A double projection method for solving variational inequalities without monotonicity*

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Abstract

We present a double projection algorithm for solving variational inequalities without monotonicity. If the solution of dual variational inequality does exist, then the sequence produced by our method is globally convergent to a solution. Under the same assumption, the sequence produced by known methods has only a subsequence converging to a solution. Numerical experiments are reported.

Key words: Variational inequality, quasimonotone, double projection method.
AMS subject classifications. 90C33; 90C25

1. Introduction

We consider the classical variational inequality problem VI(F,C), which is to find a point $x^* \in C$ such that

$$\langle F(x^*), y - x^* \rangle \geq 0 \quad \text{for all } y \in C, \tag{1}$$

where C is a nonempty closed convex subset of $\mathbb{R}^n$, F is a continuous operator from $\mathbb{R}^n$ into itself, and $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^n$.

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Let S be the solution set of VI(F,C) and S_D be the solution set of the dual variational inequality:

$$S_D := \{x \in C | \langle F(y), y - x\rangle \geq 0, \text{for all } y \in C\}.$$

Since F is continuous and C is convex, we have

$$S_D \subset S. \tag{2}$$

We use S_T and S_N to denote the trivial solution set and the nontrivial solution set of VI(F,C), respectively; that is,

$$\begin{aligned} S_T : &= \{x^* \in C | \langle F(x^*), y - x^*\rangle = 0, \text{for all } y \in C\}, \\ S_N : &= S \setminus S_T. \end{aligned}$$

The projection-type algorithms for solving nonlinear variational inequality problem have been extensively studied in the literature, such as Goldstein-Levitin-Polyak Projection methods [1, 2]; proximal point methods [3]; extragradient projection methods [4, 5, 6, 7]; combined relaxation (CR) methods [8, 10, 9]; double projection methods [11, 12, 13]; self-adaptive projection methods [14, 15, 16]. To prove the global convergence of generated sequence, all these methods have the common assumption $S \subset S_D$, that is,

$$\text{for any } \; x^* \in S, \langle F(y), y - x^*\rangle \geq 0 \; \text{ for all } y \in C. \tag{3}$$

This assumption is a direct consequence of pseudomotonicity of F on C in the sense of Karamardian [17]. F is said to be pseudomonotone on C; i.e., for all $x, y \in C$,

$$\langle F(x), y - x\rangle \geq 0 \Longrightarrow \langle F(y), y - x\rangle \geq 0.$$

In the proof of convergence of iterated sequence $\{x^k\}$ under the assumption of (3), all above methods adopt three similar steps:

(a) For any $\hat{x} \in S_D$,

$$\|x^{k+1} - \hat{x}\|^2 \leq \|x^k - \hat{x}\|^2 - \alpha_k \|r_{\mu_k}(x^k)\|^2, \tag{4}$$

where $\alpha_k > 0$ and $r_{\mu_k}(x^k)$ is the natural residual function defined in the next section.

(b) $\{\|x^{k+1} - \hat{x}\|\}$ is decreasing, $\{x^k\}$ is bounded and its cluster point, say $\bar{x}$ is a solution of variational inequality.

(c) Note that the inequality (4) holds for any $\hat{x} \in S_D$ and $\bar{x} \in S$. The assumption (3) implies that we can replace $\hat{x}$ by $\bar{x}$ in (a). It can be proved that x^k converging to $\bar{x}$.

Now we weaken the assumption (3) to the following

$$\exists\, x_0 \in S, \text{ such that } \langle F(y), y - x_0\rangle \geq 0 \text{ for all } y \in C. \tag{5}$$

Note that $x_0 \in S \cap S_D$. The inequality in the step (a) holds for the point x_0 (it is not known whether such an inequality holds for other points in S), that is

$$\|x^{k+1} - x_0\|^2 \leq \|x^k - x_0\|^2 - \alpha_k \|r_{\mu_k}(x^k)\|^2. \tag{6}$$

Thus we cannot replace x_0 by $\bar{x}$ in the step (c), because $\bar{x}$ in the step (b) is not necessarily equal to the given point x_0.

