2-Stage Robust MILP with continuous recourse variables

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Abstract

We solve a linear robust problem with mixed-integer first-stage variables and continuous second stage variables. We consider column wise uncertainty. We first focus on a problem with right hand-side uncertainty which satisfies a "full recourse property" and a specific definition of the uncertainty. We propose a solution based on a generation constraint algorithm. Then we give several generalizations of the approach: for left-hand side uncertainty, for the cases where the "full recourse property" is not satisfied and for uncertainty sets defined by a polytope.

1 Introduction

This paper deals with robust mixed-integer linear programming (MILP) to study problems with uncertain data. This is a possible alternative to two-stage stochastic linear programming introduced by Dantzig in [8]. In this framework the uncertain data of the problem are modeled by random variables, and the decision-maker looks for an optimal solution with respect to the expected objective value. He makes decisions in two stages: first before discovering the actual value taken by the random variables, second once uncertainty has been revealed. However, this approach requires to know the underlying probability distribution of the data, which is, in many cases, not available; furthermore the size of the resulting optimization model increases in such a way that the stochastic optimization problem is often not tractable. Robust optimization is a recent approach that does not rely

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*Corresponding author: E-mail address: pierre-louis.poirion@ensta-paristech.fr on a prerequisite precise probability model but on mild assumptions on the uncertainties involved in the problem, as bounds or reference values of the uncertain data. It looks for a solution that remains satisfactory for all realizations of the data (i.e. for worst scenarios). It was first explored by Soyster [13] who proposed a linear optimization model for data given in a convex set. However this is an over conservative approach that leads to optimal solutions too far from the one of the nominal problem. Robust adjustable optimization models have been proposed and studied to address this conservatism. More precisely, a lot of recent published works cover robust linear programming with row-wise uncertainty for continuous variables [3, 4, 6, 7] or discrete variables [1, 10] and, even more recently, column-wise right-hand side uncertainty [5, 11, 14].

In [9], Gabrel et al. propose a solution based on the approaches given in these last papers to solve a location transportation problem. We first show that their solution can be applied to any linear program with mixed-integer first stage variables and continuous recourse variables. We will also see that problems with left-hand side uncertainty can be solved in the same way.

To the extent of our knowledge, in all works published until now, the authors always assumed that the problem satisfies a "full recourse property" (see Section 2) which cannot be always satisfied for real problems: we show that, when this property is not verified, we can modify the objective function in order to use the previous approach to solve the problem.

Finally, we show that the method can also be used for an affine definition of the uncertainty set which is more general than the one used in [9, 11, 14].

We focus here on a linear robust problem with right-hand and left hand-side uncertainty, mixed-integer first-stage variables and continuous second-stage variables. In Section 2 we present the general problem. For the sake of clarity, we first study the robust problem with right-hand side uncertainty and full recourse property with a specific definition of the uncertainty set. In Section 3 we show how to modelize and solve the recourse problem. In Section 4 we present the solution for the robust problem. Finally, in Section 5 we show that our results can be applied in case of left-hand side uncertainty, we study the cases where the full recourse property is not verified and we extend our results to other definitions of the uncertainty set.

2 A mixed-integer linear robust problem

We consider applications requiring decision-making under uncertainty which can be modeled as a two-stage mixed-integer linear program with recourse. The set of variables is partitioned into two distinct sets: the x variables, called decision variables, concern the decisions to be taken in the first stage, before knowing the realization of the uncertain events; the second stage variables y, called recourse variables, will be fixed only after the uncertainty has been revealed.

We focus here on robust mixed-integer linear problems when the constraint coefficients are uncertain, as well on the right-hand side as on the left-hand side. In addition, we restrict our study to the case where the recourse variables y are continuous variables while the decision variables x are mixed-integer variables.

The deterministic problem can be formulated as the following MILP (in this paper we will omit the transpose sign ^{tr} when there is no possible confusion):

$$
(P) \left| \begin{array}{ll}
\min\limits_{x,y} \alpha x + \beta y & \\
\quad Ax + By \geq d & (1) \\
\quad Cx \geq b & (2) \\
\quad x_i \in \mathbb{N},\ i = 1, ..., p_1,\ x_i \in \mathbb{R}_+,\ i = (p_1 + 1), ..., p,\ y \in \mathbb{R}^q_+. & (3)
\end{array} \right.
$$

where $A \in \mathbb{Q}^{T \times p},\ B \in \mathbb{Q}^{T \times q},\ d \in \mathbb{Q}^T, C \in \mathbb{Q}^{n \times p},\ b \in \mathbb{Q}^n, \alpha \in \mathbb{Q}^p_+,\ \beta \in \mathbb{Q}^q_+$, and $\mathbb{Q}$ is the set of rational numbers.

