On the use of semi-closed sets and functions in convex analysis

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Abstract

The main aim of this short note is to show that the subdifferentiability and duality results established by Laghdir (2005), Laghdir and Benabbou (2007), and Alimohammady *et al.* (2011), stated in Fréchet spaces, are consequences of the corresponding known results using Moreau–Rockafellar type conditions.

Key words: Semi-closed convex set, semi-closed convex function, semi-closure, semi-interior, subdifferential, duality.

MSC Classification: 90C25

According to Alimohammady *et al.* (see [1] and the references therein), the subset A of the topological space (X,τ) is *semi-open* if there exists $U\in\tau$ such that $U\subset A\subset \operatorname{cl}U$; this is equivalent to $A\subset \operatorname{cl}(\operatorname{int}A)$ (and is also equivalent to $\operatorname{cl}A=\operatorname{cl}(\operatorname{int}A)$). Correspondingly, $A\subset(X,\tau)$ is *semi-closed* if $X\setminus A$ is semi-open; this is equivalent to each of the following assertions: there exists a closed set $F\subset X$ such that $\operatorname{int}F\subset A\subset F$, $A\supset\operatorname{int}(\operatorname{cl}A)$, $\operatorname{int}A=\operatorname{int}(\operatorname{cl}A)$. Moreover, the *semi-closure* of $A\subset(X,\tau)$ is the set $\text{s-cl}\,A:=\cap\{B\mid A\subset B,\ B$ is semi-closed$\}$, and the *semi-interior* of A is the set $\text{s-int}\,A:=\cup\{B\mid B\subset A,\ B$ is semi-open$\}$. The function $f:X\to\overline{\mathbb{R}}$ is *semi-closed* if its epigraph is semi-closed in $X\times\mathbb{R}$, where $\operatorname{epi}f:=\{(x,t)\in X\times\mathbb{R}\mid f(x)\le t\}$. Laghdir [2] introduced the notion of semi-closed set (function) for subsets of (functions defined on) separated topological vector spaces.

Proposition 1 *Let X be a topological vector space and $A\subset X$ be a convex set.*

(i) *Assume that* $\operatorname{int}A\neq\emptyset$. *Then A is semi-closed and semi-open; consequently,* $s\text{-int}\,A=s\text{-cl}\,A=A$.

(ii) $s\text{-int}\,A\neq\emptyset$ *if and only if* $\operatorname{int}A\neq\emptyset$. *Moreover,* $\operatorname{int}A\subset s\text{-int}\,A$.

Proof. (i) It is known (see e.g. [4, Th. 1.1.2 (iv)]) that $\operatorname{cl}A=\operatorname{cl}(\operatorname{int}A)$ and $\operatorname{int}A=\operatorname{int}(\operatorname{cl}A)$. Hence A is semi-closed and semi-open.

(ii) Assume that $\text{s-int}\,A\neq\emptyset$. Then there exists a semi-open nonempty set $B\subset A$. Since $B\subset\operatorname{cl}(\operatorname{int}B)$, we have that $\operatorname{int}B\neq\emptyset$, whence $\operatorname{int}A\neq\emptyset$.

Assume now that $\operatorname{int}A\neq\emptyset$. By (i) we have that $\text{s-int}\,A=A$, and so $\text{s-int}\,A\neq\emptyset$.

The claimed inclusion is obvious if $\operatorname{int}A=\emptyset$, while for $\operatorname{int}A\neq\emptyset$ we have that $\operatorname{int}A\subset A=\text{s-int}\,A$. □

Before presenting the next remark and results let us recall other notions and notations used in the sequel related to functions $f:X\to\overline{\mathbb{R}}$. The *domain* of f is the set $\operatorname{dom}f:=\{x\in X\mid f(x)<+\infty\}$; f is *proper* if $\operatorname{dom}f\neq\emptyset$ and $f(x)\neq-\infty$ for every $x\in X$. In the case in which X is a topological vector space (tvs for short) with topological dual X^*, the *conjugate* of f is the function $f^*:X^*\to\overline{\mathbb{R}}$ defined by $f^*(x^*):=\sup(x^*-f)$, and the *subdifferential* of

