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REDUCTION OF TWO-STAGE PROBABILISTIC OPTIMIZATION PROBLEMS WITH DISCRETE DISTRIBUTION OF RANDOM DATA TO MIXED INTEGER PROGRAMMING PROBLEMS [1]

Abstract. We consider models of two-stage stochastic programming with a quantile second stage criterion and optimization models with a chance constraint on the second stage objective function values. Such models allow to formalize requirements to reliability and safety of the system under consideration, and to optimize the system in extreme conditions. We suggest a method of equivalent transformation of such models under a discrete distribution of random parameters to mixed integer programming problems. The number of auxiliary Boolean variables in the latter problems equals to the number of possible scenarios for random data. The obtained mixed integer optimization problems are supposed to be solved by contemporary discrete optimization software. As an illustration, results of a numerical experiment with a small test problem are presented.

Key words. Stochastic programming, two-stage problems, quantile programming, chance constraints, deterministic equivalent, mixed integer programming.

1. Introduction

Commonly, in stochastic programming the mean value of a random performance indicator is optimized (Dantzig and Thapa (2003) [1], Ermoliev and Wets (1988) [2], Birge and Luveaux (1997) [3], Shapiro et al. (2009) [4], Ermoliev (1976) [5], Yudin (1979) [6]). Besides the mean value, other criteria can be used, e.g. the median or others quantiles (Malyshev and Kibzun (1987) [7], Kibzun and Kan (1996, 2009) [8, 9]). Such criteria are exploited, e.g., in aircraft control problems [7]. However the quantile optimization problem is a more complicated task than the mean value optimization, since the first one can be non-convex and discontinuous. For an approximate minimization of quantiles their upper bounds are often minimized. These bounds are obtained, e.g., by the so-called confidence method (Malyshev and Kibzun (1987) [7], Kibzun and Kan (1996, 2009) [8, 9], Kibzun and Naumov (1995) [10]), or by means of CVaR functions (Conditional Value at Risk, see Larsen et al. (2002) [11]). Other methods for approximate and global optimization of quantile functions (related to financial portfolio optimization) are reviewed in Wozabal et al. (2010) [12]. In Norkin (2010) [13], Ivanov and Naumov (2012) [14] some quantile optimization problems with discrete distribution of random parameters were reduced to mixed integer

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programming problems. Most general results in this direction were obtained in Kibzun et al. (2012) [15, 16]. Earlier, reduction of chance constrained problems, closely related to optimization of quantiles, to mixed integer programming problems was used in Korbut and Finkelstein (1969) [17], Sen (1992) [18], Ruszczyński (2002) [19], Benati and Rizzi (2007) [20], Luedtke et al. (2010) [21], Norkin and Boyko (2012) [22].

The two-stage models with the objective function in the form of a mathematical expectation belong to main settings in stochastic programming [1 – 6]. The two-stage stochastic programming model reflects decision making conditions, where firstly (at the first stage) under stochastic uncertainty a deterministic decision is made accounting for that in the future, when the situation becomes clear (at the second stage), additional optimal correcting decision is made. The first stage decision is optimized with respect to the sum of the first stage and the second stage objective functions.

On the other hand two-stage stochastic programming models are particular cases of the two-level cooperative decision making models, where the upper level objective function additively includes the mean value of the lower level objective functions.

For example, if applied to energy systems optimization, the two-stage stochastic programming problems can be interpreted as follows. The first stage decision variables may describe functioning of long term working powerful energy facilities, having relatively high starting and stopping costs. Second stage variables may describe energy production by mobile facilities, which have relatively low starting, stopping and running costs and can easily adapt to peak loads. As the objective function of the problem (under fixed energy prices) one can take average energy production costs on all kinds of facilities. The second stage objective function and variables may describe amount of energy produced on mobile facilities. The problem setting can include chance constraints, bounding below probability of satisfying peak energy demands. To feed a stochastic model of an energy system with data, one needs probabilistic scenarios of energy demands.

Instead of a mathematical expectation in two-stage stochastic programming, sometimes it makes sense to use quantile functions of the random optimal value of the second stage criterion function. This allows optimizing a stochastic system in extreme conditions. In Kibzun and Naumov (1995) [10], Bogdanov and Naumov (2003, 2006) [23, 24], Naumov (2010) [25], Naumov and Bobylev (2012) [26] the two-stage stochastic programming problems with a quantile objective function were attacked by means of the confidence method developed in Kibzun and Kan (1996, 2009) [8, 9].

In the present paper we reduce two-stage quantile stochastic programming problems and two-stage chance constrained stochastic programming problems under a discrete distribution of random data to mixed integer programming problems and thus extend results from Kibzun et al. (2012) [15, 16] to general two-stage quantile and probabilistic optimization problems with a discrete distribution. For solution of appearing auxiliary mixed integer problems contemporary discrete optimization software IBM ILOG CPLEX [27] is used.

The advantages of the proposed approach to solution of probabilistic and quantile stochastic programming problems consist in the following:

Probability and quantile functions allow to formalize requirements to reliability and safety of the optimized system, and to optimize stochastic systems in extreme conditions;

The approach admits presence both continuous and integer variable in the original problem setting;

The problem of non-convexity and discontinuity of probabilistic and quantile functions is shifted to methods of discrete optimization;

In case of a continuous distribution it always can be approximated by a discrete, e.g., empirical distribution with a sufficiently large number of realizations;

The obtained equivalent mixed integer programming problems can be solved by contemporary powerful discrete optimization software even for a very large number of problem variables and a number of scenarios for random data.

2. Notation and preliminaries

Let $(\Omega, \Sigma, \mathrm{P})$ be some probability space, $X : \Omega \to \mathrm{X}(\Omega) \subseteq \mathbb{R}^m$ be some vector random variable with values in the set $\mathrm{X}(\Omega)$, $U \subseteq \mathbb{R}^n$ be a set of admissible optimization strategies, $\Phi : U \times \mathrm{X}(\Omega) \to \mathbb{R}^1$ and $Q : U \times \mathrm{X}(\Omega) \to \mathbb{R}^1$ are Borel in $x \in \mathrm{X}(\Omega)$ functions for all $u \in U$. Let us define the mathematical expectation function

$$f_1(u) \overset{def}{=} E[\Phi(u, X)] = \int_{\Omega} \Phi(u, X(\omega)) d\mathrm{P}(\omega),$$

the probability function

$$P_{\varphi}(u) \overset{def}{=} \mathrm{P}\left\{ \Phi(u, X) \le \varphi,\ Q(u, X) \le 0 \right\} = \mathrm{P}\left\{ \omega \in \Omega :\ \Phi(u, X(\omega)) \le \varphi,\ Q(u, X(\omega)) \le 0 \right\},$$

and the quantile function

$$\varphi_\alpha(u) \overset{def}{=} \inf\left\{\varphi \in \mathbb{R}^1 : P_\varphi(u) \ge \alpha\right\} = \min\left\{\varphi \in \mathbb{R}^1 : \mathrm{P}\left\{\Phi(u,X) \le \varphi,\ Q(u,X) \le 0\right\} \ge \alpha\right\}, \qquad (1)$$

where α is a parameter, $0 < \alpha < P^*$; $\mathrm{P}\{\cdot\}$ denotes probability of the event in brackets (by definition, $\mathrm{P}\{\varnothing\} = 0$), E denotes the mathematical expectation;

$$P^* \overset{def}{=} \sup_{u \in U} \mathrm{P}\{\omega \in \Omega : Q(u, X(\omega)) \le 0\}.$$