Note that the assumption (5) is equivalent to $S_D \neq \emptyset$, by (2). Moreover, (3) implies (5), but not the converse, see Example 4.2. Assume that F is quasimonotone on C; i.e., for all $x, y \in C$,

$$\langle F(x), y - x\rangle > 0 \Longrightarrow \langle F(y), y - x\rangle \geq 0.$$

Then $S_N \neq \emptyset$ implies $S_D \neq \emptyset$; see Proposition 2.1. Recently, [18, 19] proposed an interior proximal algorithm for solving quasimonotone variational inequalities, the global convergence is obtained under more assumptions than $S_D \neq \emptyset$.

Under the assumption of $S_D \neq \emptyset$, [8] proposed a method and proved the sequence produced has a subsequence converging to a solution; see also Theorem 1(i) in [9]. Related results are contained in the monograph [10].

Our main purpose in this paper is to suggest a new method which produces a globally convergent sequence, under the only assumption $S_D \neq \emptyset$. Known methods either assume more conditions or prove only the sequence produced has a subsequence converging to a solution.

The organization of this paper is as follows. We present the algorithm in the next section and establish convergence analysis in Section 3. Numerical experiments are reported in Section 4.

2. Algorithm and preliminary results

Let int$C$ denote the interior of C. The projection from $x \in \mathbb{R}^n$ onto C is defined by $P_C(x) := \text{argmin}\{\|y - x\| \mid y \in C\}$. The distance from $x \in \mathbb{R}^n$ to C is defined by

$$\text{dist}(x, C) := \inf\{\|y - x\| \mid y \in C\}.$$

The natural residual function $r_\mu(\cdot)$ is defined by $r_\mu(x) := x - P_C(x - \mu F(x))$, where $\mu > 0$ is a parameter. If $\mu = 1$, we write $r(x)$ for $r_\mu(x)$.

Algorithm 2.1. Choose $x^0 \in C$ as an initial point, $\sigma \in (0, 1)$ and $\gamma \in (0, 1)$. Set $k = 0$. Compute $z^k := P_C(x^k - F(x^k))$

Step 1. Compute $r(x^k) = x^k - z^k$. If $r(x^k) = 0$, stop. Otherwise, go to Step 2.

Step 2. Compute $y^k = x^k - \eta_k r(x^k)$, where $\eta_k = \gamma^{m_k}$, with m_k being the smallest nonnegative integer satisfying

$$\langle F(x^k) - F(x^k - \gamma^m r(x^k)), r(x^k)\rangle \leq \sigma \|r(x^k)\|^2. \tag{7}$$

Step 3. Compute $x^{k+1} = P_{C\cap\tilde{H}_k}(x^k)$, where $\tilde{H}_k := \bigcap_{j=0}^{j=k} H_j$ with $H_j := \{v : h_j(v) \leq 0\}$ is a hyperplane defined by the function

$$h_j(v) := \langle F(y^j), v - y^j\rangle. \tag{8}$$

Let $k = k+1$ and return to Step 1.

F being continuous, Step 2 is well-defined. Moreover, if $S_D \neq \emptyset$, then Step 3 is well-defined, as $S_D \subset C \cap \tilde{H}_k$ and hence $C \cap \tilde{H}_k$ is nonempty for every k.

The following four results are well-known in the literature; see [20].

Lemma 2.1. For any $x \in \mathbb{R}^n$ and $z \in C$, $\langle P_C(x) - x, z - P_C(x)\rangle \geq 0$.

Lemma 2.2. Let $\mu > 0$. Then $x^* \in S$ if and only if $\|r_\mu(x^*)\| = 0$.