We assume that there exists (x, y) such that (1)-(3) are satisfied and we say that a solution x is feasible if x satisfies constraints (2) and (3). The uncertain coefficients are those of d (right-hand side) and a part of A (left-hand side).

Given a mathematical program π, we denote by $v(\pi)$ the value of an optimal solution. We assume that the program (P) satisfies the property $\mathcal{P}$, called "full recourse property": for any feasible values of the decision variables (here x) and for any possible value of A and d, there exist values of the recourse variables (here y) such that (1) is satisfied, that is such that there exists a feasible solution of (P). Let us notice that the property $\mathcal{P}$ is always satisfied if there is a column of B whose all terms are positive. The hypothesis that (P) satisfies the property $\mathcal{P}$ cannot be always satisfied for real problems: we show in Section 5, that we can extend our results when $\mathcal{P}$ is not satisfied.

We suppose that d belongs to a given set $\mathfrak{D}$ which defines the set of possible scenarios and for the sake of clarity we assume at first that the uncertainty concerns only the right-hand side d of (1). We show in Section 5 that our results can be extended when the matrix A is also uncertain.

Our robustness objective is to find a feasible solution x, y of (P) that minimizes the total cost involved by the worst possible scenario of d in connection with x. We can state the robust problem as the following mathematical program:

$$(PR)\left|\begin{array}{l} \min\limits_{x} \ \alpha x \ + \ \max\limits_{d\in\mathfrak{D}} \min\limits_{y} \beta y \\ \qquad\qquad\qquad\qquad\quad By \geq d - Ax \\ \qquad\qquad\qquad\qquad\quad y \in \mathbb{R}^q_+ \\ \qquad Cx \geq b \\ \qquad x_i \in \mathbb{N}, \ i = 1, ..., p_1, \ x_i \in \mathbb{R}_+, \ i = (p_1 + 1), ..., p. \end{array}\right.$$

For any feasible x, we define the following linear program $R(x)$ called "Recourse Program":

$$R(x)\left|\begin{array}{l} \max\limits_{d\in\mathfrak{D}} \ \min\limits_{y} \ \beta y \\ \qquad\qquad By \geq d - Ax \\ \qquad\qquad y \in \mathbb{R}^q_+. \end{array}\right.$$

The robust program can then be rewritten as:

$$(PR)\left|\begin{array}{l} \min\limits_{x} \ \ \alpha x + v(R(x)) \\ \qquad Cx \geq b \\ \qquad x_i \in \mathbb{N}, \ i = 1, ..., p_1, \ x_i \in \mathbb{R}_+, \ i = (p_1 + 1), ..., p. \end{array}\right.$$

Let us define more precisely the uncertainty. Following the idea proposed by Bertsimas and Sim [7] and Minoux [11], we suppose that each coefficient $d_t, \ t = 1, ..., T$ belongs to an interval $[\bar{d}_t - \Delta_t, \bar{d}_t + \Delta_t]$ where $\bar{d}_t$ is a given value and where $\Delta_t \geq 0$ is a given bound of the uncertainty of d. The uncertainty set $\mathfrak{D}$ is therefore given by:

$$\mathfrak{D} = \{d : \ d_t \in [\bar{d}_t - \Delta_t, \bar{d}_t + \Delta_t], \ \forall t = 1, ..., T\}.$$

For a fixed x, the worst scenario is obtained for $d_t = \bar{d}_t + \Delta_t, \ t = 1, ..., T$. Indeed, $By \geq \bar{d} + \Delta - Ax$ implies $By \geq d - Ax$ for all $d \in \mathfrak{D}$. Thus this uncertainty definition brings the robust problem back to a deterministic one. It provides a high "protection" against uncertainty, but it is very conservative in practice and leads to very expensive solutions. To avoid overprotecting the system, we impose, as in [14], the constraint

$$\sum_{\substack{t=1 \\ \Delta_t > 0}}^{T} \left|\frac{d_t - \bar{d}_t}{\Delta_t}\right| \leq \bar{\delta},$$

where $\bar{\delta}$ is a positive integer which bounds the total scaled deviation of d from its nominal value $\bar{d}$. Notice that there always exists a worst scenario with $d_t \geq \bar{d}_t, \ \forall t$, hence we can redefine the uncertainty set $\mathfrak{D}$ as

$$\mathfrak{D} = \{d : \ d_t = \bar{d}_t + \delta_t \Delta_t, \ \sum_{t=1}^{T} \delta_t \leq \bar{\delta}, \ 0 \leq \delta_t \leq 1, \ \forall t = 1, ..., T\}.$$