Remark that function $P_\varphi(\cdot)$ is defined on the whole set U. Probability functions were studied in Raik (1971) [28], Yudin (1979) [2], Prekopa (1995) [29], Kibzun and Kan (1996, 2009) [8, 9], and quantile functions were investigated in Raik (1971) [30], Kibzun and Kan (1996, 2009) [8, 9]. In particular, if $\Phi(u,x), Q(u,x)$ are lower semi-continuous in u for each x, then the function $P_\varphi(u)$ is upper semi-continuous in pair variables (u,φ) (Raik (1972) [31]). Besides, the function $P_\varphi(u)$ is monotonous (non-decreasing) in φ, and is right continuous, so the infinum in the definition of $\varphi_\alpha(u)$ is achieved. The quantile function is a special case of the marginal (maximum) function, so under made assumptions it is lower semi-continuous in (u,α) (Aubin and Ekeland (1988) [32, Ch. 1, sec. 1, prop. 21]).

By continuity of the probability measure, it holds true:

$$\lim_{\varphi \to +\infty} \mathrm{P}\left\{\Phi(u,X) \le \varphi, Q(u,X) \le 0\right\} = \mathrm{P}\left\{Q(u,X) \le 0\right\}, \qquad (2)$$

$$\lim_{\varphi \to -\infty} \mathrm{P}\left\{\Phi(u,X) \le \varphi, Q(u,X) \le 0\right\} = 0\,. \qquad (3)$$

3. Quantile Optimization Problems

The basic (one stage) stochastic optimization problem has the form [1 – 6]:

$$f_1(u) \overset{def}{=} \mathrm{E}[\Phi(u,X)] \to \inf_{u \in U},$$

where the mean value of the random indicator $\Phi(u,X)$ is minimized over the set U of values of the deterministic variable u. Instead of the mean value the median of the random variable $\Phi(u,X)$ or its other α-quantile can be used as the objective function, $0 < \alpha < P^*$. The quantile optimization problem has the form (Kataoka (1963) [33], Raik (1971) [30], Kibzun and Kan (1996) [8]):

$$\varphi_\alpha(u) \stackrel{def}{=} \min\left\{\varphi \in \mathbb{R}^1 :\ \mathrm{P}\left\{\Phi(u,X) \le \varphi, Q(u,X) \le 0\right\} \ge \alpha\right\} \to \inf_{u\in U}\ . \qquad (4)$$

It is known that the latter is equivalent to the following problem (see Kibzun and Kan (1996), [8, sec. 4.2]):

$$\begin{aligned} &\varphi \to \inf_{\varphi\in\mathbb{R}^1, u\in U}, \\ &P_\varphi(u) = \mathrm{P}\left\{\Phi(u,X) - \varphi \le 0,\ Q(u,X) \le 0\right\} \ge \alpha. \end{aligned} \qquad (5)$$

In what follows we shall consider also other equivalent optimization problems, where the equivalence is understood in the following sense.

Consider a general mathematical programming problem.

Definition 1. *A mathematical programming problem is defined as a problem of minimization of the objective function* $\Phi : U \to \mathbb{R}^1$ *over some feasible set* U *of points (strategies, etc.), in formal notation:*

$$\Phi(u) \to \inf_{u\in U}. \qquad (6)$$

Elements $u \in U$ *are called feasible solutions of the problem. The set* U *may be empty, then it is said that the problem has no feasible solutions.*

Definition 2. *The greatest lower bound (finite or infinite)* φ^* *of* $\Phi(u)$ *over* U, $\varphi^* = \inf_{u\in U}\Phi(u)$, *is called the optimal value of the objective function of problem (6). If* $\varphi^* > -\infty$ *and there exists a feasible point* $u^* \in U$ *such that* $\varphi^* = \Phi(u^*)$, *then it is said that the optimal value of problem (6) is achieved and the point* u^* *is called an optimal solution of the problem. In this case we also write* $\varphi^* = \min_{u\in U}\Phi(u)$. *Otherwise, i.e. if* $\varphi^* = -\infty$ *or there is no a point* $u^* \in U$ *such that* $\varphi^* = \Phi(u^*)$, *then we say that the optimal value of the problem is not achieved.*

Remark that if there exists an optimal solution u^*, then the optimal value $\varphi^* = \Phi(u^*)$ of the objective function is achieved.

Definition 3. *Two problems of form (6) are called equivalent, if the following conditions are fulfilled:*

a) either both problems have feasible solutions (with finite values of their objective functions) or both have no such solutions;

b) if these problems have feasible solutions, then optimal values of their objective functions (finite or infinite) coincide;

c) if the optimal values of their objective functions are finite, then these values in both problems either are achieved or not achieved;

d) if optimal values are achieved, then having an optimal solution of one problem one can restore by an explicit algorithm an optimal solution of the other.

Remark that the introduced equivalence relation is transitive.

Proofs of equivalence of optimization problems will be based on establishing a special correspondence between feasible sets of the problems, namely, for each feasible point of one problem we point out a feasible point of the other with the same or less value of the objective function.

Lemma 1 (Kibzun et al. (2012) [15, Lemma 1]). *The two optimization problems*

$$\Phi_1(u) \to \inf_{u \in U_1} \quad \textit{and} \quad \Phi_2(u) \to \inf_{u \in U_2} \tag{7}$$

are equivalent in the sense of Definition 3, if there are known algorithms (mappings) $A_1 : U_1 \to U_2$ *and* $A_2 : U_2 \to U_1$ *such that for every feasible point of one problem they point out a feasible point of the other with the same or less value of the objective function, i.e. for any* $u_1 \in U_1$ *and* $u_2 \in U_2$ *it holds* $\Phi_2(A_1(u_1)) \le \Phi_1(u_1)$ *and* $\Phi_1(A_2(u_2)) \le \Phi_2(u_2)$.

Besides, if u_1^* *is an optimal solution of the first problem, then* $A_1(u_1^*)$ *is an optimal solution of the second problem, and* $\Phi_1(u_1^*) = \Phi_2(A_1(u_1^*))$. *Symmetrically, if* u_2^* *is an optimal solution of the second problem, then* $A_2(u_2^*)$ *is an optimal solution of the first one, and* $\Phi_2(u_2^*) = \Phi_1(A_2(u_2^*))$.