Lemma 2.3. For every $x \in C$,

$$\langle F(x), r_\mu(x)\rangle \geq \mu^{-1}\|r_\mu(x)\|^2. \tag{9}$$

Lemma 2.4. Let K be a nonempty closed convex subset of $\mathbb{R}^n$ and $x^{k+1} = P_K(x^k)$. Then for any $x^* \in K$, we have

$$\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 - \|x^{k+1} - x^k\|^2. \tag{10}$$

Lemma 2.5. ([12]) Let $C \subset \mathbb{R}^n$ be a closed convex subset of $\mathbb{R}^n$, h be a real-valued function on $\mathbb{R}^n$, and $K := \{x \in C : h(x) \leq 0\}$. If K is nonempty and h is Lipschitz continuous on C with modulus $\theta > 0$, then

$$\operatorname{dist}(x, K) \geq \theta^{-1} \max\{h(x), 0\} \quad \text{for all } x \in C. \tag{11}$$

Lemma 2.6. ([21]) Let F be a continuous and quasimonotone operator. If for some $x_0 \in C$, we have $\langle F(y), x_0 - y\rangle \geq 0$, then at least one of the following must hold:

$$\langle F(x_0), x_0 - y\rangle \geq 0, \quad \text{or } \langle F(y), x - y\rangle \leq 0 \text{ for all } x \in C. \tag{12}$$

Lemma 2.7. Let F be a continuous and quasimonotone operator. Then $S_N \subset S_D$.

Proof. Let $x^* \in S_N$. Fix any $y \in C$. Since $S_N \subset S$, we have $\langle F(x^*), y - x^* \rangle \geq 0$. By Lemma 2.6, one of the following must hold:

$$\langle F(y), y - x^* \rangle \geq 0, \quad \text{or } \langle F(x^*), x - x^* \rangle \leq 0 \text{ for all } x \in C.$$

Since $x^* \in S$, the second inequality implies that $x^* \in S_T$, which contradicts $x^* \in S_N$. Thus the first inequality must hold: $\langle F(y), y - x^* \rangle \geq 0$. The conclusion follows as $y \in C$ is taken arbitrarily. □

Theorem 2.1. If int$C$ is nonempty, then $x^* \in S_T$ if and only if $F(x^*) = 0$.

Proof. Clearly, we need to prove only the necessary condition. Let $x^* \in S_T$. Assume that $F(x^*) \neq 0$. By the definition of S_T, we have

$$\langle F(x^*), y - x^* \rangle = 0 \text{ for all } y \in C. \tag{13}$$

Since int$C$ is nonempty, we can suppose there exists $x_0 \in \text{int}C$ and a sufficiently small positive number $t > 0$ such that $x_0 - tF(x^*) \in C$. By (13), we have

$$\begin{aligned} 0 =& \langle F(x^*), (x_0 - tF(x^*)) - x^* \rangle \\ =& - t\|F(x^*)\|^2 + \langle F(x^*), x_0 - x^* \rangle \\ =& - t\|F(x^*)\|^2. \end{aligned}$$

Which contradicts the assumption of $F(x^*) \neq 0$. This completes the proof. □

Proposition 2.1. If either

(a) F is pseudomonotone on C and $S \neq \emptyset$;

(b) F is the gradient of G, where G is a differentiable quasiconvex function on an open set $K \supset C$ and attains its global minimum on C;

(c) F is quasimonotone on C, $F \neq 0$ and C is bounded;

(d) F is quasimonotone on C, $F \neq 0$ on C and there exists a positive number r such that, for every $x \in C$ with $\|x\| \geq r$, there exists $y \in C$ such that $\|y\| \leq r$ and $\langle F(x), y - x \rangle \leq 0$;

(e) F is quasimonotone on C and $S_N \neq \emptyset$;

(f) F is quasimonotone on C, int$C$ is nonempty and there exists $x^* \in S$ such that $F(x^*) \neq 0$,

then S_D is nonempty.

Proof. (a),(b),(c) and (d) are conclusions of Proposition 1 in [22], (e) is the corollary of Lemma 2.7, (f) is the consequence of (e) and Theorem 2.1. □

Lemma 2.8. Let the function h_k be defined by (8) and $\{x^k\}$ be generated by Algorithm 2.1. If $S_D \neq \emptyset$, then $h_k(x^k) \geq (1-\sigma)\eta_k \|r(x^k)\|^2 > 0$ for all k. If $x^* \in S_D$, then $h_k(x^*) \leq 0$ for all k.