3 The recourse problem

To solve the recourse "$max\ min$" problem for given values of the decision variables, the minimization linear sub-program is transformed in a maximization program by considering its dual. But that leads to a quadratic objective function. We show that whatever the coefficients in the recourse problem, the quadratic terms can be written as products of a 0-1 variable and a continuous but bounded variable, which allows a linearization of these products.

Let x be a feasible solution and let $d \in \mathfrak{D}$, we define the following linear program

$$\hat{R}(x,d) \left| \begin{array}{l} \min\limits_{y} \ \beta y \\ \qquad By \geq d - Ax \\ \qquad y \in \mathbb{R}^q_+. \end{array} \right.$$

We notice that $\hat{R}(x,d)$ has a finite solution for all feasible x and for all possible scenario d since (P) satisfies $\mathcal{P}$, and since $\beta y \geq 0$ for any feasible solution y of $\hat{R}(x,d)$. Thus by the strong duality theorem, we have

$$v(\hat{R}(x,d)) = v(D\hat{R}(x,d)),$$

where $D\hat{R}(x,d)$ is the dual program of $\hat{R}(x,d)$:

$$D\hat{R}(x,d) \left| \begin{array}{ll} \max\limits_{\lambda} \ (d - Ax)\lambda & \\ \qquad \lambda B \leq \beta & (4) \\ \qquad \lambda \in \mathbb{R}^T_+. & (5) \end{array} \right.$$

Then, for any feasible x, $v(R(x)) = \max_{d \in \mathfrak{D}} v(D\hat{R}(x,d))$.

Hence we can reformulate $DR(x)$ as follows:

$$DR(x) \left| \ \max_{\substack{\delta:\ \sum_{t=1}^{T} \delta_t \leq \bar{\delta} \\ 0 \leq \delta_t \leq 1,\ t=1,\ldots,T}} \ \max_{\substack{\lambda:\ \lambda B \leq \beta \\ \lambda \in \mathbb{R}^T_+.}} \sum_{t=1}^{T} [(\bar{d}_t + \delta_t \Delta_t - (Ax)_t)\lambda_t] \right.$$

where for a vector (u), we denote by $(u)_t$ the t-th coordinate of (u). $DR(x)$ can be written

$$DR(x)\left|\begin{array}{ll} \max\limits_{\lambda,\delta} \sum\limits_{t=1}^{T}[(\bar{d}_t-(Ax)_t)\lambda_t+\Delta_t\delta_t\lambda_t] & \\ \lambda B \le \beta & (4)\\ \sum\limits_{t=1}^{T} \delta_t \le \bar{\delta} & (6)\\ 0\le\delta_t\le 1 \; t=1,...,T & (7) \\ \lambda\in\mathbb{R}^T_+. & (5)\end{array}\right.$$

However, this bilinear program with linear constraints is not concave. Therefore computing the optimal solution of $DR(x)$ written as above is not an easy task. We now prove that we can solve $DR(x)$ by solving an equivalent mixed-integer linear program. To prove this claim, we need the following proposition:

Proposition 1. *There is an optimal solution* λ^*,δ^* *of* $DR(x)$ *such that* $\delta^*_t\in\{0,1\},\ 1\le t\le T$.

Proof. For any fixed λ, there is an optimal solution, (λ,δ^*), of $DR(x)$, where δ^* is an extreme point of the polyhedron defined by (6) and (7), that is to say, a point such that $\delta^*_t\in\{0,1\},\ 1\le t\le T$, since $\bar{\delta}$ is an integer.
More precisely $\delta^*_t=1$ for indices corresponding to the $\bar{\delta}$ largest $\Delta_t\lambda_t$. □

Therefore we can assume that there is an optimal solution of $DR(x)$, such that $\lambda_t\delta_t$ belongs to $\{0,\lambda_t\}$. To linearize $\lambda_t\delta_t$, we now prove that we can restrict ourselves to the case where λ_t is bounded by a constant Λ, for all t.