Similar to Lemma 1 method of proving equivalence was applied, e.g., in Mihalevich et al. (1987), [34, p. 131], Pagnoncelli et al. (2009) [35], Norkin (2010) [13], Norkin and Boyko (2012) [22], Ivanov and Naumov (2012) [14]. Thus, to prove equivalence of two problems, generally one need not assume existence of solutions of the original problem, the existence or not existence can be established in the course of solution of the equivalent problem. If it is known that one of problems has an optimal solution, then the sufficient conditions of equivalence of Lemma 1 can be a bit relaxed.

Lemma 2. *Suppose that one of problems* (7) *(e.g., the first) has an optimal solution* $u_1^* \in U_1$ *and there exists a feasible point* $u_2' \in U_2$ *of the other problem such that* $\Phi_2(u_2') \le \Phi_1(u_1^*)$. *Assume also that there known an algorithm (mapping)* $A_2 : U_2 \to U_1$ *such that for any feasible point of the second problem it points out a feasible point of the first problem with the same or less value of the objective function, i.e. for any* $u_2 \in U_2$ *it holds* $\Phi_1(A_2(u_2)) \le \Phi_2(u_2)$. *Then the other problem also has optimal solutions, and* $u_2' \in U_2$ *is one of such solutions; besides,* $A_2(u_2')$ *is also an optimal solution of the first problem, and the optimal values of both problems coincide. Thus the considered optimization problems are equivalent.*

Proof. By optimality of point u_1^* we have

$$\Phi_1(u_1^*) \ge \Phi_2(u_2') \ge \Phi_1(A_2(u_2)) \ge \Phi_1(u_1^*),$$

i.e. $\Phi_1(u_1^*) = \Phi_1(A_2(u_2')) = \Phi_2(u_2')$ and thus $A_2(u_2')$ is an optimal solution of the first problem. Let us show that u_2' is an optimal solution of the second problem. Suppose the opposite. Then there exists another feasible point $u_2'' \in U_2$ with less value of the objective function, $\Phi_2(u_2'') < \Phi_2(u_2')$. But by assumption, there exists a feasible point of the first problem, $A_2(u_2'') \in U_1$, such that $\Phi_1(A_2(u_2'')) \le \Phi_2(u_2'') < \Phi_2(u_2') \le \Phi_1(u_1^*)$, that contradicts to the optimality of point u_1^*. The proof is complete.

Assumption A. *Random variable* X *takes on a finite number of values, i.e.* $X \in \mathrm{X}(\Omega) = \{x_1, x_2, \ldots, x_K\}$ *with probabilities* $p_1 > 0, p_2 > 0, \ldots, p_K > 0$, $\sum_{k=1}^{K} p_k = 1$.

Assumptions B. *Functions* $\mu_1(u,x)$ *and* $\mu_2(u,x)$ *are known such that for any* $u \in U$, $x \in \mathrm{X}(\Omega)$ *it holds true:*

$$-\infty < \mu_1(u,x) \le \inf_{x \in \mathrm{X}(\Omega)} \{\Phi(u,x) : Q(u,x) \le 0\}, \tag{8}$$

$$-\infty < \mu_2(u,x) \le \max\left\{0, \inf_{x \in \mathrm{X}(\Omega)} Q(u,x)\right\}. \tag{9}$$

Condition (9) means that either $\mu_2(u,x) \le 0$ or $\mu_2(u,x) \le \inf_{x \in \mathrm{X}(\Omega)} Q(u,x)$.

Consider the following mixed integer programming problem:

$$
\begin{cases}
\varphi \to \inf\limits_{\varphi\in\mathbb{R}^1,\, u\in U,\, w_1,\dots,w_K}, \\
\Phi(u,x_k)-\varphi \leqslant (\Phi(u,x_k)-\mu_1(u,x_k))w_k, \ \ k=\overline{1,K}; \\
Q(u,x_k) \leqslant (Q(u,x_k)-\mu_2(u,x_k))w_k, \ \ k=\overline{1,K}; \\
\sum_{k=1}^{K} w_k p_k \leqslant 1-\alpha \ ; \\
w_k \in \{0,1\}, \quad k=\overline{1,K}.
\end{cases}
\tag{10}
$$

The following result on reduction of quantile optimization problem (4) to mixed integer programming problems was obtained in Kibzun et al. (2012) [15].

Theorem 1 [15]. *Under Assumptions A and B mixed integer optimization problem (10) is equivalent in the sense of Definition 3 to each of problems (4), (5). Besides, if* $(\varphi^*, u^*, w_1^*, ..., w_K^*)$ *is an optimal solution of problem (10), then* u^*, (φ^*, u^*) *are optimal solutions of problems (4), (5) respectively.*

Remark 1. In paper [15] instead of (8) a stronger condition was used: $-\infty < \mu_1(u,x) \le \inf\limits_{x\in X(\Omega)} \Phi(u,x)$ for any $u\in U$. However the proof and the result of Theorem 1 hold true under this weaker condition (8).

Remark 2. Problem (10) in general is a mixed integer nonlinear programming one even if functions Φ, Q are linear in u. However, due to certain freedom of choice of functions $\mu_1(\cdot)$, $\mu_2(\cdot)$, often the latter can be chosen in such a way that problem (10) becomes convex (even linear) mixed integer one. For solving problem (10) one can apply methods of pruned enumerative search, e.g. those implemented in mixed integer programming software, e.g., IBM ILOG CPLEX [27]. In a general case, solution of problem (10) is reduced to enumerative examining subsets $I_1 \subseteq \{1,...,K\}$ such that $\sum_{k\in I_1} p_k \le 1-\alpha$, solving corresponding mathematical programming subproblems of the form:

$$
\begin{cases}
\varphi \to \inf\limits_{\varphi\in\mathbb{R}^1, u\in U}, \\
\Phi(u,x_k) \le \varphi, \ \ k\in\{1,2,\dots,K\}\setminus I_1; \\
\mu_1(u,x_k) \le \varphi, \ \ k\in I_1; \\
Q(u,x_k) \le 0, \quad k\in\{1,2,\dots,K\}\setminus I_1; \\
\mu_2(u,x_k) \le 0, \ \ k\in I_1;
\end{cases}
\tag{11}
$$

and selecting a variant with the minimal value of the objective function. If the set U is convex and functions Φ, Q, μ_1, μ_2 are convex in $u \in U$, then (11) is a convex programming problem. If, besides, Φ, Q, μ_1, μ_2 are piece-wise linear in u, and U is given by linear constraints, then (11) is reduced to a linear programming problem.