Proof. By the definition of y^k, we have

$$\begin{aligned} h_k(x^k) &= \eta_k \langle F(y^k), r(x^k)\rangle \\ &\geq \eta_k(\langle F(x^k), r(x^k)\rangle - \sigma\|r(x^k)\|^2) \\ &\geq (1-\sigma)\eta_k\|r(x^k)\|^2 > 0, \end{aligned}$$

where the first inequality is obtained by (7) and the second inequality is obtained by (9). If $x^* \in S_D$, so

$$h_k(x^*) = \langle F(y^k), x^* - y^k\rangle \leq 0 \text{ for all } k. \tag{14}$$

□

Lemma 2.9. If $\{x^k\}$ is an infinite sequence generated by Algorithm 2.1 and $\tilde{x}$ is any accumulation point of $\{x^k\}$, then $\tilde{x} \in \bigcap_{k=1}^{\infty} H_k$.

Proof. Let l be any given nonnegative integer and $\tilde{x}$ be an accumulation point of $\{x^k\}$. There exists a subsequence $\{x^{k_m}\}$ of $\{x^k\}$, such that $\lim_{m\to\infty} x^{k_m} = \tilde{x}$. By the definition of $x^{k_m} = P_{C\cap \tilde{H}_{k_m-1}}(x^{k_m-1})$ and $\tilde{H}_{k_m-1} = \bigcap_{j=1}^{j=k_m-1} H_j$, we have $x^{k_m} \in H_l$ for all $m \geq l+1$. Since H_l is closed and $\lim_{m\to\infty} x^{k_m} = \tilde{x}$, we have $\tilde{x} \in H_l$. This completes the proof. □

3. Convergence analysis

Theorem 3.1. If $S_D \neq \emptyset$, then the infinite sequence $\{x^k\}$ generated by the Algorithm 2.1 converges to a solution of VI(F,C).

Proof. We assume that $\{x^k\}$ is an infinite sequence generated by the Algorithm 2.1, then $r(x^k) \neq 0$ for every k. By the definition of $x^{k+1} = P_{C\cap\tilde{H}_k}(x^k)$ and Lemma 2.4, for every $x^* \in \bigcap_{k=0}^{\infty}(H_k \cap C)$ we have

$$\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 - \|x^{k+1} - x^k\|^2 = \|x^k - x^*\|^2 - \text{dist}^2(x^k, C\cap\tilde{H}_k). \tag{15}$$

It follows that the sequence $\{\|x^k - x^*\|^2\}$ is nonincreasing, and hence is a convergent sequence. This implies that $\{x^k\}$ is bounded and

$$\lim_{k\to\infty} \text{dist}(x^k, C\cap\tilde{H}_k) = 0. \tag{16}$$

Since $F(x)$ and $r(x)$ are continuous, the sequence $\{z^k\}$ is bounded, and so are $\{r(x^k)\}$ and $\{y^k\}$. Similarly, the continuity of F implies that $\{F(y^k)\}$ is a bounded sequence, that is, for some $M > 0$,

$$\|F(y^k)\| \leq M \quad \text{for all } k.$$

By the definition of $\tilde{H}_k$, we have $\tilde{H}_k \subseteq H_k$ for all k. It follows that

$$\operatorname{dist}(x^k, C \cap \tilde{H}_k) \geq \operatorname{dist}(x^k, C \cap H_k). \tag{17}$$

Therefore (16) implies that

$$\lim_{k\to\infty} \operatorname{dist}(x^k, C \cap H_k) = 0. \tag{18}$$

Clearly each function h_k is Lipschitz continuous on C with modulus M. By Lemma 2.5 and Lemma 2.8, we have

$$\operatorname{dist}(x^k, C \cap H_k) \geq M^{-1} h_k(x^k) \geq M^{-1}(1-\sigma)\eta_k \|r(x^k)\|^2. \tag{19}$$

Thus (18) and (19) imply that $\lim_{k\to\infty} \eta_k \|r(x^k)\|^2 = 0$.