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f at x with $f(x) \in \mathbb{R}$ is the set $\partial f(x) := \{x^* \in X^* \mid \langle x' - x, x^* \rangle \leq f(x') - f(x)\ \forall x' \in X\}$, where $\langle x, x^* \rangle := x^*(x)$, and $\partial f(x) := \emptyset$ if $f(x) \notin \mathbb{R}$. Moreover, $\overline{f}$ (resp. $\overline{\operatorname{conv}} f$) is the lsc (resp. lsc convex) hull of f, where lsc means lower semicontinuous. Having $A \subset X$, the *indicator* function of $A \subset X$ is the function $\iota_A : X \to \overline{\mathbb{R}}$ defined by $\iota_A(x) := 0$ if $x \in A$ and $\iota_A(x) := +\infty$ if $x \in X \setminus A$; $N_A(x) := \partial \iota_A(x)$.

Remark 1 (a) Having in view the above proposition, the separation theorem [1, Th. 2.10] is weaker than the usual separation theorem for two nonempty convex sets because one asks the space to be Fréchet and the semi-interior of one of the sets be nonempty and disjoint of the other set. A similar remark is valid for [1, Cor. 2.11].

(b) Theorem 2.12 in [1] is false. To prove this consider $X = \mathbb{R}$, $C := (-\infty, 0]$ and $f : X \to \overline{\mathbb{R}}$ defined by $f(x) := -\sqrt{x}$ for $x \geq 0$, $f(x) := +\infty$ for $x < 0$. Then $\operatorname{s-int} C = C$, and so $\operatorname{s-int} C \cap \operatorname{dom} f = \{0\} \neq \emptyset$; moreover $\gamma := \inf_{x \in C} f(x) = f(0) = 0$. The conclusion that there exists an affine function α such that $\alpha \leq f$ and $\gamma = \inf_{x \in C} \alpha(x)$ is false; indeed, taking $\alpha(x) = ax + b$ we get $0 = \inf_{x \in C} \alpha(x)$, and so $a \leq 0$ and $b = 0$, and $ax \leq -\sqrt{x}$ for every $x \geq 0$ (equivalently $a\sqrt{x} \leq -1$ for every $x \geq 0$), whence the contradiction $0 \leq -1$. Since $0 \in C$ is a solution of the problem $\min f(x)$ s.t. $x \in C$, the fact that $\partial f(0) + N_C(0) = \emptyset \not\ni 0$, shows that the second conclusion of [1, Th. 2.12] is false, too.

Laghdir [2, p. 151] says: "One may ask a natural question if a lower cs-closed function (see below the definition) is semi-closed?". We give a partial answer to this question. First recall that the subset A of the topological vector space X is *cs-closed* if any convergent series of the form $\sum_{n=1}^{\infty} \lambda_n x_n$ with $(\lambda_n)_{n \geq 1} \subset \mathbb{R}_+$, $\sum_{n=1}^{\infty} \lambda_n = 1$, $(x_n)_{n \geq 1} \subset A$, has the sum in A; A is *lower cs-closed* if there exists a Fréchet space Y and a cs-closed set $C \subset X \times Y$ such that $A = \operatorname{Pr}_X(C)$. The set A is (*lower*) *ideally convex* if in the preceding definition one asks $(x_n)_{n \geq 1} \subset A$ to be bounded. Of course, any (lower) cs-closed set is (lower) ideally convex, and any (lower) ideally convex set is convex. The function $f : X \to \overline{\mathbb{R}}$ is *cs-closed* (*lower cs-closed, ideally convex, lower ideally convex*) if $\operatorname{epi} f$ is so. Clearly any lower ideally convex set (function) is convex. Recall that the *core* (or *algebraic interior*) of the subset A of the real linear space X is the set $\operatorname{core} A := \{a \in X \mid \forall x \in X,\ \exists \delta > 0,\ \forall t \in [0, \delta] : a + tx \in A\}$.