Proposition 2. *There exists* $\Lambda>0$ *such that the conditions* $\lambda_t\le\Lambda,\ t=1,...,T$, *can be added to* $DR(x)$ *without loss of generality.*

Proof. Let x be feasible. Let us rewrite $DR(x)$ with the slack variables $\lambda'_t\ge 0,\ t=1,...,T$. The constraints (4) become: $B^{tr}\lambda+\lambda'=\beta$. Let $(\lambda^*,\lambda'^*,\delta^*)$ be an optimal solution of $DR(x)$, we can assume w.l.o.g. that (λ^*,λ'^*) is an optimal basic solution of $DR(x)$ when δ is set to δ^*.
Therefore, there exists a basic matrix $E=(e_{ij})$ of $(B^{tr}\ I_T)$ and basic vectors λ^*_E,λ'^*_E such that: $(\lambda^*_E\ \lambda'^*_E)^{tr}=E^{-1}\beta$. Let $\hat{e}$ be an upper bound on the absolute value of the coefficients of E^{-1} for all basic matrices E of $(B^{tr}\ I_T)$, and let $\hat{\beta}=\max\limits_{i=1,...,q}\beta_i$, we have $\lambda^*_t\le\hat{e}\hat{\beta}q,\ t=1,...,T$. Therefore there exists an optimal solution (λ^*,δ^*) of $DR(x)$ such that λ^*_t is bounded by $\Lambda=\hat{e}\hat{\beta}q$ for any $t=1,...,T$. □

We can now linearize $DR(x)$ by substituting the new variables ν_t to the products $\lambda_t\delta_t$ and by adding the constraints: $\nu_t \leq \lambda_t$, $\nu_t \leq \Lambda\delta_t$, $\nu_t \geq \lambda_t - \Lambda(1-\delta_t)$, $\nu_t \geq 0$.

$DR(x)$ is equivalent to the following mixed-integer linear program:

$$LDR(x) \left| \begin{array}{l} \max\limits_{\lambda,\delta,\nu} \sum_{t=1}^{T}[(\bar{d}_t - (Ax)_t)\lambda_t + \Delta_t\nu_t] \\ \lambda B \leq \beta \\ \sum_{t=1}^{T}\delta_t \leq \bar{\delta} \\ \nu_t \leq \lambda_t,\ t=1,...,T \\ \nu_t \leq \Lambda\delta_t,\ t=1,...,T \\ \lambda,\nu \in \mathbb{R}_+^T \\ \delta_t \in \{0,1\},\ t=1,...,T. \end{array} \right.$$

Notice that the linearization constraints, $\nu_t \geq \lambda_t - \Lambda(1-\delta_t),\ t=1,...,T$, can be omitted since the coefficients of ν_t in the objective function to maximize are positive.

4 Solving the robust problem

In order to solve the robust problem (PR), we will first reformulate it as a linear program and then use a constraint generation algorithm. In the previous section, we proved that the recourse problem is equivalent to the linear program $LDR(x)$. Thus the robust problem can be reformulated as:

$$(PR) \left| \begin{array}{ll} \min\limits_{x} & \alpha x + v(LDR(x)) \\ & Cx \geq b \\ & x_i \in \mathbb{N},\ i=1,...,p_1,\ x_i \in \mathbb{R}_+,\ i=(p_1+1),...,p. \end{array} \right.$$

Let $\mathcal{P}_Q$ be the polyhedron defined by the constraints of $LDR(x)$ where we replace $\delta_t \in \{0,1\}$ by $0 \leq \delta_t \leq 1$, and let $(\mathcal{P}_Q)_I = conv(\mathcal{P}_Q \cap \{\delta \in \mathbb{N}^m\})$, be the convex hull of the feasible solution of $LDR(x)$. Notice that this convex hull does not depend on x. $(\mathcal{P}_Q)_I$ is a polyhedron, thus we have

$$LDR(x) \left| \begin{array}{l} \max\limits_{\lambda,\delta,\nu} \sum_{t=1}^{T}[(\bar{d}_t - (Ax)_t)\lambda_t + \Delta_t\nu_t] \\ \begin{pmatrix} \lambda \\ \delta \\ \nu \end{pmatrix} \in (\mathcal{P}_Q)_I, \end{array} \right.$$

Let $\mathcal{S} = \{(\lambda^s, \delta^s, \nu^s)_{1\leq s\leq S}\}$, be the set of extreme points of $(\mathcal{P}_Q)_I$. For any feasible x, there is $s \in \{1,...,S\}$ such that $(\lambda^s, \delta^s, \nu^s)$ is an optimal solution of