Remark 3. The result of Theorem 1 can be extended to quantile optimization problems with functions of discrete maximum. Namely, let in problems (4), (5) functions Φ, Q have the form:

$$\Phi(u,x) = \max_{i \in I} \Phi_i(u,x), \qquad Q(u,x) = \max_{j \in J} Q_j(u,x), \tag{12}$$

where I, J are finite sets of indexes. Suppose, there exist functions $\mu_{1i}(u,x)$, $\mu_{2j}(u,x)$, satisfying at each $u \in U, x \in \mathrm{X}(\Omega)$ and $i \in I$, $j \in J$ the conditions:

$$-\infty < \mu_{1i}(u,x) \le \inf_{x \in \mathrm{X}(\Omega)} \Phi_i(u,x), \tag{13}$$

$$-\infty < \mu_{2j}(u,x) \le \max\left\{0, \inf_{x \in \mathrm{X}(\Omega)} Q_j(u,x)\right\}. \tag{14}$$

Let us compose the following mixed integer programming problem:

$$\begin{cases} \varphi \to \inf\limits_{\varphi \in \mathbb{R}^1,\, u \in U,\, w_1,\dots,w_K}, \\ \Phi_i(u,x_k) - \varphi \leqslant (\Phi_i(u,x_k) - \mu_{1i}(u,x_k))w_k, \quad i \in I, \ k = \overline{1,K}; \\ Q_j(u,x_k) \leqslant (Q_j(u,x_k) - \mu_{2j}(u,x_k))w_k, \quad j \in J, \ k = \overline{1,K}; \\ \sum_{k=1}^{K} w_k p_k \leqslant 1 - \alpha; \\ w_k \in \{0,1\}, \ k = \overline{1,K}. \end{cases} \tag{15}$$

Corollary 1 [15]**.** *Problems (4), (5) under conditions (12), (13), (14) are equivalent to mixed integer programming problem (15).*

Remark 4. Although problems (10), (15) are nonlinear in (u, w_k), but due to certain freedom of choice of functions $\mu_1, \mu_2, \mu_{1i}, \mu_{2j}$ one can select them in such a way that coefficients at w_k do not depend on u and thus problems (10), (15) become linear in w_k. The linearity in w_k allows to use a continuous relaxation of problems (10), (15) to obtain lower bounds for optimal values of these problems. And if, besides, functions Φ, Q are maximums of linear functions and the set U is polyhedral, then we come to problem (15) of mixed integer linear programming problem, that can be solved by standard optimization software, e.g., by IBM ILOG CPLEX [27]. The following examples illustrate these opportunities.

Example 1 [15]**.** Let us make the following **Assumptions B1', B2'** on the ranges of problem functions. Let there are known a constant μ and functions $M(x)$ and $N(x)$ such that functions Φ, Q from (4), (10) satisfy the conditions:

B1': $-\infty < \mu \le \inf_{u\in U,\, x\in \mathrm{X}(\Omega)} \{\Phi(u,x) : Q(u,x) \le 0\},$

and for all $x \in \mathrm{X}(\Omega)$ it holds true

B2': $\sup_{u\in U} \Phi(u,x) \le M(x) < \infty, \quad \sup_{u\in U} Q(u,x) \le N(x) < \infty.$

Let us consider functions

$$\mu_1(u,x) = \Phi(u,x) - M(x) + \mu, \qquad \mu_2(u,x) = Q(u,x) - N(x). \tag{16}$$

They satisfy Assumption B. Indeed, for any $u \in U$ by B1', B2' it holds

$$\mu_1(u,x) \le \mu \le \inf_{u\in U,\, x\in \mathrm{X}(\Omega)} \{\Phi(u,x) : Q(u,x) \le 0\} \le \inf_{x\in \mathrm{X}(\Omega)} \{\Phi(u,x) : Q(u,x) \le 0\},$$

$$\mu_2(u,x) = Q(u,x) - N(x) \le 0.$$

Such functions $M(x)$, $N(x)$ and the number μ certainly exist if functions $\Phi(u,x)$, $Q(u,x)$ are continuous in u for each $x \in \mathrm{X}(\Omega)$ and U is a compact set.

Substituting (16) into (10), we come to the following linear in Boolean variables mixed integer programming problem, that is equivalent under assumptions A and B1', B2' to original problems (4), (5):

$$\begin{cases} \varphi \to \inf\limits_{\varphi\in\mathbb{R}^1,\, u\in U,\, w_1\in\{0,1\},\dots,w_K\in\{0,1\}}, \\ \sum_{k=1}^{K} p_k w_k \le 1-\alpha; \\ \Phi(u,x_k) - \varphi \le \left(M(x_k) - \mu\right) w_k, \quad k = 1,2,\dots,K; \\ Q(u,x_k) \le N(x_k) w_k, \quad k = 1,2,\dots,K. \end{cases} \tag{17}$$

Example 2 [15]**.** Consider a particular case of problem (4), where functions $\Phi(u,x)$ and $Q(u,x)$ are separable and have the form:

$$\Phi(u,x) = \Phi_1(u) + \Phi_2(x), \quad Q(u,x) = Q_1(u) + Q_2(x).$$

Suppose that there exists a constant μ_3 such that

$$-\infty < \mu_3 \le \min\left\{ \inf_{x \in X(\Omega)} \Phi_2(x), \inf_{x \in X(\Omega)} Q_2(x) \right\}.$$

Let us take

$$\mu_1(u,x) = \Phi_1(u) + \mu_3, \qquad \mu_1(u,x) = Q_1(u) + \mu_3. \tag{18}$$

These functions satisfy Assumption B. Indeed,

$$\mu_1(u,x) = \Phi_1(u) + \mu_3 \le \Phi_1(u) + \inf_{x \in X(\Omega)} \Phi_2(x) \le$$
$$\le \inf_{x \in X(\Omega)} \{\Phi_1(u) + \Phi_2(x)\} \le \inf_{x \in X(\Omega)} \{\Phi(u,x) : Q(u,x) \le 0\}.$$

The second condition B is checked in a similar way. Substituting (18) into (10), we come to one more mixed integer programming problem, that is equivalent under made assumptions to problem (4):

$$\begin{cases} \varphi \to \inf\limits_{\varphi \in \mathbb{R}^1,\, u \in U,\; w_1,\dots,w_K}, \\ \Phi_1(u) + \Phi_2(x_k) - w_k(\Phi_2(x_k) - \mu_3) \leqslant \varphi, \quad k = \overline{1,K}; \\ Q_1(u) + Q_2(x_k) - w_k(Q_2(x_k) - \mu_3) \leqslant 0, \quad k = \overline{1,K}; \\ \sum_{k=1}^{K} w_k p_k \leqslant 1 - \alpha; \\ w_k \in \{0,1\}, \quad k = \overline{1,K}. \end{cases} \tag{19}$$

Remark that if functions $\Phi_1(u)$ and $Q_1(u)$ are convex in $u \in U$, then problem (19) becomes a mixed integer convex programming problem.