Proposition 2 *Let X be a complete semi-metrizable locally convex space (for example a Fréchet space), $A \subset X$ and $f : X \to \overline{\mathbb{R}}$ a proper function.*

(i) *If A is lower ideally convex and* $\operatorname{core} A \neq \emptyset$ *then A is semi-closed.*

(ii) *If f is lower ideally convex and* $\operatorname{core}(\operatorname{dom} f) \neq \emptyset$ *then f is semi-closed.*

Proof. (i) By [4, Cor. 1.3.8] we have that $\operatorname{core} A = \operatorname{int} A \neq \emptyset$. Since A is convex, by Proposition 1 we have that A is semi-closed.

(ii) Since f is convex and $\operatorname{core}(\operatorname{dom} f) \neq \emptyset$ we have that $\operatorname{core}(\operatorname{epi} f) \neq \emptyset$; indeed, if $\overline{x} \in \operatorname{core}(\operatorname{dom} f)$ then $(f(\overline{x}), f(\overline{x}) + \alpha) \in \operatorname{core}(\operatorname{epi} f)$ for every $\alpha > 0$ (see e.g. [5, Lem. 12 (i)]). By (i) we have that $\operatorname{epi} f$ is semi-closed, and so f is semi-closed. □

In [3] (see also [1]) one uses the next condition (extracted from $(C.Q_1)$ in [3]):

(H) X is a Fréchet space, $g : X \to \mathbb{R} \cup \{+\infty\}$ is convex, proper and semi-closed, $\mathbb{R}_+[\operatorname{dom} g - \overline{x}] = X$.

Note that for g convex and $\overline{x} \in \operatorname{dom} g$, g is continuous at $\overline{x}$ if and only if $\overline{x} \in \operatorname{core}(\operatorname{dom} g)$ and $\operatorname{int}(\operatorname{epi} g) \neq \emptyset$ (see e.g. [4, Cor. 2.2.10]). Because the condition $\mathbb{R}_+[\operatorname{dom} g - \overline{x}] = X$ in (H) is equivalent to $\overline{x} \in \operatorname{core}(\operatorname{dom} g)$, the role of the other two conditions in (H) is to ensure that $\operatorname{int}(\operatorname{epi} g) \neq \emptyset$. In this sense we have the following result.

Proposition 3 *Assume that (H) holds. Then* $\overline{x} \in \operatorname{int}(\operatorname{dom} g)$ *and* g *is continuous on* $\operatorname{int}(\operatorname{dom} g)$.

Proof. As seen above, $\overline{x} \in \operatorname{core}(\operatorname{dom} g)$. Consider $\overline{g}$ the lsc hull of g. Then $\operatorname{epi} \overline{g} = \operatorname{cl}(\operatorname{epi} g)$, and so $\overline{g}$ is a lsc convex function; moreover, because $\overline{g} \leq g$, $\overline{x} \in \operatorname{core}(\operatorname{dom} g) \subset \operatorname{core}(\operatorname{dom} \overline{g})$. Using for example [4, Th. 2.2.20] we obtain that $\operatorname{core}(\operatorname{dom} \overline{g}) = \operatorname{int}(\operatorname{dom} \overline{g})$ $(\neq \emptyset)$ and $\overline{g}$ is continuous on $\operatorname{int}(\operatorname{dom} \overline{g})$. By [4, Cor. 2.2.10] and the fact that g is semi-closed we have that $\emptyset \neq \operatorname{int}(\operatorname{epi} \overline{g}) = \operatorname{int}(\operatorname{epi} g)$, and so, using again [4, Th. 2.2.20] we have that g is continuous on $\operatorname{int}(\operatorname{dom} g) = \operatorname{core}(\operatorname{dom} g)$; of course, $\overline{x} \in \operatorname{int}(\operatorname{dom} g)$. The proof is complete. □

Related to [2, Th. 3.2] we have the following result.

Proposition 4 *Assume that* X *is a separated locally convex space and* $f : X \to \overline{\mathbb{R}}$ *is a proper function. If* $\partial f(\overline{x}) \neq \emptyset$ *then* $\overline{\operatorname{conv}} f(\overline{x}) = \overline{f}(\overline{x}) = f(\overline{x})$ *and* $\partial \overline{\operatorname{conv}} f(\overline{x}) = \partial \overline{f}(\overline{x}) = \partial f(\overline{x})$ *(in particular* f *is lsc at* $\overline{x}$*). Conversely, if* f *is convex and lsc at* $\overline{x} \in \operatorname{dom} f$*, then* $\partial \overline{f}(\overline{x}) = \partial f(\overline{x})$.

