Since $C_1, \mathcal{A}(C_2) \subseteq \mathbb{R}^n_+$ are nonempty, we have $0 \leq \alpha < +\infty$.

If $\alpha = 0$, then for every $\epsilon > 0$, there exist $a^\epsilon \in \mathcal{A}(C_2)$ and $x^\epsilon \in C_1$ such that $(a^\epsilon)^T x^\epsilon < \epsilon$. Equivalently, $(a^\epsilon)^T \left(\frac{1}{\epsilon} x^\epsilon\right) < 1$. The latter implies, there does not exist $\beta > 0$ such that $\beta C_1 \subseteq C_2$. We have defined $\rho[C_1, C_2]$ to be $+\infty$ in this case. Therefore the theorem holds when $\alpha = 0$.

Hence, we may assume $\alpha \in (0, +\infty)$. Let $\bar{x} \in C_1$. Then,

$$a^T \bar{x} \geq \alpha, \quad \forall a \in \mathcal{A}(C_2).$$

The latter is equivalent to

$$a^T \left(\frac{1}{\alpha} \bar{x} \right) \geq 1, \quad \forall a \in \mathcal{A}(C_2).$$

Since in addition, $\frac{1}{\alpha}\bar{x} \in \mathbb{R}^n_+$, we have $\frac{1}{\alpha}\bar{x} \in C_2$. Hence, $C_1 \subseteq \alpha C_2$.

Second, we prove that there does not exist $\bar{\alpha} \in (\alpha, +\infty)$ such that $C_1 \subseteq \bar{\alpha} C_2$. Suppose there is such a $\bar{\alpha}$ (we are seeking a contradiction). By the definition of α, for every $\epsilon > 0$, there exist $x^\epsilon \in C_1$ and $a^\epsilon \in \mathcal{A}(C_2)$ such that

$$(a^\epsilon)^T x^\epsilon < \alpha + \epsilon. \tag{4}$$

Since $C_1 \subseteq \bar{\alpha} C_2$, we must have

$$(a^\epsilon)^T \left(\frac{1}{\bar{\alpha}} x^\epsilon \right) \geq 1. \tag{5}$$

Now, the relations (4) and (5) imply

$$\alpha + \epsilon > (a^\epsilon)^T x^\epsilon \geq \bar{\alpha}, \quad \text{for every } \epsilon > 0;$$

i.e., $\alpha = \bar{\alpha}$, a contradiction. Therefore, the above characterization of ρ is correct. $\square$

Lemma 3.6. *Let $\mathcal{L}$ denote a family of lattice-free convex sets in $\mathbb{R}^n$. Let $f \in \mathbb{Q}^n \setminus \mathbb{Z}^n$ and consider Γ, Γ' finite sequences of vectors in $\mathbb{Q}^n \setminus \{0\}$. Suppose that $\Gamma' \subseteq \Gamma$ and that every vector in Γ is a convex combination of vectors in Γ'. Then*

$$\inf \left\{ \mathbb{1}^T s : s \in \mathcal{L}(f; \Gamma) \right\} = \inf \left\{ \mathbb{1}^T s : s \in \mathcal{L}(f; \Gamma') \right\}.$$

For the case where $\mathcal{L}$ is finite the above lemma specializes to Theorem 4.2 in [3].

Proof. Suppose every vector in Γ is a convex combination of vectors in $\Gamma' = r^1, r^2, \ldots, r^k$. So, we may assume, $\Gamma = r^1, r^2, \ldots, r^{k+\ell}$, and $r^{k+i} = \sum_{j=1}^{k} \lambda_{ij} r^j$ with $\sum_{j=1}^{k} \lambda_{ij} = 1$, $\forall i \in [\ell]$, where $\lambda_{ij} \geq 0, \forall i \in [\ell], j \in [k]$.

Every feasible solution of the semi-infinite linear program in the RHS may be appended by ℓ zeros to make up a feasible solution of the semi-infinite linear program in the LHS with the same objective value. Thus, LHS $\leq$ RHS.

Let $\bar{s} \in \mathbb{R}_+^{k+\ell}$ be a feasible solution of the problem in the LHS. Define

$$\hat{s}_j := \bar{s}_j + \sum_{i=1}^{\ell} \lambda_{ij} \bar{s}_{k+i}, \quad \forall j \in [k].$$

Note that

$$\mathbb{1}^T \hat{s} = \sum_{j=1}^{k} \left(\bar{s}_j + \sum_{i=1}^{\ell} \lambda_{ij} \bar{s}_{k+i} \right) = \sum_{j=1}^{k} \bar{s}_j + \sum_{i=1}^{\ell} \bar{s}_{k+i} = \mathbb{1}^T \bar{s}.$$

Consider an arbitrary inequality among the constraints in the LHS:

$$\sum_{j=1}^{k+\ell} \psi_{f;B}(r^j) s_j \geq 1.$$

We have

$$\begin{aligned}
1 &\leq \sum_{j=1}^{k+\ell} \psi_{f;B}(r^j) \bar{s}_j \\
&= \sum_{j=1}^{k} \psi_{f;B}(r^j) \bar{s}_j + \sum_{i=1}^{\ell} \psi_{f;B} \left(\sum_{j=1}^{k} \lambda_{ij} r^j \right) \bar{s}_{k+i} \\
&\leq \sum_{j=1}^{k} \psi_{f;B}(r^j) \left(\bar{s}_j + \sum_{i=1}^{\ell} \lambda_{ij} \bar{s}_{k+i} \right) \\
&= \sum_{j=1}^{k} \psi_{f;B}(r^j) \hat{s}_j,
\end{aligned}$$

where the second inequality above uses the convexity of $\psi_{f;B}(\cdot)$ (see [5] for example). Hence, $\hat{s}$ is a feasible solution of the problem in the RHS with the same objective value as $\bar{s}$ (in the LHS). Therefore, LHS $\geq$ RHS. □

The next remark characterizes the effect of scaling vectors.

Remark 3.7. *Let $\mathcal{L}$ denote a family of lattice-free convex sets in $\mathbb{R}^n$. Let $f \in \mathbb{Q}^n \setminus \mathbb{Z}^n$ and let $r^1, \ldots, r^k \in \mathbb{Q}^n \setminus \{0\}$ such that* $\operatorname{cone}\{r^1, \ldots r^k\} = \mathbb{R}^n$. *Consider $\mu_j > 0$ for all $j \in [k]$ and let $D = \operatorname{Diag}(\mu_1, \ldots, \mu_k)$. Suppose that $a^T s \geq 1$ is a minimal constraint of $\mathcal{L}(f; r^1, \ldots, r^k)$ and let $s \in \mathcal{L}(f; r^1, \ldots, r^k)$. Then*

1. *$(Da)^T s \geq 1$ is a minimal constraint of $\mathcal{L}(f; \mu_1 r^1, \ldots, \mu_k r^k)$, and*

2. $D^{-1}s \in \mathcal{L}(f;\mu_1 r^1,\dots,\mu_k r^k)$.

Proof. (1) By definition of $\mathcal{L}(f;r^1,\dots,r^k)$, $a^T s \geq 1$ is an intersection cut for $B \in \mathcal{L}$, f and $r^1,\dots,r^k$, i.e. for all $j \in [k]$, $a_j = \psi_{f;B}(r^j)$. It suffices to show that $(Da)^T s \geq 1$ is an intersection cut for B, f, and $\mu_1 r^1,\dots,\mu_k r^k$, i.e. that for all $j \in [k]$, $\psi_{f;B}(\mu_j r^j) = \mu_j a_j = \mu_j \psi_{f;B}(r^j)$. If there is no positive scalar λ such that $f+\lambda r^j$ is on the boundary of B then $\psi_{f;B}(r^j) = \psi_{f;B}(\mu_j r^j) = 0$ as required. Otherwise, $f+\lambda r^j$ is on the boundary of B, and $\psi_{f;B}(r^j) = \frac{1}{\lambda}$. Hence, $f + \frac{\lambda}{\mu_j}(\mu_j r^j)$ is on the boundary of B, and $\psi_{f;B}(\mu_j r^j) = \frac{\mu_j}{\lambda}$ as required. (2) follows from (1) as every minimal constraint of $\mathcal{L}(f;r^1,\dots,r^k)$ distinct from $s \geq 0$ is of the form $a^T s \geq 1$, for $a \geq 0$. □

The next remark shows invariance under affine unimodular transformations.