Example 3. In Ivanov and Naumov (2012) [14] there was considered one more particular case of problem (4), where functions $\Phi(u,x)$ and $Q(u,x)$ are piece-wise linear convex maximum functions of the form:

$$\Phi(u,x) = \max_{i=1,\dots,k_1} \left(A_{1i}u + B_{1i}x + b_{1i} \right),$$
$$Q(u,x) = \max_{j=1,\dots,k_2} \left(A_{2j}u + B_{2j}x + b_{2j} \right), \tag{20}$$

where $u \in [0,\bar{u}]^n$, $x \in \mathbb{R}^m$, $A_{1i}, A_{2j}, B_{1i}, B_{2j}$ are rows of matrices A_1, A_2, B_1, B_2 respectively; b_{1i}, b_{2j} are components of vectors b_1, b_2. Let the constant μ_3 satisfies the following inequality:

$$\mu_3 \leqslant \min\left\{ \min_{i=\overline{1,k_1},k=\overline{1,K}} B_{1i}x_k,\ \min_{j=\overline{1,k_2},k=\overline{1,K}} B_{2j}x_k \right\}.$$

In the mentioned paper it was shown that in this case for a polyhedral convex set U problem (5) is equivalent to the next mixed integer programming problem:

$$\begin{cases} \varphi \to \inf\limits_{\varphi\in\mathbb{R}^1,\, u\in U,\, w_1,\dots,w_K}, \\ A_{1i}u + \mu_3 + w_k(B_{1i}x_k - \mu_3) + b_{1i} \leqslant \varphi,\ \ i=\overline{1,k_1},\ \ k=\overline{1,K}; \\ A_{2j}u + \mu_3 + w_k(B_{2j}x_k - \mu_3) + b_{2j} \leqslant 0,\ \ j=\overline{1,k_2},\ \ k=\overline{1,K}; \\ \sum_{k=1}^{K} p_k w_k \geqslant \alpha, \\ w_k \in \{0,1\},\ \ k=\overline{1,K}. \end{cases} \tag{21}$$

This problem is a particular case of (15).

Remark 5. For checking the correctness of setting of problems (4), (5), by (2), it is necessary to find

$$P^* = \sup_{u\in U} \mathrm{P}\left\{ Q(u,X) \le 0 \right\}. \tag{22}$$

The probability maximization problem (22) under assumption (9) is reduced to the next mixed integer programming problem (see Kibzun et al. (2012) [15], and also Norkin and Boyko (2012) [22], Ivanov and Naumov (2012) [14]):

$$\sum_{k=1}^{K} p_k w_k \to \sup_{u\in U,\, w_1\in\{0,1\},\dots,w_K\in\{0,1\}},$$
$$Q(u,x_k) \le (Q(u,x_k) - \mu_2(u,x_k))w_k,\ k=1,\dots,K. \tag{23}$$

4. Two-stage quantile stochastic programming problems

Two-stage stochastic programming problem with the expectation criterion has the following form (see [1, Ch. 12], [3, sec. 3.4], [4, Ch. 2], [5, Ch. IV, § 3], [6, гл. 6]):

$$f_1(u) + \mathrm{E}\left[\Phi(u, X)\right] \to \min_{u \in U},$$

where

$$\Phi(u,x) = \begin{cases} f_2^*(u,x) = \inf\limits_{v \in W(u,x)} f_2(u,v,x), & W(u,x) \neq \varnothing, \\ +\infty, & W(u,x) = \varnothing; \end{cases} \qquad (24)$$
$$W(u,x) = \left\{v \in V(x) : Q_2(u,v,x) \le 0\right\};$$

$U \subseteq \mathbb{R}^n$ is the set of feasible strategies of the first stage; $u \in U$ is a particular deterministic strategy of the first stage; $f_1(u)$ is the first stage objective function; X is a random parameter taking (in the present paper) a finite set of values $\mathrm{X}(\Omega) = \left\{x_1, \ldots, x_K\right\}$ with probabilities $p_1, \ldots, p_K$ (Assumption A); $V(x) \subset \mathbb{R}^m$ is the set of feasible strategies at the second stage for $X = x$; $v \in V(x)$ is a particular strategy (correction) at the second stage; $f_2(u,v,x)$ is the objective function of the second stage; $Q_2(u,v,x)$ is the constraint function at the second stage; E denotes the mathematical expectation.

Remark that the feasible set of the first stage may be narrower than U, because it includes an implicit constraint $\mathrm{E}\left[\Phi(u,X)\right] < +\infty$. Nonintegrability of $\Phi(u,X)$ may arise both from nonintegrability of random parameters of the problem and due to possible infinite values of $\Phi(u,X)$.

In the two-stage problem instead of mean value $\mathrm{E}\left[\Phi(u,X)\right]$ the median or other quantiles of $\Phi(u,X)$ can be used (Kibzun and Naumov (1995) [10], Bogdanov and Naumov (2003, 2006) [23, 24], Naumov (2010) [25]).

Define the probability and quantile functions:

$$P_\varphi(u) = \mathrm{P}\left\{\Phi(u,X) \le \varphi\right\}, \quad \varphi_\alpha(u) = \min_\varphi \left\{\varphi : P_\varphi(u) \ge \alpha\right\}, \quad 0 < \alpha < 1.$$

Then two-stage quantile optimization problem has the form:

$$\left\{f_1(u) + \varphi_\alpha(u)\right\} \to \min_{u \in U}. \qquad (25)$$

It is equivalent to the following problem:

$$f_1(u) + \varphi \to \min_{u \in U,\ \varphi \in \mathbb{R}^1}, \qquad (26)$$
$$\mathrm{P}\left\{\Phi(u,X) \le \varphi\right\} \ge \alpha.$$

Problem (26) with a fixed parameter φ, e.g., $\varphi = 0$, and $\Phi(u,x)$ from (24) has its own value and is called two-stage chance constrained problem (i.e. with a probabilistic constraint on the second stage objective function values).

Lemma 3. *Problems (25) and (26) are equivalent.*

Proof. Let u' be a feasible solution of problem (25), then $\varphi' = \varphi_\alpha(u')$ is finite. By properties of quantile functions

$$\mathrm{P}\left\{\Phi(u', X) \le \varphi'\right\} = \mathrm{P}\left\{\Phi(u', X) \le \varphi_\alpha(u')\right\} \ge \alpha.$$

Thus (φ', u') is a feasible solution of problem (26) with the same value $f_1(u') + \varphi_\alpha(u')$ of the objective function.

Conversely, let (φ', u') be a feasible solution of problem (26), i.e. $u' \in U$, $\mathrm{P}\left\{\Phi(u', X) \le \varphi'\right\} \ge \alpha > 0$ and $-\infty < f_1(u') + \varphi' < +\infty$. By continuity of the probability measure $\lim_{\varphi \to -\infty} \mathrm{P}\left\{\Phi(u', X) \le \varphi\right\} = 0$. So $-\infty < \varphi_\alpha(u') \le \varphi'$ and hence for the objective function values of problems (26), (25) we have the inequality $f' = f_1(u') + \varphi' \ge f_1(u') + \varphi_\alpha(u') > -\infty$, i.e. u' is a feasible solution for problem (25). Thus, by Lemma 1, problems (25) and (26) are equivalent. The proof is complete.