If $\limsup_{k\to\infty} \eta_k > 0$, then we must have $\liminf_{k\to\infty} \|r(x^k)\| = 0$. Since $\{x^k\}$ is bounded and $r(x)$ is continuous, there exists an accumulation point $\hat{x}$ of $\{x^k\}$ such that $r(\hat{x}) = 0$. By Lemma 2.2 and Lemma 2.9, we have $\hat{x} \in \bigcap_{k=1}^{\infty}(H_k \cap S)$. Replace x^* by $\hat{x}$ in (15). We obtain that the sequence $\{\|x^k - \hat{x}\|^2\}$ is nonincreasing and hence is convergent. Note that $\hat{x}$ is an accumulation point of $\{x^k\}$. It follows that $\{x^k\}$ converges to $\hat{x}$.

If $\limsup_{k\to\infty} \eta_k = 0$, then $\lim_{k\to\infty} \eta_k = 0$. Let $\bar{x}$ be any accumulation point of $\{x^k\}$. Then there exists a subsequence $\{x^{k_j}\}$ converges to $\bar{x}$. By the choice of η_k, (7) is not satisfied for $m_k - 1$, that is,

$$\langle F(x^{k_j}) - F(x^{k_j} - \gamma^{-1}\eta_{k_j} r(x^{k_j})), r(x^{k_j})\rangle > \sigma \|r(x^{k_j})\|^2. \tag{20}$$

Since $F(x)$ and $r(x)$ are continuous, passing onto the limit in (20), we have

$$0 \geq \sigma \|r(\bar{x})\|^2 \geq 0. \tag{21}$$

Thus (21) implies that $r(\bar{x}) = 0$. Therefore, $\bar{x} \in \bigcap_{k=1}^{\infty}(H_k \cap S)$. Applying the similar argument in the previous case, we obtain $\{x^k\}$ converges to $\bar{x} \in S$. This completes the proof. □

Remark 3.1. Under the assumption of $S_D \neq \emptyset$, the sequence produced by methods in [1]-[16] has a subsequence converging to a solution while our method generates an sequence globally converging to a solution. Note that $S = S_D$ when F is pseudomonotone on C. Thus our method not only can apply to solve pseudomonotone variational inequalities under the assumption of $S \neq \emptyset$ but also can apply to solve quasimonotone variational inequalities under the assumption of $S_D \neq \emptyset$.

4. Numerical experiments

In this section, we use some numerical experiments to test Algorithm 2.1. The MATLAB codes are run on a PC (with CPU AMD(Athlon) Core(tm)X2 Dual) under MATLAB Version 7.1.0.246(R14) Service Pack 3 which contains Optimization ToolboxVersion 3.0. We take $\|r(x)\| \leq 10^{-4}$ as the termination criterion. That means when $\|r(x)\| \leq 10^{-4}$, the procedure stops. We choose $\gamma = 0.4$, $\sigma = 0.99$ for our algorithm. We denote by x^0 the initial point of the test problem and by x the solution of VI(F,C). We use nf for the total number of times that F is evaluated.

Table 1: Results for Example 4.1 and Example 4.2

Had-Sch-Problem				SQM1-problem			
x^0	iter(nf)	time	x	x^0	iter(nf)	time	x
$(0,1)$	2(3)	1.14063	(1,1)	1	95(97)	5.73438	0.00995113
$(0,0)$	2(3)	1.0625	(1,1)	0.5	94(95)	6.5	0.00996295
$(1,0)$	3(4)	1.20313	(1,1)	0.1	88(89)	5.4375	0.00996464
$(0.5,0.5)$	1(2)	0.703125	(1,1)	-0.1	12(13)	1.84375	-1
$(0.2,0.7)$	2(3)	1.109381	(1,1)	-0.5	2(3)	1.29688	-1
$(0.1,0.6)$	2(3)	1.21875	(1,1)	-0.8	1(2)	0.90625	-1