Remark 3.8. *Let $\mathcal{L}$ denote a family of lattice-free convex sets in $\mathbb{R}^n$ that is closed under unimodular transformations. Let $f \in \mathbb{Q}^n \setminus \mathbb{Z}^n$ and let $r^1,\dots,r^k \in \mathbb{Q}^n \setminus \{0\}$ such that* $\operatorname{cone}\{r^1,\dots r^k\} = \mathbb{R}^n$*. Let $M \in \mathbb{Z}^n \times \mathbb{Z}^n$ be a unimodular matrix and let $c \in \mathbb{Z}^n$. Then*

$$\mathcal{L}(f;r^1,\dots,r^k) = \mathcal{L}(c+Mf;Mr^1,\dots,Mr^k).$$

Proof. Since M^{-1} is unimodular, it suffices to show that every intersection cut for $f, B \in \mathcal{L}$ and $r^1,\dots,r^k$, is an intersection cut for f, B' and $Mr^1,\dots,Mr^k$ where B' is some set in $\mathcal{L}$. This follows immediately from the fact that $f+\lambda r^j$ is on the boundary of B, if and only if $(c+Mf)+\lambda(Mr^j) = c + M(f+\lambda r^j)$ is on the boundary of $B' = c + \{Mb : b \in B\}$. Finally, as $\mathcal{L}$ is closed under unimodular transformations, $B' \in \mathcal{L}$ as well. □

Suppose $B \subset \mathbb{R}^n$ is a lattice-free convex set and let f be in the interior of B. We say that $r \in \mathbb{Q}^n \setminus \{0\}$ is a *boundary ray* (for f, B) if $f+r$ is on the boundary of B, and that r is a *corner ray* if $f+r$ is an extreme point of B.

Proof of Theorem 3.1. By definition of ρ, see (3), $\frac{1}{\rho[\mathcal{L}_1,\mathcal{L}_2]}$ is equal to,

$$\inf\left\{\frac{1}{\rho\big[\mathcal{L}_1(f;\Gamma),\mathcal{L}_2(f;\Gamma)\big]} : f \in \mathbb{Q}^n \setminus \mathbb{Z}^n; \Gamma = r^1,\dots,r^k \in \mathbb{Q}^n \setminus \{0\}; k \geq 1; \operatorname{cone}\{\Gamma\} = \mathbb{R}^n\right\}. \tag{6}$$

It follows from Theorem 3.5 and the fact that every minimal constraint of $\mathcal{L}_2(f;\Gamma)$ is an intersection cut that (6) can be written as,

$$\begin{aligned} \inf \quad & \tau\big[a,\mathcal{L}_1(f;\Gamma)\big] \\ \text{subject to} \quad & \\ & a^T s \geq 1 \text{ intersection cut for } f, B \in \mathcal{L}_2 \text{ and } \Gamma \\ & f \in \mathbb{Q}^n \setminus \mathbb{Z}^n; \Gamma = r^1,\dots,r^k \in \mathbb{Q}^n \setminus \{0\}; k \geq 1; \operatorname{cone}\{\Gamma\} = \mathbb{R}^n. \end{aligned} \tag{7}$$

Because of Remark 3.8 we may restrict in (6) $B \in \mathcal{L}_2$ to $B \in \widehat{\mathcal{L}}_2$. Since $a^T s \geq 1$ is an intersection cut and f is in the interior of B, $a_j > 0$ for all $j \in [k]$. Let $D = \operatorname{Diag}(a_1,\dots,a_k)$.

Then

$$\begin{aligned}\tau\big[a,\mathcal{L}_1(f;\Gamma)\big] &= \inf\left\{a^T s : s \in \mathcal{L}_1(f;r^1,\ldots,r^k)\right\}\\ &= \inf\left\{\mathbb{1}^T Ds : Ds \in \mathcal{L}_1\Big(f;\frac{1}{a_1}r^1,\ldots,\frac{1}{a_k}r^k\Big)\right\},\end{aligned}$$

where the first equality follows from the definition of τ and the second from Remark 3.7(2). Moreover, by Remark 3.7(1), $\mathbb{1}^T s \geq 1$ is a minimal constraint of $\mathcal{L}_1\Big(f;\frac{1}{a_1}r^1,\ldots,\frac{1}{a_k}r^k\Big)$. Since in (7) the infimum is taken over all $r^1,\ldots,r^k \in \mathbb{Q}^n \setminus \{0\}$, after redefining Ds as s, we can rewrite (7) as follows,

$$\begin{aligned} \inf \quad & \mathbb{1}^T s \\ \text{subject to} \quad & \\ & s \in \mathcal{L}_1\big(f;r^1,\ldots,r^k\big) \\ & \mathbb{1}^T s \geq 1 \text{ intersection cut for } f, B \in \widehat{\mathcal{L}}_2 \text{ and } \Gamma \\ & f \in \mathbb{Q}^n \setminus \mathbb{Z}^n; \Gamma = r^1,\ldots,r^k \in \mathbb{Q}^n \setminus \{0\}; k \geq 1; \text{cone}\{\Gamma\} = \mathbb{R}^n. \end{aligned} \tag{8}$$

Let $B \in \widehat{\mathcal{L}}_2$ for which the intersection cut for f, B, Γ is of the form $\mathbb{1}^T s \geq 1$. Then all vectors of Γ must be boundary rays for f, B. Let Γ'' be the corner rays for B and f, and let $\Gamma' = \Gamma \cup \Gamma''$. Then

$$\inf\{\mathbb{1}^T s : s \in \mathcal{L}_1(f;\Gamma)\} \geq \inf\{\mathbb{1}^T s : s \in \mathcal{L}_1(f;\Gamma')\} = \inf\{\mathbb{1}^T s : s \in \mathcal{L}_1(f;\Gamma'')\}.$$

The inequality arises from the fact that the second optimization problem is a relaxation of the first as setting variables for the second problem to zero for all vectors in $\Gamma' \setminus \Gamma$ yields the first problem. The equality follows by Lemma 3.6. Thus, in (8) we can restrict Γ to correspond to the set of all corner rays of B, i.e. that $B = f + \text{conv}\{r^1,\ldots,r^\ell\}$ where ℓ is the number of extreme points of each $B \in \mathcal{L}_2$. Then, the resulting problem is as required. □

4 Upper bound for $\rho[\triangle_2, \square]$

Let Q be a normalized quadrilateral with corners v^1, v^2, v^3, v^4. We may assume (after possibly relabelling v^1, v^2, v^3, v^4) that (see Figure 4),

$$\begin{aligned} &\text{edge } v_1, v_2 \text{ contains } (1,0)^T \text{ and has slope } \beta, \\ &\text{edge } v_2, v_3 \text{ contains } (0,0)^T \text{ and has slope } -\alpha, \\ &\text{edge } v_3, v_4 \text{ contains } (0,1)^T \text{ and has slope } \gamma, \text{ and} \\ &\text{edge } v_4, v_1 \text{ contains } (1,1)^T \text{ and has slope } -\delta, \\ &\text{where } \alpha,\beta,\gamma,\delta > 0. \end{aligned} \tag{9}$$

Note that Q is completely described by $\alpha, \beta, \gamma, \delta$. The following can be readily checked.

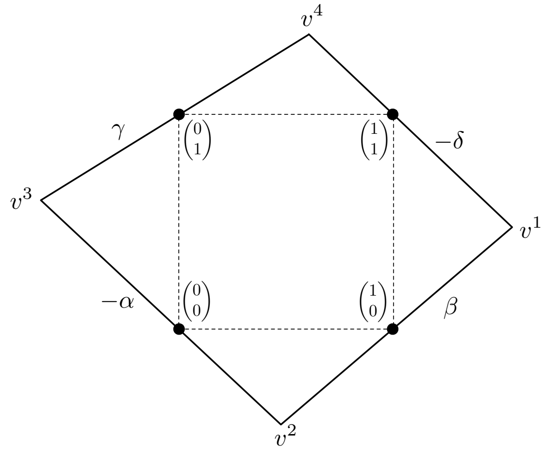

Figure 4: Quadrilateral with corners v^1, v^2, v^3, v^4.

Remark 4.1. *Let Q be a normalized quadrilateral as in* (9)*. Then*

$$v^1 = \left(1 + \frac{1}{\beta+\delta}, \frac{\beta}{\beta+\delta}\right)^T \qquad v^2 = \left(\frac{\beta}{\alpha+\beta}, \frac{-\alpha\beta}{\alpha+\beta}\right)^T$$
$$v^3 = \left(\frac{-1}{\alpha+\gamma}, \frac{\alpha}{\alpha+\gamma}\right)^T \qquad v^4 = \left(\frac{\delta}{\delta+\gamma}, 1 + \frac{\gamma\delta}{\delta+\gamma}\right)^T.$$

Let Q be a normalized quadrilateral as in (9) and let $\ell_1, \ell_2, \ell_3, \ell_4$ be distinct elements of [4]. The *fixed triangle T_{ℓ_1} associated with Q* is the unique maximal lattice-free triangle that has $v^{\ell_2}, v^{\ell_3}, v^{\ell_4}$ on the boundary of T_{ℓ_1} and that contains all of $(0,0)^T$, $(0,1)^T$, $(1,0)^T$, $(1,1)^T$. See Figure 5.