Proof. The first part is known (see, e.g., [4, Th. 2.4.1 (ii)]). Assume that f is convex and lsc at $\overline{x} \in \operatorname{dom} f$. Then $\overline{f}$ is convex (because f is so) and $\overline{f}(\overline{x}) = f(\overline{x})$. Since for $g \leq h$ with $g(\overline{x}) = h(\overline{x}) \in \mathbb{R}$ (for arbitrary $g, h : X \to \overline{\mathbb{R}}$) one has $\partial g(\overline{x}) \subset \partial h(\overline{x})$, we have that $\partial \overline{f}(\overline{x}) \subset \partial f(\overline{x})$. The converse inclusion being true by the first part, we get the conclusion. □

The next result is [2, Cor. 3.2]; taking $\overline{x} = 0$ it is [2, Th. 3.2].

Corollary 5 *Assume that* X *is a Fréchet space,* $f : X \to \overline{\mathbb{R}}$ *is a proper convex function and* $\overline{x} \in X$ *is such that* $\mathbb{R}_+[\operatorname{dom} f - \overline{x}]$ *is a closed linear subspace of* X*. Then* $\partial f(\overline{x}) \neq \emptyset$ *if and only if* f *is lsc at* $\overline{x}$.

Proof. Of course, as seen in Proposition 4, $\partial f(\overline{x}) \neq \emptyset$ implies that f is lsc at $\overline{x}$ (without any condition on X or f).

Assume now that f is lsc at $\overline{x}$. By Proposition 4 we have that $\partial f(\overline{x}) = \partial \overline{f}(\overline{x})$. Without loss of generality we take $\overline{x} = 0$. By hypothesis, $X_0 := \mathbb{R}_+ \operatorname{dom} f$ is a closed linear subspace of X, and so X_0 is a Fréchet space. Then $\operatorname{dom} f \subset \operatorname{dom} \overline{f} \subset X_0$. Since $0 \in \operatorname{core}(\operatorname{dom} f|_{X_0})$, we get $0 \in \operatorname{core}(\operatorname{dom} \overline{f}|_{X_0})$. Because $\overline{f}|_{X_0}$ is lsc, using [4, Th. 2.2.20], we obtain that $\overline{f}|_{X_0}$ is continuous at 0. Using now [4, Th. 2.4.12], we have that $\partial \overline{f}(0) \neq \emptyset$ and so $\partial f(0) \neq \emptyset$. □

Remark 2 Using Proposition 3, all the subdifferentiability and duality results in [2], [3], [1] (stated in Fréchet spaces) are consequences of the corresponding results using Moreau–Rockafellar type conditions.

Indeed, using Proposition 3 we have that condition $(C.Q_1)$ in [3] (considered in [1] as condition (2.1)) implies that $f, g : X \to \overline{\mathbb{R}}$ are proper convex functions such that g is continuous at some $\overline{x} \in \operatorname{dom} f \cap \operatorname{dom} g$, while condition $(C.Q_2)$ in [3] implies that $f : X \to \overline{\mathbb{R}}$, $g : Y \to \overline{\mathbb{R}}$ are proper convex functions, $h : X \to Y \cup \{+\infty\}$ is a Y_+-convex proper function such that g is continuous at $h(\overline{x})$, where $\overline{x} \in \operatorname{dom} f \cap h^{-1}(\operatorname{dom} g)$ $(\subset \operatorname{dom} h)$; condition (2.2) in [1] is obtained from condition $(C.Q_2)$ in [3] replacing h by T with the property that $T|_{\operatorname{dom} T} : \operatorname{dom} T \to Y$ is linear and continuous. So, one recognizes the usual Moreau–Rockafellar type conditions.

Other remarks:

Remark 3 (a) Note that in the proof of [2, Th. 3.1] one obtained that the function f is finite at 0 and bounded above on a neighborhood of 0, but, instead of concluding that f is continuous at 0, the conclusion was only that $\partial f(0) \neq \emptyset$.