Consider also the following problem:

$$\begin{aligned} & f_1(u) + \varphi \to \min_{u \in U, \varphi \in \mathbb{R}^1, v_1 \in V(x_1), \ldots, v_K \in V(x_K)}, \\ & \sum_{k=1}^{K} p_k I\left\{ f_2(u, v_k, x_k) \le \varphi,\, Q_2(u, v_k, x_k) \le 0 \right\} \ge \alpha, \end{aligned} \qquad (27)$$

where $I\{\cdot\}$ is the indicator of conditions in the brackets, equal to one if all conditions in the brackets are fulfilled, and equaled to zero otherwise.

Assumption C *(on existence of solutions of the second stage problems). If the set* $W(u,x) = \left\{v \in V(x) : Q_2(u,v,x) \le 0\right\}$ *is nonempty, then* inf *in (24) is achieved, and thus there exists* $v(u,x) \in W(u,x)$ *such that* $\Phi(u,x) = f_2(u, v(u,x), x)$.

This assumption is fulfilled, e.g., if for any $u \in U$, $x \in X(\Omega)$ function $Q_2(u,v,x)$ is lower semicontinuous in v on a compact set $V(x)$.

Lemma 4. *Under assumptions A, C problems (26) and (27) are equivalent.*

Proof. Let (φ', u') be a feasible solution for problem (26), i.e. $u' \in U$, $\mathrm{P}\{\Phi(u', X) \le \varphi'\} \ge \alpha$, where function $\Phi(u, x)$ is defined in (24). Denote I'_α the set of indexes k such that $\Phi(u', x_k) \le \varphi'$. Obviously, $\sum_{k \in I'_\alpha} p_k \ge \alpha$. For $k \in I'_\alpha$ the set $W(u', x_k)$ is nonempty and by Assumption C there exist $v'_k \in V(x_k)$ such that $f_2(u', v'_k, x_k) = \Phi(u', x_k) \le \varphi'$ and $Q_2(u', v'_k, x_k) \le 0$. For $k \notin I'_\alpha$ choose an arbitrary $v'_k \in V(x_k)$. Obviously, the set $(\varphi', u', v'_1, \ldots, v'_K)$ is feasible for problem (27) with the same value $f' = f_1(u') + \varphi'$ of the objective function.

Conversely, let the set $(\varphi', u', \{v'_k\}_1^K)$ be feasible for problem (27). Let us show that (φ', u') is a feasible solution for problem (26). Denote I'_α the set of indexes k such that $v'_k \in V(x_k)$, $f_2(u', v'_k, x_k) \le \varphi'$ and $Q_2(u', v'_k, x_k) \le 0$. Obviously, $\sum_{k \in I'_\alpha} p_k \ge \alpha$. For $k \in I'_\alpha$ it holds $\Phi(u', x_k) \le f_2(u', v'_k, x_k) \le \varphi'$, so $\mathrm{P}\{\Phi(u', X) \le \varphi'\} \ge \sum_{k \in I'_\alpha} p_k \ge \alpha$, and thus (f', φ', u') is a feasible solution for problem (26) with the same value $f' = f_1(u') + \varphi'$ of the objective function. Hence, by Lemma 1 problems (26) and (27) are equivalent. The proof is complete.

By transitivity of the equivalence relation all problems (25), (26) and (27) are equivalent under Assumptions A, C. Problem (27) is of the same type as (5), for which the possibility of reduction to mixed integer equivalent (15) under Assumptions A, B was shown in Theorem 1. Let us make the following additional assumptions regarding problem (25) and, hence, relative to (27).

Assumptions D. There are known functions $\mu_1(u, v, x)$ and $\mu_2(u, v, x)$ such that for any $u \in U$, $x \in \mathrm{X}(\Omega)$, $v \in V(x)$ it holds true:

D1. $-\infty < \mu_1(u, v, x) \le \inf_{x \in \mathrm{X}(\Omega)} \inf_{v \in V(x)} \{ f_2(u, v, x) : Q_2(u, v, x) \le 0 \}$;

D2. $-\infty < \mu_2(u, v, x) \le \max \left\{ 0, \inf_{x \in \mathrm{X}(\Omega)} \inf_{v \in V(x)} Q_2(u, v, x) \right\}$.

A bit weaker assumption is the following one, although assuming existence of a solution of problem (25).

Assumption D'. Conditions D1, D2 are fulfilled not for all $u \in U$, but only for some optimal solution u^* of problem (25) and for all $x \in \mathrm{X}(\Omega)$, $v \in V(x)$.

Denote $w_1, \ldots, w_K$ a set of Boolean variables. Consider the following mixed integer programming problem:

$$\begin{cases} f_1(u) + \varphi \to \min_{u \in U,\, \varphi \in \mathbb{R}^1, v_1,\ldots,v_K;\, w_1,\ldots,w_K}, \\ \sum_{k=1}^{K} p_k w_k \le 1-\alpha, \\ f_2(u, v_k, x_k) - \varphi \le \left(f_2(u, v_k, x_k) - \mu_1(u, v_k, x_k)\right) w_k, \quad k = 1, 2, \ldots, K; \\ Q_2(u, v_k, x_k) \le \left(Q_2(u, v_k, x_k) - \mu_2(u, v_k, x_k)\right) w_k, \quad k = 1, 2, \ldots, K; \\ v_k \in V(x_k), \quad w_k \in \{0, 1\}, \quad k = 1, 2, \ldots, K. \end{cases} \tag{28}$$

The main result of the present paper is the following theorem on a possibility of reduction of two-stage quantile optimization problem (25) with a discrete distribution of random data to mixed integer programming problem (28).

Theorem 2. *Under assumptions A, C, D problems (25) –* (28) *are equivalent in the sense of Definition 3. If there exist solution of problem (25) and assumptions A, C, D' are fulfilled, then problems (25) – (28) are also equivalent. From here it follows that if* $(\varphi^*, u^*, v_1^*, \ldots, v_K^*, w_1^*, \ldots, w_K^*)$ *is an optimal solution of problem (28), then* u^* *is an optimal solution of problem (25),* (φ^*, u^*) *is an optimal solution of problem (26), and* $(\varphi^*, u^*, v_1^*, \ldots, v_K^*)$ *is an optimal solution of problem (27).*

Proof. The equivalence of problems (25) – (27) was proved in Lemmas 3, 4. Let us prove equivalence of problems (27) an (28) by means of Lemma 1.