Remark 4.2. *Let Q be a normalized quadrilateral as in* (9)*. For all $i \in [4]$, let T_i denote a fixed triangle associated with Q and let $r^i = v^i - f$ where $f \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$ is an interior point of $T_i \cap Q$. For all $i, j \in [4]$, $i \neq j$, $\psi_{f;T_i}(r^j) = 1$. Moreover,*

$$\psi_{f;T_1}(r^1) = 1 + \frac{1}{a}, \quad \psi_{f;T_2}(r^2) = 1 + \frac{1}{b}, \quad \psi_{f;T_3}(r^3) = 1 + \frac{1}{c}, \quad \psi_{f;T_4}(r^4) = 1 + \frac{1}{d},$$

where

$$\begin{aligned} a &= (1-f_1)(\beta+\delta) & \qquad b &= f_2(\tfrac{1}{\alpha} + \tfrac{1}{\beta}) \\ c &= f_1(\alpha+\gamma) & \qquad d &= (1-f_2)(\tfrac{1}{\delta} + \tfrac{1}{\gamma}). \end{aligned} \tag{10}$$

Proof. Suppose $j \neq i$. Then $f + r^j$ is on the boundary of T_i, and by definition, $\psi_{f;T_i}(r^j) = 1$ as required. Suppose $i = j$. We only consider the case where $i = 1$ as the other cases are analogous. For some scalar $\lambda > 0$, $f + \lambda r^1$ is on the boundary of T_1. In particular, $f_1 + \lambda r^1_1 = 1$, thus

$$\psi_{f;T_1}(r^1) = \frac{1}{\lambda} = \frac{r^1_1}{1-f_1} = \frac{v^1_1 - f_1}{1-f_1} = 1 + \frac{v^1_1 - 1}{1-f_1}.$$

By Remark 4.1, $v^1_1 = 1 + \frac{1}{\beta+\delta}$, and the result follows. ∎

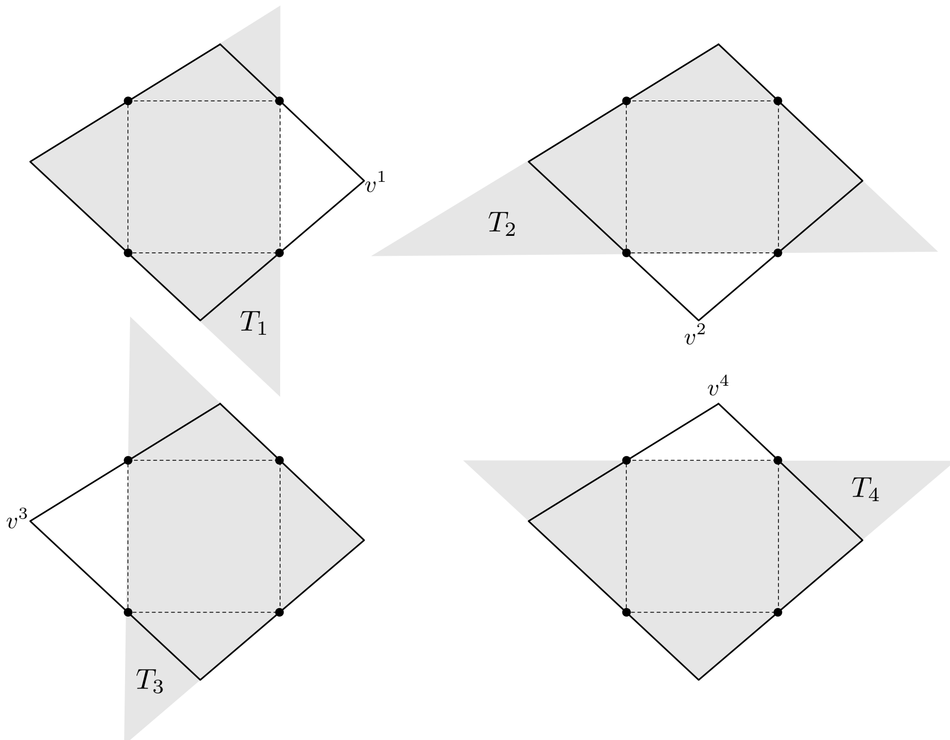

Figure 5: Associated triangles T_1, T_2, T_3 and T_4.

We consider the functions $h, h' : \mathbb{R}^6 \to \mathbb{R}$ with variables $f_1, f_2, \alpha, \beta, \delta, \gamma$ where,

$$h(f_1, f_2, \alpha, \beta, \delta, \gamma) = a + b + c + d \qquad \text{and} \qquad h'(f_1, f_2, \alpha, \beta, \delta, \gamma) = b + c + d, \tag{11}$$

where a, b, c, d are defined as in (10).

Lemma 4.3. *For h, h' defined as in* (11)*,*

1. $\inf \{h : f_1, f_2 \in [0,1], \alpha, \beta, \gamma, \delta > 0\} \geq 2$*, and*
2. $\inf \big\{h' : f_1 \geq 1, f_2 \in [0,1], \alpha, \beta, \gamma, \delta > 0\big\} \geq 2$.

The proof will require the following observation.

Remark 4.4. *For every pair of constants $c_1, c_2 > 0$, the function $f : \mathbb{R}_{++} \to \mathbb{R}$ where $f(x) = c_1 \frac{1}{x} + c_2 x$ attains it minimum value $2\sqrt{c_1 c_2}$ uniquely at $x = \sqrt{\frac{c_1}{c_2}}$. In particular,*

$$\min \left\{ x + \frac{1}{x} : x > 0 \right\} \geq 2.$$

Proof of Lemma 4.3. Consider part 1. Let us define the function $g : (0,1) \to \mathbb{R}^2$ where

$$g(\epsilon) = \inf \left\{ h : f_1, f_2 \in [0,1], \alpha, \beta, \gamma, \delta \in \left[\epsilon, \frac{1}{\epsilon}\right] \right\}.$$

[2] (a, b) denotes the open interval between a and b.

Claim. *For all $\epsilon \in (0,1)$, $g(\epsilon) \geq 2$.*

Proof of claim: Consider a fixed $\epsilon > 0$. Then $g(\epsilon)$ is obtained by minimizing the continuous function h, over the compact set $\{(f_1, f_2, \alpha, \beta, \gamma, \delta) : f_1, f_2 \in [0,1], \alpha, \beta, \gamma, \delta \in \left[\epsilon, \frac{1}{\epsilon}\right]\}$. It follows that h attains its minimum for say values $\hat{f}_1, \hat{f}_2, \hat{\alpha}, \hat{\beta}, \hat{\delta}, \hat{\gamma}$. For fixed values $\hat{\alpha}, \hat{\beta}, \hat{\delta}, \hat{\gamma}$, h is a linear function in f_1, f_2. Thus it attains its minimum for one of the following values (f_1, f_2): $(0,0)^T$, $(1,0)^T$, $(0,1)^T$, $(1,1)^T$. Observe that by symmetry of h it suffices to consider the case $(f_1, f_2) = (0,0)^T$. In that case h becomes,

$$\hat{\beta} + \hat{\delta} + \frac{1}{\hat{\delta}} + \frac{1}{\hat{\gamma}} \geq \hat{\delta} + \frac{1}{\hat{\delta}} \geq 2,$$

where the first inequality follows from $\hat{\beta}, \hat{\gamma} \geq \epsilon$ and the second from $\hat{\delta} > \epsilon$ and Remark 4.4. $\diamond$

Finally, if for some $\hat{f}_1, \hat{f}_2 \in [0,1]$ and $\hat{\alpha}, \hat{\beta}, \hat{\delta}, \hat{\gamma} > 0$, h has value < 2, then for $\epsilon > 0$ small enough, $g(\epsilon) < 2$, contradicting the Claim. Hence, part 1. holds.

For part 2., consider $\hat{\alpha}, \hat{\beta}, \hat{\delta}, \hat{\gamma} > 0$ and $f_1 \geq 1$, $f_2 \in [0,1]$. Then

$$\begin{aligned} h'(f_1, f_2, \alpha, \beta, \delta, \gamma) &= f_2(\frac{1}{\alpha} + \frac{1}{\beta}) + f_1(\alpha + \gamma) + (1 - f_2)(\frac{1}{\delta} + \frac{1}{\gamma}) \\ &\geq f_2(\frac{1}{\alpha} + \frac{1}{\beta}) + (\alpha + \gamma) + (1 - f_2)(\frac{1}{\delta} + \frac{1}{\gamma}) \\ &= h(1, f_2, \alpha, \beta, \delta, \gamma) \geq 2, \end{aligned}$$

where the first inequality follows from $\delta, \gamma > 0$ and the second from part (1). $\square$

We are now ready for the main proof of this section,

Proof of Theorem 1.7. Choose arbitrary fixed $r^1, r^2, r^3, r^4 \in \mathbb{Q}^2 \backslash \{0\}$ and $f \in \mathbb{Q}^2 \backslash \mathbb{Z}^2$ such that $Q = f + conv\{r^1, r^2, r^3, r^4\}$ is a normalized quadrilateral. Consider the following minimization problem,

$$\begin{aligned} \inf \quad & s_1 + s_2 + s_3 + s_4 \\ \text{subject to} \quad & \\ & s \in \triangle_2(f; r^1, r^2, r^3, r^4). \end{aligned} \tag{12}$$

It suffices to show that some relaxation of (12) has a lower bound of $\frac{2}{3}$, for as r^1, r^2, r^3, r^4, f were chosen arbitrarily, it implies by Corollary 3.3 that $\frac{2}{3}$ is a lower bound for $\frac{1}{\rho[\triangle_2, \square]}$; i.e., that $\rho[\triangle_2, \square] \leq 1.5$ as required. For $i \in [4]$, $v^i = f + r^i$ is a corner of Q. We may assume that Q is described by $\alpha, \beta, \gamma, \delta$ as in (9). After possibly rotating by multiples of $\pi/2$ (see Remark 3.8) either, $0 < f_1, f_2 < 1$ or $f_1 \geq 1$ and $0 < f_2 < 1$. Thus, it will suffice to consider Case 1 and Case 2.