(b) In [2, Rem. 3.1 3°)] one says: 'Note that for a convex set A of X one has $\mathbb{R}_+ A = X$ if, and only if, 0 is in the interior of A. So the condition "$\mathbb{R}_+[\operatorname{dom} f - \overline{x}] = X$" is equivalent to "$x$ is the interior of dom f" (for f convex , which is the case throughout the paper), condition which is much older than the Attouch-Brézis condition'.

In fact the condition $\mathbb{R}_+ A = X$ for A convex is equivalent to $0 \in \operatorname{core} A$, and this is far from being equivalent to $0 \in \operatorname{int} A$ for X an infinite dimensional tvs, even if X is a Fréchet space (which is not assumed here). One of the most general sufficient condition for $\operatorname{core} A = \operatorname{int} A$ is provided by [4, Cor. 1.3.8] (used in the proof of Proposition 2).

(c) Remark 2.1 6) in [3] asserts that "In [10], Laghdir studied the subdifferentiability of a convex semi-closed function, i.e. $\partial f(\overline{x}) \neq \emptyset$ whenever $\overline{x} \in \operatorname{dom} f$, $\mathbb{R}_+[\operatorname{dom} f - \overline{x}] = X$ and X is a Féchet space. It was proved in [10], that this result falses under the weakened condition: $\mathbb{R}_+[\operatorname{dom} f - \overline{x}]$ is a closed vector subspace". (Here [10] means our reference [2].) This assertion is quite strange because it contradicts [2, Cor. 3.2], that is Corollary 5 above.

(d) In Corollaries 3.5, 3.6, 3.7 of [3] one must assume that X is a Fréchet space, while in Corollary 4.5 one must assume that Y is a Fréchet space. On the other hand, in the results of Section 4 of [3] it is not needed to assume that X is a Fréchet space.

(e) I suppose that by cone one means convex cone in [1] (otherwise the relation $\leq_k$ defined on line 2 of page 1290 is not transitive.

(f) In condition (2.2) of [1] one assumes "$T : X \to Y \cup \{+\infty\}$ is linear and bounded"; this sounds strange because $Y \cup \{+\infty\}$ is not a (topological) vector space. Probably the authors wished to take $T_0 \in L(X_0, Y)$ with $X_0 \subset X$ a linear subspace and $T : X \to Y \cup \{+\infty\}$ defined by $T(x) := T_0(x)$ for $x \in X_0$, $T(x) := +\infty$ for $x \in X \setminus X_0$; said differently, the quoted text probably means that $T|_{\operatorname{dom} T} : \operatorname{dom} T \to Y$ is a continuous linear operator. Observe also that in Th. 2.5 and Th. 2.7 of [1] one speaks about (the adjoint?) T^* of such an operator T!

(g) The conclusion of [1, Th. 2.2] is equivalent to the existence of ($x^* = 0$ and) $y^* \in K^+$ such that $\gamma := \inf_{x \in X}\{f(x) + g(T(x))\} \leq f(x) + g(y) - \langle y, y^* \rangle + y^*(T(x))$ for all $x \in X$ and all $y \in Y$, where $y^*(+\infty) := +\infty =: g(+\infty)$. Since $\gamma = -(f + g \circ T)^*(0)$, the conclusion of [1, Th. 2.2] is equivalent the existence of $y^* \in K^+$ such that $(f + g \circ T)^*(0) \geq (f + y^* \circ T)^*(0) + g^*(y^*)$. Of course, this conclusion can be obtained from [4, Th. 2.8.10 (iii)] taking $x^* = 0$ in (2.66).

Remarks 1, 2 and 3 (f), (g) show that the claim of Alimohammady *et al.* in [1, Rem. 2.13] that "Our results extend and improve many known theorems of convex analysis and variational analysis as well as some results in functional analysis, the original forms of which can be found in [12, 7, 1, 2, 13, 9, 3, 11, 10, 8, 4–6] and the references cited therein" is exaggerated. In fact the results are particular cases of known results, one of them being even false.

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