Under assumptions A, C, D1, D2 let us prove the first statement of the theorem. Let $(\varphi', u', v_1', \ldots v_K')$ be a feasible solution of problem (27) with value $f_1(u') + \varphi'$ of the objective function. Let us define Boolean variables:

$$w_k' = \begin{cases} 0, & f_2(u', v_k', x_k) - \varphi' \le 0, \ Q_2(u', v_k', x_k) \le 0; \\ 1, & \textit{otherwise.} \end{cases}$$

Let us show that $(\varphi', u', v_1', \ldots v_K', w_1', \ldots, w_K')$ is a feasible solution of problem (28). Denote I_0' and I_1' sets of indexes k such that $w_k' = 0$ and $w_k' = 1$, respectively. Then from constrain inequality (27) it follows

$$\sum_{k=1}^{K} p_k w'_k = 1 - \sum_{k=1}^{K} p_k (1 - w'_k) = 1 - \sum_{k \in I'_0} p_k =$$

$$= 1 - \sum_{k=1}^{K} p_k I\{ f_2(u', v'_k, x_k) - \varphi' \le 0, Q_2(u', v'_k, x_k) \le 0\} \le 1 - \alpha.$$

Thus the constraint $\sum_{k=1}^{K} p_k w'_k \le 1-\alpha$ in (28) is fulfilled. Let us check the two next groups of inequalities in (28). For $k \in I'_0$ these constraints take on the form: $f_2(u', v'_k, x_k) - \varphi' \le 0$, $Q_2(u', v'_k, x_k) \le 0$; they are fulfilled by the definition of the index set I'_0. For $k \in I'_1$ these constraints take on the form: $\mu_1(u', v'_k, x_k) - \varphi' \le 0$, $\mu_2(u', v'_k, x_k) - \varphi' \le 0$. Let us check their fulfillment. By feasibility of $(\varphi', u', v'_1, ... v'_K)$ from (27) it follows that $I'_0 \ne \varnothing$ and there exists $x_{k'}$, $v'_{k'} \in V(x_{k'})$ such that $f_2(u', v'_{k'}, x_{k'}) - \varphi' \le 0$, $Q_2(u', v'_{k'}, x_{k'}) \le 0$. By assumption D1 for $k \in I'_1$ it takes place

$$\mu_1(u', v'_k, x_k) \le \inf_{x \in \mathrm{X}(\Omega)} \inf_{v \in V(x)} \left\{ f_2(u', v, x) : Q_2(u', v, x) \le 0 \right\} \le f_2(u', v'_{k'}, x_{k'}) \le \varphi'$$

and thus the inequality $\mu_1(u', v'_k, x_k) - \varphi' \le 0$ holds true. By assumption D2 it takes place either $\mu_2(u', v'_k, x_k) \le 0$ (that is required) or

$$\mu_2(u', v'_k, x_k) \le \inf_{x \in \mathrm{X}(\Omega)} \inf_{v \in V(x)} Q_2(u', v, x) \le Q_2(u', v'_{k'}, x_{k'}) \le 0 .$$

Thus, the second inequality $\mu_2(u', v'_k, x_k) \le 0$ is also holds. Hence, $(\varphi', u', v'_1, ... v'_K, w'_1, ..., w'_K)$ is a feasible solution of problem (28) with same value $f_1(u') + \varphi'$ of the objective function.

Let now $(\varphi', u', v'_1, ... v'_K, w'_1, ..., w'_K)$ be a feasible solution of problem (28) with objective function value $f_1(u') + \varphi'$. Let us check that $(\varphi', u', v'_1, ... v'_K)$ is a feasible solution of problem (27). Denote I'_0 the set of indexes k such that $w'_k = 0$. By feasibility, it holds true $\sum_{k=1}^{K} p_k w'_k \le 1-\alpha$, which is equivalent to the inequality $\sum_{k \in I'_0} p_k \ge \alpha$. For $k \in I'_0$ it takes place $w'_k = 0$, and hence from (28) we have: $f_2(u', v'_k, x_k) - \varphi' \le 0$, $Q_2(u', v'_k, x_k) \le 0$. Now let us check fulfillment of the probabilistic constraint in (27)

$$\sum_{k=1}^{K} p_k I\{ f_2(u',v'_k,x_k) - \varphi' \le 0, Q_2(u',v'_k,x_k) \le 0\} \ge$$
$$\ge \sum_{k\in I'_0} p_k I\{ f_2(u',v'_k,x_k) - \varphi' \le 0, Q_2(u',v'_k,x_k) \le 0\} = \sum_{k\in I'_0} p_k \ge \alpha.$$

Thus, $(\varphi',u',v'_1,...v'_K)$ is a feasible solution of problem (27) with the same value $f_1(u')+\varphi'$ of the objective function. Now the first statement of the theorem follows from Lemma 1. The second statement is proved similarly, but on the basis of Lemma 2. The proof is complete.

Assumptions D admit large freedom in choice of functions $\mu_1(\cdot)$, $\mu_2(\cdot)$. Let us choose them in a way to simplify problem (28).

Assumptions E (on the ranges of the second-stage problem functions). Let there exist (known) a constant μ and functions $M_2(x)$ and $N_2(x)$ such that

E1. $\inf_{u\in U,\ x\in X(\Omega), v\in V(x)} \left\{ f_2(u,v,x):\ Q_2(u,v,x) \le 0 \right\} \ge \mu > -\infty$.

E2. $\sup_{u\in U, v\in V(x)} f_2(u,v,x) \le M_2(x) < \infty$, $\quad \sup_{u\in U, v\in V(x)} Q_2(u,v,x) \le N_2(x) < \infty$.

Such functions $M_2(x)$, $N_2(x)$ and the number μ certainly exist if functions $f_2(u,v,x)$, $Q_2(u,v,x)$ are continuous in $(u,v) \in U \times V(x)$ for each $x \in \mathrm{X}(\Omega)$, sets $U, V(x)$ are compact and the set $\mathrm{X}(\Omega)$ is finite.