Case 1: $0 < f_1, f_2 < 1$.

Let T_1, T_2, T_3, T_4 be the fixed triangles associated with Q (see Figure 5). The following linear program is a relaxation of (12),

$$\begin{aligned} \min \quad & s_1 + s_2 + s_3 + s_4 \\ \text{subject to} \quad & \\ & \sum_{j=1}^{4} \psi_{f;T_i}(r^j) s_j \geq 1 \qquad i \in [4] \\ & s \in \mathbb{R}^4_+. \end{aligned} \tag{13}$$

Since $0 < f_1, f_2 < 1$ and $\alpha, \beta, \gamma, \delta > 0$ we have, $a, b, c, d > 0$, where a, b, c, d are defined as in (10). Remark 4.2 implies that (13) can be written as,

$$\begin{aligned} \min \quad & s_1 + s_2 + s_3 + s_4 \\ \text{subject to} \quad & \\ & \begin{pmatrix} 1+\frac{1}{a} & 1 & 1 & 1 \\ 1 & 1+\frac{1}{b} & 1 & 1 \\ 1 & 1 & 1+\frac{1}{c} & 1 \\ 1 & 1 & 1 & 1+\frac{1}{d} \end{pmatrix} s \geq \mathbb{1} \\ & s \geq 0. \end{aligned} \tag{14}$$

The dual of (14) is given by,

$$\begin{aligned} \max \quad & \nu_1 + \nu_2 + \nu_3 + \nu_4 \\ \text{subject to} \quad & \\ & \begin{pmatrix} 1+\frac{1}{a} & 1 & 1 & 1 \\ 1 & 1+\frac{1}{b} & 1 & 1 \\ 1 & 1 & 1+\frac{1}{c} & 1 \\ 1 & 1 & 1 & 1+\frac{1}{d} \end{pmatrix} \nu \leq \mathbb{1} \\ & \nu \geq 0. \end{aligned} \tag{15}$$

Consider,

$$\hat{\nu} = \frac{1}{1+a+b+c+d} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.$$

As $a, b, c, d > 0$, $\hat{\nu} \geq 0$ and it can be readily checked that constraints of (15) corresponding to each of the primal variables s_1, s_2, s_3, s_4 are satisfied with equality. Thus $\hat{\nu}$ is feasible for (15). Moreover, it has objective value,

$$\hat{\nu}^T \mathbb{1} = 1 - \frac{1}{1+a+b+c+d}.$$

By (11) $h = a + b + c + d$. Lemma 4.3 part 1. shows that for all $f_1, f_2, \alpha, \beta, \gamma, \delta$ where $0 < f_1, f_2 < 1$ and $\alpha, \beta, \gamma, \delta > 0$, we have $h \geq 2$. Thus $\hat{\nu}^T \mathbb{1} \geq \frac{2}{3}$. In particular, $\frac{2}{3}$ is a lower bound for (15) and in turn by weak duality to (13) as required.

Case 2: $f_1 \geq 1$ **and** $0 < f_2 < 1$**.**

Let T_2, T_3, T_4 be the fixed triangles associated with Q (see Figure 5). Note, that f is in the interior of T_2, T_3 and T_4. The following linear program is a relaxation of (12),

$$\begin{aligned} \min \quad & s_1 + s_2 + s_3 + s_4 \\ \text{subject to} \quad & \\ & \sum_{j=1}^{4} \psi_{f;T_i}(r^j) s_j \geq 1 \qquad i \in \{2,3,4\} \\ & s \in \mathbb{R}^4_+. \end{aligned} \tag{16}$$

This differs from (13) as we omitted the constraint corresponding to T_1. Since $f_1 \geq 1$, $0 < f_2 < 1$ and $\alpha, \beta, \gamma, \delta > 0$ we have, $b, c, d > 0$, where b, c, d are defined as in (10). Remark 4.2 implies that (16) can be written as,

$$\begin{aligned} \min \quad & s_1 + s_2 + s_3 + s_4 \\ \text{subject to} \quad & \\ & \begin{pmatrix} 1 & 1+\frac{1}{b} & 1 & 1 \\ 1 & 1 & 1+\frac{1}{c} & 1 \\ 1 & 1 & 1 & 1+\frac{1}{d} \end{pmatrix} s \geq \mathbb{1} \\ & s \geq 0. \end{aligned} \tag{17}$$

The dual of (17) is given by,

$$\begin{aligned} \max \quad & \nu_1 + \nu_2 + \nu_3 \\ \text{subject to} \quad & \\ & \begin{pmatrix} 1 & 1 & 1 \\ 1+\frac{1}{b} & 1 & 1 \\ 1 & 1+\frac{1}{c} & 1 \\ 1 & 1 & 1+\frac{1}{d} \end{pmatrix} \nu \leq \mathbb{1} \\ & \nu \geq 0. \end{aligned} \tag{18}$$

Consider,

$$\hat{\nu} = \frac{1}{1+b+c+d} \begin{pmatrix} b \\ c \\ d \end{pmatrix}.$$

As $b, c, d > 0$, $\hat{\nu} \geq 0$ and it can be readily checked that constraints of (18) corresponding to each of primal variables s_2, s_3, s_4 are satisfied with equality, and that since $b, c, d > 0$ the constraint corresponding to s_1 also holds. Thus $\hat{\nu}$ is feasible for (18). Moreover, it has objective value,

$$\hat{\nu}^T \mathbb{1} = 1 - \frac{1}{1+b+c+d}.$$

By (11), $h' = b + c + d$. Lemma 4.3 part 2. shows that for all $f_1, f_2, \alpha, \beta, \gamma, \delta$ where $f_1 \geq 1$, $0 < f_2 < 1$ and $\alpha, \beta, \gamma, \delta > 0$, we have $h' = b + c + d \geq 2$. Thus $\hat{\nu}^T \mathbb{1} \geq \frac{2}{3}$. In particular, $\frac{2}{3}$ is a lower bound for (18) and in turn by weak duality to (16) as required. ☐

5 Upper bound for $\rho[\triangle_2, \triangle_3]$

Let T be a normalized Type 3 triangle with corners v^1, v^2, v^3. We may assume (after possibly relabelling v^1, v^2, v^3) that (see Figure 6),

$$\begin{array}{l} \text{edge } v_1, v_2 \text{ contains } (0,1)^T \text{ and has slope } \delta, \\ \text{edge } v_2, v_3 \text{ contains } (0,0)^T \text{ and has slope } -\alpha, \text{ and} \\ \text{edge } v_3, v_1 \text{ contains } (1,0)^T \text{ and has slope } -\beta, \\ \text{where } \beta > 1,\ \alpha, \delta > 0, \text{ and } \alpha < 1. \end{array} \tag{19}$$

Note, by definition, as T is a normalized Type 3 triangle, $(0,0)^T$ and $(1,1)^T$ are on different sides of the line containing v^1, v^3. This implies $\beta > 1$. Since $(1,0)^T$ is an interior point of the edge v_1, v_3 of T, $\alpha > 0$. As $(1,-1)^T \notin T$, $\alpha < 1$. Finally, since $(-1,1)^T \notin T$, $\delta > 0$.