Algorithm 1 Constraint generation algorithm

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1: $(\lambda^0, \delta^0, \nu^0) = (0, 0, 0)$. Set $L \leftarrow -\infty, U \leftarrow +\infty, k \leftarrow 1$.
2: Solve the master problem :
       $(PR)^k \left| \begin{array}{l} \min\limits_{x,z} \ \alpha x + z \\ z \geq \sum_{t=1}^{T} (\bar{d}_t - (Ax)_t)\lambda_t^s + \Delta_t \nu_t^s, \ 0 \leq s \leq k-1 \\ Cx \geq b \\ x_i \in \mathbb{N}, \ i = 1, ..., p_1, \ x_i \in \mathbb{R}_+, \ i = (p_1 + 1), ..., p \\ z \in \mathbb{R} \end{array} \right.$
   Let $(x^k, z^k)$ be the obtained solution.
   $L \leftarrow \alpha x^k + z^k$.
3: Solve $LDR(x^k)$. Let $(\lambda^k, \delta^k, \nu^k)$ be the optimal solution.
       $U \leftarrow \min\{U, \alpha x^k + v(DR(x^k))\}$.
   if $U = L$, then return $(x^k, z^k)$ else go to 4.
4: Add the constraint
       $z \geq \sum_{t=1}^{T} (\bar{d}_t - (Ax)_t)\lambda_t^k + \Delta_t \nu_t^k$,
   to the master problem $(PR)^k$, $k \leftarrow k + 1$ and go to 2.
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$LDR(x)$.
Thus the robust problem can be reformulated as the linear program:

$$(PR) \left| \begin{array}{l} \min\limits_{x,z} \alpha x + z \\ \\ z \geq \sum\limits_{t=1}^{T} [(\bar{d}_t - (Ax)_t)\lambda_t^s + \Delta_t \nu_t^s], \ 1 \leq s \leq S \qquad (8) \\ Cx \geq b \\ x_i \in \mathbb{N}, \ i = 1, ..., p_1, \ x_i \in \mathbb{R}_+, \ i = (p_1 + 1), ..., p, \ z \in \mathbb{R} \end{array} \right.$$

However, due to the potentially tremendous number of constraints, we solve (PR) by a constraint generation algorithm as in [14] or [9]. Initially, we consider a subset $\mathcal{S}_0$ of $\mathcal{S}$; at a step k, we consider a subset $\mathcal{S}^k$ of $\mathcal{S}$ and we solve a relaxed program $(PR)^k$ of (PR), called *master problem*, which consists in solving (PR) with the subset of constraints (8) corresponding to $\mathcal{S}^k$. The obtained solution in denoted by (x^k, z^k).
Then we solve $DR(x^k)$, called *slave problem*, to check if (x^k, z^k) is optimal. If not, then a new constraint is added, i.e. an extreme point is added to $\mathcal{S}^k$ (See Algorithm 1).
On the basis that the number of extreme points of $(\mathcal{P}_Q)_I$ is finite, one can prove that this algorithm converges in a finite number of steps.

5 Some generalizations

5.1 Left-hand side uncertainty

In the previous sections, we assumed that the uncertainty concerned only the right-hand side d of constraints (1). We now prove that our approach can be generalized to the case where the constraint coefficients ($A = (a_{ti})_{1\le t\le T,\ 1\le i\le p}$), are also likely to be uncertain. As before, we assume that each coefficient a_{ti} belongs to an interval $[\bar{a}_{ti} - \Gamma_{ti}, \bar{a}_{ti} + \Gamma_{ti}]$, where $\bar{a}_{ti}$ is a given value and where Γ_{ti} is a given bound of the uncertainty of a_{ti}.

Furthermore, in order to avoid overprotecting the system, we assume that the total scaled deviation of the uncertainty of the i-th column of A, $a_i = (a_{ti},\ t = 1, ..., T)$, is bounded. Similarly to $\mathfrak{D}$, the uncertainty set $\mathcal{A}_i$ of a_i is defined as :

$$\mathcal{A}_i = \{a_i :\ a_{ti} = \bar{a}_{ti} - \gamma_{ti}\Gamma_{ti},\ \textstyle\sum_{t=1}^{T}\gamma_{ti} \le \bar{\gamma}_i,\ 0 \le \gamma_{ti} \le 1\},$$

where $\bar{\gamma}_i$ is a given integer.