Table 2: Results for Example 4.3

SQM2-Problem				
x^0	a	iter(nf)	time	x
$(0,0,0,0,5)$	5	33(34)	0.53125	(0.9991,0.9991,0.9991,0.9991,1.0034)
$(0,0,5,0,0)$	5	52(53)	0.6875	(0.9999,0.9999,1.0006,0.9999,0.9999)
$(5,0,0,0,5)$	5	31(32)	0.875	(1.0003,0.9999,0.9999,0.9999,0.9999)
$(1,1,1,1,6)$	10	103(104)	1.20313	(1.9997,1.9997,1.9997,1.9997,2.0011)
$(1,1,6,1,1)$	10	79(80)	0.953125	(1.9998,1.9998,2.0008,1.9998,1.9998)
$(6,1,1,1,1)$	10	63(64)	1.23438	(2.0007,1.9998,1.9998,1.9998,1.9998)

Example 4.1. Let $C = [0,1] \times [0,1]$ and $t = (x_1 + \sqrt{x_1^2 + 4x_2})/2$. We define

$$F(x_1, x_2) = \begin{cases} (-t/(1+t), -1/(1+t)) & \text{if } (x_1, x_2) \neq (0,0) \\ (0,-1) & \text{if } (x_1, x_2) = (0,0). \end{cases}$$

This example was proposed by Hadjisavvas and Schaible in [21], where F is quasimonotone. We call this test problem Had-Sch-Problem.

Table 3: Results for Example 4.4

Dimension	Algorithm 2.1		Algorithm 2.2 in [11]	
n	iter(nf)	time	iter(nf)	time
50	17(52)	1.21875	23(71)	1.04688
100	17(52)	1.90625	23(71)	1.53125
200	18(55)	5.67188	25(77)	6.4375
500	19(58)	62.0781	25(77)	80.3438

Example 4.2. Let $C = [-1, 1]$ and $F(x) = x^2$. Then VI(F,C) is a quasimonotone variational inequality with $S_N = \{-1\}$, $S_T = \{0\}$ and $S_D = \{-1\}$. Thus $S = S_N \cup S_T \not\subset S_D$. We call this test problem SQM1-problem.

Example 4.3. Let $C = \{x \in \mathbb{R}^5 : x_i \geq 0, i = 1, \cdots, 5, \sum_{i=1}^{i=5} x_i = a\}$, $a > 0$ and $G(x) = \frac{\frac{1}{2}x^T Hx + q^T x + r}{\sum_{i=1}^{i=5} x_i}$. Thus G is a smooth quasiconvex function and can attain its minimum value on C (see Exercise 4.7 in [23]), where $q = (-1, \cdots, -1)^T \in \mathbb{R}^5$, $r = 1 \in \mathbb{R}$ and H is a positive diagonal matrix with elements uniformly drawn from the $(0.1, 1.6)$. Let $F(x) = (F_1(x), \cdots, F_5(x))^T$ be the derivative of $G(x)$. Then $F_i(x) = \frac{hx_i \sum_{i=1}^{i=5} x_i - \frac{1}{2}h \sum_{i=1}^{i=5} x_i^2 - 1}{(\sum_{i=1}^{i=5} x_i)^2}$ and VI(F,C) is a qusimonotone variational inequality with $S_D = \{(\frac{1}{5}a, \cdots, \frac{1}{5}a)\}$, where h is the diagonal elements of H. We call this test problem SQM2-problem.

Example 4.4. Consider the affine variational inequality problem with $C = [0, 1]^n$ and $F(x) = Mx + d$ where

$$M = \begin{pmatrix} 4 & -2 & & & \\ 1 & 4 & -2 & & \\ & 1 & 4 & -2 & \\ & & \cdot & \cdot & \cdot \\ & & & 1 & 4 \end{pmatrix} \quad \text{and } d = \begin{pmatrix} -1 \\ -1 \\ \cdots \\ -1 \end{pmatrix}.$$

The initial point $x^0 = (0, \ldots, 0)$. This problem is tested in [24].

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