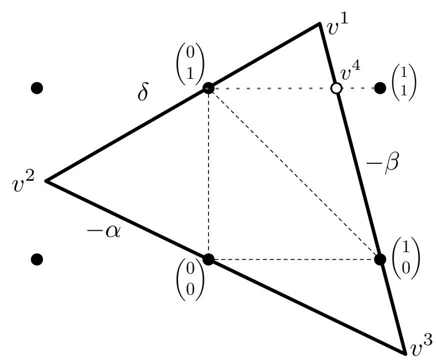

Figure 6: Normalized triangle of Type 3

Remark 5.1. *Let T be a normalized Type 3 triangle as in* (19)*. Then*

$$v^1 = \left(\frac{\beta-1}{\beta+\delta}, \frac{\beta(\delta+1)}{\delta+\beta}\right)^T \quad v^2 = \left(\frac{-1}{\delta+\alpha}, \frac{\alpha}{\delta+\alpha}\right)^T \quad v^3 = \left(\frac{\beta}{\beta-\alpha}, \frac{\alpha\beta}{\alpha-\beta}\right)^T.$$

Let T be a normalized Type 3 triangle as in (19). The *fixed triangle* T_1 associated with T has vertices corresponding to the pairwise intersection of the lines going through respectively: v^3, v^1; v^3, v^2; and $(0,1)^T, (-1,1)^T$. The *fixed triangle* T_2 associated with T has vertices corresponding to the pairwise intersection of the lines going through respectively: v^1, v^2; v^1, v^3; and $(0,0)^T, (1,-1)^T$. The *fixed triangle* T_3 associated with T has vertices corresponding to the pairwise intersection of the lines going through respectively: v^2, v^1; v^2, v^3; and $(1,1)^T, (1,0)^T$. See Figure 7. As T is a lattice-free convex set, so is the triangle with corners v_3, $(0,0)^T$ and $(1,0)^T$. As T_1 is obtained from that triangle by sliding the line L_1 going through $(0,0)^T, (1,0)^T$ to the line L_2 going through $(0,1)^T, (1,1)^T$ and as there is no integer point in the interior of the region between L_1 and L_2, it follows that T_1 is a lattice-free convex set. Moreover, as it has exactly one integer point in the interior of edges v^3, v^1 and v^3, v^2 and at least two integer points on the third edge, it is a triangle of Type 2. Similarly, we can show that T_2, T_3 are of Type 2.

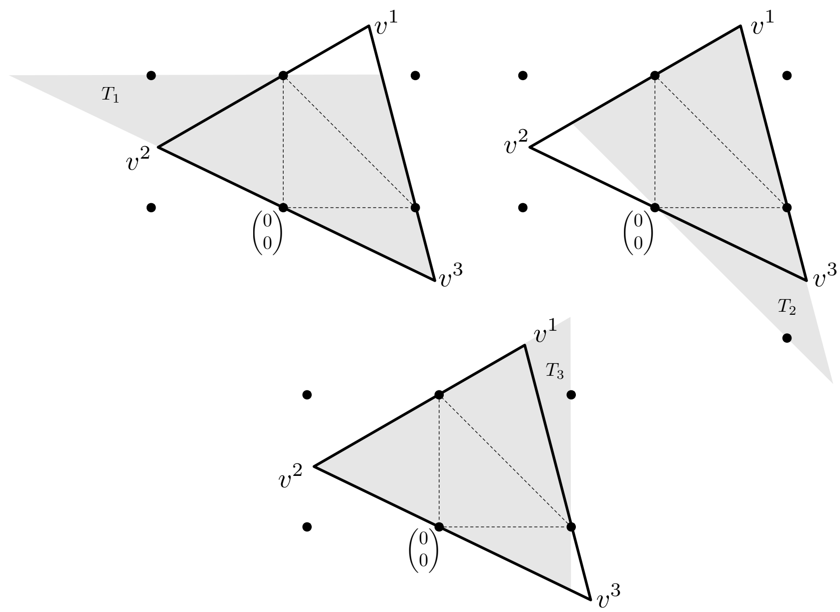

Figure 7: Triangle of Type 3 and associated triangles

Remark 5.2. *Let T be a normalized Type 3 triangle as in* (19)*. For all $i \in [3]$, let T_i denote a fixed triangle associated with T and let $r^i = v^i - f$ where $f \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$ is an interior point of $T_i \cap Q$. For all $i, j \in [3]$, $i \neq j$, $\psi_{f;T_i}(r^j) = 1$. Moreover,*

$$\psi_{f;T_1}(r^1) = 1 + \frac{1}{a}, \quad \psi_{f;T_2}(r^2) = 1 + \frac{1}{b}, \quad \psi_{f;T_3}(r^3) = 1 + \frac{1}{c},$$

where

$$a = (1 - f_2)\frac{\delta + \beta}{\delta(\beta - 1)} \qquad b = (f_1 + f_2)\frac{\alpha + \delta}{1 - \alpha} \qquad c = (1 - f_1)\frac{\beta - \alpha}{\alpha}. \tag{20}$$

Proof. Suppose $i \neq j$. Then $f + r^j$ is on the boundary of T_i, and by definition, $\psi_{f;T_i}(r^j) = 1$ as required. Thus, we may assume $i = j$. The proof for $i = 1, 3$ is similar to that of the proof of Remark 4.2, so we shall omit it. Suppose $i = 2$. Then, by definition, $\psi_{f;T_2}(r^2) = \frac{1}{\lambda}$ such that $\bar{v} = f + \lambda(v^2 - f)$ is on the boundary of T_2. Then, $\bar{v}$ is on the line segment going through $(1, -1)^T, (-1, 1)^T$, i.e. $f + \lambda(v^2 - f) = (1, -1)^T + t(-1, 1)^T$ for some $t \geq 0$, or equivalently,

$$\begin{pmatrix} f_1 - v_1^2 & -1 \\ f_2 - v_2^2 & 1 \end{pmatrix} \begin{pmatrix} \lambda \\ t \end{pmatrix} = \begin{pmatrix} f_1 - 1 \\ f_2 + 1 \end{pmatrix}.$$

Solving the system we get,

$$\frac{1}{\lambda} = \frac{f_1 + f_2 - (v_1^2 + v_2^2)}{f_1 + f_2} = 1 + \frac{-v_1^2 - v_2^2}{f_1 + f_2} = 1 + \frac{1}{b},$$

where the last equality follows from Remark 5.1. □

We consider the functions $h, h' : \mathbb{R}^5 \to \mathbb{R}$ with variables $\alpha, \beta, \delta, f_1, f_2$ where,

$$h(\alpha, \beta, \delta, f_1, f_2) = a + b + c \qquad \text{and} \qquad h'(\alpha, \beta, \delta, f_1, f_2) = b + c, \tag{21}$$

where a, b, c are defined as in (20).

Lemma 5.3. *Let $v^4 = \left(\frac{\beta-1}{\beta}, 1\right)$ (see Figure 6). For h, h' defined as in (21),*

1. $\inf\left\{h : \beta > 1, \alpha, \delta > 0, \alpha < 1; f \in \operatorname{conv}\left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, v^4\right\}\right\} \geq 2,$ *and*

2. $\inf\left\{h' : \beta > 1, \alpha, \delta > 0, \alpha < 1; f \in \operatorname{conv}\left\{\begin{pmatrix}0\\1\end{pmatrix}, v^1, v^4\right\}\right\} \geq 2.$

The proof will require the following easy observation.

Remark 5.4.

$$\min\left\{\frac{1}{x} + \frac{1}{1-x} : 0 < x < 1\right\} = 4.$$

Proof of Lemma 5.3. Consider part 1. Let us define the function $g : (0,1) \to \mathbb{R}$ where

$$g(\epsilon) = \inf\left\{h : (f_1, f_2, \alpha, \beta, \delta) \in S\right\},$$

where S is the set of tuples $(f_1, f_2, \alpha, \beta, \delta)$ that satisfy,

$$f \in \operatorname{conv}\left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, v^4\right\} \quad \text{and} \quad 1+\epsilon \leq \beta \leq \frac{1}{\epsilon}, \alpha, \delta \geq \epsilon, \alpha \leq 1-\epsilon, \delta \leq \frac{1}{\epsilon}.$$

Claim. *For all $\epsilon \in (0,1)$, $g(\epsilon) \geq 2$.*

Proof of claim: Then, $g(\epsilon)$ is obtained by minimizing the continuous function h, over the compact set S. It follows that h attains its minimum for say values $\hat{f}_1, \hat{f}_2, \hat{\alpha}, \hat{\beta}, \hat{\delta}$. For fixed values $\hat{\alpha}, \hat{\beta}, \hat{\delta}$, h is a linear function in f_1, f_2. Thus, it attains its minimum for one of the following values (f_1, f_2): $(0,0)^T$, $(0,1)^T$, $(1,0)^T$, v^4.

Case 1: $f_1 = f_2 = 0$**.** Then h can be rewritten as,

$$h = \frac{\hat{\delta} + \hat{\beta}}{\hat{\delta}(\hat{\beta}-1)} + \frac{\hat{\beta} - \hat{\alpha}}{\hat{\alpha}} = \frac{1}{\hat{\beta}-1} + \frac{\hat{\beta}}{\hat{\delta}(\hat{\beta}-1)} + \frac{\hat{\beta}}{\hat{\alpha}} - 1 \geq \frac{1}{\hat{\beta}-1} + (\hat{\beta}-1) \geq 2,$$

where the first inequality follows from $\hat{\delta}, \hat{\beta}, \hat{\beta} - 1 > 0$ and $\hat{\alpha} < 1$ and the second inequality from Remark 4.4 and $\hat{\beta} - 1 > 0$.