The robust problem can thus be formulated as:

$$(PR')\quad \left|\begin{array}{l} \min\limits_{x}\ \alpha x + \max\limits_{\substack{a_i\in\mathcal{A}_i,\\ \forall i=1,\dots,p\\ d\in\mathfrak{D}}}\ \min\limits_{y}\beta y \\ \qquad\qquad\qquad By \ge d - (a_1, ..., a_p)x \\ \qquad\qquad\qquad y \in \mathbb{R}^q_+ \\ Cx \ge b \\ x_i \in \mathbb{N},\ i = 1, ..., p_1 \\ x_i \in \mathbb{R}_+,\ i = (p_1 + 1), ..., p. \end{array}\right.$$

And the recourse problem becomes:

$$DR'(x)\quad \left|\begin{array}{l} \max\limits_{\lambda,\delta,\gamma}\ \sum_{t=1}^{T}[(\bar{d}_t - \sum_{i=1}^{p}\bar{a}_{ti}x_i)\lambda_t + \Delta_t\delta_t\lambda_t + \sum_{i=1}^{p}\Gamma_{ti}x_i\gamma_{ti}\lambda_t] \\ \lambda B \le \beta \\ \sum_{t=1}^{T}\delta_t \le \bar{\delta} \\ 0 \le \delta_t \le 1,\ t = 1, .., T \\ \lambda \in \mathbb{R}^T_+ \\ \sum_{t=1}^{m}\gamma_{ti} \le \bar{\gamma}_i,\ i = 1, ..., p \\ 0 \le \gamma_{ti} \le 1,\ i = 1, .., p,\ t = 1, ..., T. \end{array}\right.$$

We can then linearize the quadratic terms ($\delta_t\lambda_t$ and $\gamma_{ti}\lambda_t$), to obtain a mixed-integer linear recourse problem and then solve the robust problem as we did in the previous sections.

5.2 Solving the problem without the full recourse property

In the previous sections, we assumed that the deterministic problem satisfied the property $\mathcal{P}$. We now prove that we can extend our results to the case where we only assume that the robust problem (PR) have a finite optimal solution, i.e. there exists M such that $v(PR) \leq M$. In addition, the method detects if the problem has no solutions. For the sake of clarity, we give the proof for the case where all the decision variables are integer. In fact, the proof for mixed integer variables is based on the same ideas but is more complicated; it is given in [12].
First let us show how to obtain a new MILP, denoted $(\tilde{P})$ such that the robust associated problem has the same optimal solution as the initial robust problem and $(\tilde{P})$ satisfies $\mathcal{P}$. To obtain $(\tilde{P})$, we add new recourse variables $w_t,\ t = 1, ..., T$. As in the sections 2, 3 and 4, for the sake of clarity and w.l.o.g., we consider only right-hand side uncertainty.

Let ε be a given strictly positive value, we define the following MILP:

$$\tilde{P}_\varepsilon \left| \begin{array}{ll} \min\limits_{x,y,w} \alpha x + \beta y + \dfrac{M}{\varepsilon} \sum\limits_{t=1}^{T} w_t & \\ \quad Ax + By + w \geq d & (1\varepsilon) \\ \quad Cx \geq b & (2) \\ \quad x_i \in \mathbb{N},\ i = 1, ..., p & (3\varepsilon) \\ \quad y \in \mathbb{R}^q_+,\ w \in \mathbb{R}^T_+ & (4\varepsilon) \end{array} \right.$$

We notice that since the variables $w_t,\ t = 1, ..., T$, are not bounded, $(\tilde{P_\varepsilon})$ satisfies the property $\mathcal{P}$.

We denote by $(\tilde{PR})_\varepsilon$, the robust problem associated to $(\tilde{P_\varepsilon})$, and by $(\tilde{R}_\varepsilon(x))$, $(\tilde{R}_\varepsilon(x, d))$ and $(\tilde{DR}_\varepsilon(x))$ the associated subproblems as those defined in Section 3. Notice that since all the inputs, $A, B, C, b, d, \alpha, \beta, \Delta$, of (PR) have rational coefficients, we can reduce $(\tilde{PR})_\varepsilon$ and all the corresponding subproblems to programs where all the inputs are integer. Therefore we assume from now that all the inputs are integer.