Let us take the functions $\mu_1(\cdot)$, $\mu_2(\cdot)$ in the form:

$$\mu_1(u,v,x) = f_2(u,v,x) - M_2(x) + \mu, \qquad \mu_2(u,v,x) = Q_2(u,v,x) - N_2(x).$$

They satisfy Assumptions D. Indeed, for any $u \in U$ by assumptions E1, E2 it holds true:

$$\mu_1(u,v,x) == f_2(u,v,x) - M_2(x) + \mu \le \mu \le \inf_{u\in U,\ x\in X(\Omega), v\in V(x)} \left\{ f_2(u,v,x):\ Q_2(u,v,x) \le 0 \right\},$$

$$\mu_2(u,v_2,x) = Q_2(u,v,x) - N_2(x) \le 0.$$

Substituting these $\mu_1(u,v,x) = f_2(u,v,x) - M_2(x) + \mu$ and $\mu_2(u,v,x) = Q_2(u,v,x) - N_2(x)$ into (28), we come to the following linear in Boolean variables mixed integer programming problem [16], equivalent to (25) (under assumptions E):

$$\begin{cases} f_1(u) + \varphi \to \min_{u\in U,\ \varphi\in\mathbb{R}^1;\ v_1,\dots,v_K;\ w_1,\dots,w_K}, \\ \sum_{k=1}^{K} p_k w_k \le 1-\alpha, \\ f_2(u,v_k,x_k) - \varphi \le \left(M(x_k)-\mu\right) w_k, \quad k=1,2,\dots,K; \\ Q_2(u,v_k,x_k) \le N(x_k) w_k, \quad k=1,2,\dots,K; \\ v_k \in V(x_k), \quad w_k \in \{0,1\}, \quad k=1,2,\dots,K. \end{cases} \tag{29}$$

Remark that if functions $f_2(u,v,x)$, $Q_2(u,v,x)$ are piece-wise linear in (u,v), e.g., are maximums of linear in (u,v) functions, and sets U, $V(x)$ are given by linear constraints, then problem (29) is reduced to linear mixed integer programming problem.

Remark 6. Let us consider problem (26) with fixed φ, e.g., $\varphi \equiv 0$; in this case it is called a two-stage chance constrained stochastic programming problem. Consider also problems (27) – (29) with the same fixed φ. In such a case Lemma 4 and Theorem 2 hold true. Thus two-stage chance constrained problem (26) with $\varphi \equiv 0$ under discrete distribution of random data is equivalently reduced to mixed integer programming problem (29) with $\varphi \equiv 0$.

5. Numerical experiments

Naumov and Bobylev (2012) [26] considered a numerical example of a two-stage linear quantile optimization problem of form (25), where at the first stage the following problem is solved:

$$c_1^T u + \varphi_\alpha(u) \to \min_{u\in U}, \qquad U = \{u : A_1 u \ge b_1, u \ge 0\},$$

and the second stage the problem has the form:

$$\Phi(u,x) = \min_{y\in Y(u,x)} c_2^T v, \qquad Y(u,x) = \{v : v \ge 0, B_2 v \ge x - A_2 u\}.$$

Parameters of the problems take on the following values: $\alpha = 0.5$,

$$c_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix},\ c_2 = \begin{pmatrix} 0.1 \\ 0.36 \end{pmatrix},\ B_2 = \begin{pmatrix} -0.875 & 1.8 \\ 1 & -1 \end{pmatrix},\ A_2 = \begin{pmatrix} 1.1 & 1.7 \\ 2.8 & 2.4 \end{pmatrix},\ A_1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},\ b_1 = \begin{pmatrix} -10 \\ -10 \end{pmatrix}.$$

The random (right hand side) vector X takes on the following values with corresponding probabilities:

x_k	$\begin{pmatrix}2.5\\4\end{pmatrix}$	$\begin{pmatrix}1.5\\2.1\end{pmatrix}$	$\begin{pmatrix}2.3\\3\end{pmatrix}$	$\begin{pmatrix}3.2\\1\end{pmatrix}$	$\begin{pmatrix}7\\4\end{pmatrix}$	$\begin{pmatrix}3\\4\end{pmatrix}$	$\begin{pmatrix}2\\5\end{pmatrix}$	$\begin{pmatrix}8\\9\end{pmatrix}$
p_k	0.05	0.10	0.10	0.25	0.15	0.15	0.15	0.05

An approximate solution $u' = (u_1', u_2')$, found by the algorithm from Naumov and Bobylev (2012) [26], has the form:

u_1'	u_2'	$\Phi_\alpha(u')$	$c_1^T u' + \Phi_\alpha(u')$
0	1.56	0.1	1.67

Let us solve this problem by reducing it to a mixed integer linear programming problem of form (29):

$$c_1^T u + \varphi \to \min_{0 \le u_1, u_2 \le \bar{U};\ \varphi \ge \mu;\ 0 \le v_1, \dots, v_K \le \bar{V};\ w_1 \in \{0,1\}, \dots, w_K \in \{0,1\}},$$

$$\sum_{k=1}^{K} p_k w_k \le 1 - \alpha, \qquad A_1 u \ge b_1,$$

$$c_2^T v_k - \varphi \le \left(M_2(x_k) - \mu\right) w_k, \ \ k = 1, 2, \dots, K;$$

$$-A_{2i} u - B_{2i} v_k + x_{ki} \le N_2(x_k) w_k, \qquad i = 1, 2, \qquad k = 1, 2, \dots, K;$$

where A_{2i}, B_{2i} are i-th rows of matrices A_2, B_2 respectively; $x_k = \{x_{ki}, i = 1, 2\}$; $\mu = 0$; $\bar{U} = \bar{V} = 100$;

$$M_2(x_k) \equiv M = \bar{V}(c_{21} + c_{22}) = 46;$$

$$N_2(x_k) \equiv N = \max_{i=1,2} \left(\bar{U} \sum_{j=1,2} \left|(A_2)_{ij}\right| + \bar{V} \sum_{j=1,2} \left|(B_2)_{ij}\right| + \max_{k=1,\dots,K} |x_{ki}| \right) = 303.5.$$

The latter problem contains 11 continuous variables, 8 Boolean variables, one Boolean inequality constraint, two continuous inequality constraints, and 24 mixed inequality constraints. The exact solution of this problem $u^* = (u_1^*, u_2^*)$ was obtained by software IBM ILOG CPLEX V12.1 [27] (with default parameters, on an ordinary personal computer AMD Athlon 64 3200+, 200MHz Bus Frequency, 1.50 GB DDR1 RAM) within 0.05 seconds:

$$u_1^* = 0, \ u_2^* = 1.5331, \ \varphi^* = 0.1188, \ c_1^T u^* + \varphi^* = 1.6518.$$

Obviously, the obtained optimal solution u^* is better that an approximate one u'.

6. Conclusions

Quantile functions can be used instead of mean value functions in stochastic programming problems. They describe a stochastic system behavior in extreme conditions. This is a popular approach in contemporary financial optimization. However quantile functions are nonconvex as a rule. So the quantile optimization problems are solved either by means of heuristic approximate methods or by global optimization methods. In the present paper we study two-stage quantile optimization problems with discrete distribution of random data. It appears that as in case of one-stage quantile optimization problems, they can be equivalently reduced to deterministic mixed integer programming problems. The number of auxiliary Boolean variables in the latter problems equals to the number of scenarios for random data in the original problem. A similar transformation is applicable to two-stage problems with a chance constraint on the random optimal value of the second stage problem. The obtained mixed integer problems are supposed to be solved by standard discrete optimization software. The results are illustrated by a test numerical example. Thus, a practical method for solution of two-stage quantile stochastic optimization problems has been validated. Further work will be directed to development of special methods accounting for the structure of appearing mixed integer problems.

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