Case 2: $f_1 = 0, f_2 = 1$**.** Then h can be rewritten as,

$$h = \frac{\hat{\alpha} + \hat{\delta}}{1 - \hat{\alpha}} + \frac{\hat{\beta} - \hat{\alpha}}{\hat{\alpha}} = \frac{\hat{\beta}}{\hat{\alpha}} - 1 + \frac{\hat{\alpha}}{1-\hat{\alpha}} + \frac{\hat{\delta}}{1-\hat{\alpha}} \geq \frac{1}{\hat{\alpha}} + \frac{1}{1-\hat{\alpha}} - 2 \geq 2,$$

where the first inequality follows from $\hat{\beta} > 1, 1 - \hat{\alpha}, \hat{\delta} > 0$, and the second inequality from Remark 5.4 and $0 < \hat{\alpha} < 1$.

Case 3: $f_1 = 1, f_2 = 0$**.** Then h can be rewritten as,

$$h = \frac{\hat{\delta} + \hat{\beta}}{\hat{\delta}(\hat{\beta}-1)} + \frac{\hat{\alpha} + \hat{\delta}}{1-\hat{\alpha}} = \left(\frac{1}{\hat{\beta}-1} + \frac{\hat{\alpha}}{1-\hat{\alpha}}\right) + \frac{1}{\hat{\delta}}\left(\frac{\hat{\beta}}{\hat{\beta}-1}\right) + \hat{\delta}\left(\frac{1}{1-\hat{\alpha}}\right). \qquad (\star)$$

Keeping $\hat{\alpha}, \hat{\beta}$ fixed, we optimize over δ. Then by Remark 4.4 the optimal value for δ is,

$$\hat{\delta} = \sqrt{\frac{\hat{\beta}(1-\hat{\alpha})}{\hat{\beta}-1}} > 0.$$

Note, the square root is well defined as $\hat{\beta}-1, 1-\hat{\alpha} > 0$. Substituting into $(\star)$ yields,

$$h = \frac{1}{\hat{\beta}-1} + \frac{\hat{\alpha}}{1-\hat{\alpha}} + 2\sqrt{\frac{\hat{\beta}}{(\hat{\beta}-1)(1-\hat{\alpha})}} \geq 2,$$

where the inequality follows from $\hat{\beta} > 1$ and $0 < \hat{\alpha} < 1$.

Case 4: $f_1 = \frac{\hat{\beta}-1}{\hat{\beta}}, f_2 = 1$**.** Then $1 - f_1 = \frac{1}{\hat{\beta}}$ and h can be rewritten as,

$$\begin{aligned} h =& (f_1+f_2)\frac{\hat{\alpha}+\hat{\delta}}{1-\hat{\alpha}} + (1-f_1)\frac{\hat{\beta}-\hat{\alpha}}{\hat{\alpha}} = (f_1+f_2)\frac{\hat{\alpha}}{1-\hat{\alpha}} + (f_1+f_2)\frac{\hat{\delta}}{1-\hat{\alpha}} + \frac{\hat{\beta}-\hat{\alpha}}{\hat{\alpha}\hat{\beta}} \\ \geq& \frac{\hat{\alpha}}{1-\hat{\alpha}} + \frac{1}{\hat{\alpha}} - \frac{1}{\hat{\beta}} \geq \frac{\hat{\alpha}}{1-\hat{\alpha}} + \frac{1}{\hat{\alpha}} - 1 = \frac{1}{1-\hat{\alpha}} + \frac{1}{\hat{\alpha}} - 2 \geq 2, \end{aligned}$$

where the first inequality follows from $f_1 + f_2 \geq 1$, $\hat{\delta}, 1-\hat{\alpha} > 0$, $\hat{\delta} > 0$, the second inequality from $\hat{\beta} > 1$ and the third inequality from Remark 5.4 and $0 < \hat{\alpha} < 1$. $\diamond$

Finally, if for some $\hat{f}_1, \hat{f}_2 \in [0,1]$ and $\hat{\alpha}, \hat{\beta}, \hat{\delta}$, h has value < 2, then for $\epsilon > 0$ small enough, $g(\epsilon) < 2$, contradicting the Claim.

For part 2., consider $\hat{\alpha}, \hat{\beta}, \hat{\delta}$ with $\hat{\beta} > 1, \hat{\alpha}, \hat{\delta} > 0, \hat{\alpha} < 1$ and $\hat{f} \in \text{conv}\{(0,1)^T, v^1, v^4\}$. Then

$$\begin{aligned} h' =& (\hat{f}_1+\hat{f}_2)\frac{\hat{\alpha}+\hat{\delta}}{1-\hat{\alpha}} + (1-\hat{f}_1)\frac{\hat{\beta}-\hat{\alpha}}{\hat{\alpha}} \\ \geq& (\hat{f}_1+1)\frac{\hat{\alpha}+\hat{\delta}}{1-\hat{\alpha}} + (1-\hat{f}_1)\frac{\hat{\beta}-\hat{\alpha}}{\hat{\alpha}} = h(\hat{f}_1, 1, \hat{\alpha}, \hat{\beta}, \hat{\delta}) \geq 2, \end{aligned}$$

where the first inequality follows from $\hat{f}_2 \geq 1, \hat{\alpha}, \hat{\beta}, 1-\hat{\alpha} > 0$, the second equality from the definitions of h, h', and the second inequality from part (1). $\square$

We are now ready for the main proof of this section.

Proof of Theorem 1.8. Choose arbitrary fixed $r^1, r^2, r^3 \in \mathbb{Q}^2 \setminus \{0\}$ and $f \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$ such that $T = f + \text{conv}\{r^1, r^2, r^3\}$ is a normalized Type 3 triangle. Consider the following minimization problem,

$$\begin{aligned} \inf \quad & s_1 + s_2 + s_3 \\ \text{subject to} \quad & \\ & s \in \triangle_2(f; r^1, r^2, r^3). \end{aligned} \tag{22}$$

It suffices to show that some relaxation of (22) has a lower bound of $\frac{2}{3}$, for as r^1, r^2, r^3, f were chosen arbitrarily it implies by Corollary 3.4 that $\frac{2}{3}$ is a lower bound for $\frac{1}{\rho[\triangle_2, \triangle_3]}$; i.e., that

$\rho[\triangle_2, \triangle_3] \leq 1.5$ as required. For $i \in [3]$, $v^i = f + r^i$ is a corner of T. We may assume that T is described by α, β, δ as in (19). After possibly applying a unimodular transformation (see Remark 3.8), we may assume that

$$f \in \operatorname{conv}\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, v^1 \right\} \cap \mathbb{R}^2_{++}.$$

As f is in the interior of T, we are either in Case 1 or Case 2.

Case 1: f is in the interior of $\operatorname{conv}\{(0,0)^T, (1,0)^T, (0,1)^T, v^4\}$.

Let T_1, T_2, T_3 be the fixed triangles associated with T (see Figure 7). The following linear program is a relaxation of (22),

$$\begin{aligned} \min \quad & s_1 + s_2 + s_3 \\ \text{subject to} \quad & \\ & \sum_{j=1}^{3} \psi_{f;T_i}(r^j) s_j \geq 1 \qquad i \in [3] \\ & s \in \mathbb{R}^3_+. \end{aligned} \tag{23}$$

Since $f \in \operatorname{conv}\{(0,0)^T, (1,0)^T, (0,1)^T, v^4\}$ and $\beta > 1$, $\alpha, \delta > 0$ and $\alpha < 1$, it follows that $a, b, c > 0$, where a, b, c are defined as in (20). Remark 5.2 implies that (23) can be written as,

$$\begin{aligned} \min \quad & s_1 + s_2 + s_3 \\ \text{subject to} \quad & \\ & \begin{pmatrix} 1 + \frac{1}{a} & 1 & 1 \\ 1 & 1 + \frac{1}{b} & 1 \\ 1 & 1 & 1 + \frac{1}{c} \end{pmatrix} s \geq \mathbb{1} \\ & s \geq 0. \end{aligned} \tag{24}$$

The dual of (24) is given by,

$$\begin{aligned} \max \quad & \nu_1 + \nu_2 + \nu_3 \\ \text{subject to} \quad & \\ & \begin{pmatrix} 1 + \frac{1}{a} & 1 & 1 \\ 1 & 1 + \frac{1}{b} & 1 \\ 1 & 1 & 1 + \frac{1}{c} \end{pmatrix} \nu \leq \mathbb{1} \\ & \nu \geq 0. \end{aligned} \tag{25}$$

Consider,

$$\hat{\nu} = \frac{1}{1 + a + b + c} \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

As $a, b, c > 0$, $\hat{\nu} \geq 0$ and it can be readily checked that constraints of (25) corresponding to each of primal variables s_1, s_2, s_3 are satisfied with equality. Thus $\hat{\nu}$ is feasible for (25). Moreover, it has objective value,

$$\hat{\nu}^T \mathbb{1} = 1 - \frac{1}{1+a+b+c}$$

By (21), $h = a+b+c$. Lemma 5.3 part 1. implies that $h \geq 2$. Thus, $\hat{\nu}^T \mathbb{1} \geq \frac{2}{3}$. In particular, $\frac{2}{3}$ is a lower bound for (25) and in turn by weak duality to (23), as required.