Proposition 3. *$v(\tilde{PR_\varepsilon})$ satisfies $0 \leq v(\tilde{PR_\varepsilon}) \leq v(PR) \leq M$.*

Proof. Let $(\hat{x}, \hat{y})$ be an optimal solution of (PR), By hypothesis, $v(PR) \leq M$ thus $v(R(\hat{x})) \leq M$. Let $\bar{d}$ be a scenario in $\mathfrak{D}$. Since $v(R(\hat{x})) = \max\limits_{d \in \mathfrak{D}} v(\hat{R}(\hat{x}, d))$, we have $v(R(\hat{x}, \bar{d})) \leq M$. Let $\bar{y}$ be an optimal solution of $R(\hat{x}, \bar{d})$, we notice that $(y, w) = (\bar{y}, 0)$ is a feasible solution of $(\tilde{R}_\varepsilon(\hat{x}, \bar{d}))$ with the same cost. Thus $v(\tilde{R}_\varepsilon(\hat{x}, \bar{d})) \leq v(\hat{R}(\hat{x}, \bar{d}))$, for any $\bar{d} \in \mathfrak{D}$, which implies $v(\tilde{R}_\varepsilon(\hat{x})) \leq v(R(\hat{x}))$, and $0 \leq v(\tilde{PR}_\varepsilon) \leq v(PR) \leq M$. $\square$

Let (x^*, y^*, w^*) be an optimal solution of $(\tilde{PR}_\varepsilon)$, and let d^* be the worst scenario for x^*. Notice that Proposition 1 is valid for $(\tilde{DR}_\varepsilon(x^*))$. Therefore $d^*_t = \bar{d}_t$ or $d^*_t = \bar{d}_t + \Delta_t$, and d^*_t is an integer. From proposition 3, we have

$$\alpha x^* + \beta y^* + \frac{M}{\varepsilon} \sum_{t=1}^{T} w^*_t \leq M.$$

Since $\alpha x^* + \beta y^* \geq 0$, we have $\sum_{t=1}^{T} w^*_t \leq \varepsilon$, and thus $w^*_t \leq \varepsilon,\ \forall t = 1, .., T$. We now prove that if (y^*, w^*) is a basic optimal solution of $(\tilde{R}_\varepsilon(x^*, d^*))$, then for ε small enough, we have $w^*_t = 0,\ \forall t$; and then (x^*, y^*) is admissible for (PR), and therefore from proposition 3, $v(PR) = v((\tilde{PR})_\varepsilon)$.

Let us rewrite $(\tilde{PR})_\varepsilon$ with the positive slack variables $s = (s_t,\ t = 1, ..., T)$: the constraint (1ε) becomes $Ax + By + w - s = d$. Let (x^*, y^*, w^*, s^*) be an optimal solution where (y^*, w^*, s^*) is a basic optimal solution of the program:

$$\tilde{R}_\varepsilon(x^*, d^*) \left| \begin{array}{l} \min\limits_{y,w,s} \beta y + \dfrac{M}{\varepsilon} \displaystyle\sum_{t=1}^{T} w_t \\ \qquad By + w - s = d^* - Ax^* \\ \qquad y \in \mathbb{R}^q_+,\ s, w \in \mathbb{R}^T_+. \end{array} \right.$$

which is equivalent to

$$\tilde{R}_\varepsilon(x^*, d^*) \left| \begin{array}{l} \min\limits_{y,w,s} \beta y + \dfrac{M}{\varepsilon} \displaystyle\sum_{t=1}^{T} w_t \\ \qquad \begin{pmatrix} B & I_T & -I_T \end{pmatrix} \begin{pmatrix} y \\ w \\ s \end{pmatrix} = d^* - Ax^* \\ \qquad y \in \mathbb{R}^q_+,\ s, w \in \mathbb{R}^T_+. \end{array} \right.$$

Let $L = \begin{pmatrix} B & I_T & -I_T \end{pmatrix} = (l_{ij}) \in \mathbb{Z}^{T \times (q+2T)}$ and $l_M = \max_{i,j} |l_{ij}|$. We notice that L has rank T.