Case 2: $f \in \text{conv}\{(0,1)^T, v^1, v^4\}$.

Let T_2, T_3 be the fixed triangles associated with T (see Figure 7). Note, that f is in the interior of T_2 and T_3. The following linear program is a relaxation of (22),

$$\begin{aligned} \min \quad & s_1 + s_2 + s_3 \\ \text{subject to} \quad & \\ & \sum_{j=1}^{3} \psi_{f;T_i}(r^j) s_j \geq 1 \qquad i \in \{2,3\} \\ & s \in \mathbb{R}^3_+. \end{aligned} \tag{26}$$

It can be readily checked that in this case $b, c > 0$. Remark 5.2 implies that (26) can be written as,

$$\begin{aligned} \min \quad & s_1 + s_2 + s_3 \\ \text{subject to} \quad & \\ & \begin{pmatrix} 1 & 1+\frac{1}{b} & 1 \\ 1 & 1 & 1+\frac{1}{c} \end{pmatrix} s \geq \mathbb{1} \\ & s \geq 0. \end{aligned} \tag{27}$$

The dual of (27) is given by,

$$\begin{aligned} \max \quad & \nu_1 + \nu_2 \\ \text{subject to} \quad & \\ & \begin{pmatrix} 1 & 1 \\ 1+\frac{1}{b} & 1 \\ 1 & 1+\frac{1}{c} \end{pmatrix} \nu \leq \mathbb{1} \\ & \nu \geq 0. \end{aligned} \tag{28}$$

Consider,

$$\hat{\nu} = \frac{1}{1+b+c} \begin{pmatrix} b \\ c \end{pmatrix}.$$

As $b, c > 0$, $\hat{\nu}$ is feasible solution to (28) with objective value $1 - \frac{1}{1+b+c}$. Lemma 5.3 part 2. shows that $h' = b + c \geq 2$. In particular, $\frac{2}{3}$ is a lower bound for (28) and in turn by weak duality to (26) as required. □

6 Lower bounds

We start this section with a proof of Theorem 1.5(1): $\rho[\triangle_1, \mathrm{S}] = +\infty$. To prove this, it suffices to exhibit an instance $f \in \mathbb{Q}^n \setminus \mathbb{Z}^n$, $\Gamma = r^1, \ldots, r^k \in \mathbb{Q}^n \setminus \{0\}, k \geq 1$ for which there is no $\alpha > 0$ satisfying $\alpha \mathrm{S}(f;\Gamma) \supseteq \triangle_1(f;\Gamma)$.

Let $f := (\frac{1}{2}, 0)^T$, $k := 2$, $r^1 := (1,0)^T$, $r^2 := (0,1)^T$.

Because the integer point $(1,0)^T$ is the midpoint between f and $f + r^1$, every intersection cut $\psi(r^1)s_1 + \psi(r^2)s_2 \geq 1$ satisfies $\psi(r^1) \geq 2$. The split cut generated by the split $0 \leq x_1 \leq 1$ is the inequality $2s_1 \geq 1$. Therefore it dominates all other intersection cuts. In particular, we have shown that $\mathrm{S}(f;r^1,r^2) = \{s \in \mathbb{R}^2_+ : \ s_1 \geq \frac{1}{2}\}$.

We claim that the point $(0, \frac{3}{2})^T$ is in $\triangle_1(f;r^1,r^2)$. This will complete the proof since $(0, \frac{3}{2})^T \notin \alpha\{s \in \mathbb{R}^2_+ : \ s_1 \geq \frac{1}{2}\}$ for any $\alpha > 0$. To prove the claim, consider any Type 1 triangle T. Not all three vertices of T can lie on the two lines $x_1 = 0$ and $x_1 = 1$. Therefore T has a vertex with coordinate $x_1 \leq -1$ or $x_1 \geq 2$, say $x_1 \geq 2$. Because T is lattice-free, the intersection of T with the line $x_1 = 1$ has length at most 1, and therefore its intersection with the line $x_1 = \frac{1}{2}$ has length at most $\frac{3}{2}$. Since f is in the interior of this segment, we conclude that $\psi(r^2) \geq \frac{2}{3}$ in all intersection cuts arising from Type 1 triangles. This proves $(0, \frac{3}{2})^T \in \triangle_1(f;r^1,r^2)$.

Proof of Theorem 1.6 in [3] contains a proof of Theorem 1.5(2).

Next we prove Theorem 1.6(1), namely that $\rho[\triangle_3, \square] \geq 1.125$. Because of Remark 1.9 it will suffice to show the following result,

Theorem 6.1. $\rho[\triangle, \square] \geq 1.125$.

Proof. Define, $f := (0.5, 0.5)^T$ and

$$r^1 := (0.9, 0.3)^T \quad r^2 := (0.3, -0.9)^T \quad r^3 := (-0.9, -0.3)^T \quad r^4 := (-0.3, 0.9)^T.$$

We will show $\rho\big[\triangle(f;r^1,r^2,r^3,r^4), \square(f;r^1,r^2,r^3,r^4)\big] \geq 9/8 = 1.125$. We first claim that it suffices to show that $\bar{s} := (2/9, 2/9, 2/9, 2/9)^T \in \triangle(f;r^1,r^2,r^3,r^4)$. Let Q denote the square with vertices $v^i = f + r^i$ for $i = 1,2,3,4$ (See Figure 8.) Since r^1, r^2, r^3, r^4 are corner rays

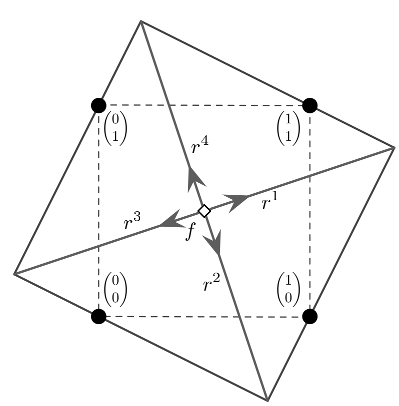

Figure 8: Square Q and vectors f, r^1, r^2, r^3, r^4.

of Q, the intersection cut for f and Q is given by $s_1 + s_2 + s_3 + s_4 \geq 1$, in particular, that constraint is valid for $\square(f; r^1, r^2, r^3, r^4)$. It follows that if $\bar{s} \in \alpha\, \square(f; r^1, r^2, r^3)$ then $\alpha \leq 8/9$. Thus, $\rho\big[\triangle(f; r^1, r^2, r^3, r^4), \square(f; r^1, r^2, r^3, r^4)\big] \geq 9/8$, as required.

Let T be an arbitrary maximal lattice free triangle containing f in its interior. It suffices to show that $\bar{s}$ satisfies the intersection cut for f and T. Consider the three lines defined by the edges of T. At least one of these lines must have two of the points $(0,0)^T$, $(0,1)^T$, $(1,0)^T$, $(1,1)^T$ on one side and f on the opposite side. Without loss of generality, assume that an edge of T defines a line L with $(1,0)^T$ and $(1,1)^T$ on the opposite side of f. This implies

$$\psi_{f;T}(r^1) \geq \frac{9}{5}. \tag{29}$$

Let $r := \frac{1}{3}r^1 + \frac{2}{3}r^2$. Then,

$$\frac{1}{3}\psi_{f;T}(r^1) + \frac{2}{3}\psi_{f;T}(r^2) \geq \psi_{f;T}(r) \geq 1, \tag{30}$$

where the first inequality arises from convexity of $\psi_{f;T}$, and the second one from the fact that $f + r = (1,0)^T$ is not in the interior of T.

Let $r' := \frac{1}{3}r^3 + \frac{2}{3}r^4$. Then,

$$\frac{1}{3}\psi_{f;T}(r^3) + \frac{2}{3}\psi_{f;T}(r^4) \geq \psi_{f,T}(r') \geq 1, \tag{31}$$

where the first inequality arises from convexity of $\psi_{f;T}$, and the second one from the fact that $f + r = (0,1)^T$ is not in the interior of T.

We break the remainder of the proof in two cases.

<u>Case 1.</u> $\psi_{f;T}(r^3) \geq \frac{6}{5}$.