Proposition 4. *If* $\varepsilon < \frac{1}{(l_M)^T T^{T/2}}$, *then* $w^*_t = 0,\ t = 1, ..., T$, *for any optimal solution* (x^*, y^*, w^*, s^*) *of* $(\tilde{PR}_\varepsilon)$

Proof. Assume that (y^*, w^*, s^*) is a basic optimal solution of $(\tilde{R}_\varepsilon(x^*, d^*))$, where d^* is the worst scenario for x^*. There exists a basic matrix $E \in \mathbb{Z}^{T\times T}$ of L and basic vectors y^*_E, w^*_E, s^*_E such that: $E\,(y^*_E\; w^*_E\; s^*_E)^{tr} = d^* - Ax^*$. The matrix $E = (e_{kj})$ is invertible, and $E^{-1} = \frac{1}{\det(E)} adj(E)$, where $adj(E)$ is the adjugate matrix of E. Therefore $(y^*_E\; w^*_E\; s^*_E)^{tr} = \frac{1}{\det(E)} adj(E)(d^* - Ax^*)$,
and $0 \le w^*_t = \frac{1}{\det(E)}(adj(E)(d^* - Ax^*))_{t'} \le \varepsilon$, where w^*_t is a basic variable and where t' is the associated index in (y^*_E, w^*_E, s^*_E). Thus

$$|adj(E)(d^* - Ax^*))_{t'}| \le \varepsilon |\det(E)|. \tag{9}$$

Since E is a sub-matrix of L, we can, according to Hadamard's inequality, bound $|\det(E)|$ by $(l_M)^T T^{T/2}$. If $\varepsilon < \frac{1}{(l_M)^T T^{T/2}}$, then according to (9), $|(adj(E)(d^* - Ax^*))_{t'}| < 1$. Since $|(adj(E)(d^* - Ax^*))_{t'}| \in \mathbb{N}$, we have $|(adj(E)(d^* - Ax^*))_{t'}| = 0$, therefore $w_t = 0$ for any basic variable w_t and thus $w_t = 0$ for all $t = 1, ..., T$. Thus for any optimal solution (x^*, y^*, w^*, s^*) of $(\tilde{PR}_\varepsilon)$, $w^* = 0$. $\square$

Eventually, if we fix $\varepsilon < \frac{1}{(l_M)^T T^{T/2}}$, then the optimal solution (x^*, y^*, w^*) of $(\tilde{PR})_\varepsilon$, verifies $w^* = 0$, (x^*, y^*) is an optimal solution of (PR), and $v(PR) = v((\tilde{PR})_\varepsilon)$.
Notice that this method can detect if $v(PR)$ is finite or not. Indeed if the optimal solution (x^*, y^*, w^*) of $(\tilde{PR})_\varepsilon$, does not satisfy $w^* = 0$, then $v(PR) = \infty$.

5.3 Generalization to other uncertainty sets

In the previous sections, we assumed that uncertain coefficients could be written as $d_t = \bar{d}_t + \delta_t \Delta_t\ \forall t$, where δ_t expresses the uncertainty on d_t and satisfies $\sum_{t=1}^T \delta_t \le \bar{\delta}$. Now we generalize our results when the vector d can be written as $d = \bar{d} + D\delta$, where the vector $\bar{d}$ and the matrix D are given and where δ belongs to a bounded polyhedron $\mathcal{D}$ whose extreme points $(d^1, ..., d^S)$ are known. Notice that this definition of the uncertainty covers the one given by Babonneau et al. in [2] Let us rewrite the recourse problem:

$$DR'(x)\left|\begin{array}{l} \max\limits_{\lambda,\delta}\ (\bar{d} + D\delta - Ax)\lambda \\ \qquad \lambda B \le \beta \\ \qquad \delta \in \mathcal{D} \\ \qquad \lambda \in \mathbb{R}^T_+. \end{array}\right.$$

Let $v^1, ..., v^S \in [0, 1]$ be variables such that $\delta = \sum_{s=1}^S d^s v^s$ and $\sum_{s=1}^S v^s = 1$. We can rewrite the recourse problem as

$$
DR'(x)\ \left|\ \begin{array}{l}
\max\limits_{\lambda,v}\ (\bar{d}-Ax)\lambda+\sum\limits_{s=1}^{S}(v^s(Dd^s)\lambda) \\
\quad \lambda B \leq \beta \\
\quad \sum\limits_{s=1}^{S} v^s = 1 \\
\quad 0 \leq v^s \leq 1,\ s=1,...,S \\
\quad \lambda \in \mathbb{R}_{+}^{T}.
\end{array}\right.
$$

Using the same argument as in Proposition 1, we can prove that there exists an optimal solution (λ^*, v^*) of $DR'(x)$ such that either $v^{s*}=1$ or $v^{s*}=0,\ s=1,...,S$. Therefore we can linearize the quadratic terms $v^s\lambda_t$, for all s and for all t, as we did in Section 3, to obtain a mixed-integer linear recourse problem, and finally we can solve the robust problem by using Algorithm 1.

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