Then using (30) and (31), we get

$$\frac{2}{3}\left[\psi_{f;T}(r^1) + \psi_{f;T}(r^2) + \psi_{f;T}(r^3) + \psi_{f;T}(r^4)\right] \geq 2 + \frac{1}{3}(\psi_{f;T}(r^1) + \psi_{f;T}(r^3)).$$

Using (29) and $\psi_{f;T}(r^3) \geq \frac{6}{5}$, we get

$$\frac{2}{9}\left[\psi_{f;T}(r^1) + \psi_{f;T}(r^2) + \psi_{f;T}(r^3) + \psi_{f;T}(r^4)\right] \geq 1. \tag{32}$$

It follows that $\bar{s}$ satisfies the intersection cut for f and T.

<u>Case 2.</u> $\psi_{f;T}(r^3) \leq \frac{6}{5}$.

Then, the intersection point w of the triangle T with the half-line H defined by $f + \lambda r^3$ with $\lambda \geq 0$, has negative first coordinate x_1. By convexity of T, two distinct sides of T separate the segment wf from the points $(0,0)$ and $(0,1)$. Let L_1 be the line containing the side of T that separates $(0,0)$ and L_2 the line containing the side of T that separates $(0,1)$. We may assume that w is the vertex of T at the intersection of L_1 and L_2 since, otherwise, we can modify the triangle by changing L_1 or L_2 and get an inequality at least as strong.

Notice that the line passing through w and $(0,0)$ intersects $f+\lambda r^2$ with $\lambda \geq 0$, at a point with first coordinate at most 1, by our assumption on $\psi_{f;T}(r^3)$. So, we may also assume without loss of generality that the line L defined earlier is the line $x_1 = 1$. It follows that

$$\psi_{f;T}(r^1) = \frac{9}{5}. \tag{33}$$

Furthermore, by maximality of T, the lines L_1 and L_2 go through $(0,0)^T$ and $(0,1)^T$ respectively. Let $r'' := \frac{1}{3}r^2 + \frac{2}{3}r^3$. Then,

$$\frac{1}{3}\psi_{f;T}(r^2) + \frac{2}{3}\psi_{f;T}(r^3) = \psi_{f;T}(r'') = 1, \tag{34}$$

where the first equality arises from convexity of $\psi_{f;T}$ and the fact that r^2, r^3, r'' are all on the line L_1 and on the boundary of T; and the second equality follows from the fact that r'' is on the boundary of T. Similarly, as r^3, r^4, r are on the line L_2 and on the boundary of T, equality holds throughout in (31). Together with (34) we get,

$$\psi_{f;T}(r^2) + \psi_{f;T}(r^3) + \psi_{f;T}(r^4) = \frac{9}{2} - \frac{3}{2}\psi_{f;T}(r^3).$$

Using (33) we get

$$\frac{2}{9}(\psi_{f;T}(r^1) + \psi_{f;T}(r^2) + \psi_{f;T}(r^3) + \psi_{f;T}(r^4)) = \frac{7}{5} - \frac{1}{3}\psi_{f;T}(r^3) \geq 1.$$

It follows again that $\bar{s}$ satisfies the intersection cut for f and T. $\square$

Next we prove Theorem 1.6(2).

Theorem 6.2. $\rho[\square, \triangle_3] \geq 1.125$.

Proof. Define, $f := (1/3, 1/3)^T$ and

$$r^1 := (1,-1)^T \quad r^2 := (0,1)^T \quad r^3 := (-1,0)^T.$$

We will show $\rho\big[\square(f;r^1,r^2,r^3), \triangle_3(f;r^1,r^2,r^3)\big] \geq 9/8 = 1.125$. We first claim that it suffices to show that $\bar{s} := (8/27, 8/27, 8/27)^T \in \square(f;r^1,r^2,r^3)$. Let T denote the triangle of Type 3 with vertices $v^i = f + r^i$ for $i = 1,2,3$. (See Figure 9.) Since r^1, r^2, r^3 are corner rays of T, the intersection cut for f and Q is given by $s_1 + s_2 + s_3 \geq 1$, in particular, that constraint is valid for $\triangle_3(f;r^1,r^2,r^3)$. It follows that if $\bar{s} \in \alpha\, \triangle_3(f;r^1,r^2,r^3)$ then $\alpha \leq 24/27$. Thus $\rho\big[\square(f;r^1,r^2,r^3), \triangle_3(f;r^1,r^2,r^3)\big] \geq 27/24 = 9/8$, as required.

Let Q be an arbitrary maximal lattice free quadrilateral containing f in its interior. It suffices to show that $\bar{s}$ satisfies the intersection cut for f and Q.

Let $r := \frac{2}{3}r^1 + \frac{1}{3}r^2$. Then,

$$\frac{2}{3}\psi_{f;Q}(r^1) + \frac{1}{3}\psi_{f;Q}(r^2) \geq \psi_{f;Q}(r) \geq 1, \tag{35}$$

where the first inequality arises from convexity of $\psi_{f;Q}$, and the second one from the fact that $f + r = (1,0)^T$ is not in the interior of Q.

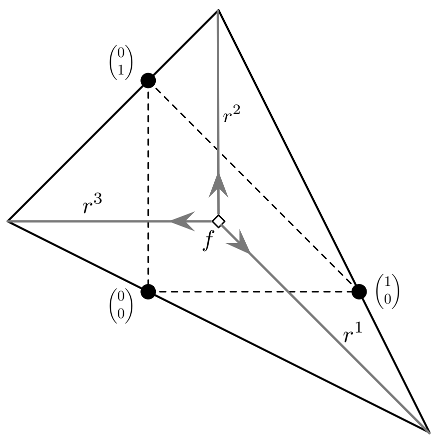

Figure 9: Triangle T and vectors f, r^1, r^2, r^3.

Similarly

$$\frac{2}{3}\psi_{f;Q}(r^2) + \frac{1}{3}\psi_{f;Q}(r^3) \geq 1, \tag{36}$$

and

$$\frac{2}{3}\psi_{f;Q}(r^3) + \frac{1}{3}\psi_{f;Q}(r^1) \geq 1. \tag{37}$$

<u>Case 1.</u> $\psi_{f;Q}(r^i) \geq \frac{3}{2}$ for $i \in [3]$.

We may assume,

$$\psi_{f;Q}(r^1) \geq 3/2. \tag{38}$$

Then, $\frac{3}{2}\times(36)+\frac{3}{4}\times(37)+\frac{3}{4}\times(38)$ gives

$$\psi_{f;Q}(r^1) + \psi_{f;Q}(r^2) + \psi_{f;Q}(r^3) \geq 27/8.$$

Therefore, the point $\bar{s} = (\frac{8}{27}, \frac{8}{27}, \frac{8}{27})^T$ satisfies the intersection cut for f and Q. Note that it is possible to satisfy all three inequalities (36), (37) and (38) at equality. So, the bound of 9/8 can be achieved.

<u>Case 2.</u> $\psi_{f;Q}(r^i) < \frac{3}{2}$ for all $i \in [3]$.

By convexity of Q, the only integer points that can be on the boundary of Q are $(0,0)^T$, $(0,1)^T$, $(1,0)^T$. However, this contradicts the property that maximal lattice-free quadrilaterals contain four integral points on their boundary. □

7 Concluding remarks

We refer the reader to Figure 2 and Table 1 in the Introduction. In Table 2, we give lower and upper bounds on $\rho[\#_1, \#_2]$ values for every pair of families of intersection cuts. If a cell contains only one value, then that is the exact value of the approximation ratio ρ. So, Table 1 corresponds to the last column of Table 2. Those entries in the table which are stated as some

$\rho[\#_1, \#_2]$	S	$\triangle_1$	$\triangle_2$	$\triangle_3$	$\square$	R
S	–	2	$+\infty$	$+\infty$	$+\infty$	$+\infty$
$\triangle_1$	$+\infty$	–	$+\infty$	$+\infty$	$+\infty$	$+\infty$
$\triangle_2$	1	1	–	1.125, **1.5**	1.125, **1.5**	1.125, 1.5
$\triangle_3$	1	1	1	–	**1.125**, 1.5	1.125, 1.5
$\square$	1	1	1	**1.125**, 1.5	–	1.125, 1.5

Table 2: Lower bounds and upper bounds on ρ[row set, column set] values.

of the main results in this paper are in bold face. The other numbers were either proved in previous work or are elementary consequences of our results (e.g., by utilizing Remark 1.9).

We proved that Type 2 triangle closure is within 50% of the convex hull of integer points, R, and no single family (among the five families) can guarantee better than a 12.5% approximation to R. Moreover, the inclusion lattice Figure 2 together with the facts that Split and Type 1 closures can give arbitrarily bad approximations of R, and to close in on R with a tighter than 12% approximation, one needs *both* Type 3 triangle closure and the quadrilateral closure, indicate that Type 2 triangles provide a natural compromise for implementation. The additional fact that one needs fewer parameters to describe Type 2 triangles compared to the union of Type 3 triangles and quadrilaterals, adds to the argument for focusing on Type 2 triangles for implementations.

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